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On kernels of Toeplitz operators

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Abstract
We apply the theory of de Branges–Rovnyak spaces to describe kernels of some Toeplitz operators on the classical Hardy space H^2. In particular, we discuss the kernels of the operators $T_{\bar{f}/f}$ and $T_{\bar{I}\bar{f}/f}$, where f is an outer function in H^2 and I is inner such that $I(0) = 0$. We also obtain a result on the structure of de Branges–Rovnyak spaces generated by nonextreme functions.

Keywords Toeplitz operators · de Branges–Rovnyak spaces · Nearly invariant subspaces · Rigid functions · Nonextreme functions · Kernel functions

Mathematics Subject Classification 47B32 · 46E22 · 30H10

1 Introduction

Let H^2 denote the standard Hardy space on the unit disk $\mathbb{D}$. For $\varphi \in L^\infty(\partial\mathbb{D})$ the Toeplitz operator on H^2 is given by $T_\varphi f = P_+(\varphi f)$, where P_+ is the orthogonal projection of $L^2(\partial\mathbb{D})$ onto H^2. We will denote by $\mathcal{M}(\varphi)$ the range of T_φ equipped with the range norm, that is, the norm that makes the operator T_φ a coisometry of H^2 onto $\mathcal{M}(\varphi)$. For a nonconstant function b in the unit ball of H^∞ the de Branges–

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Rovnyak space $\mathcal{H}(b)$ is the image of H^2 under the operator $(1 - T_b T_{\bar{b}})^{1/2}$ with the corresponding range norm. The norm and the inner product in $\mathcal{H}(b)$ will be denoted by $\|\cdot\|_b$ and $\langle\cdot,\cdot\rangle_b$. The space $\mathcal{H}(b)$ is a Hilbert space with the reproducing kernel

$$k_w^b(z) = \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z} \qquad (z, w \in \mathbb{D}).$$

In the case when b is an inner function the space $\mathcal{H}(b)$ is the well-known model space $K_b = H^2 \ominus bH^2$.

If the function b fails to be an extreme point of the unit ball in H^∞, that is, when $\log(1 - |b|) \in L^1(\partial\mathbb{D})$, we will say simply that b is nonextreme. In this case one can define an outer function a whose modulus on $\partial\mathbb{D}$ equals $\left(1 - |b|^2\right)^{1/2}$. Then we say that the functions b and a form a *pair* (b, a). By the Herglotz representation theorem there exists a positive measure μ on $\partial\mathbb{D}$ such that

$$\frac{1 + b(z)}{1 - b(z)} = \int_{\partial\mathbb{D}} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z}\, d\mu(e^{i\theta}) + i \operatorname{Im} \frac{1 + b(0)}{1 - b(0)}, \quad z \in \mathbb{D}. \tag{1}$$

Moreover the function $\left|\frac{a}{1-b}\right|^2$ is the Radon-Nikodym derivative of the absolutely continuous component of μ with respect to the normalized Lebesgue measure. If the measure μ is absolutely continuous the pair (b, a) is called *special*.

Recall that a function $f \in H^1$ is called *rigid* if and only if no other functions in H^1, except for positive scalar multiples of f have the same argument as f a.e. on $\partial\mathbb{D}$.

If (b, a) is a pair, then $\mathcal{M}(a)$ is contained contractively in $\mathcal{H}(b)$. If a pair (b, a) is special and $f = \frac{a}{1-b}$, then $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$ if and only if f^2 is rigid ([20]). Spaces $\mathcal{H}(b)$ for nonextreme b have been studied in [2,3,5,15,16,22], and [23].

The kernels of Toeplitz operators have been studied since the late 80's. We mention that two recent survey articles [1,8] and the book [4] contain a number of results on this topic.

The Hayashi theorem [12] (see also [21]) states that the kernel of a Toeplitz operator T_φ is a subspace of H^2 of the form $\ker T_\varphi = f K_I$, where $K_I = H^2 \ominus I H^2$ is the model space corresponding to the inner function I such that $I(0) = 0$ and f is an outer function of unit H^2 norm that acts as an isometric multiplier from K_I onto $f K_I$. Moreover, f can be expressed as $f = \frac{a}{1-Ib_0}$, where (b_0, a) is a special pair and $\left(\frac{a}{1-b_0}\right)^2$ is a rigid function in H^1. Then we also have $\ker T_{\frac{\bar{I}\bar{f}}{f}} = f K_I$. In the recent paper [6] the authors considered the Toeplitz operator $T_{\frac{\bar{g}}{g}}$ where $g \in H^\infty$ is outer. Among other results, they described all outer functions g such that $\ker T_{\frac{\bar{g}}{g}} = K_I$. In Sect. 2 we describe all such functions g for which $\ker T_{\frac{\bar{g}}{g}} = f K_I$.

If (b, a) is a special pair, $f = \frac{a}{1-b}$ and $b = Ib_0$, where I as above, then $f K_I \subset \ker T_{\frac{\bar{I}\bar{f}}{f}}$. In the next two sections we study the space $\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus f K_I$ and show that it is isometrically isomorphic to the orthogonal complement of $\mathcal{M}(a)$ in the de Branges–Rovnyak space $\mathcal{H}(b_0)$. We also give an example of a function f for which the space

$\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus f K_I$ is one dimensional. In the last section we discuss the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ and get a generalization of results obtained in [15] and [5] for the case when pairs are rational.

2 The kernel of $T_{\frac{\bar{g}}{g}}$

It is known that if g is an outer function in H^2, then the kernel of $T_{\frac{\bar{g}}{g}}$ is trivial if and only if g^2 is rigid (see e.g. [18]).

The finite dimensional kernels of Toeplitz operators were described by Nakazi [17]. Nakazi's theorem says that $\dim \ker T_\varphi = n$ if and only if there exists an outer function $f \in H^2$ such that f^2 is rigid and $\ker T_\varphi = \{fp\colon\ p \in \mathcal{P}_{n-1}\}$, where $\mathcal{P}_{n-1}$ denotes the set of all polynomials of degree at most $n-1$.

Consider the following example.

Example For $\alpha > -\frac{1}{2}$ set $g(z) = (1-z)^\alpha$, $z \in \mathbb{D}$. Then the kernel of $T_{\frac{\bar{g}}{g}}$ is trivial for $\alpha \in (-\frac{1}{2}, \frac{1}{2}]$ and dimension of the kernel of $T_{\frac{\bar{g}}{g}}$ is n for $\alpha \in (n-\frac{1}{2}, n+\frac{1}{2}]$, $n = 1, 2, \ldots$, and

$$\ker T_{\frac{\overline{(1-z)^\alpha}}{(1-z)^\alpha}} = (1-z)^{\alpha-n} K_{z^n}.$$

In the general case the kernels of Toeplitz operators are characterized by Hayashi's theorem. To state this theorem we need some notation. We note that an outer function f having unit norm in H^2 ($\|f\|_2 = 1$) can be written as

$$f = \frac{a}{1-b},$$

where a is an outer function, b is a function from the unit ball of H^∞ such that $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$. Following Sarason [20, p. 156] we call (b, a) the *pair associated with* f. Note also that b is a nonextreme point of the closed unit ball of H^∞ and is given by

$$\frac{1+b(z)}{1-b(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}|f(e^{i\theta})|^2 d\theta, \quad z \in \mathbb{D}. \tag{2}$$

Let S denote the unilateral shift operator on H^2, i.e., $S = T_z$. A closed subspace M of H^2 is said to be *nearly S^*-invariant* if for every $f \in M$ vanishing at 0, we also have $S^* f \in M$. It is known that the kernels of Toeplitz operators are nearly S^*-invariant.

Nearly S^*-invariant spaces are characterized by Hitt's theorem [14].

Hitt's Theorem *The closed subspace M of H^2 is nearly S^*-invariant if and only if there exists a function f of unit norm and a model space $K_I = H^2 \ominus IH^2$ such that $M = T_f K_I$, where I is an inner function vanishing at the origin, and T_f acts isometrically on K_I.*

It has been proved by D. Sarason [18] that T_f acts isometrically on K_I if and only if I divides b (the first function in the pair associated with f). Consequently, the function f in Hitt's theorem can be written as

$$f = \frac{a}{1 - Ib_0}.$$

The function $\frac{1+b_0(z)}{1-b_0(z)}$ has a positive real part and is the Herglotz integral of a positive measure on $\partial\mathbb{D}$ up to an additive imaginary constant,

$$\frac{1 + b_0(z)}{1 - b_0(z)} = \int_{\partial\mathbb{D}} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z}\, d\mu(e^{i\theta}) + ic. \tag{3}$$

Clearly b_0 is also a nonextreme point of the closed unit ball of H^∞ and $|a|^2 + |b_0|^2 = 1$ a.e. on $\partial\mathbb{D}$. We remark that in view of (2) the pair (b, a) associated with an outer function $f \in H^2$ is special, while the pair (b_0, a) need not to be special. We also put

$$f_0 = \frac{a}{1 - b_0}$$

and note that $f_0 \in H^2$ (see, e.g. [4, Theorem 23.1]).

Under the above notations Hayashi's theorem reads as follows:

Hayashi's Theorem *The nearly S^*-invariant space $M = T_f K_I$ is the kernel of a Toeplitz operator if and only if the pair (b_0, a) is special and f_0^2 is a rigid function.*

Moreover, it follows from Sarason's proof of Hayashi's theorem that if $M = T_f K_I$ is the kernel of a Toeplitz operator then it is the kernel of $T_{\frac{\bar{I}\bar{f}}{f}}$.

Recently E. Fricain, A. Hartmann and W. T. Ross [6] considered the Toeplitz operators $T_{\frac{\bar{g}}{g}}$ where $g \in H^\infty$ is outer. If $\ker T_{\frac{\bar{g}}{g}}$ is non-trivial, then by Hayashi's theorem there exist the outer function f and the inner function I, $I(0) = 0$, such that

$$\ker T_{\frac{\bar{g}}{g}} = f K_I.$$

In the above mentioned paper [6] the authors described all outer functions $g \in H^\infty$ for which

$$\ker T_{\frac{\bar{g}}{g}} = K_I,$$

where I is an inner function not necessarily satisfying $I(0) = 0$.

We prove the following

Theorem 1 *Assume that $g \in H^2$ is outer and $M = T_f K_I$ is the nearly S^*-invariant space, where I is an inner function such that $I(0) = 0$, $(b_0 I, a)$ is the pair associated* *with the outer function* f, (b_0, a) *is special, and* f_0^2 *is rigid. Then* $\ker T_{\frac{\bar{g}}{g}} = M$ *if and only if*

$$g = i\frac{I_1 + I_2}{I_1 - I_2}(1 + I)f,$$

where I_1 *and* I_2 *are inner and* $I_1 - I_2$ *is outer.*

Recall that the *Smirnov class* $\mathcal{N}^+$ consists of those holomorphic functions in $\mathbb{D}$ that are quotients of functions in H^∞ in which the denominators are outer functions.

In the proof of Theorem 1, similarly to [6], we use the following result due to H. Helson [13].

Helson's Theorem *The functions* $f \in \mathcal{N}^+$ *that are real almost everywhere on* $\partial\mathbb{D}$ *can be written as*

$$f = i\,\frac{I_1 + I_2}{I_1 - I_2},$$

where I_1 *and* I_2 *are inner and* $I_1 - I_2$ *is outer.*

We also apply a description of kernels in terms of S^*-invariant subspaces $K_I^p(|f|^p)$ of weighted Hardy spaces (in the case when $p = 2$) considered by A. Hartmann and K. Seip in their paper [10] (see also [7]). For an outer function f in H^2 the weighted Hardy space is defined as follows

$$H^2(|f|^2) = \{g \in \mathcal{N}^+ \colon \ \|g\|_{2,f}^2 = \frac{1}{2\pi}\int_0^{2\pi} |g(e^{it})|^2 |f(e^{it})|^2 dt < \infty\}$$

and, for an inner function I, $K_I^2(|f|^2) = K_I(|f|^2)$ is given by

$$K_I(|f|^2) = \{g = I\overline{\psi} \in H^2(|f|^2) \colon \ \psi \in H_0^2(|f|^2)\},$$

where $H_0^2(|f|^2) = zH^2(|f|^2)$.

Then $K_I(|f|^2)$ is S^*-invariant and $fK_I(|f|^2) = \ker T_{\frac{\bar{I}\bar{f}}{f}}$ (see [10]).

Proof of Theorem 1. Assume that $\ker T_{\frac{\bar{g}}{g}} = fK_I$. Then

$$fK_I = \ker T_{\frac{\bar{I}\bar{f}}{f}} = \ker T_{\frac{\bar{g}}{g}}.$$

Since $f \in \ker T_{\frac{\bar{I}\bar{f}}{f}}$, the last equalities imply that

$$\frac{\overline{g}f}{g} = \overline{I}_0\bar{h},$$

where I_0 is an inner function such that $I_0(0) = 0$, and $h \in H^2$ is outer. This means that $|f(z)| = |h(z)|$ a.e. on $|z| = 1$ and consequently $h(z) = cf(z)$, where c is a unimodular constant. Replacing cI_0 by I_0, we get

$$\frac{\bar{g}}{g} = \bar{I}_0 \frac{\overline{f}}{f}. \tag{4}$$

It then follows

$$fK_I = \ker T_{\frac{\bar{g}}{g}} = \ker T_{\frac{\bar{I}_0 \bar{f}}{f}} = fK_{I_0}(|f|^2),$$

which implies $I = I_0$ up to a unimodular constant. Indeed, these equalities imply that an analytic function h can be written in the form $h = fI_0\overline{\psi}_0$, where $\psi_0 \in H_0^2(|f|^2)$, if and only if $h = fI\overline{\psi}$, where $\psi \in H_0^2$. Since $|\psi_0| = |\psi|$ a.e. on $|z| = 1$ and $\psi_0 \in \mathcal{N}^+$, we see that also $\psi_0 \in H_0^2$. Hence $K_I = K_{I_0}$.

Consequently, equality (4) can be written as

$$\frac{\bar{g}}{g} = \frac{\overline{f(1+I)}}{f(1+I)} \quad \text{a.e. on } \partial\mathbb{D},$$

which means that the function $\frac{g}{f(1+I)}$ is real a.e. on $\partial\mathbb{D}$. Since this function is in the Smirnov class $\mathcal{N}^+$, our claim follows from Helson's theorem. To prove the other implication it is enough to observe that if

$$g = i\frac{I_1 + I_2}{I_1 - I_2}(1 + I)f,$$

then

$$\frac{\bar{g}}{g} = \frac{\bar{I}\bar{f}}{f}.$$

□

3 The complement of fK_I in $\ker T_{\frac{\bar{I}\bar{f}}{f}}$

It was noticed in [4, Corollary 30.21] that if f is an outer function of the unit norm, (b, a) is the pair associated with f, and I is an inner function vanishing at the origin that divides b, then

$$fK_I \subset \ker T_{\frac{\bar{I}\bar{f}}{f}}$$

and, according to Hayashi's theorem, the equality holds if and only if the pair (b_0, a) is special and f_0^2 is rigid.

Recall that $\mathcal{M}(a)$ is dense in $\mathcal{H}(b_0)$ if and only if the pair (b_0, a) is special and f_0^2 is a rigid function.

Theorem 2 *Assume that (Ib_0, a), where I is inner, and $I(0) = 0$, is the pair associated with an outer function f. If the pair (b_0, a) is not special or the function f_0^2 is not rigid, then for a positive integer k,*

$$\dim\big(\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus fK_I\big) = k$$

if and only if the codimension of $\overline{\mathcal{M}(a)}$ in the de Branges–Rovnyak space $\mathcal{H}(b_0)$ is k.

In the proof of this theorem we use some ideas from Sarason's proof of Hayashi's theorem. If a positive measure μ on the unit circle $\partial\mathbb{D}$ is as in (1) and $H^2(\mu)$ is the closure of the polynomials in $L^2(\mu)$, then an operator V_b given by

$$(V_b q)(z) = (1 - b(z)) \int_{\partial\mathbb{D}} \frac{q(e^{i\theta})}{1 - e^{-i\theta}z}\, d\mu(e^{i\theta}) \tag{5}$$

is an isometry of $H^2(\mu)$ onto $\mathcal{H}(b)$. Furthermore, if (b, a) is a pair and $f = \frac{a}{1-b}$, then the operator $T_{1-b}T_{\bar{f}}$ is an isometry of H^2 into $\mathcal{H}(b)$. Its range is all of $\mathcal{H}(b)$ if and only if the pair (b, a) is special ([21, III-6,7] and [4, Theorem 24.26]). We note that in the last case $d\mu(e^{i\theta}) = \frac{1}{2\pi}|f(e^{i\theta})|^2 d\theta$.

Proof of Theorem 2. Since the pair (b, a) is special, the operator $T_{1-b}T_{\bar{f}}$ is an isometry of H^2 onto $\mathcal{H}(b)$. Moreover, since I divides b, T_f acts as an isometry on K_I and $T_{1-b}T_{\bar{f}}(fK_I) = K_I$ ([20]). Hence

$$\begin{aligned}\mathcal{H}(b) = T_{1-b}T_{\bar{f}}(H^2) &= T_{1-b}T_{\bar{f}}\Big(\overline{T_{\frac{If}{\bar{f}}}(H^2)} \oplus \big(T_{\frac{If}{\bar{f}}}(H^2)\big)^{\perp}\Big)\\ &= \overline{I\mathcal{M}(a)}^{b} \oplus T_{1-b}T_{\bar{f}}(\ker T_{\frac{\bar{I}\bar{f}}{f}})\\ &= \overline{I\mathcal{M}(a)}^{b} \oplus T_{1-b}T_{\bar{f}}(fK_I) \oplus T_{1-b}T_{\bar{f}}(\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus fK_I)\\ &= \overline{I\mathcal{M}(a)}^{b} \oplus K_I \oplus T_{1-b}T_{\bar{f}}(\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus fK_I),\end{aligned}$$

where $\overline{T_{\frac{If}{\bar{f}}}(H^2)}$ denotes the closure of $T_{\frac{If}{\bar{f}}}(H^2)$ in H^2 and $\overline{I\mathcal{M}(a)}^{b}$ denotes the closure of $I\mathcal{M}(a)$ in $\mathcal{H}(b)$. On the other hand,

$$\mathcal{H}(b) = \mathcal{H}(b_0 I) = K_I \oplus I\mathcal{H}(b_0) = K_I \oplus I(\mathcal{H}(b_0) \ominus \overline{\mathcal{M}(a)}^{b_0}) \oplus I\overline{\mathcal{M}(a)}^{b_0}.$$

Since $T_I\colon \mathcal{H}(b_0) \to \mathcal{H}(Ib_0)$ is an isometry ([20, Proposition 4]), $I\overline{(\mathcal{M}(a))}^{b_0} = \overline{I\mathcal{M}(a)}^{b}$. It then follows,

$$T_{1-b}T_{\bar{f}}(\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus fK_I) = I(\mathcal{H}(b_0) \ominus \overline{\mathcal{M}(a)}^{b_0}). \tag{6}$$

□

We remark that the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is discussed in Sect. 5.

4 The example

Let, as in the previous sections, f be an outer function in H^2 and let (b, a) be the pair associated with f. Let $b = Ib_0$, where I is an inner function such that $I(0) = 0$ and $f_0 = \frac{a}{1-b_0}$. Then $fK_I \subset \ker T_{\frac{\bar{I}\bar{f}}{f}}$ and equality holds if and only if the pair (b_0, a) is special and f_0^2 is rigid. Moreover, if the pair (b_0, a) is special and f_0^2 is rigid, then (b, a) is special and f^2 is rigid but the converse implication fails ([18, p. 158]).

In [4, vol. 2, pp. 541–542] the authors constructed a function h in $\ker T_{\frac{\bar{I}\bar{f}}{f}}$ which is not in fK_I under the assumption that f^2 is not rigid. Here we consider the function f such that f^2 is rigid, the pair (b_0, a) is special but f_0^2 is not rigid, and describe the space $\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus fK_I$.

Our example is a slight modification of the one given in [19, p. 491], see also [4, vol. 2, p. 494]. The corresponding functions f and f_0 are defined by taking $a(z) = \frac{1}{2}(1+z)$, $b_0(z) = \frac{1}{2}z(1-z)$, and $I(z) = zB(z)$, where $B(z)$ is a Blaschke product with zero sequence $\{r_n\}_{n=1}^{\infty}$ lying in $(-1, 0)$ and converging to -1. It has been proved in [19, pp. 491–492] (see also [4, vol. 2, pp. 494–496] that f^2 is rigid while f_0^2 is not. Notice that the pair (b_0, a) is rational and the point -1 is the only zero of the function a. It then follows from [15, Theorem 4.1] (see also [5]) that $\mathcal{M}(a)$ is a closed subspace of $\mathcal{H}(b_0)$ and

$$\mathcal{H}(b_0) = \mathcal{M}(a) \oplus \mathbb{C}k_{-1}^{b_0},$$

where

$$k_{-1}^{b_0}(z) = \frac{1 - \overline{b_0(-1)}b_0(z)}{1+z} = \frac{2-z}{2}.$$

Thus we see that

$$\mathcal{H}(b_0) \ominus \mathcal{M}(a) = \mathbb{C}k_{-1}^{b_0}.$$

Moreover (6) implies that

$$T_{1-b}T_{\bar{f}}(\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus fK_I) = \mathbb{C}Ik_{-1}^{b_0}.$$

Our aim is to prove that

$$\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus fK_I = \mathbb{C}g, \tag{7}$$

where the function $g \in H^2$ is given by $g = fk_{-1}(I+1)$, with $k_{-1}(z) = (1+z)^{-1}$, $z \in \mathbb{D}$.

For λ in $\mathbb{D}$ let k_λ denote the kernel function in H^2 for the functional of evaluation at λ, $k_\lambda(z) = (1 - \bar{\lambda}z)^{-1}$. In the proof of (7) we will apply the following

Lemma [9, Lemma 2]

(i) $P_+\left(|f|^2 I k_\lambda\right) = \dfrac{I k_\lambda}{1-b} + \dfrac{\overline{b_0(\lambda)}k_\lambda}{1-\overline{b(\lambda)}}.$

(ii) $P_+\left(|f|^2 k_\lambda\right) = \dfrac{k_\lambda}{1-b} + \dfrac{\overline{b(\lambda)}k_\lambda}{1-\overline{b(\lambda)}}.$

Since $I(r_n) = 0$, (i) and (ii) in the Lemma yield

$$
\begin{aligned}
T_{1-b}T_{\bar{f}}(f I k_{r_n}) &= I k_{r_n}(1 - \overline{b_0(r_n)}b_0) + \overline{b_0(r_n)}k_{r_n},\\
T_{1-b}T_{\bar{f}}(f k_{r_n}) &= k_{r_n}.
\end{aligned}
$$

Hence

$$
T_{1-b}T_{\bar{f}}(f k_{r_n}(I - \overline{b_0(r_n)})) = I k_{r_n}(1 - \overline{b_0(r_n)}b_0) = I k_{r_n}^{b_0}. \tag{8}
$$

It follows from [4, Theorem 21.1] that

$$
\|k_{r_n}^{b_0} - k_{-1}^{b_0}\|_{b_0} \underset{n\to\infty}{\longrightarrow} 0.
$$

Next, since $T_I\colon \mathcal{H}(b_0) \to \mathcal{H}(Ib_0) = \mathcal{H}(b)$ is an isometry and $T_{1-b}T_{\bar{f}}$ is an isometry of H^2 onto $\mathcal{H}(b)$, we see that $\{f k_{r_n}(I - \overline{b_0(r_n)})\}_{n\in\mathbb{N}}$ is a bounded sequence in H^2. So it contains a subsequence that converges weakly, say, to a function $g \in H^2$. Without loss of generality, we may assume that the sequence $\{f k_{r_n}(I - \overline{b_0(r_n)})\}$ itself converges weakly to g. Then for any point $z \in \mathbb{D}$,

$$
\begin{aligned}
g(z) = \langle g, k_z\rangle &= \lim_{n\to\infty}\langle f k_{r_n}(I - \overline{b_0(r_n)}), k_z\rangle\\
&= \lim_{n\to\infty}\frac{f(z)(I(z) - \overline{b_0(r_n)})}{1 - r_n z} = \frac{f(z)(I(z)+1)}{1+z}.
\end{aligned}
$$

Now observe that since

$$
\|f k_{r_n}(I - \overline{b_0(r_n)})\|_2 = \|k_{r_n}^{b_0}\|_{b_0} \quad \text{and} \quad \|f k_{-1}(I+1)\|_2 = \|k_{-1}^{b_0}\|_{b_0},
$$

$f k_{r_n}(I - \overline{b_0(r_n)}) \to f k_{-1}(I+1)$ in H^2 strongly. Finally, passing to the limit in (8) gives

$$
T_{1-b}T_{\bar{f}}(f k_{-1}(I+1)) = I k_{-1}(1 + b_0) = I k_{-1}^{b_0},
$$

which proves (7).

Remark One can check directly that the function $g = f k_{-1}(I+1)$ is in $\ker T_{\frac{\bar{I}\bar{f}}{f}} \ominus f K_I$.

Indeed, we have

$$T_{\frac{\bar{I}\bar{f}}{f}}(fk_{-1}(I+1)) = P_+\left(\bar{f}\bar{I}\frac{I+1}{1+z}\right)$$
$$= P_+\left(\bar{f}\frac{\bar{z}(\bar{I}+1)}{\bar{z}+1}\right) = P_+\left(\bar{z}\overline{fk_{-1}}(\bar{I}+1)\right) = 0.$$

To see that the functions $(fk_{r_n}I - b_0(r_n)fk_{r_n})$ are orthogonal to fK_I note that a function $h \in H^2$ is in K_I if and only if $h = h - IP_+(\bar{I}h)$. So, we have to check that for any $h \in H^2$,

$$\langle fk_{r_n}I - b_0(r_n)fk_{r_n}, f(h - IP_+(\bar{I}h))\rangle = 0.$$

Since the functions $\{k_\lambda, \lambda \in \mathbb{D}\}$ are dense in H^2, it is enough to show that for any $\lambda \in \mathbb{D}$,

$$\langle fk_{r_n}I - b_0(r_n)fk_{r_n}, f(k_\lambda - IP_+(\bar{I}k_\lambda))\rangle$$
$$= \langle fk_{r_n}I - b_0(r_n)fk_{r_n}, f(k_\lambda - \overline{I(\lambda)}Ik_\lambda)\rangle = 0.$$

Finally, the last equality follows from

$$\langle fk_{r_n}I, fk_\lambda\rangle = \frac{I(\lambda)k_{r_n}(\lambda)}{1-b(\lambda)} + b_0(r_n)k_{r_n}(\lambda),$$
$$\langle fk_{r_n}I, -\overline{I(\lambda)}fIk_\lambda\rangle = -\frac{I(\lambda)k_{r_n}(\lambda)}{1-b(\lambda)},$$
$$\langle -b_0(r_n)fk_{r_n}, fk_\lambda\rangle = -\frac{b_0(r_n)k_{r_n}(\lambda)}{1-b(\lambda)},$$

and

$$\langle -b_0(r_n)fk_{r_n}, -\overline{I(\lambda)}fIk_\lambda\rangle = b_0(r_n)I(\lambda)\frac{b_0(\lambda)k_{r_n}(\lambda)}{1-b(\lambda)} = \frac{b_0(r_n)b(\lambda)k_{r_n}(\lambda)}{1-b(\lambda)}.$$

5 A remark on orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$

In this section we continue to assume that b is nonextreme. Recall that if the pair (b, a) is special and $f^2 = \left(\frac{a}{1-b}\right)^2$ is not rigid or the pair (b, a) is not special, then the space $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$ ([20]). In such a case let $\mathcal{H}_0(b)$ denote the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$. Let Y be the restriction of the shift operator S to $\mathcal{H}(b)$ and let Y_0 be the compression of Y to the subspace $\mathcal{H}_0(b)$. Then the spectrum of Y_0 is contained in the unit circle. Moreover, if $z_0 \in \partial\mathbb{D}$ and k is a positive integer, then $\ker(Y^* - \bar{z_0})^k$, which actually equals $\ker(Y_0^* - \bar{z_0})^k$, lies in $\mathcal{H}_0(b)$. The necessary and sufficient conditions for $\mathcal{H}_0(b)$ to have finite dimension are given in Chapter X of [21] (see also [4, Theorem 29.11]). In particular, the dimension of $\mathcal{H}_0(b)$ is N if and only if the operator Y_0 has distinct eigenvalues $z_1, z_2, \ldots, z_s$ with their algebraic multiplicities $n_1, \ldots, n_s$, $N = n_1 + n_2 + \cdots + n_s$. Then $\bar{z_1}, \bar{z_2}, \ldots, \bar{z_s}$ are the eigenvalues of Y_0^* with the same multiplicities, i.e., $\dim\ker(Y_0 - z_j)^{n_j} = \dim\ker(Y_0^* - \bar{z_j})^{n_j}$ and $\mathcal{H}_0(b)$ is the direct sum of the subspaces $\ker(Y_0^* - \bar{z_j})^{n_j}$, $j = 1, 2, \ldots, s$.

On the other hand, if z_0 is a point of $\partial\mathbb{D}$ and b has an angular derivative in the sense of Carathéodory at z_0, then the function given by

$$k_{z_0}^b(z) = \frac{1 - \overline{b(z_0)}b(z)}{1 - \bar{z_0}z}, \tag{9}$$

where $b(z_0)$ is the nontangential limit of b at z_0, is in $\mathcal{H}(b)$ (see [21, VI-4,5], [4, Theorem 21.1]). In this section we actually show that $k_{z_0}^b$ is in $\mathcal{H}_0(b)$.

Here we consider the case when the eigenspaces corresponding to eigenvalues z_1, $z_2, \ldots, z_s$ are one dimensional and show that then the space $\mathcal{H}_0(b)$ is spanned by the functions $k_{z_1}^b, k_{z_2}^b, \ldots, k_{z_s}^b$.

For $|\lambda| = 1$ let μ_λ denote the measure for which equality in (1) holds when b is replaced by $\bar{\lambda}b$. If we put $F_\lambda(z) = \frac{a}{1-\bar{\lambda}b}$, then the Radon-Nikodym derivative of the absolutely continuous component of μ_λ is $|F_\lambda|^2$. Note also that $\mathcal{H}(b) = \mathcal{H}(\bar{\lambda}b)$.

In the proof of our main result in this section we use the following theorem proved in [21, X-13].

Sarason's Theorem *Let z_0 be a point of $\partial\mathbb{D}$ and λ a point of $\partial\mathbb{D}$ such that the measure μ_λ is absolutely continuous. The following conditions are equivalent.*

(i) *$\bar{z_0}$ is an eigenvalue of Y^*.*

(ii) *The function $\dfrac{F_\lambda(z)}{1 - \bar{z_0}z}$ is in H^2.*

(iii) *The function b has an angular derivative in the sense of Carathéodory at z_0.*

In view of remark in Sect. 3 under the assumption of Sarason's Theorem the operator $T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}$ is an isometry of H^2 onto $\mathcal{H}(b)$. Let A_λ be an operator on H^2 that intertwines $T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}$ with the operator Y^*, i.e.,

$$T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}A_\lambda = Y^* T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}. \tag{10}$$

The operator A_λ is given by

$$A_\lambda = S^* - F_\lambda(0)^{-1}(S^*F_\lambda \otimes 1).$$

It follows from the proof of Sarason's theorem that if one of conditions (i)–(iii) holds true, then the space $\ker(A_\lambda - \bar{z_0})$ is one dimensional and is spanned by the function

$$g(z) = \frac{F_\lambda(z)}{1 - \bar{z_0}z} = F_\lambda(z)k_{z_0}(z). \tag{11}$$

We also note that condition (iii) in Sarason's Theorem is equivalent to the fact that the function $k_{z_0}^b$ given by (9) is in $\mathcal{H}(b)$.

Theorem 3 *If the assumptions of Sarason's theorem are satisfied and $\bar{z}_0$ is an eigenvalue of Y_0^*, then* $\ker(Y_0^* - \bar{z}_0)$ *is spanned by $k_{z_0}^b$.*

Proof According to the remark at the beginning of this section $\ker(Y_0^* - \bar{z}_0)$ is equal to $\ker(Y^* - \bar{z}_0)$. Since the operator $T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}$ is an isometry of H^2 onto $\mathcal{H}(b)$, (11) and (10) imply that the space $\ker(Y^* - \bar{z}_0)$ is spanned by

$$h = T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}g = T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}(F_\lambda k_{z_0}).$$

We will show that $h = Ck_{z_0}^b$. To this end we use the operator V_b given by (5). We know that for $w \in \mathbb{D}$,

$$V_b((1 - \overline{b(w)})k_w) = k_w^b$$

(see [21, III-7], [4, Theorem 20.5]). Since $\mathcal{H}(b) = \mathcal{H}(\bar{\lambda}b)$, we have

$$\begin{aligned} V_{\bar{\lambda}b}((1 - \lambda\overline{b(w)})k_w)(z) &= (1 - \bar{\lambda}b(z))(1 - \lambda\overline{b(w)}) \int_{\partial\mathbb{D}} \frac{|F_\lambda(e^{i\theta})|^2 d\theta}{(1 - \bar{w}e^{i\theta})(1 - ze^{-i\theta})} \\ &= (1 - \bar{\lambda}b(z))T_{\overline{F_\lambda}}((1 - \lambda\overline{b(w)})F_\lambda k_w)(z) = k_w^b(z). \end{aligned}$$

Let $\{z_n\}$ be a sequence in $\mathbb{D}$ converging nontangentially to z_0. Then

$$T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}((1 - \lambda\overline{b(z_n)})F_\lambda k_{z_n}) = k_{z_n}^b.$$

Observe also that since μ_λ is absolutely continuous, $b(z_0) \neq \lambda$ ([21, VI-7, VI-9]). Moreover, k_z^b tends to $k_{z_0}^b$ in norm as z tends to z_0 nontangentially (see [21, VI-4,5], [4, Theorem 21.1]). It then follows that the sequence $\{(1 - \lambda\overline{b(z_n)})F_\lambda k_{z_n}\}$ converges in H^2, which in turn implies compact and pointwise convergence. Hence passing to the limit in the last equality yields

$$T_{1-\bar{\lambda}b}T_{\overline{F_\lambda}}(F_\lambda k_{z_0}) = Ck_{z_0}^b,$$

where $C = (1 - \lambda\overline{b(z_0)})^{-1}$. □

Our last theorem is an immediate consequence of Theorem 3.

Theorem 4 *If $z_1, z_2, \ldots, z_s$ are the only eigenvalues of Y_0 and each of them is of multiplicity one, then $\mathcal{H}_0(b)$ is spanned by the functions $k_{z_1}^b, k_{z_2}^b, \ldots, k_{z_s}^b$.*

Finally, we remark that this theorem generalizes results obtained in [5] and in [15] for the case when pairs (b, a) are rational.

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