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Classes of meromorphic harmonic functions and duality principle

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Abstract
We introduce new classes of meromorphic harmonic univalent functions. Using the duality principle, we obtain the duals of such classes of functions leading to coefficient bounds, extreme points and some applications for these functions.

Keywords Subordination · Duality · Extreme points · Meromorphic harmonic functions · Varying coefficients

Mathematics Subject Classification Primary 30C45; Secondary 30C80

1 Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain D, we write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally one-to-one and orientation preserving in D is that $|g'(z)| < |h'(z)|$ for $z \in D$ (see Clunie and Sheil-Small [4]). Functions that are harmonic and univalent in $\mathbb{D} = \{z : |z| > 1\}$ are investigated by Hengartner and Schober [7]. In particular, it was shown in [7] that a complex-valued, harmonic, orientation preserving univalent mapping f, defined on $\mathbb{D}$ and satisfying $f(\infty) = \infty$, must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \tag{1}$$

where $h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}$, $g(z) = \beta z + \sum_{n=0}^{\infty} b_n z^{-n}$, $0 \le |\beta| < |\alpha|$, and $\omega = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfies $|\omega(z)| < 1$ for $z \in \mathbb{D}$. We remove the logarithmic singularity in (1) by letting $A = 0$ and also let $\alpha = 1$ and $\beta = 0$ and focus on the family $\Sigma_{\mathcal{H}}$ of meromorphic harmonic orientation preserving univalent mappings of the form

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$$f(z)=h(z)+\overline{g(z)}$$

where

$$h(z)=z+\sum\nolimits_{n=1}^{\infty}a_nz^{-n}\ ,\quad g(z)=\sum\nolimits_{n=1}^{\infty}b_nz^{-n}\ . \tag{2}$$

We say that a function $f\in\Sigma_{\mathcal{H}}$ is harmonic starlike in $\mathbb{D}$ if

$$\frac{\partial}{\partial t}\left(\arg f\left(re^{it}\right)\right)\geq 0$$

or

$$\operatorname{Re}\frac{D_{\mathcal{H}}f\ (z)}{f\ (z)}\geq 0$$

where $z=re^{it}\in\mathbb{D}$, $0\leq t\leq 2\pi$, and

$$D_{\mathcal{H}}f(z)=zh'(z)-\overline{zg'(z)}=z-\sum_{n=1}^{\infty}n\left(a_nz^{-n}-\overline{b_nz^{-n}}\right).$$

For $l=1,2$ and functions $f_l\in\Sigma_{\mathcal{H}}$ of the form

$$f_l(z)=z+\sum_{n=1}^{\infty}\left(a_{l,n}z^{-n}+\overline{b_{l,n}z^{-n}}\right) \tag{3}$$

we define*the convolution* of f_1 and f_2 by

$$(f_1*f_2)\,(z)=f_1(z)*f_2(z)=z+\sum_{n=1}^{\infty}\left(a_{1,n}a_{2,n}z^{-n}+\overline{b_{1,n}b_{2,n}z^{-n}}\right).$$

For $m\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}=\{1,2,...\}$ and $f=h+\overline{g}\in\Sigma_{\mathcal{H}}$ we define the linear operator $D_{\mathcal{H}}^m:\Sigma_{\mathcal{H}}\to\Sigma_{\mathcal{H}}$ by

$$\begin{aligned}D_{\mathcal{H}}^m:=D_{\mathcal{H}}^mf(z)&=\mathcal{D}^mh(z)+(-1)^m\,\overline{\mathcal{D}^mg(z)}\\&=z+\sum_{n=1}^{\infty}m_n\left\{a_nz^{-n}+(-1)^m\,\overline{b_nz^{-n}}\right\}\end{aligned}$$

where

$$\mathcal{D}^mh(z)=h(z)*\left(z+\frac{(-1)^m}{z(1-\frac{1}{z})^{m+1}}\right)=h(z)*\left(z+\frac{(-z)^m}{(z-1)^{m+1}}\right),$$

$$m_1=1,m_n=\frac{(m+1)\cdot\ldots\cdot(m+n-1)}{(n-1)!};\ (n=2,3,...)\,.$$

We note that $D_{\mathcal{H}}^0f=f$ and $D_{\mathcal{H}}^1f=D_{\mathcal{H}}f$.

In 2008 Mauir [13] considered a weak subordination for complex-valued harmonic functions defined in the open unit disk $\Delta := \{z : |z| < 1\}$. In [8] Jahangiri (see also [10]) investigated the classes of harmonic meromorphic starlike and convex functions of order γ. To obtain some generalizations of these classes we introduce definition of weak subordination for complex-valued functions in $\mathbb{D}$.

A complex-valued function f in $\mathbb{D}$ is said to be *weakly subordinate* to a complex-valued function F in $\mathbb{D}$, and we write $f(z) \preceq F(z)$ (or simply $f \preceq F$), if $f(\infty) = F(\infty)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

If $f \preceq F$ and F is univalent in $\mathbb{D}$, then we can consider the function $\omega(z) = F^{-1}(f(z)), z \in \mathbb{D}$ which maps $\mathbb{D}$ into oneself with $\omega(\infty) = \infty$. Conversely, if $\omega(z) = F^{-1}(f(z)), z \in \mathbb{D}$, maps $\mathbb{D}$ into oneself with $\omega(\infty) = \infty$, then $f \preceq F$. Thus, we have the following equivalence.

Lemma 1 *A complex-valued function f in $\mathbb{D}$ is weakly subordinate to a function complex-valued function F in $\mathbb{D}$ if and only if there exists a complex-valued function ω which maps $\mathbb{D}$ into oneself with $\omega(\infty) = \infty$ such that $f(z) = F(\omega(z)), z \in \mathbb{D}$.*

The following two subclasses of $\Sigma_{\mathcal{H}}$ are the main focus of this paper.
For $-B \le A < B \le 1$ let $\Sigma^m_{\mathcal{H}}(A, B)$ be the class of functions $f \in \Sigma_{\mathcal{H}}$ so that

$$\frac{D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f(z)\right)}{D^m_{\mathcal{H}} f(z)} \preceq \frac{A+z}{B+z}$$

and $\mathbb{V}^m_{\mathcal{H}}(A, B)$ be the class of functions $f \in \Sigma_{\mathcal{H}}$ so that

$$\frac{D^m_{\mathcal{H}} f(z)}{z} \preceq \frac{A+z}{B+z}.$$

In particular, for $m = 0, m = 1$, we obtain classes studied in [5]. The classes $\Sigma^*_{\mathcal{H}}(\gamma) := \Sigma^1_{\mathcal{H}}(2\gamma - 1, 1)$ and $\Sigma^c_{\mathcal{H}}(\gamma) := \Sigma^1_{\mathcal{H}}(2\gamma - 1, 1)$ are investigated by Jahangiri [8]. The classes $\Sigma^*_{\mathcal{H}} := \Sigma^*_{\mathcal{H}}(0)$ and $\Sigma^c_{\mathcal{H}} := \Sigma^c_{\mathcal{H}}(0)$ are the classes of functions $f \in \Sigma_{\mathcal{H}}(k)$ which are starlike in $\mathbb{U}(r)$ or convex in $\mathbb{U}(r)$, respectively, for all $r > 1$ (see [10]).

We further let $\mathcal{T}_\eta$ be the class of functions $f = h + \overline{g} \in \Sigma_{\mathcal{H}}$ with varying coefficients (e.g. see [9]) for which there exists a real number η so that

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} e^{i(n+1)\eta} |a_n| z^{-n} + (-1)^{m+1} \overline{\sum_{n=2}^{\infty} e^{i(n-1)\eta} |b_n| z^{-n}} \tag{4}$$

and $\Sigma^m_\eta(A, B) := \mathcal{T}_\eta \cap \Sigma^m_{\mathcal{H}}(A, B)$ and $V^m_\eta(A, B) := \mathcal{T}_\eta \cap V^m_{\mathcal{H}}(A, B)$.

2 Dual sets and coefficient bounds

Here we make use of the "duality principle" defined by Ruscheweyh ([14], Chapter 1). For the set $\mathcal{V} \subset \Sigma_{\mathcal{H}}$ we define the dual set of $\mathcal{V}$ by

$$\mathcal{V}^* = \{f_2 \in \Sigma_{\mathcal{H}} | \forall f_1 \in \mathcal{V} : (f_1 * f_2)(z) \ne 0; z \in \mathbb{D}\}.$$

Consequently, the second dual of $\mathcal{V}$ or the dual of $\mathcal{V}^*$ is defined as

$$\mathcal{V}^{**} = (\mathcal{V}^*)^* = \left\{ f_3 \in \Sigma_{\mathcal{H}} | \forall f_2 \in \mathcal{V}^* : (f_2 * f_3)(z) \neq 0; z \in \mathbb{D} \right\}.$$

This duality principle indicates that under fairly weak conditions on $\mathcal{V}$ which is a subset of $\Sigma_{\mathcal{H}}$, many linear and other extremal problems in the second dual of $\mathcal{V}$ are solved in $\mathcal{V}$. This is a very useful tool since in many cases of interest (such as convex, starlike or close-to-convex harmonic univalent functions), the set $\mathcal{V}^{**}$ is much larger than the set $\mathcal{V}$. This, on its own right, would be a separate endeavor and research topic that can be explored further which is not the focus of the present paper. The following theorem presents a duality condition for the set $\Sigma_{\mathcal{H}}^m(A, B)$.

Theorem 1

$$\Sigma_{\mathcal{H}}^m(A, B) = \left\{ \psi_\xi : |\xi| = 1 \right\}^*,$$

where

$$\begin{aligned} \psi_\xi(z) := {} & (B - A) z + (-1)^{m+1} \frac{(B + A + 2\xi) z - (1 + m)\xi + (\lambda B + A)}{(1 - z)^{m+2}} z^m \\ & + (-1)^{m+1} (2\xi + B + A) \overline{z} \\ & + \frac{(B - A)\overline{z} + (1 - m)\xi - (\lambda B - A)}{(1 - \overline{z})^{m+2}} \overline{z}^m; \ (z \in \mathbb{D}). \end{aligned}$$

Proof Let $f \in \Sigma_{\mathcal{H}}$. Then $f \in \Sigma_{\mathcal{H}}^m(A, B)$ if and only if

$$\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z)}{D_{\mathcal{H}}^m f(z)} \neq \frac{A + \xi}{B + \xi}; \ (z \in \mathbb{D}, \ |\xi| = 1). \tag{5}$$

Since

$$\begin{aligned} D_{\mathcal{H}}\left(D_{\mathcal{H}}^m h\right)(z) &= z\left(D_{\mathcal{H}}^m h\right)'(z) = h(z) * \left\{ z\left(z + \frac{(-z)^m}{(z-1)^{m+1}}\right)' \right\} \\ &= h(z) * \left(z - \frac{(z - m)(-z)^m}{(z-1)^{m+2}}\right), \end{aligned}$$

the above inequality (5) yields

$$\begin{aligned} & (B + \xi) D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z) - (A + \xi) D_{\mathcal{H}}^m f(z) \\ & = (B + \xi) D_{\mathcal{H}}\left(D_{\mathcal{H}}^m h\right)(z) - (A + \xi) D_{\mathcal{H}}^m h(z) \\ & \quad - (-1)^m \left[(B + \xi) \overline{D_{\mathcal{H}}\left(D_{\mathcal{H}}^m g\right)(z)} + (A + \xi) \overline{D_{\mathcal{H}}^m h(z)} \right] \\ & = h(z) * \left((B + \xi)\left(z - \frac{(z - m)(-z)^m}{(z-1)^{m+2}}\right) - (A + \xi)\left(z + \frac{(-z)^m}{(z-1)^{m+1}}\right) \right) \\ & \quad - (-1)^m \overline{g(z)} * \left((B + \xi)\left(\overline{z} - \frac{(\overline{z} - m)(\overline{z})^m}{(\overline{z} - 1)^{m+2}}\right) \right. \end{aligned}$$

$$+ (A+\xi)\left(\overline{z}+\frac{(-\overline{z})^m}{(\overline{z}-1)^{m+1}}\right)\Bigg)$$
$$= f(z) * \psi_\xi(z) \neq 0.$$

Thus, $f \in \Sigma_{\mathcal{H}}^m(A,B)$ if and only if $f(z) * \psi_\xi(z) \neq 0$ *i.e.* $\Sigma_{\mathcal{H}}^m(A,B) = \left\{\psi_\xi : |\xi| = 1\right\}^*$. □

A similar argument can be used to obtain a duality condition for the set $\mathbb{V}_{\mathcal{H}}^m(A,B)$.

Theorem 2

$$\mathbb{V}_{\mathcal{H}}^m(A,B) = \left\{\delta_\xi : |\xi| = 1\right\}^*,$$

where

$$\delta_\xi(z) := (B-A)z + (-1)^{m+1}\frac{(1+B\xi)z^m}{(z-1)^{m+1}} + (-1)^m(B+\xi)\overline{z} - \frac{(B+\xi)\overline{z}^m}{(\overline{z}-1)^{m+1}} \quad (z \in \mathbb{D}).$$

Next we determine sufficient coefficient bounds for function in $\Sigma_{\mathcal{H}}^m(A,B)$.

Theorem 3 *Let $f = h + \overline{g}$ be of the form (2) and $-B \le A < B \le 1$. If*

$$\sum_{n=1}^{\infty} m_n\left\{(n(1+B)+(1+A))|a_n| + (n(1+B)-(1+A))|b_n|\right\} \le B - A,$$

then f is univalent and orientation preserving in $\mathbb{D}$ and $f \in \Sigma_{\mathcal{H}}^m(A,B)$.

Proof The univalency and orientation preserving of the function f follows by a result of Jahangiri and Silverman ([10], Theorem 1) since

$$n(B-A) \le m_n(n(1+B)-(1+A)) \le m_n(n(1+B)+(1+A)).$$

Therefore, by Lemma 1 $f \in \Sigma_{\mathcal{H}}^m(A,B)$ if and only if there exists a complex-valued function ω where $\omega(\infty) = \infty$, $|\omega(z)| > 1$ and $z \in \mathbb{D}$ such that

$$\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z)}{D_{\mathcal{H}}^m f(z)} = \frac{A+\omega(z)}{B+\omega(z)}$$

or equivalently

$$\left|\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z) - D_{\mathcal{H}}^m f(z)}{B D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z) - A\left(D_{\mathcal{H}}^m f(z)\right)(z)}\right| < 1. \tag{6}$$

Thus for $z \in \mathbb{D}$ it suffices to show that

$$\left|D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z) - D_{\mathcal{H}}^m f(z)\right| - \left|B D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z) - D_{\mathcal{H}}^m f(z)\right| < 0.$$

Indeed, letting $|z| = r \ (r > 1)$ we have

$$\begin{aligned}
&\left|D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - D^m_{\mathcal{H}} f(z)\right| - \left|B D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - D^m_{\mathcal{H}} f(z)\right| \\
&\quad = \left|\sum_{n=1}^{\infty}(n+1) m_n a_n z^{-n} - (-1)^m \sum_{n=1}^{\infty}(n-1) m_n \overline{b_n} \overline{z}^{-n}\right| \\
&\qquad - \left|(B-A) z + \sum_{n=1}^{\infty}(Bn+A) m_n a_n z^{-n} + (-1)^m \sum_{n=1}^{\infty}(Bn-A) m_n \overline{b_n} \overline{z}^{-n}\right| \\
&\quad \leq \sum_{n=1}^{\infty}(n+1) m_n |a_n| r^{-n} + \sum_{n=1}^{\infty}(n-1) m_n |b_n| r^{-n} - (B-A) r \\
&\qquad + \sum_{n=1}^{\infty}(Bn+A) m_n |a_n| r^{-n} + \sum_{n=1}^{\infty}(Bn-A) m_n |b_n| r^{-n} \\
&\quad \leq r \left\{\sum_{n=1}^{\infty}(\alpha_n |a_n| + \beta_n |b_n|) r^{-n-1} - (B-A)\right\} < 0
\end{aligned}$$

where $\alpha_n = m_n(n(1+B) + (1+A))$ and $\beta_n = m_n(n(1+B) - (1+A)))$. Thus, according to the hypothesis of the theorem, $f \in \Sigma^m_{\mathcal{H}}(A, B)$. □

The sufficient coefficient bound given in Theorem 3 is also necessary for functions to be in the class $\Sigma^m_\eta(A, B)$ as stated in the following theorem.

Theorem 4 *Let* $f = h + \overline{g}$ *be of the form (4) and* $-B \leq A < B \leq 1$. *Then* $f \in \Sigma^m_\eta(A, B)$ *if and only if*

$$\sum_{n=1}^{\infty} m_n \left\{(n(1+B) + (1+A)) |a_n| + (n(1+B) - (1+A)) |b_n|\right\} \leq B - A.$$

Proof The "*if*" part follows from Theorem 3. For the "*only-if*" part, assume that $f \in \Sigma^m_\eta(A, B)$, then by (6) we have

$$\left|\frac{\sum\limits_{n=1}^{\infty} m_n \left\{(n+1) a_n z^{-n} - (-1)^m (n-1) \overline{b_n} \overline{z}^{-n}\right\}}{(B-A) z - \sum\limits_{n=1}^{\infty} m_n \left\{(Bn+A) a_n z^{-n} - (-1)^m (Bn-A) \overline{b_n} \overline{z}^{-n}\right\}}\right| < 1 \ (z \in \mathbb{D}).$$

Therefore, by (4) for $z = re^{i\eta}$ $(r > 1)$, we obtain

$$\frac{\sum_{n=1}^{\infty} \left\{(n+1) |m_n| |a_n| + (n-1) |m_n| |b_n|\right\} r^{-n-1}}{(B-A) - \sum_{n=1}^{\infty} \left\{(Bn+A) |m_n| |a_n| + (Bn-A) |m_n| |b_n|\right\} r^{-n-1}} < 1. \tag{7}$$

It is clear that the denominator of the ratio in the inequality (7) cannot vanish for $r > 1$. Moreover, it is positive as $r \to \infty$ and consequently for $r > 1$. Thus, we must have

$$\sum_{n=1}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) r^{-n-1} < B - A (r > 1) \tag{8}$$

where $\alpha_n = m_n(n(1 + B) + (1 + A))$ and $\beta_n = m_n(n(1 + B) - (1 + A)))$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=1}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|)$ is a non-decreasing sequence. Moreover, by (8), it is bounded above by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=1}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) = \lim_{n \to \infty} \{S_n\} \leq B - A.$$

□

A similar argument can be used to prove the following theorem.

Theorem 5 *Let $f = h + \overline{g}$ be of the form (2) and $-B \leq A < B \leq 1$. Then $f \in \mathbb{V}_{\eta}^{m}(A, B)$ if and only if*

$$\sum_{n=1}^{\infty} m_n (|a_n| + |b_n|) \leq \frac{B - A}{1 + B}.$$

As a consequence of Theorems 4 and 5 we have the following corollary.

Corollary 1 *For $-B \leq A < B \leq 1$ let $a = \frac{1+A}{1+B}$,*

$$\phi(z) = z + \sum_{n=1}^{\infty} \left(\frac{1}{n - a} z^{-n} + \frac{1}{n + a} \overline{z^{-n}} \right) (z \in \mathbb{D}),$$
$$\omega(z) = z + \sum_{n=1}^{\infty} \left((n - a) z^{-n} + (n + a) \overline{z^{-n}} \right) (z \in \mathbb{D}).$$

Then

$$f \in \mathbb{V}_{\eta}^{m}(A, B) \Leftrightarrow f * \phi \in \Sigma_{\eta}^{m}(A, B),$$
$$f \in \Sigma_{\eta}^{m}(A, B) \Leftrightarrow f * \omega \in \mathbb{V}_{\eta}^{m}(A, B).$$

3 Extreme points

The Krein-Milman theorem (see [11]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 2 *Let $\mathcal{F}$ be a non-empty compact convex subset of the set $\Sigma_{\mathcal{H}}$ and $\mathcal{J} : \Sigma_{\mathcal{H}} \to \mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then*

$$\max\{\mathcal{J}(f) : f \in \mathcal{F}\} = \max\{\mathcal{J}(f) : f \in E\mathcal{F}\},$$

where $E\mathcal{F}$ denotes the set of extreme points of $\mathcal{F}$.

Since $\Sigma_{\mathcal{H}}$ is a complete metric space, Montel's theorem [12] implies the following lemma.

Lemma 3 *A set $\mathcal{F} \subset \Sigma_{\mathcal{H}}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Now we are equipped to state and prove the two main theorems in this section.

Theorem 6 *The set $\Sigma_{\eta}^{m}(A, B)$ is a convex and compact subset of $\Sigma_{\mathcal{H}}$.*

Proof For $l = 1, 2$ let $f_l \in \Sigma_{\eta}^{m}(A, B)$ be functions of the form (3), $0 \le \gamma \le 1$. Since

$$\gamma f_1(z) + (1-\gamma) f_2(z) = z + \sum_{n=1}^{\infty} \left\{ \left(\gamma a_{1,n} + (1-\gamma) a_{2,n}\right) z^{-n} + \overline{\left(\gamma b_{1,n} + (1-\gamma) b_{2,n}\right) z^{-n}} \right\},$$

and since for

$$\alpha_n = m_n(n(1+B) + (1+A)), \beta_n = m_n(n(1+B) - (1+A))) \tag{9}$$

(by Theorem 4), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ \alpha_n \left|\gamma a_{1,n} + (1-\gamma) a_{2,n}\right| + \beta_n \left|\gamma b_{1,n} + (1-\gamma) b_{2,n}\right| \right\} \\ &\le \gamma \sum_{n=1}^{\infty} \left\{ \alpha_n \left|a_{1,n}\right| + \beta_n \left|b_{1,n}\right| \right\} + (1-\gamma) \sum_{n=1}^{\infty} \left\{ \alpha_n \left|a_{2,n}\right| + \beta_n \left|b_{2,n}\right| \right\} \\ &\le \gamma (B - A) + (1-\gamma)(B - A) = B - A, \end{aligned}$$

the function $\phi = \gamma f_1 + (1-\gamma) f_2$ belongs to the class $\Sigma_{\eta}^{m}(A, B)$. Hence, the set $\Sigma_{\eta}^{m}(A, B)$ is convex. Furthermore, for $f \in \Sigma_{\eta}^{m}(A, B)$ and $|z| = r > 1$ we have

$$|f(z)| \le r + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^{-n} \le r + \sum_{n=1}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) \le r + (B - A).$$

Thus, the class $\Sigma_\eta^m(A, B)$ is locally uniformly bounded. Now, by Lemma 3, we only need to show that it is closed *i.e.* if $f_l \in \Sigma_\eta^m(A, B)$ and $f_l \to f$ then $f \in \Sigma_\eta^m(A, B)$. Let f_l and f be given by (3) and (4), respectively. Using Theorem 4 we have

$$\sum_{n=1}^{\infty}\left(\alpha_n\left|a_{l,n}\right| + \beta_n\left|b_{l,n}\right|\right) \leq B - A \ \ (l \in \mathbb{N}). \tag{10}$$

Since $f_l \to f$, we conclude that $a_{l,n} \to a_n$ and $b_{l,n} \to b_n$ as $l \to \infty$ $(n \in \mathbb{N})$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=1}^{\infty}(\alpha_n|a_n| + \beta_n|b_n|)$ is a non-decreasing sequence. Moreover, by (10), it is bounded above by $B - A$. Therefore, the sequence$\{S_n\}$ is convergent and

$$\sum_{n=1}^{\infty}(\alpha_n|a_n| + \beta_n|b_n|) = \lim_{n\to\infty} S_n \leq B - A.$$

This implies that $f \in \Sigma_\eta^m(A, B)$ which completes the proof. □

In the following theorem we determine the extreme points of $\Sigma_\eta^m(A, B)$.

Theorem 7

$$E\Sigma_\eta^m(A, B) = \{h_n : \ n \in \mathbb{N}_0\} \cup \{g_n : \ n \in \mathbb{N}\},$$

where

$$h_0(z) = z, \ \ h_n(z) = z + \frac{(B-A)\,e^{i(1+n)\eta}}{m_n(n(1+B)+(1+A))}z^{-n},$$
$$g_n(z) = z + \frac{(-1)^{m+1}(B-A)\,e^{i(1-n)\eta}}{m_n(n(1+B)-(1+A)))}\overline{z^{-n}} \quad (z \in \mathbb{D}). \tag{11}$$

Proof Suppose that $0 < \gamma < 1$ and $g_n = \gamma f_1 + (1-\gamma) f_2$ where $f_1, f_2 \in \Sigma_\eta^m(A, B)$ are functions of the form (3) and let α_n, β_n be defined by (9). Then $\left|b_{1,n}\right| = \left|b_{2,n}\right| = \frac{B-A}{\beta_n}$ and consequently $a_{1,n} = a_{2,n} = 0$ and $b_{1,k} = b_{2,k} = 0$ for $k \in \mathbb{N}\diagdown\{n\}$. It follows that $g_n = f_1 = f_2$, and so $g_n \in \Sigma_\eta^m(A, B)$. Similarly, we can verify that the functions h_n of the form (11) are the extreme points of the class $\Sigma_\eta^m(A, B)$. Now, suppose that the function f belongs to the set $E\Sigma_\eta^m(A, B)$ and f is not of the form (11). Then there exists $s \in \mathbb{N}$ such that

$$0 < |a_s| < \frac{B-A}{\alpha_s} \ \text{ or } \ 0 < |b_s| < \frac{B-A}{\beta_s}.$$

If $0 < |a_s| < \frac{B-A}{\alpha_s}$, then setting

$$\gamma = \frac{\alpha_s|a_s|}{B-A} \ \textit{and} \ \varphi = \frac{1}{1-\gamma}(f - \gamma h_s),$$

we have that $h_s \neq \varphi$ and so $f = \gamma h_s + (1-\gamma)\varphi$, $0 < \gamma < 1$. Thus, $f \notin E\Sigma_\eta^m(A, B)$.

Similarly, if $0 < |b_s| < \frac{B-A}{\beta_n}$, then setting

$$\gamma = \frac{\beta_s |b_s|}{B-A}, \ \phi = \frac{1}{1-\gamma}(f - \gamma g_s),$$

we have that $g_s \neq \phi$ and so $f = \gamma g_s + (1-\gamma)\phi, \ 0 < \gamma < 1$.

It follows that $f \notin E\Sigma_\eta^m(A, B)$ and so the proof is completed. □

4 Applications

It is clear that if the class

$$\mathcal{F} = \{f_n \in \Sigma_{\mathcal{H}} : \ n \in \mathbb{N}\},$$

is locally uniformly bounded, then

$$\overline{co}\mathcal{F} = \left\{ \sum_{n=1}^{\infty} \gamma_n f_n : \ \sum_{n=1}^{\infty} \gamma_n = 1, \ \gamma_n \geq 0 \ (n \in \mathbb{N}) \right\}.$$

Thus, by Theorem 7, we have the following corollary.

Corollary 2

$$\Sigma_\eta^m(A, B) := \left\{ \sum_{n=0}^{\infty} \gamma_n h_n + \sum_{n=1}^{\infty} \delta_n g_n : \ \sum_{n=0}^{\infty} \gamma_n + \sum_{n=1}^{\infty} \delta_n = 1, \ \gamma_n \geq 0, \ \delta_n \geq 0 \right\}$$

where h_n and g_n are given by (11).

For $f \in \Sigma_{\mathcal{H}}$, $\gamma \geq 1$ and $r > 1$ the real-valued functional

$$\mathcal{J}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^\gamma d\theta \right)^{1/\gamma}$$

is continuous and convex on $\Sigma_{\mathcal{H}}$.

Moreover, for $f \in \Sigma_{\mathcal{H}}$, $z \in \mathbb{D}$ and each fixed $n \in \mathbb{N}$ the real-valued functionals $\mathcal{J}_1(f) = a_n$, $\mathcal{J}_2(f) = b_n$, $\mathcal{J}_3(f) = |f(z)|$ and $\mathcal{J}_4(f) = |D_{\mathcal{H}} f(z)|$ are also continuous and convex on $\Sigma_{\mathcal{H}}$.

These in conjunction with Lemma 2 yield the following corollaries.

Corollary 3 *Let $f \in \Sigma_\eta^m(A, B)$. Then*

$$|a_n| \leq \frac{B-A}{m_n(n(1+B)+(1+A))} \textit{and } |b_n| \leq \frac{B-A}{m_n(n(1+B)-(1+A)))}.$$

The result is sharp and the functions h_n and g_n of the form (11) *are the extremal functions.*

Corollary 4 *Let* $f \in \Sigma_{\eta}^{m}(A, B)$. *Then*

$$r - \frac{1}{r} \leq |f(z)| \leq r + \frac{1}{r}$$

and

$$r - \frac{1}{r} \leq |D_{\mathcal{H}} f(z)| \leq r + \frac{1}{r}.$$

The result is sharp and the extremal function is given by (11).

The following covering result follows from Corollary 4.

Corollary 5 *If* $f \in \Sigma_{\eta}^{m}(A, B)$, *then* $\mathbb{D}(2) \subset f(\mathbb{D})$.

Corollary 6 *Let* $r > 1$ *and* $\gamma \geq 1$. *If* $f \in \Sigma_{\eta}^{m}(A, B)$, *then*

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left| f(re^{i\theta}) \right|^{\gamma} d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| r - \frac{1}{re^{i\theta}} \right|^{\gamma} d\theta,$$
$$\frac{1}{2\pi} \int_{0}^{2\pi} |D_{\mathcal{H}} f(z)|^{\gamma} d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| r - \frac{1}{re^{i\theta}} \right|^{\gamma} d\theta.$$

Corollary 7 *The class* $\mathbb{V}_{\eta}^{m}(A, B)$ *is a convex compact subset of* $\Sigma_{\mathcal{H}}$. *Moreover,*

$$E\mathbb{V}_{\eta}^{m}(A, B) = \{h_n : n \in \mathbb{N}_0\} \cup \{g_n : n \in \mathbb{N}\}$$

and

$$\mathbb{V}_{\eta}^{m}(A, B) = \left\{ \sum_{n=0}^{\infty} \gamma_n h_n + \sum_{n=1}^{\infty} \delta_n g_n : \sum\nolimits_{n=0}^{\infty} \gamma_n + \sum\nolimits_{n=1}^{\infty} \delta_n = 1, \ \gamma_n \geq 0, \ \delta_n \geq 0 \right\}$$

where $h_0(z) = z$, *and*

$$h_n(z) = z + \frac{(B - A)\, e^{i(1+n)\eta}}{(1 + B)\, m_n} z^{-n},$$
$$g_n(z) = z + (-1)^{m+1} \frac{(B - A)\, e^{i(1-n)\eta}}{(1 + B)\, m_n} \overline{z^{-n}}. \tag{12}$$

Corollary 8 *Let* $f \in \mathbb{V}_{\eta}^{m}(A, B)$ *be a function of the form* (4) . *Then*

$$|a_n| \leq \frac{B - A}{(1 + B)\, m_n}, \ |b_n| \leq \frac{B - A}{(1 + B)\, m_n} \quad (n \in \mathbb{N}),$$
$$r - \frac{B - A}{(1 + B)\, r} \leq |f(z)| \leq r + \frac{B - A}{(1 + B)\, r},$$

$$r-\frac{B-A}{(1+B)\,r}\le |D_{\mathcal{H}}f(z)|\le r+\frac{B-A}{(1+B)\,r},$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\gamma}d\theta\le\frac{1}{2\pi}\int_0^{2\pi}\left|r-\frac{B-A}{(1+B)\,re^{i\theta}}\right|^{\gamma}d\theta,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left|D_{\mathcal{H}}f(re^{i\theta})\right|^{\gamma}d\theta\le\frac{1}{2\pi}\int_0^{2\pi}\left|r-\frac{B-A}{(1+B)\,re^{i\theta}}\right|^{\gamma}d\theta.$$

The results are sharp and the functions h_n and g_n of the form (12) are the extremal functions.

Corollary 9 *If $f\in\mathbb{V}^m_\eta(A,B)$, then $\mathbb{D}(r)\subset f(\mathbb{D})$ where*

$$r=1+\frac{B-A}{1+B}.$$

Remark 1 The classes $\Sigma^m_{\mathcal{H}}(A,B)$ and $\mathbb{V}^m_{\mathcal{H}}(A,B)$ for different values of m, A and B give various well-known as well as new classes of meromorphic harmonic univalent functions (see for example [2,3,5,6,8,10]).

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