Conjugations in L^2 and their invariants

M. Cristina Câmara¹ · Kamila Kliś–Garlicka² · Bartosz Łanucha³ · Marek Ptak²

Received: 3 April 2019 / Accepted: 1 April 2020 / Published online: 10 April 2020
© The Author(s) 2020

Abstract

Conjugations in space L^2 of the unit circle commuting with multiplication by z or intertwining multiplications by z and $\bar{z}$ are characterized. We also study their behaviour with respect to the Hardy space, subspaces invariant for the unilateral shift and model spaces.

Keywords Conjugation · C –symmetric operator · Hardy space · Invariant subspaces for the unilateral shift · Model space · Truncated Toeplitz operator

Mathematics Subject Classification Primary 47B35; Secondary 30D20 · 30H10

The work of the first author was partially supported by FCT/Portugal through UID/MAT/04459/2019 and the research of the second and the fourth authors was financed by the Ministry of Science and Higher Education of the Republic of Poland.

✉ Kamila Kliś–Garlicka
rmklis@cyfronet.pl

M. Cristina Câmara
cristina.camara@tecnico.ulisboa.pt

Bartosz Łanucha
bartosz.lanucha@poczta.umcs.lublin.pl

Marek Ptak
rmptak@cyf-kr.edu.pl

¹ Center for Mathematical Analysis, Geometry and Dynamical Systems, Mathematics Department, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049–001 Lisboa, Portugal

² Department of Applied Mathematics, University of Agriculture, ul. Balicka 253c, 30-198 Kraków, Poland

³ Department of Mathematics, Maria Curie-Skłodowska University, Maria Curie-Skłodowska Square 1, 20-031 Lublin, Poland

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A *conjugation* C in $\mathcal{H}$ is an antilinear isometric involution, i.e., $C^2 = id_{\mathcal{H}}$ and

$$\langle Cg, Ch \rangle = \langle h, g \rangle \quad \text{for } g, h \in \mathcal{H}. \quad (1.1)$$

Conjugations have recently been intensively studied and the roots of this subject comes from physics. An operator $A \in B(\mathcal{H})$ is called *C-symmetric* if $CAC = A^*$ (or equivalently $AC = CA^*$). A strong motivation to study conjugations comes from the study of complex symmetric operators, i.e., those operators that are *C-symmetric* with respect to some conjugation C . For references see for instance [2,3,7–10]. Hence obtaining the full description of conjugations with certain properties is of great interest.

Let $\mathbb{T}$ denote the unit circle, and let m be the normalized Lebesgue measure on $\mathbb{T}$. Consider the spaces $L^2 = L^2(\mathbb{T}, m)$, $L^\infty = L^\infty(\mathbb{T}, m)$, the classical Hardy space H^2 on the unit disc $\mathbb{D}$ identified with a subspace of L^2 , and the Hardy space H^∞ of all analytic and bounded functions in $\mathbb{D}$ identified with a subspace of L^∞ . Denote by M_φ the operator defined on L^2 of multiplication by a function $\varphi \in L^\infty$.

The most natural conjugation in L^2 is J defined by $Jf = \bar{f}$, for $f \in L^2$. This conjugation has two natural properties: the operator M_z is J -symmetric, i.e., $M_z J = J M_{\bar{z}}$, and J maps an analytic function into a co-analytic one, i.e., $JH^2 = \overline{H^2}$.

Another natural conjugation in L^2 is $J^* f = f^\#$ with $f^\#(z) \stackrel{df}{=} \overline{f(\bar{z})}$. The conjugation J^* has a different behaviour: it commutes with multiplication by z ($M_z J^* = J^* M_z$) and leaves analytic functions invariant, $J^* H^2 \subset H^2$. The map J^* appears for example in connection with Hankel operators (see [11, pp. 146–147]). Its connection with model spaces was studied in [4] (Lemma 4.4, see also [1, p. 37]). Hence a natural question is to characterize conjugations with respect to these properties. The first step was done in [2] where all conjugations in L^2 with respect to which the operator M_z is *C-symmetric* were characterized, see Theorem 2.2. In Sect. 2 we give a characterization of all conjugations which commute with M_z , Theorem 2.4. In Sect. 3, using the above characterizations we show that there are no conjugations in L^2 leaving H^2 invariant, with respect to which the operator M_z is *C-symmetric*. We also show that J^* is the only conjugation commuting with M_z and leaving H^2 invariant.

Beurling's theorem makes subspaces of H^2 of the type θH^2 (θ inner function, i.e., $\theta \in H^\infty$, $|\theta| = 1$ a.e. on $\mathbb{T}$) exceptionally interesting, as the only invariant subspaces for the unilateral shift S , $Sf(z) = zf(z)$ for $f \in H^2$. On the other hand, model spaces (subspaces of the type $K_\theta = H^2 \ominus \theta H^2$), which are invariant for the adjoint of the unilateral shift, are important in model theory, [12].

In [2] all conjugations C with respect to which the operator M_z is *C-symmetric* and mapping a model space K_α into another model space K_θ were characterized, with the assumption that α divides θ ($\alpha \leq \theta$). Recall that for α and θ inner, $\alpha \leq \theta$ means that θ/α is also an inner function. In what follows we will show that the result holds without the assumption $\alpha \leq \theta$.

In Sect. 4 conjugations commuting with M_z and preserving model spaces are described. Section 5 is devoted to conjugations between S -invariant subspaces (i.e., subspaces of the form θH^2 with θ an inner function). In the last section we deal with conjugations commuting with the truncated shift A_z^θ ($A_z^\theta = P_\theta M_z|_{K_\theta}$ where P_θ is the orthogonal projection from L^2 onto K_θ) or conjugations such that A_z^θ is C -symmetric with respect to them.

2 M_z and $M_{\bar{z}}$ -commuting conjugations in L^2

Denote by J the conjugation in L^2 defined as $Jf = \bar{f}$, for $f \in L^2$. This conjugation has the following obvious properties:

- Proposition 2.1** (1) $M_z J = J M_{\bar{z}}$;
 (2) $M_\varphi J = J M_{\bar{\varphi}}$ for all $\varphi \in L^\infty$;
 (3) $J H^2 = \overline{H^2}$.

Let us consider all conjugations C in L^2 satisfying the condition

$$M_z C = C M_{\bar{z}}. \quad (2.1)$$

Such conjugations were studied in [2] and are called M_z -conjugations. The following theorem characterizes all M_z -conjugations in L^2 .

Theorem 2.2 [2] *Let C be a conjugation in L^2 . Then the following are equivalent:*

- (1) $M_z C = C M_{\bar{z}}$,
- (2) $M_\varphi C = C M_{\bar{\varphi}}$ for all $\varphi \in L^\infty$,
- (3) there is $\psi \in L^\infty$, with $|\psi| = 1$, such that $C = M_\psi J$,
- (4) there is $\psi' \in L^\infty$, with $|\psi'| = 1$, such that $C = J M_{\psi'}$.

Another natural conjugation in L^2 is defined as

$$J^* f = f^\# \quad \text{with} \quad f^\#(z) \stackrel{df}{=} \overline{f(\bar{z})}. \quad (2.2)$$

The basic properties of J^* are the following:

Proposition 2.3

- (1) $M_z J^* = J^* M_z$;
- (2) $M_{\bar{z}} J^* = J^* M_{\bar{z}}$;
- (3) $M_\varphi J^* = J^* M_{\varphi^\#}$ for all $\varphi \in L^\infty$;
- (4) $J^* H^2 = H^2$.

In the context of the Proposition 2.3 it seems natural to consider all conjugations C in L^2 commuting with M_z , i.e.,

$$M_z C = C M_z. \quad (2.3)$$

Such conjugations will be called M_z -commuting. In what follows we will often deal with functions $f \in L^2$ such that $f(z) = f(\bar{z})$ a.e. on $\mathbb{T}$, which will be called *symmetric*. Observe that if f is symmetric and $f \in H^2$, then we also have $f \in \overline{H^2}$ and so it is a constant function.

Theorem 2.4 *Let C be a conjugation in L^2 . Then the following are equivalent:*

- (1) $M_z C = C M_z$,
- (2) $M_\varphi C = C M_{\varphi^\#}$ for all $\varphi \in L^\infty$,
- (3) there is a symmetric unimodular function $\psi \in L^\infty$ such that $C = M_\psi J^\star$,
- (4) there is a symmetric unimodular function $\psi' \in L^\infty$ such that $C = J^\star M_{\psi'}$.

Proof We will show that (1) $\Rightarrow$ (3). The other implications are straightforward. Assume that $M_z C = C M_z$. Then $M_z C J^\star = C M_z J^\star = C J^\star M_z$. It follows that the linear operator $C J^\star$ commutes with M_z . By [13, Theorem 3.2], there is $\psi \in L^\infty$ such that $C J^\star = M_\psi$. Hence $C = M_\psi J^\star = J^\star M_{\psi^\#}$.

By (1.1) for any $f, g \in L^2$ we have

$$\begin{aligned} \int_{\mathbb{T}} g \bar{f} dm(z) &= \langle g, f \rangle = \langle Cf, Cg \rangle \\ &= \langle \psi f^\#, \psi g^\# \rangle = \int_{\mathbb{T}} |\psi(z)|^2 \overline{f(\bar{z})} g(\bar{z}) dm(z) \\ &= \int_{\mathbb{T}} |\psi(\bar{z})|^2 \overline{f(\bar{z})} g(\bar{z}) dm(z). \end{aligned}$$

Hence $|\psi| = 1$ a.e. on $\mathbb{T}$. On the other hand, since $C^2 = I_{L^2}$, for all $f \in L^2$ we have

$$f = C^2 f = M_\psi J^\star M_\psi J^\star f = M_\psi J^\star (\psi f^\#) = \psi \psi^\# f,$$

which implies that $\psi \psi^\# = 1$ a.e. on $\mathbb{T}$. Therefore ψ is symmetric, i.e., $\psi(z) = \psi(\bar{z})$ a.e. on $\mathbb{T}$. $\square$

3 Conjugations preserving H^2

In the previous section all M_z -conjugations C in L^2 , i.e., such that

$$M_z C = C M_{\bar{z}}$$

or M_z -commuting conjugations, i.e., such that

$$M_z C = C M_z$$

were characterized. Let us now consider the question which of them preserve H^2 . Clearly, if C is a conjugation in L^2 and $C(H^2) \subset H^2$, then $C(H^2) = H^2$. Since $J^\star$ preserves H^2 , it can be considered as a conjugation in H^2 . The following result shows that $J^\star$ is in that sense unique.

Corollary 3.1 *Let C be an M_z -commuting conjugation in L^2 . If $C(H^2) \subset H^2$, then $C = \lambda J^*$ for some $\lambda \in \mathbb{T}$.*

Proof By Theorem 2.4 we have that $C = M_\psi J^*$ for some $\psi \in L^\infty$ with $|\psi| = 1$ and $\psi(z) = \psi(\bar{z})$ a.e. on $\mathbb{T}$. Since C preserves H^2 , we have

$$\psi = M_\psi J^*(1) = C(1) \in L^\infty \cap H^2 = H^\infty.$$

Thus ψ is analytic. Since it is symmetric, it is also co-analytic. Hence ψ must be a constant function, so $\psi = \lambda \in \mathbb{C}$ and $|\lambda| = |\psi| = 1$. $\square$

Corollary 3.2 *There are no M_z -conjugations in L^2 which preserve H^2 .*

Proof If C is an M_z -conjugation in L^2 , then by Theorem 2.2 it follows that $C = M_\psi J$ for some $\psi \in L^\infty$ with $|\psi| = 1$. As in the proof of Corollary 3.1 the assumption $C(H^2) \subset H^2$ implies that $\psi \in H^\infty$, which in turn means that ψ is an inner function. Moreover, for $n = 0, 1, 2, \dots$ we have

$$0 = \langle Cz^{n+1}, \bar{z} \rangle = \langle \psi \bar{z}^{n+1}, \bar{z} \rangle = \langle \psi, z^n \rangle = 0.$$

So $\psi = 0$ which is a contradiction. $\square$

The following example shows that not all conjugations in L^2 satisfy either (2.1) or (2.3).

Example 3.3 There is a set of naturally defined conjugations. For $k, l \in \mathbb{Z}$, $k < l$, define $C_{k,l} : L^2 \rightarrow L^2$ by

$$C_{k,l} \left(\sum_{n \in \mathbb{Z}} a_n z^n \right) = \bar{a}_l z^k + \bar{a}_k z^l + \sum_{n \notin \{k,l\}} \bar{a}_n z^n, \quad (3.1)$$

where $\{z^n\}$ is the standard basis in L^2 . Then (2.1) and (2.3) are not satisfied since

$$M_z C_{k,l}(z^k) = M_z(z^l) = z^{l+1}, \quad C_{k,l} M_{\bar{z}}(z^k) = C_{k,l}(z^{k-1}) = z^{k-1}$$

and

$$C_{k,l} M_z(z^k) = C_{k,l}(z^{k+1}) = \begin{cases} z^{k+1} & \text{if } k+1 \neq l, \\ z^k & \text{if } k+1 = l. \end{cases}$$

Note that, on the other hand, $C_{k,l}$ preserves H^2 whenever $k \geq 0$ or $l < 0$.

4 Conjugations preserving model spaces

There is another class of conjugations in L^2 which appear naturally in connection with model spaces. For a nonconstant inner function θ , denote by K_θ the so called *model space* of the form $H^2 \ominus \theta H^2$. The conjugation C_θ defined in L^2 by

$$C_\theta f = \theta \bar{z} \bar{f}$$

has the important property that it preserves the model space K_θ , i.e., $C_\theta K_\theta = K_\theta$. Thus C_θ can be considered as a conjugation in K_θ . Such conjugations are important in connection with truncated Toeplitz operators (see for instance [6]). Here we present several simple properties of such conjugations, which we will use later.

Proposition 4.1 *Let α, β, γ be nonconstant inner functions. Then*

- (1) $C_\beta C_\alpha = M_{\beta\bar{\alpha}}$,
- (2) $M_\gamma C_\alpha M_{\bar{\gamma}}$ is a conjugation in L^2 ,
- (3) $C_\beta M_\gamma = M_{\bar{\gamma}} C_\beta$.

Now we will consider relations between M_z -conjugations and model spaces. The theorem below was proved in [2, Theorem 4.2] with the additional assumption that $\alpha \leq \theta$. As we prove here, this assumption is not necessary.

Theorem 4.2 *Let α, γ, θ be inner functions (α, θ nonconstant). Let C be a conjugation in L^2 such that $M_z C = C M_{\bar{z}}$. Assume that $C(\gamma K_\alpha) \subset K_\theta$. Then there is an inner function β such that $C = C_\beta$, with $\gamma\alpha \leq \beta \leq \gamma\theta$ and $\alpha \leq \theta$.*

Proof Recall the standard notation for the reproducing kernel function at 0 in K_α , namely, $k_0^\alpha = 1 - \bar{\alpha}(0)\alpha$ and its conjugate $\bar{k}_0^\alpha = C_\alpha k_0^\alpha = \bar{z}(\alpha - \alpha(0))$. By Theorem 2.2 we know that $C = M_\psi J$ for some function $\psi \in L^\infty$, $|\psi| = 1$. Hence

$$K_\theta \ni C(\gamma \bar{k}_0^\alpha) = M_\psi J(\gamma \bar{k}_0^\alpha) = \psi \overline{\gamma \bar{z}(\alpha - \alpha(0))} = \bar{\gamma} \bar{\alpha} z \psi (1 - \overline{\alpha(0)\alpha}).$$

Thus there is $h \in K_\theta$ such that $h = \bar{\gamma} \bar{\alpha} z \psi (1 - \overline{\alpha(0)\alpha})$. Since $(1 - \overline{\alpha(0)\alpha})^{-1}$ is a bounded analytic function, we have

$$\bar{\gamma} \bar{\alpha} z \psi = h(1 - \overline{\alpha(0)\alpha})^{-1} \in H^2.$$

Since $\bar{\gamma} \bar{\alpha} z \psi \in H^2$ and $|\bar{\gamma} \bar{\alpha} z \psi| = 1$ a.e. on $\mathbb{T}$, it has to be an inner function. Moreover $\beta = z\psi$ has to be inner and divisible by $\gamma\alpha$, i.e., $\gamma\alpha \leq \beta$.

On the other hand, we have similarly

$$K_\theta \ni C_\theta C(\gamma k_0^\alpha) = C_\theta (\psi \gamma (1 - \overline{\alpha(0)\alpha})) = \overline{\theta \gamma \bar{z} \bar{\psi} (1 - \overline{\alpha(0)\alpha})},$$

and $\theta \gamma \bar{\beta} = \theta \gamma \bar{z} \bar{\psi} \in H^2$. Hence β divides $\theta \gamma$, i.e., $\beta \leq \gamma\theta$. It is clear that $C = C_\beta$. Finally, we have $\alpha \leq \theta$ as a consequence of $\gamma\alpha \leq \beta \leq \gamma\theta$. $\square$

Note that if $\alpha \leq \theta$ and $C = C_\beta$ for some inner β with $\gamma\alpha \leq \beta \leq \gamma\theta$, then $K_\alpha \subset K_{\frac{\beta}{\gamma}} \subset K_\theta$, $C_\beta M_\gamma = C_{\frac{\beta}{\gamma}}$ and

$$C(\gamma K_\alpha) = C_\beta M_\gamma(K_\alpha) = C_{\frac{\beta}{\gamma}}(K_\alpha) \subset K_{\frac{\beta}{\gamma}} \subset K_\theta.$$

Hence the implication in Theorem 4.2 is actually an equivalence.

The corollary below strengthens [2, Proposition 4.5].

Corollary 4.3 *Let α, θ be nonconstant inner functions, and let C be a conjugation in L^2 such that $M_z C = C M_{\bar{z}}$. Assume that $C(K_\alpha) \subset K_\theta$. Then $\alpha \leq \theta$ and there is an inner function β such that $C = C_\beta$, with $\alpha \leq \beta \leq \theta$.*

Let us turn to discussing the relations between M_z -commuting conjugations and model spaces. The following proposition describes some more properties of J^* .

Proposition 4.4 *Let α be an inner function. Then*

- (1) $J^*(\alpha H^2) = \alpha^\# H^2$;
- (2) $J^*(K_\alpha) = K_{\alpha^\#}$;
- (3) $J^* C_\alpha = C_{\alpha^\#} J^*$.

Proof The condition (1) is clear, (2) and (3) were proved in [4, Lemma 4.4]. □

Hence the conjugation J^* has a nice behaviour in connection with model spaces, namely $J^*(K_\alpha) = K_{\alpha^\#}$. Theorem 4.6 below says that the conjugation J^* is, in some sense the only M_z -commuting conjugation with this property.

We start with the following:

Proposition 4.5 *Let α, γ, θ be inner functions (α, θ nonconstant). Let C be an M_z -commuting conjugation in L^2 . Assume that $C(\gamma K_\alpha) \subset K_\theta$. Then $\alpha \leq \theta^\#$ and there is an inner function β with $\gamma\alpha \leq \beta \leq \gamma\theta^\#$ such that $C = J^* M_{\frac{\beta}{\gamma\alpha}} \bar{\gamma}$.*

Proof Observe that since C is an M_z -commuting conjugation, taking antilinear adjoints and applying [2, Proposition 2.1] we get $M_{\bar{z}} C = C M_z$. Since by Proposition 4.1 the antilinear operator $M_\gamma C_\alpha M_{\bar{\gamma}}$ is a conjugation, then $J^* C M_\gamma C_\alpha M_{\bar{\gamma}}$ is also a conjugation. Note also that $M_\gamma C_\alpha M_{\bar{\gamma}} M_z = M_{\bar{z}} M_\gamma C_\alpha M_{\bar{\gamma}}$. Hence

$$J^* C M_\gamma C_\alpha M_{\bar{\gamma}} M_z = M_{\bar{z}} J^* C M_\gamma C_\alpha M_{\bar{\gamma}}. \quad (4.1)$$

On the other hand,

$$\begin{aligned} J^* C M_\gamma C_\alpha M_{\bar{\gamma}}(\gamma K_\alpha) &\subset J^* C M_\gamma C_\alpha(K_\alpha) \\ &\subset J^* C(\gamma K_\alpha) \subset J^*(K_\theta) \subset K_{\theta^\#}. \end{aligned}$$

By Theorem 4.2 there is an inner function β such that $J^* C M_\gamma C_\alpha M_{\bar{\gamma}} = C_\beta$, with $\gamma\alpha \leq \beta \leq \gamma\theta^\#$ and $\alpha \leq \theta^\#$. Hence $C = J^* C_\beta M_\gamma C_\alpha M_{\bar{\gamma}}$. Therefore

$$C = J^* M_{\frac{\beta}{\gamma\alpha}} \bar{\gamma}$$

□

As in Theorem 4.2 the implication in Proposition 4.5 can be reversed. Indeed, if $\alpha \leq \theta^\#$ and $C = J^* M_{\frac{\beta}{\gamma\alpha}} \bar{\gamma}$ for some inner function β with $\gamma\alpha \leq \beta \leq \gamma\theta^\#$, then $K_\alpha \subset K_{\frac{\beta}{\gamma}} \subset K_{\theta^\#}$ and

$$C(\gamma K_\alpha) = J^* M_{\frac{\beta}{\gamma\alpha}}(K_\alpha) = J^* C_{\frac{\beta}{\gamma}}(K_\alpha) \subset J^*(K_{\frac{\beta}{\gamma}}) \subset J^*(K_{\theta^\#}) = K_\theta.$$

Theorem 4.6 *Let α, θ be nonconstant inner functions, and let C be an M_z -commuting conjugation in L^2 , i.e., $M_z C = C M_z$. Assume that $C(K_\alpha) \subset K_\theta$. Then $\alpha \leq \theta^\#$ and $C = \lambda J^*$ with $\lambda \in \mathbb{T}$.*

Corollary 4.7 *Let C be an M_z -commuting conjugation in L^2 . Assume that there is some nonconstant inner function θ such that $C(K_\theta) \subset K_{\theta^\#}$. Then $C = \lambda J^*$ with $\lambda \in \mathbb{T}$.*

Proof of Theorem 4.6 By Proposition 4.5 there is an inner function β with $\alpha \leq \beta \leq \theta^\#$ such that $C = J^* M_{\frac{\beta}{\gamma\alpha}} \bar{\gamma}$. The function $\frac{\beta}{\alpha}$ is inner and by Theorem 2.4 it is symmetric. As observed before it follows that it is constant. Hence $C = J^*$ up to multiplication by a constant of modulus 1. □

5 Conjugations preserving S -invariant subspaces of H^2

Beurling's theorem says that all invariant subspaces for the unilateral shift S are of the form θH^2 with θ inner. We will now investigate conjugations in L^2 which preserve subspaces of this form. Since C_θ transforms θH^2 onto $\overline{z} H^2$, the operator

$$C_\theta J^* C_\theta = M_\theta J^* M_{\bar{\theta}}$$

is an example of such a conjugation. Note that

$$(C_\theta J^* C_\theta) M_z = M_z (C_\theta J^* C_\theta).$$

Let α, θ be two inner functions. Then the operator $C_\theta J^* C_\alpha : L^2 \rightarrow L^2$ is an antilinear isometry which maps αH^2 onto θH^2 and commutes with M_z . This operator however does not have to be an involution.

Lemma 5.1 *Let α, θ be two inner functions. The operator $C_\theta J^* C_\alpha$ is an involution (and hence a conjugation in L^2) if and only if the function $\theta\alpha^\#$ is symmetric (or equivalently $\alpha\alpha^\# = \theta\theta^\#$).*

Proof Note that by Proposition 4.1,

$$(C_\theta J^\star C_\alpha)(C_\theta J^\star C_\alpha) = C_\theta C_{\alpha^\#} C_{\theta^\#} C_\alpha = M_{\theta\alpha^\#} \overline{M_{\theta^\# \alpha}} = M_{\theta\alpha^\# \theta^\# \alpha}.$$

Therefore $C_\theta J^\star C_\alpha$ is an involution if and only if

$$\overline{\theta\alpha^\# \theta^\# \alpha} = 1 \quad \text{a.e. on } \mathbb{T}, \quad \text{i.e., } \theta\theta^\# = \alpha\alpha^\#,$$

which means that

$$(\overline{\theta\alpha^\#})(z) = \overline{(\theta^\# \alpha)}(z) = \theta(\bar{z}) \overline{\alpha^\#(\bar{z})} = (\overline{\theta\alpha^\#})(\bar{z}) \quad \text{a.e. on } \mathbb{T}.$$

□

The theorem bellow characterizes all M_z -commuting conjugations mapping one S -invariant subspace into another S -invariant subspace.

Theorem 5.2 *Let θ and α be two inner functions and let C be a conjugation in L^2 such that $CM_z = M_z C$. Then $C(\alpha H^2) \subset \theta H^2$ if and only if $\theta\theta^\# \leq \alpha\alpha^\#$ and $C = C_\beta J^\star C_\alpha$, where β is an inner function such that $\theta \leq \beta$, $\beta\beta^\# = \alpha\alpha^\#$. Moreover, in that case $C(\alpha H^2) = \beta H^2$.*

Let α be a fixed inner function. By Lemma 5.1, for each inner function β with $\beta\beta^\# = \alpha\alpha^\#$ there exists an M_z -commuting conjugation C which maps αH^2 onto βH^2 , namely $C = C_\beta J^\star C_\alpha$. On the other hand, if β is an inner function and there exists an M_z -commuting conjugation C which maps αH^2 onto βH^2 , then by Theorem 5.2, $\beta\beta^\# \leq \alpha\alpha^\#$ and $C = C_\gamma J^\star C_\alpha$ for some inner function γ such that $\beta \leq \gamma$, $\gamma\gamma^\# = \alpha\alpha^\#$. In particular, $C(\alpha H^2) = \gamma H^2 = \beta H^2$ and so γ is a constant multiple of β , $\beta\beta^\# = \alpha\alpha^\#$.

It follows from the above that Lemma 5.3 characterizes all possible spaces of type βH^2 such that for a given S -invariant subspace αH^2 there is an M_z -commuting conjugation mapping αH^2 onto βH^2 .

Lemma 5.3 *Let α be a nonconstant inner function. Then*

$$\begin{aligned} & \{\beta : \beta \text{ is inner, } \alpha\alpha^\# = \beta\beta^\#\} \\ &= \{\lambda uv^\# : u, v \text{ are inner, } \alpha = uv, \lambda \in \mathbb{T}\}. \end{aligned} \quad (5.1)$$

For two inner functions α and β denote by $\alpha \wedge \beta$ the greatest common divisor of α and β . We will write $\alpha \wedge \beta = 1$ if the only common divisor of α and β is a constant function.

Proof Note that for $\alpha = uv$ and $\beta = \lambda uv^\#$ we have $\alpha\alpha^\# = \beta\beta^\#$, hence one inclusion is proved. For the other inclusion let $u = \alpha \wedge \beta$ and we can write $\alpha = uv$ and $\beta = uv_1$. From the condition $\alpha\alpha^\# = \beta\beta^\#$ it follows that

$$uvu^\#v^\# = uv_1u^\#v_1^\#.$$

Hence $vv^\# = v_1v_1^\#$. Since $v \wedge v_1 = 1$, we have that v divides $v_1^\#$ and v_1 divides $v^\#$ and vice-versa. Thus we can take $v_1 = \lambda v^\#$ with $\lambda \in \mathbb{T}$, and so $\beta = \lambda uv^\#$. $\square$

Proof of Theorem 5.2 Assume firstly that $CM_z = M_zC$ and $C(\alpha H^2) \subset \theta H^2$. By Theorem 2.4, $C = M_\psi J^\star$ for some unimodular symmetric function $\psi \in L^\infty$. In particular,

$$\psi \alpha^\# = M_\psi J^\star(\alpha) = C(\alpha) \in \theta H^2,$$

and there exists $u \in H^2$ such that $\psi \alpha^\# = \theta u$. Note that u must be inner and so $\psi = \beta \alpha^\#$ with $\beta = \theta u$, $\theta \leq \beta$. Clearly $\beta \alpha^\#$ is symmetric, i.e., $\beta \beta^\# = \alpha \alpha^\#$. Hence $\theta \theta^\# \leq \alpha \alpha^\#$.

Assume now that $\theta \theta^\# \leq \alpha \alpha^\#$, and let $\alpha = \alpha_1 \cdot (\alpha \wedge \theta)$ and $\theta = \theta_1 \cdot (\alpha \wedge \theta)$. Since $(\alpha \wedge \theta)^\# = \alpha^\# \wedge \theta^\#$, we get $\alpha^\# = \alpha_1^\# \cdot (\alpha^\# \wedge \theta^\#)$ and $\theta^\# = \theta_1^\# \cdot (\alpha^\# \wedge \theta^\#)$. Note also that $\theta_1 \theta_1^\# \leq \alpha_1 \alpha_1^\#$ and $\theta_1 \wedge \alpha_1 = 1$, so $\theta_1 \leq \alpha_1^\#$. Thus

$$\frac{\alpha \alpha^\#}{\theta \theta^\#} = \frac{\alpha_1 \alpha_1^\#}{\theta_1 \theta_1^\#} = \frac{\alpha_1^\#}{\theta_1} \frac{\alpha_1}{\theta_1^\#} = uu^\#,$$

where $u = \frac{\alpha_1^\#}{\theta_1}$ is an inner function. Now we may take $\beta = \theta u$. Since $\theta \leq \beta$ and $\beta \beta^\# = \alpha \alpha^\#$, by Lemma 5.1 and by Proposition 4.1, $C = M_{\beta \alpha^\#} J^\star = C_\beta J^\star C_\alpha$ is a conjugation which maps αH^2 onto $\beta H^2 \subset \theta H^2$. $\square$

Corollary 5.4 *Let θ be an inner function and let C be an M_z -commuting conjugation in L^2 . Then*

- (1) $C(\theta H^2) \subset \theta H^2$ if and only if $C = \lambda C_\theta J^\star C_\theta$ with $\lambda \in \mathbb{T}$;
- (2) $C(\theta H^2) \subset \theta^\# H^2$ if and only if $C = \lambda J^\star$ with $\lambda \in \mathbb{T}$.

Proof By Theorem 5.2, $C(\theta H^2) \subset \theta H^2$ if and only if there exists an inner function β such that $\theta \leq \beta$ and $\beta \beta^\# = \theta \theta^\#$. This is only possible if β is constant multiple of θ and (1) is proved. The proof of (2) is similar. $\square$

Note that by Theorem 5.2 (Lemma 5.1, actually) if $\overline{\theta \alpha^\#}$ is symmetric, then there exists an M_z -commuting conjugation from αH^2 into θH^2 . The following example shows that in that case there may be no such conjugation between the corresponding model spaces K_α and K_θ .

Example 5.5 Fix $a, b \in \mathbb{D}$ such that $a \neq b$, $a \neq \bar{a}$ and $b \neq \bar{b}$, and put

$$\alpha(z) = \frac{a-z}{1-\bar{a}z} \frac{b-z}{1-\bar{b}z} \quad \text{and} \quad \theta(z) = \frac{a-z}{1-\bar{a}z} \frac{\bar{b}-z}{1-\bar{b}z}.$$

Then

$$\alpha^\#(z) = \frac{\bar{a}-z}{1-\bar{a}z} \frac{\bar{b}-z}{1-\bar{b}z} \quad \text{and} \quad \theta^\#(z) = \frac{\bar{a}-z}{1-\bar{a}z} \frac{b-z}{1-\bar{b}z}$$

and so $\alpha\alpha^\# = \theta\theta^\#$. Thus there exists an M_z -commuting conjugation from αH^2 onto θH^2 . In this case however neither $\alpha \leq \theta^\#$ nor $\theta \leq \alpha^\#$, so by Theorem 4.6 no M_z -commuting conjugation between K_α and K_θ exists. Here also neither $\alpha \leq \theta$ nor $\theta \leq \alpha$, and so by Theorem 4.2 no M_z -conjugation between K_α and K_θ exists.

Finally, consider M_z -conjugations preserving S -invariant subspaces.

Proposition 5.6 *Let θ and α be two inner functions. There are no M_z -conjugations in L^2 which map αH^2 into θH^2 .*

Proof If C was such a conjugation, then by Theorem 2.2, $C = M_\psi J$ for some unimodular function $\psi \in L^\infty$ and, in particular,

$$C(\alpha) = \psi \bar{\alpha} = \theta g \quad \text{for some } g \in H^2.$$

Clearly g must be an inner function and $\psi = \alpha \theta g$. Then, for every $h \in H^2$,

$$C(\alpha h) = \alpha \theta g \bar{\alpha h} = \theta g \bar{h} \in \theta H^2,$$

and so $g \bar{h} \in H^2$. It follows that $g = 0$ and $C(\alpha) = 0$ which is a contradiction. $\square$

6 Conjugations and truncated Toeplitz operators

For $\varphi \in L^2$ define the *truncated Toeplitz operator* A_φ^θ by

$$A_\varphi^\theta f = P_\theta(\varphi f), \quad \text{for } f \in H^\infty \cap K_\theta,$$

where $P_\theta: L^2 \rightarrow K_\theta$ is the orthogonal projection (see [14]). The operator A_φ^θ is closed and densely defined, and if it is bounded, it admits a unique bounded extension to K_θ . The set of all bounded truncated Toeplitz operators on K_θ is denoted by $\mathcal{T}(\theta)$. Note that $A_\varphi^\theta \in \mathcal{T}(\theta)$ for $\varphi \in L^\infty$. It is known that every operator from $\mathcal{T}(\theta)$ is C_θ -symmetric (see [14, Lemma 2.1]).

Observe that if $k \geq 0$, then the conjugation $C_{k,l}$ defined by (3.1) satisfies neither $M_z C_{k,l} = C_{k,l} M_z$ nor $S C_{k,l} = C_{k,l} S$. However, for $0 \leq n < k$ and $\theta(z) = z^n$,

$$C_{k,l}(K_\theta) = K_\theta \quad \text{and} \quad A_z^\theta C_{k,l} = C_{k,l} A_z^\theta$$

(since here $C_{k,l}|_{K_\theta} = J_{|K_\theta}^*$ and $\theta^\# = \theta$).

Theorem below characterizes conjugations intertwining truncated shifts A_z^θ and $A_z^{\theta^\#}$.

Theorem 6.1 *Let θ be a nonconstant inner function and let C be a conjugation in L^2 such that $C(K_\theta) \subset K_{\theta^\#}$. Then the following are equivalent:*

- (1) $A_{\varphi^\#}^{\theta^\#} C = C A_\varphi^\theta$ on K_θ for all $\varphi \in H^\infty$,
- (2) $A_z^{\theta^\#} C = C A_z^\theta$ on K_θ ,

- (3) there is a function $\psi \in H^\infty$ such that $C|_{K_\theta} = J^* A_\psi^\theta$ and A_ψ^θ is an isometry,
 (4) there is a function $\psi' \in H^\infty$ such that $C|_{K_\theta} = A_{\psi'}^{\theta^\#} J_{|K_\theta}^*$ and $A_{\psi'}^{\theta^\#}$ is an isometry.

Proof We will only prove that (2) $\Rightarrow$ (3). Since $J^*(K_{\theta^\#}) = K_\theta$ and $J^* A_{\varphi^\#}^{\theta^\#} = A_\varphi^\theta J^*$ for all $\varphi \in H^\infty$ (see [4, Lemma 4.5]), we have

$$J^* C A_z^\theta = J^* A_z^{\theta^\#} C = A_z^\theta J^* C$$

on K_θ and so $J^* C|_{K_\theta} = A_\psi^\theta$ for some $\psi \in H^\infty$ ([5, Theorem 14.38]). Hence A_ψ^θ is an isometry and

$$C|_{K_\theta} = J^* A_\psi^\theta.$$

□

It is much more restrictive if $\theta^\# = \theta$.

Proposition 6.2 *Let θ be an inner function such that $\theta^\# = \theta$ and let C be a conjugation in K_θ . Then the following are equivalent:*

- (1) $A_{\varphi^\#}^\theta C = C A_\varphi^\theta$ for all $\varphi \in H^\infty$,
 (2) $A_z^\theta C = C A_z^\theta$,
 (3) $C = \lambda J_{|K_\theta}^*$ with $\lambda \in \mathbb{T}$.

Proof Implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear. To prove (2) $\Rightarrow$ (3) apply Theorem 6.1 to the conjugation $\tilde{C}$ in L^2 defined by

$$\tilde{C} = C \oplus C_\theta : K_\theta \oplus (K_\theta)^\perp \rightarrow K_\theta \oplus (K_\theta)^\perp.$$

It follows that $\tilde{C}|_{K_\theta} = C = J^* A_\psi^\theta$ for $\psi \in H^\infty$ such that A_ψ^θ is an isometry. Since $C(K_\theta) = K_\theta$ and $J^*(K_\theta) = K_{\theta^\#} = K_\theta$, we see that $A_\psi^\theta = J^* C$ maps K_θ onto K_θ and is in fact unitary. Thus we have

$$A_\psi^\theta A_\psi^\theta = A_\psi^\theta A_\psi^\theta = I_{K_\theta}.$$

On the other hand, $C^2 = I_{K_\theta}$ so

$$C^2 = J^* A_\psi^\theta J^* A_\psi^\theta = A_{\psi^\#}^\theta A_\psi^\theta = A_\psi^\theta A_{\psi^\#}^\theta = I_{K_\theta}.$$

Hence $A_\psi^\theta = A_{\psi^\#}^\theta$ and $A_{\psi - \psi^\#}^\theta = 0$, which gives $\overline{\psi} - \psi^\# \in \overline{\theta H^2} + \theta H^2$ (see [14]). In other words, $\overline{\psi} - \psi^\# = \overline{\theta h_1} + \theta h_2$ for some functions $h_1, h_2 \in H^2$. Thus there exists a constant λ such that

$$\psi - \theta h_1 = \overline{\psi^\# + \theta h_2} = \bar{\lambda}.$$

We now have

$$A_{\psi}^{\theta} = A_{\theta h_1 + \bar{\lambda}}^{\theta} = \bar{\lambda} I_{K_{\theta}}.$$

Moreover $\lambda \in \mathbb{T}$, since A_{ψ}^{θ} is unitary. Hence

$$C = J^{\star} A_{\psi}^{\theta} = \lambda J_{|K_{\theta}}^{\star}.$$

□

Now we characterize conjugations intertwining the truncated shifts A_z^{θ} and $A_{\bar{z}}^{\theta}$.

Theorem 6.3 *Let θ be an inner function and let C be a conjugation in K_{θ} . Then the following are equivalent:*

- (1) $A_{\varphi}^{\theta} C = C A_{\bar{\varphi}}^{\theta}$ for all $\varphi \in H^{\infty}$,
- (2) $A_z^{\theta} C = C A_{\bar{z}}^{\theta}$,
- (3) there is a function $\psi \in H^{\infty}$ such that $C = A_{\psi}^{\theta} C_{\theta}$ and A_{ψ}^{θ} is unitary,
- (4) there is a function $\psi' \in H^{\infty}$ such that $C = C_{\theta} A_{\bar{\psi}'}^{\theta}$ and $A_{\psi'}^{\theta}$ is unitary.

Proof Let us start with (2) $\Rightarrow$ (3). Since A_z^{θ} is C_{θ} -symmetric,

$$A_z^{\theta} C C_{\theta} = C A_{\bar{z}}^{\theta} C_{\theta} = C C_{\theta} A_z^{\theta}.$$

Hence, by [1, Proposition 1.21], $C C_{\theta} = A_{\psi}^{\theta}$ for some $\psi \in H^{\infty}$. Clearly, A_{ψ}^{θ} is unitary and

$$C = A_{\psi}^{\theta} C_{\theta} = C_{\theta} (A_{\psi}^{\theta})^{\star} = C_{\theta} A_{\bar{\psi}}^{\theta}.$$

To prove that (4) $\Rightarrow$ (1) note that, since $A_{\bar{\psi}'}^{\theta}$ and $A_{\bar{\varphi}}^{\theta}$ commute, we have

$$A_{\varphi}^{\theta} C = A_{\varphi}^{\theta} C_{\theta} A_{\bar{\psi}'}^{\theta} = C_{\theta} A_{\bar{\varphi}}^{\theta} A_{\bar{\psi}'}^{\theta} = C_{\theta} A_{\bar{\psi}'}^{\theta} A_{\bar{\varphi}}^{\theta} = C A_{\bar{\varphi}}^{\theta} = C (A_{\varphi}^{\theta})^{\star}.$$

All other implications are straightforward. □

Corollary 6.4 *If C is a conjugation in K_{θ} and every $A \in \mathcal{T}(\theta)$ is C -symmetric, then $C = A_{\psi}^{\theta} C_{\theta}$ for some $\psi \in H^{\infty}$ such that A_{ψ}^{θ} is unitary.*

For a complete description of unitary operators from $\mathcal{T}(\theta)$ see [15, Proposition 6.5].

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Bercovici, H.: Operator Theory and Arithmetic in H^∞ , Mathematical Surveys and Monographs, vol. 26. American Mathematical Society, Providence (1988)
2. Câmara, C., Kliś-Garlicka, K., Ptak, M.: Asymmetric truncated Toeplitz operators and conjugations. *Filomat* **33**, 3697–3710 (2019)
3. Chevrot, N., Frician, E., Timotin, D.: The characteristic function of a complex symmetric contraction. *Proc. Am. Math. Soc.* **135**, 2877–2886 (2007)
4. Cima, J.A., Garcia, S.R., Ross, W.T., Wogen, W.R.: Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity. *Indiana Univ. Math. J.* **59**(2), 595–620 (2010)
5. Fricain, E., Mashreghi, J.: The Theory of $\mathcal{H}(b)$ Spaces, vol. 1. Cambridge University Press, Cambridge (2016)
6. García, S.R., Mashreghi, J., Ross, W.T.: Introduction to Model Spaces and Their Operators. Cambridge University Press, Cambridge (2016)
7. García, S.R., Putinar, M.: Complex symmetric operators and applications. *Trans. Am. Math. Soc.* **358**, 1285–1315 (2006)
8. García, S.R., Prodan, E., Putinar, M.: Mathematical and physical aspects of complex symmetric operators. *J. Phys. A Math. Theor.* **47**, 353001 (2014)
9. García, S.R., Putinar, M.: Complex symmetric operators and applications II. *Trans. Am. Math. Soc.* **359**, 3913–3931 (2007)
10. Ko, E., Lee, J.E.: Remark on complex symmetric operator matrices. *Linear Multilinear Algebra* **67**(6), 1198–1216 (2019). <https://doi.org/10.1080/03081087.2018.1450350>
11. Martínez-Aveñano, R.A., Rosenthal, P.: An Introduction to Operators on the Hardy–Hilbert Space. Springer, New York (2007)
12. Sz.-Nagy, B., Foias, C., Bercovici, H., Kérchy, L.: Harmonic Analysis of Operators on a Hilbert Space. Springer, London (2010)
13. Radjavi, H., Rosenthal, P.: Invariant Subspaces. Springer, New York (1973)
14. Sarason, D.: Algebraic properties of truncated Toeplitz operators. *Oper. Matrices* **1**, 491–526 (2007)
15. Sedlock, N.A.: Algebras of truncated Toeplitz operators. *Oper. Matrices* **5**, 309–326 (2011)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.