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Paley–Wiener properties for spaces of entire functions

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Abstract
We deduce Paley–Wiener results in the Bargmann setting. At the same time we deduce characterisations of Pilipović spaces of low orders. In particular we improve the characterisation of the Gröchenig test function space $\mathcal{H}_{\flat_1} = \mathcal{S}_C$, deduced in Toft (J Pseudo-Differ Oper Appl 8:83–139, 2017).

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1 Introduction

Paley–Wiener theorems characterize functions and distributions with certain restricted supports in terms of estimates of their Fourier–Laplace transforms. For example, let f be a distribution on $\mathbf{R}^d$ and let $B_{r_0}(0) \subseteq \mathbf{R}^d$ be the ball with center at origin and radius r_0. Then f is supported in $B_{r_0}(0)$ if and only if

$$|\widehat{f}(\zeta)| \lesssim \langle \zeta \rangle^{N} e^{r_0 |\operatorname{Im}(\zeta)|}, \quad \zeta \in \mathbf{C}^d,$$

for some $N \geq 0$. Furthermore, f is supported in $B_{r_0}(0)$ and smooth, if and only if

$$|\widehat{f}(\zeta)| \lesssim \langle \zeta \rangle^{-N} e^{r_0 |\operatorname{Im}(\zeta)|}, \quad \zeta \in \mathbf{C}^d,$$

for every $N \geq 0$.

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A similar approach for ultra-regular functions of Gevrey types and corresponding ultra-distribution spaces can be done. In fact, let $s > 1$, $\mathcal{D}_s'(\mathbf{R}^d)$ be the set of all Gevrey distributions of order s and let $\mathcal{E}_s(\mathbf{R}^d)$ be the set of all smooth functions with Gevrey regularity s. (See [6] and Sect. 2 for notations.) Then it can be proved that $f \in \mathcal{D}_s'(\mathbf{R}^d)$ is supported in $B_{r_0}(0)$, if and only if

$$|\widehat{f}(\zeta)| \lesssim e^{r_0|\operatorname{Im}(\zeta)|+r|\zeta|^{\frac{1}{s}}}, \quad \zeta \in \mathbf{C}^d,$$

for every $r > 0$. Furthermore $f \in \mathcal{E}_s(\mathbf{R}^d)$ is supported in $B_{r_0}(0)$, if and only if

$$|\widehat{f}(\zeta)| \lesssim e^{r_0|\operatorname{Im}(\zeta)|-r|\zeta|^{\frac{1}{s}}}, \quad \zeta \in \mathbf{C}^d,$$

for some $r > 0$.

We observe that s in the latter result can not be pushed to be smaller, because if $s \le 1$, it does not make any sense to discuss compact support properties of $\mathcal{D}_s'(\mathbf{R}^d)$ and $\mathcal{E}_s(\mathbf{R}^d)$.

In the paper we consider analogous Paley–Wiener properties after the Fourier–Laplace transform above is replaced by the reproducing kernel Π_A for the Bargmann transform, and the image spaces are replaced by suitable subspaces of entire functions on $\mathbf{C}^d$. These subspaces were considered in [4,12] and are given by

$$\begin{aligned} \mathcal{A}_{\flat_\sigma}(\mathbf{C}^d) &= \bigcup_{r>0} \mathcal{A}_{r,\flat_\sigma}(\mathbf{C}^d), & \mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d) &= \bigcap_{r>0} \mathcal{A}_{r,\flat_\sigma}(\mathbf{C}^d), \\ \mathcal{A}_{s}(\mathbf{C}^d) &= \bigcup_{r>0} \mathcal{A}_{r,s}(\mathbf{C}^d), & \mathcal{A}_{0,s}(\mathbf{C}^d) &= \bigcap_{r>0} \mathcal{A}_{r,s}(\mathbf{C}^d), \end{aligned} \tag{1.1}$$

when $\sigma > 0$ and $0 < s < \frac{1}{2}$, where $\mathcal{A}_{r,\flat_\sigma}(\mathbf{C}^d)$ and $\mathcal{A}_{r,s}(\mathbf{C}^d)$ are the sets of all entire F on $\mathbf{C}^d$ such that

$$|F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma+1}}} \quad \text{respective} \quad |F(z)| \lesssim e^{r(\log\langle z\rangle)^{\frac{1}{1-2s}}}.$$

The spaces in (1.1) appear naturally when considering the Bargmann transform images of extended classes of Fourier invariant Gelfand–Shilov spaces, called Pilipović spaces (see [4,12]).

If (z, w) is the scalar product of $z, w \in \mathbf{C}^d$, then the reproducing kernel of the Bargmann transform is given by

$$(\Pi_A F)(z) = \pi^{-d}\langle F, e^{(z,\,\cdot\,)-|\,\cdot\,|^2}\rangle$$

when F is a suitable function or (ultra-)distribution. If

$$z \mapsto F(z)e^{R|z|-|z|^2} \in L^1(\mathbf{C}^d), \quad R > 0 \tag{1.2}$$

holds and $d\lambda(w)$ is the Lebesgue measure on $\mathbf{C}^d$, then

$$(\Pi_A F)(z) = \pi^{-d} \int_{\mathbf{C}^d} F(w) e^{(z,w)-|w|^2}\, d\lambda(w).$$

A recent Paley–Wiener result with respect to the transform Π_A and image spaces (1.1) is given in [12], where it is proved that if $L_c^\infty(\mathbf{C}^d) = \mathscr{E}'(\mathbf{C}^d) \cap L^\infty(\mathbf{C}^d)$, then

$$\Pi_A(\mathscr{E}'(\mathbf{C}^d)) = \Pi_A(L_c^\infty(\mathbf{C}^d)) = \mathcal{A}_{\flat_1}(\mathbf{C}^d). \tag{1.3}$$

Evidently, $L_c^\infty(\mathbf{C}^d) \subseteq \mathscr{E}'(\mathbf{C}^d)$, and the gap between these spaces are rather large. It might therefore be somewhat surprising that the first equality holds in (1.3).

In Sect. 3 we improve (1.3) in different ways. Firstly we show that we may replace $L_c^\infty(\mathbf{C}^d)$ in (1.3) with the smaller space $L_{A,c}^\infty(\mathbf{C}^d)$ given by

$$L_{A,c}^\infty(\mathbf{C}^d) = \bigcup L_{A,c}^\infty(K),$$

where $L_{A,c}^\infty(K)$ is the set of all $F \cdot \chi_K$, where F is analytic in a neighbourhood of the compact set K and χ_K is the characteristic function of K. Secondly, we may replace $\mathscr{E}'(\mathbf{C}^d)$ with the set $\mathcal{E}_s'(\mathbf{C}^d)$ of all compactly supported Gevrey distributions of order $s > 1$. Summing up we improve (1.3) into

$$\Pi_A(\mathcal{E}_s'(\mathbf{C}^d)) = \Pi_A(L_{A,c}^\infty(\mathbf{C}^d)) = \mathcal{A}_{\flat_1}(\mathbf{C}^d), \quad s > 1. \tag{1.3$'$}$$

In Sect. 3 we also deduce various kinds of related mapping properties when $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$ in (1.3) is replaced by any of the spaces in (1.1). More precisely, let $\chi \in L_c^\infty(\mathbf{C}^d)$ be non-negative, radial symmetric in each complex variable z_j and bounded from below by a positive constant near origin. Then we prove

$$\begin{aligned}
\Pi_A(\mathcal{A}'_{0,\flat_{\sigma_0}}(\mathbf{C}^d) \cdot \chi) &= \mathcal{A}_{\flat_\sigma}(\mathbf{C}^d), \quad \sigma \in (\tfrac{1}{2}, 1),\ \sigma_0 = \frac{\sigma}{2\sigma - 1},\\
\Pi_A(\mathcal{A}_{\flat_{\sigma_0}}(\mathbf{C}^d) \cdot \chi) &= \mathcal{A}_{\flat_\sigma}(\mathbf{C}^d), \quad \sigma \in (0, \tfrac{1}{2}),\ \sigma_0 = \frac{\sigma}{1 - 2\sigma},\\
\Pi_A(\mathcal{A}_s(\mathbf{C}^d) \cdot \chi) &= \mathcal{A}_s(\mathbf{C}^d), \quad s \in [0, \tfrac{1}{2}),
\end{aligned}$$

and similarly for $\sigma = \frac{1}{2}$ and when $\mathcal{A}_{\flat_\sigma}$ and $\mathcal{A}_s$ are replaced by $\mathcal{A}_{0,\flat_\sigma}$ and $\mathcal{A}_{0,s}$, respectively. (Cf. Theorems 3.2–3.8 and Propositions 3.9–3.11.)

Finally, in Sect. 4 we use the results in Sect. 3 to deduce characterizations of Pilipović spaces of small orders. We remark that some continuations of the present analysis, involving larger spaces of analytic functions or formal power series expansions, can be found in [13].

2 Preliminaries

In this section we recall some basic facts. We start by discussing Pilipović spaces and some of their properties. Then we recall some facts on modulation spaces. Finally we recall the Bargmann transform and some of its mapping properties, and introduce suitable classes of entire functions on $\mathbf{C}^d$.

2.1 The Pilipović spaces

In order to define Pilipović spaces we recall the definition of Hermite functions. We recall that the Hermite function of order $\alpha \in \mathbf{N}^d$ is defined by

$$h_\alpha(x) = \pi^{-\frac{d}{4}}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{\frac{|x|^2}{2}}(\partial^\alpha e^{-|x|^2}).$$

It follows that

$$h_\alpha(x) = \pi^{-\frac{d}{4}}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{-\frac{|x|^2}{2}}p_\alpha(x),$$

for some polynomial p_α on $\mathbf{R}^d$, which is called the Hermite polynomial of order α. The Hermite functions are eigenfunctions to the Fourier transform, and to the Harmonic oscillator $H_d \equiv |x|^2 - \Delta$ which acts on functions and (ultra-)distributions defined on $\mathbf{R}^d$. More precisely, we have

$$H_d h_\alpha = (2|\alpha| + d)h_\alpha, \qquad H_d \equiv |x|^2 - \Delta.$$

It is well-known that the set of Hermite functions is a basis for $\mathscr{S}(\mathbf{R}^d)$ and an orthonormal basis for $L^2(\mathbf{R}^d)$ (cf. [10]). In particular, if $f \in L^2(\mathbf{R}^d)$, then

$$\|f\|^2_{L^2(\mathbf{R}^d)} = \sum_{\alpha \in \mathbf{N}^d} |c_h(f,\alpha)|^2,$$

where

$$f(x) = \sum_{\alpha \in \mathbf{N}^d} c_h(f,\alpha)h_\alpha \tag{2.1}$$

is the Hermite series expansion of f, and

$$c_h(f,\alpha) = (f, h_\alpha)_{L^2(\mathbf{R}^d)} \tag{2.2}$$

is the Hermite coefficient of f of order $\alpha \in \mathbf{R}^d$.

In order to define the full scale of Pilipović spaces, their order s should belong to the extended set

$$\mathbf{R}_\flat = \mathbf{R}_+ \cup \{\, \flat_\sigma \,;\, \sigma \in \mathbf{R}_+ \,\},$$

of $\mathbf{R}_+$, with extended inequality relations as

$$s_1 < \flat_\sigma < s_2 \quad \text{and} \quad \flat_{\sigma_1} < \flat_{\sigma_2}$$

when $s_1 < \frac{1}{2} \le s_2$ and $\sigma_1 < \sigma_2$. (Cf. [12].)

For such s and $r \in \mathbf{R}_+^d$ we set

$$\vartheta_{r,s}(\alpha) \equiv \begin{cases} e^{-(\frac{1}{r_1}\cdot\alpha_1^{\frac{1}{2s}}+\cdots+\frac{1}{r_d}\cdot\alpha_d^{\frac{1}{2s}})}, & s \in \mathbf{R}_+\backslash\{\frac{1}{2}\}, \\ r^\alpha \alpha!^{-\frac{1}{2\sigma}}, & s = \flat_\sigma, \\ r^\alpha, & s = \frac{1}{2}, \qquad \alpha \in \mathbf{N}^d \end{cases} \tag{2.3}$$

and

$$\vartheta'_{r,s}(\alpha) \equiv \begin{cases} e^{(\frac{1}{r_1}\cdot\alpha_1^{\frac{1}{2s}}+\cdots+\frac{1}{r_d}\cdot\alpha_d^{\frac{1}{2s}})}, & s \in \mathbf{R}_+\backslash\{\frac{1}{2}\}, \\ r^\alpha \alpha!^{\frac{1}{2\sigma}}, & s = \flat_\sigma, \\ r^\alpha, & s = \frac{1}{2}, \qquad \alpha \in \mathbf{N}^d. \end{cases} \tag{2.4}$$

Definition 2.1 Let $s \in \overline{\mathbf{R}_\flat} = \mathbf{R}_\flat \cup \{0\}$, and let $\vartheta_{r,s}$ and $\vartheta'_{r,s}$ be as in (2.3) and (2.4).

(1) $\mathcal{H}_0(\mathbf{R}^d)$ consists of all finite Hermite series expansions (2.1), and $\mathcal{H}'_0(\mathbf{R}^d)$ consists of all formal Hermite series expansions in (2.1);

(2) if $s \in \mathbf{R}_\flat$, then $\mathcal{H}_s(\mathbf{R}^d)$ ($\mathcal{H}_{0,s}(\mathbf{R}^d)$) consists of all $f \in L^2(\mathbf{R}^d)$ such that

$$|c_h(f, h_\alpha)| \lesssim \vartheta_{r,s}(\alpha)$$

holds true for some $r \in \mathbf{R}_+^d$ (for every $r \in \mathbf{R}_+^d$);

(3) if $s \in \mathbf{R}_\flat$, then $\mathcal{H}'_s(\mathbf{R}^d)$ ($\mathcal{H}'_{0,s}(\mathbf{R}^d)$) consists of all formal Hermite series expansions in (2.1) such that

$$|c_h(f, h_\alpha)| \lesssim \vartheta'_{r,s}(\alpha)$$

holds true for every $r \in \mathbf{R}_+^d$ (for some $r \in \mathbf{R}_+^d$).

The spaces $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ are called *Pilipović spaces of Roumieu respectively Beurling type* of order s, and $\mathcal{H}'_s(\mathbf{R}^d)$ and $\mathcal{H}'_{0,s}(\mathbf{R}^d)$ are called *Pilipović distribution spaces of Roumieu respectively Beurling type* of order s.

Remark 2.2 Let $\mathscr{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ be the Fourier invariant Gelfand–Shilov spaces of order $s \in \mathbf{R}_+$ and of Rourmeu and Beurling types respectively (see [12] for notations). Then it is proved in [8,9] that

$$\mathcal{H}_{0,s}(\mathbf{R}^d) = \Sigma_s(\mathbf{R}^d) \neq \{0\}, \quad s > \frac{1}{2},$$
$$\mathcal{H}_{0,s}(\mathbf{R}^d) \neq \Sigma_s(\mathbf{R}^d) = \{0\}, \quad s \le \frac{1}{2},$$

$$\mathcal{H}_s(\mathbf{R}^d) = \mathcal{S}_s(\mathbf{R}^d) \neq \{0\}, \quad s \geq \frac{1}{2}$$

and

$$\mathcal{H}_s(\mathbf{R}^d) \neq \mathcal{S}_s(\mathbf{R}^d) = \{0\}, \quad s < \frac{1}{2}.$$

In Proposition 2.3 below we give further characterisations of Pilipović spaces.

Next we recall the topologies for Pilipović spaces. Let $s \in \mathbf{R}_\flat$, $r > 0$, and let $\|f\|_{\mathcal{H}_{s;r}}$ and $\|f\|_{\mathcal{H}'_{s;r}}$ be given by

$$\|f\|_{\mathcal{H}_{s;r}} \equiv \sup_{\alpha \in \mathbf{N}^d} |c_h(f,\alpha)\vartheta'_{r,s}(\alpha)|, \quad s \in \mathbf{R}_\flat, \tag{2.5}$$

and

$$\|f\|_{\mathcal{H}'_{s;r}} \equiv \sup_{\alpha \in \mathbf{N}^d} |c_h(f,\alpha)\vartheta_{r,s}(\alpha)|, \quad s \in \mathbf{R}_\flat. \tag{2.6}$$

when f is a formal expansion in (2.1). Then $\mathcal{H}_{s;r}(\mathbf{R}^d)$ consists of all expansions (2.1) such that $\|f\|_{\mathcal{H}_{s;r}}$ is finite, and $\mathcal{H}'_{s;r}(\mathbf{R}^d)$ consists of all expansions (2.1) such that $\|f\|_{\mathcal{H}'_{s;r}}$ is finite. It follows that both $\mathcal{H}_{s;r}(\mathbf{R}^d)$ and $\mathcal{H}'_{s;r}(\mathbf{R}^d)$ are Banach spaces under the norms $f \mapsto \|f\|_{\mathcal{H}_{s;r}}$ and $f \mapsto \|f\|_{\mathcal{H}'_{s;r}}$, respectively.

We let the topologies of $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ be the inductive respectively projective limit topology of $\mathcal{H}_{s;r}(\mathbf{R}^d)$ with respect to $r > 0$. In the same way, the topologies of $\mathcal{H}'_s(\mathbf{R}^d)$ and $\mathcal{H}'_{0,s}(\mathbf{R}^d)$ are the projective respectively inductive limit topology of $\mathcal{H}'_{s;r}(\mathbf{R}^d)$ with respect to $r > 0$. It follows that all the spaces in Definition 2.1 are complete, and that $\mathcal{H}_{0,s}(\mathbf{R}^d)$ and $\mathcal{H}'_s(\mathbf{R}^d)$ are Fréchet spaces with semi-norms $f \mapsto \|f\|_{\mathcal{H}_{s;r}}$ and $f \mapsto \|f\|_{\mathcal{H}'_{s;r}}$, respectively.

The following characterisations for Pilipović spaces can be found in [12]. The proof is therefore omitted.

Proposition 2.3 *Let $s \in \mathbf{R}_+ \cup \{0\}$ and let $f \in \mathcal{H}'_0(\mathbf{R}^d)$. Then $f \in \mathcal{H}_{0,s}(\mathbf{R}^d)$ ($f \in \mathcal{H}_s(\mathbf{R}^d)$), if and only if $f \in C^\infty(\mathbf{R}^d)$ and satisfies*

$$\sup_{N \in \mathbf{N}} \left(\frac{\|H_d^N f\|_{L^\infty}}{h^N N!^{2s}} \right) < \infty$$

for every $h > 0$ (for some $h > 0$).

From now on we let

$$\phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}. \tag{2.7}$$

2.2 Spaces of entire functions and the Bargmann transform

Let $\Omega \subseteq \mathbf{C}^d$ be open. Then $A(\Omega)$ is the set of all analytic functions in Ω. If instead $\Omega \subseteq \mathbf{C}^d$ is closed, then $A(\Omega)$ is the set of all functions which are analytic in an open neighbourhood of Ω. In particular, if $z_0 \in \mathbf{C}^d$ is fixed, then $A(\{z_0\})$ is the set of all complex-valued functions which are defined and analytic near z_0.

We shall now consider the Bargmann transform which is defined by the formula

$$(\mathfrak{V}_d f)(z) = \pi^{-\frac{d}{4}} \int_{\mathbf{R}^d} \exp\Big(-\frac{1}{2}(\langle z, z\rangle + |y|^2) + 2^{\frac{1}{2}}\langle z, y\rangle\Big) f(y)\, dy,$$

when $f \in L^2(\mathbf{R}^d)$ (cf. [1]). We note that for all $f \in L^2(\mathbf{R}^d)$, then the Bargmann transform $\mathfrak{V}_d f$ of f is the entire function on $\mathbf{C}^d$, given by

$$(\mathfrak{V}_d f)(z) = \int_{\mathbf{R}^d} \mathfrak{A}_d(z, y) f(y)\, dy,$$

or

$$(\mathfrak{V}_d f)(z) = \langle f, \mathfrak{A}_d(z, \cdot\,)\rangle, \tag{2.8}$$

where the Bargmann kernel $\mathfrak{A}_d$ is given by

$$\mathfrak{A}_d(z, y) = \pi^{-\frac{d}{4}} \exp\left(-\frac{1}{2}(\langle z, z\rangle + |y|^2) + 2^{\frac{1}{2}}\langle z, y\rangle\right).$$

Here

$$\langle z, w\rangle = \sum_{j=1}^{d} z_j w_j, \quad \text{when } z = (z_1, \dots, z_d) \in \mathbf{C}^d \text{ and } w = (w_1, \dots, w_d) \in \mathbf{C}^d,$$

and otherwise $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality between test function spaces and their corresponding duals. We note that the right-hand side in (2.8) makes sense when $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ and defines an element in $A(\mathbf{C}^d)$, since $y \mapsto \mathfrak{A}_d(z, y)$ can be interpreted as an element in $\mathcal{S}_{1/2}(\mathbf{R}^d)$ with values in $A(\mathbf{C}^d)$.

It was proved in [1] that $f \mapsto \mathfrak{V}_d f$ is a bijective and isometric map from $L^2(\mathbf{R}^d)$ to the Hilbert space $A^2(\mathbf{C}^d) \equiv B^2(\mathbf{C}^d) \cap A(\mathbf{C}^d)$, where $B^2(\mathbf{C}^d)$ consists of all measurable functions F on $\mathbf{C}^d$ such that

$$\|F\|_{B^2} \equiv \left(\int_{\mathbf{C}^d} |F(z)|^2 d\mu(z)\right)^{\frac{1}{2}} < \infty. \tag{2.9}$$

Here $d\mu(z) = \pi^{-d} e^{-|z|^2}\, d\lambda(z)$, where $d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}^d$. We recall that $A^2(\mathbf{C}^d)$ and $B^2(\mathbf{C}^d)$ are Hilbert spaces, where the scalar products are given by

$$(F, G)_{B^2} \equiv \int_{\mathbf{C}^d} F(z)\overline{G(z)}\, d\mu(z), \quad F, G \in B^2(\mathbf{C}^d). \tag{2.10}$$

If $F, G \in A^2(\mathbf{C}^d)$, then we set $\|F\|_{A^2} = \|F\|_{B^2}$ and $(F,G)_{A^2} = (F,G)_{B^2}$.

Furthermore, Bargmann showed that there is a convenient reproducing formula on $A^2(\mathbf{C}^d)$. More precisely, let

$$(\Pi_A F)(z) \equiv \int_{\mathbf{C}^d} F(w) e^{(z,w)}\, d\mu(w), \tag{2.11}$$

when $z \mapsto F(z)e^{R|z|-|z|^2}$ belongs to $L^1(\mathbf{C}^d)$ for every $R \ge 0$. Here

$$(z, w) = \sum_{j=1}^{d} z_j \overline{w_j}, \quad \text{when } z = (z_1, \dots, z_d) \in \mathbf{C}^d \text{ and } w = (w_1, \dots, w_d) \in \mathbf{C}^d,$$

is the scalar product of $z \in \mathbf{C}^d$ and $w \in \mathbf{C}^d$. Then it is proved in [1,2] that Π_A is the orthogonal projection of $B^2(\mathbf{C}^d)$ onto $A^2(\mathbf{C}^d)$. In particular, $\Pi_A F = F$ when $F \in A^2(\mathbf{C}^d)$.

In [1] it is also proved that

$$\mathfrak{V}_d h_\alpha = e_\alpha, \quad \text{where } e_\alpha(z) \equiv \frac{z^\alpha}{\sqrt{\alpha!}}, \ z \in \mathbf{C}^d. \tag{2.12}$$

In particular, the Bargmann transform maps the orthonormal basis $\{h_\alpha\}_{\alpha \in \mathbf{N}^d}$ in $L^2(\mathbf{R}^d)$ bijectively into the orthonormal basis $\{e_\alpha\}_{\alpha \in \mathbf{N}^d}$ of monomials in $A^2(\mathbf{C}^d)$. Hence, there is a natural way to identify formal Hermite series expansion by formal power series expansions

$$F(z) = \sum_{\alpha \in \mathbf{N}^d} c(F, \alpha) e_\alpha(z), \tag{2.13}$$

by letting the series (2.1) be mapped into

$$\sum_{\alpha \in \mathbf{N}^d} c_h(f, \alpha) e_\alpha(z). \tag{2.14}$$

It follows that if $f, g \in L^2(\mathbf{R}^d)$ and $F, G \in A^2(\mathbf{C}^d)$, then

$$\begin{aligned} (f, g)_{L^2(\mathbf{R}^d)} &= \sum_{\alpha \in \mathbf{N}^d} c_h(f, \alpha) \overline{c_h(g, \alpha)}, \\ (F, G)_{A^2(\mathbf{C}^d)} &= \sum_{\alpha \in \mathbf{N}^d} c(F, \alpha) \overline{c(G, \alpha)}. \end{aligned} \tag{2.15}$$

Here and in what follows, $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbf{R}^d)}$ and $(\,\cdot\,,\,\cdot\,)_{A^2(\mathbf{C}^d)}$ denote the scalar products in $L^2(\mathbf{R}^d)$ and $A^2(\mathbf{C}^d)$, respectively. Furthermore,

$$c_h(f, \alpha) = c(F, \alpha) \quad \text{when } F = \mathfrak{V}_d f. \tag{2.16}$$

We now recall the following spaces of power series expansions given in [12].

Definition 2.4 Let $s \in \overline{\mathbf{R}_\flat} = \mathbf{R}_\flat \cup \{0\}$, and let $\vartheta_{r,s}$ and $\vartheta'_{r,s}$ be as in (2.3) and (2.4).

(1) $\mathcal{A}_0(\mathbf{C}^d)$ consists of all analytic polynomials on $\mathbf{C}^d$, and $\mathcal{A}'_0(\mathbf{C}^d)$ consists of all formal power series expansions on $\mathbf{C}^d$ in (2.13);

(2) if $s \in \mathbf{R}_\flat$, then $\mathcal{A}_s(\mathbf{C}^d)$ ($\mathcal{A}_{0,s}(\mathbf{C}^d)$) consists of all $F \in L^2(\mathbf{C}^d)$ such that

$$|c(F, h_\alpha)| \lesssim \vartheta_{r,s}(\alpha)$$

holds true for some $r > 0$ (for every $r > 0$);

(3) if $s \in \mathbf{R}_\flat$, then $\mathcal{A}'_s(\mathbf{C}^d)$ ($\mathcal{A}'_{0,s}(\mathbf{C}^d)$) consists of all formal power series expansions in (2.13) such that

$$|c(F, h_\alpha)| \lesssim \vartheta'_{r,s}(\alpha)$$

holds true for every $r > 0$ (for some $r > 0$).

Let $f \in \mathcal{H}'_0(\mathbf{R}^d)$ with formal Hermite series expansion (2.1). Then the Bargmann transform $\mathfrak{V}_d f$ of f is defined to be the formal power series expansion (2.14). It follows that $\mathfrak{V}_d$ agrees with the earlier definition when acting on $L^2(\mathbf{R}^d)$, that $\mathfrak{V}_d$ is linear and bijective from $\mathcal{H}'_0(\mathbf{R}^d)$ to $\mathcal{A}'_0(\mathbf{C}^d)$, and restricts to bijections from the spaces

$$\mathcal{H}_{0,s}(\mathbf{R}^d), \quad \mathcal{H}_s(\mathbf{R}^d), \quad \mathcal{H}'_s(\mathbf{R}^d) \quad \text{and} \quad \mathcal{H}'_{0,s}(\mathbf{R}^d) \tag{2.17}$$

to

$$\mathcal{A}_{0,s}(\mathbf{C}^d), \quad \mathcal{A}_s(\mathbf{C}^d), \quad \mathcal{A}'_s(\mathbf{C}^d) \quad \text{and} \quad \mathcal{A}'_{0,s}(\mathbf{C}^d) \tag{2.18}$$

respectively, when $s \in \mathbf{R}_\flat$. We also let the topologies of the spaces in (2.18) be inherited from the spaces in (2.17).

If $s \in \overline{\mathbf{R}_\flat}$, $f \in \mathcal{H}_s(\mathbf{R}^d)$, $g \in \mathcal{H}'_s(\mathbf{R}^d)$, $F \in \mathcal{A}_s(\mathbf{C}^d)$ and $G \in \mathcal{A}'_s(\mathbf{C}^d)$, then $(f, g)_{L^2(\mathbf{R}^d)}$ and $(F, G)_{A^2(\mathbf{C}^d)}$ are defined by the formula (2.15). It follows that (2.16) holds for such choices of f and F. Furthermore, the duals of $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{A}_s(\mathbf{C}^d)$ can be identified with $\mathcal{H}'_s(\mathbf{R}^d)$ and $\mathcal{A}'_s(\mathbf{C}^d)$, respectively, through the forms in (2.15). The same holds true with

$$\mathcal{H}_{0,s}, \quad \mathcal{H}'_{0,s}, \quad \mathcal{A}_{0,s}, \quad \text{and} \quad \mathcal{A}'_{0,s}$$

in place of

$$\mathcal{H}_s, \quad \mathcal{H}'_s, \quad \mathcal{A}_s, \quad \text{and} \quad \mathcal{A}'_s,$$

respectively, at each occurrence.

In order to identify the spaces of power series expansions above with spaces of analytic functions, we let

$$M_{1,r,s}(z) = \begin{cases} r_1(\log\langle z_1\rangle)^{\frac{1}{1-2s}} + \cdots + r_d(\log\langle z_d\rangle)^{\frac{1}{1-2s}}, & s < \frac{1}{2}, \\ r_1|z_1|^{\frac{2\sigma}{\sigma+1}} + \cdots + r_d|z_d|^{\frac{2\sigma}{\sigma+1}}, & s = \flat_\sigma, \ \sigma > 0, \\ \frac{|z|^2}{2} - (r_1|z_1|^{\frac{1}{s}} + \cdots + r_d|z_d|^{\frac{1}{s}}), & s \geq \frac{1}{2}, \end{cases}$$

$$M^0_{1,r,s}(z) = \begin{cases} M_{1,r,s}(z), & s \neq \frac{1}{2}, \\ r_1|z_1|^2 + \cdots + r_d|z_d|^2, & s = \frac{1}{2}, \end{cases}$$

$$M_{2,r,s}(z) = \begin{cases} r_1|z_1|^{\frac{2\sigma}{\sigma-1}} + \cdots + r_d|z_d|^{\frac{2\sigma}{\sigma-1}}, & s = \flat_\sigma,\ \sigma > 1, \\ \frac{|z|^2}{2} + (r_1|z_1|^{\frac{1}{s}} + \cdots + r_d|z_d|^{\frac{1}{s}}), & s \geq \frac{1}{2}, \end{cases}$$

$$M^0_{2,r,s}(z) = \begin{cases} M_{2,r,s}(z), & s \neq \frac{1}{2}, \\ r_1|z_1|^2 + \cdots + r_d|z_d|^2, & s = \frac{1}{2}, \end{cases} \tag{2.19}$$

when $r \in \mathbf{R}^d_+$ and $z \in \mathbf{C}^d$. For conveniency we set $M_r = M_{1,\flat_1,r}$.

By [12] we have the following. The proof is therefore omitted.

Proposition 2.5 *Let $M_{1,r,s}$, $M^0_{1,r,s}$, $M_{2,r,s}$ and $M^0_{2,r,s}$ be as in (2.19) when $s \in \overline{\mathbf{R}_\flat}$ and $r \in \mathbf{R}^d_+$. Then*

$$\begin{aligned}
\mathcal{A}_{0,s}(\mathbf{C}^d) &= \{\, F \in A(\mathbf{C}^d)\,;\ Fe^{-M^0_{1,r,s}} \in L^\infty(\mathbf{C}^d) \textit{ for every } r \in \mathbf{R}^d_+ \,\}, \qquad s > 0,\\
\mathcal{A}_{s}(\mathbf{C}^d) &= \{\, F \in A(\mathbf{C}^d)\,;\ Fe^{-M_{1,r,s}} \in L^\infty(\mathbf{C}^d) \textit{ for some } r \in \mathbf{R}^d_+ \,\}, \qquad s \geq 0,\\
\mathcal{A}'_{s}(\mathbf{C}^d) &= \{\, F \in A(\mathbf{C}^d)\,;\ Fe^{-M_{2,r,s}} \in L^\infty(\mathbf{C}^d) \textit{ for every } r \in \mathbf{R}^d_+ \,\}, \qquad s > \flat_1,\\
\mathcal{A}'_{0,s}(\mathbf{C}^d) &= \{\, F \in A(\mathbf{C}^d)\,;\ Fe^{-M^0_{2,r,s}} \in L^\infty(\mathbf{C}^d) \textit{ for some } r \in \mathbf{R}^d_+ \,\}, \qquad s > \flat_1,\\
\mathcal{A}'_{\flat_1}(\mathbf{C}^d) &= A(\mathbf{C}^d) \quad \textit{and} \quad \mathcal{A}'_{0,\flat_1}(\mathbf{C}^d) = A(\{0\}).
\end{aligned}$$

Next we recall the link between the Bargmann transform and the short-time Fourier transform $f \mapsto V_\phi f$ with window function ϕ given by (2.7), defined by

$$V_\phi f(x,\xi) \equiv \langle f, \overline{\phi(\,\cdot - x)} e^{-i\langle\,\cdot\,,\xi\rangle}\rangle.$$

Let S be the dilation operator given by

$$(SF)(x,\xi) = F(2^{-\frac{1}{2}}x, -2^{-\frac{1}{2}}\xi), \tag{2.20}$$

when $F \in L^1_{loc}(\mathbf{R}^{2d})$. Then it follows by straight-forward computations that

$$\begin{aligned}
(\mathfrak{V}_d f)(z) = (\mathfrak{V}_d f)(x+i\xi) &= (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}(|x|^2+|\xi|^2)} e^{-i\langle x,\xi\rangle} V_\phi f(2^{\frac{1}{2}}x, -2^{\frac{1}{2}}\xi)\\
&= (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}(|x|^2+|\xi|^2)} e^{-i\langle x,\xi\rangle} (S^{-1}(V_\phi f))(x,\xi),
\end{aligned} \tag{2.21}$$

or equivalently,

$$\begin{aligned}
V_\phi f(x,\xi) &= (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{4}(|x|^2+|\xi|^2)} e^{-i\langle x,\xi\rangle/2} (\mathfrak{V}_d f)(2^{-\frac{1}{2}}x, -2^{-\frac{1}{2}}\xi)\\
&= (2\pi)^{-\frac{d}{2}} e^{-i\langle x,\xi\rangle/2} S(e^{-\frac{|\cdot|^2}{2}}(\mathfrak{V}_d f))(x,\xi)
\end{aligned} \tag{2.22}$$

We observe that (2.21) and (2.22) can be formulated as

$$\mathfrak{V}_d = U_{\mathfrak{V}} \circ V_\phi, \quad \text{and} \quad U_{\mathfrak{V}}^{-1} \circ \mathfrak{V}_d = V_\phi,$$

where $U_{\mathfrak{V}}$ is the linear, continuous and bijective operator on $\mathscr{D}'(\mathbf{R}^{2d}) \simeq \mathscr{D}'(\mathbf{C}^d)$, given by

$$(U_{\mathfrak{V}}F)(x,\xi) = (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}(|x|^2+|\xi|^2)} e^{-i\langle x,\xi\rangle} F(2^{\frac{1}{2}}x, -2^{\frac{1}{2}}\xi). \tag{2.23}$$

Let $D_{d,r}(z_0)$ be the polydisc

$$\{ z = (z_1,\dots,z_d) \in \mathbf{C}^d \, ; \, |z_j - z_{0,j}| < r_j, \ j = 1,\dots,d \},$$

with center and radii given by

$$z_0 = (z_{0,1},\dots,z_{0,d}) \in \mathbf{C}^d \quad \text{and} \quad r = (r_1,\dots,r_d) \in (0,\infty)^d.$$

Then

$$A(\mathbf{C}^d) = \bigcap_{r\in(0,\infty)^d} A(D_{d,r}(z)), \qquad A(\{0\}) = \bigcup_{r\in(0,\infty)^d} A(D_{d,r}(z_0)).$$

2.3 Hilbert spaces of power series expansions and analytic functions

The spaces in Definition 2.4 can also be described by related unions and intersections of Hilbert spaces of analytic functions and power series expansions as follows. (See also [12].)

Let ϑ be a weight on $\mathbf{N}^d$, ω be a weight on $\mathbf{C}^d$, and let

$$\|F\|_{\mathcal{A}^2_{[\vartheta]}(\mathbf{C}^d)} \equiv \left(\sum_{\alpha\in\mathbf{N}^d} |c(F,\alpha)\vartheta(\alpha)|^2 \right)^{\frac{1}{2}} \tag{2.24}$$

when $F \in \mathcal{A}_0'(\mathbf{C}^d)$ is given by (2.13), and

$$\|F\|_{A^2_{(\omega)}(\mathbf{C}^d)} \equiv \left(\int_{\mathbf{C}^d} |F(z)\omega(2^{\frac{1}{2}}\overline{z})|^2 \, d\mu(z) \right)^{\frac{1}{2}} \tag{2.25}$$

when $F \in A(\mathbf{C}^d)$. We let $\mathcal{A}^2_{[\vartheta]}(\mathbf{C}^d)$ be the set of all $F \in \mathcal{A}_0'(\mathbf{C}^d)$ such that $\|F\|_{\mathcal{A}^2_{[\vartheta]}}$ is finite, and $A^2_{(\omega)}(\mathbf{C}^d)$ be the set of all $F \in A(\mathbf{C}^d)$ such that $\|F\|_{A^2_{(\omega)}}$ is finite. It follows that these spaces are Hilbert spaces under these norms.

If ϑ and ω are related to each others as

$$\vartheta(\alpha) = \left(\frac{1}{\alpha!} \int_{\mathbf{R}^d_+} \omega_0(r)^2 r^\alpha \, dr \right)^{\frac{1}{2}} \tag{2.26}$$

and

$$\omega(z) = e^{\frac{|z|^2}{2}} \omega_0(|z_1|^2,\dots,|z_d|^2), \tag{2.27}$$

for some suitable weight ω_0 on $\mathbf{R}_+^d$, then the following multi-dimensional version of [7, Theorem (4.1)] shows that $\mathcal{A}^2_{[\vartheta]}(\mathbf{C}^d) = A^2_{(\omega)}(\mathbf{C}^d)$ with equal norms. Here we identify entire functions with their power series expansions at origin. Consequently, the Bargmann transform is bijective and isometric from $\mathcal{H}^2_{[\vartheta]}(\mathbf{R}^d)$ to $A^2_{(\omega)}(\mathbf{C}^d)$ for such choices of ϑ and ω. We omit the proof since the result is an immediate consequence of [12, Theorem 3.5].

Theorem 2.6 *Let $\alpha \in \mathbf{N}^d$ and ω_0 be a positive measurable function on $\mathbf{R}_+^d$. Also let ϑ and ω be weights on $\mathbf{N}^d$ and $\mathbf{C}^d$, respectively, related to each others by (2.26) and (2.27), and such that*

$$\frac{r^{|\alpha|}}{(\alpha!)^{\frac{1}{2}}} \lesssim \vartheta(\alpha), \quad \alpha \in \mathbf{N}^d, \tag{2.28}$$

holds for every $r > 0$. Then $\mathcal{A}^2_{[\vartheta]}(\mathbf{C}^d) = A^2_{(\omega)}(\mathbf{C}^d)$ with equality in norms.

In our situation, the involved weights should satisfy a split condition. In one dimension, (2.26), (2.27) and (2.28) take the forms

$$\vartheta_j(\alpha_j) = \left(\frac{1}{\alpha_j!}\int_{\mathbf{R}_+} \omega_{0,j}(r)^2 r^{\alpha_j}\, dr\right)^{\frac{1}{2}}, \quad \alpha_j \in \mathbf{N}, \tag{2.26$'$}$$

$$\omega_j(z_j) = e^{\frac{|z_j|^2}{2}}\omega_{0,j}(|z_j|^2), \quad z_j \in \mathbf{C} \tag{2.27$'$}$$

and

$$\frac{r^{|\alpha_j|}}{(\alpha_j!)^{\frac{1}{2}}} \lesssim \vartheta_j(\alpha_j), \quad r > 0,\ \alpha_j \in \mathbf{N}. \tag{2.28$'$}$$

Lemma 2.7 *Let $\omega_{0,j}$ be weights on $\overline{\mathbf{R}_+}$, ω_j be weights on $\mathbf{C}$ and ϑ_j be weights on $\mathbf{N}$ such that (2.26)$'$–(2.28)$'$ hold, $j = 1, \dots, d$, and set $\omega_0(z) \equiv \prod_{j=1}^d \omega_{0,j}(z_j)$, $z \equiv (z_1, \dots, z_d) \in \mathbf{C}^d$. If ϑ and ω are given by (2.26), and (2.27), then*

$$\vartheta(\alpha) = \prod_{j=1}^d \vartheta_j(\alpha_j), \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d \tag{2.29}$$

and

$$\omega(z) = \prod_{j=1}^d \omega_j(z_j), \quad z = (z_1, \dots, z_d) \in \mathbf{C}^d. \tag{2.30}$$

Proof By [12, Theorem 3.5] and its proof, it follows that $\omega_{0,j} \cdot r^{\alpha_j} \in L^1(\mathbf{R}_+)$ for all $j \in \{1, \dots, d\}$ and $\alpha_j \in \mathbf{N}$. Hence, Fubini's theorem gives

$$\vartheta(\alpha) = \left(\frac{1}{\alpha!} \int_{\mathbf{R}_+^d} \omega_0(r)^2 r^\alpha \, dr \right)^{\frac{1}{2}} = \left(\frac{1}{\alpha!} \prod_{j=1}^d \int_0^\infty \omega_{0,j}(r_j)^2 r_j^{\alpha_j} \, dr_j \right)^{\frac{1}{2}}$$
$$= \prod_{j=1}^d \vartheta_j(\alpha_j),$$

and (2.29) follows. The assertion (2.30) follows from the definitions. □

2.4 A test function space introduced by Gröchenig

In this section we recall some comparison results deduced in [12], between a test function space, $\mathcal{S}_C(\mathbf{R}^d)$, introduced by Gröchenig in [5] to handle modulation spaces with elements in spaces of ultra-distributions.

The definition of $\mathcal{S}_C(\mathbf{R}^d)$ is given as follows.

Definition 2.8 Let $\phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}$. Then $\mathcal{S}_C(\mathbf{R}^d)$ and $\mathcal{S}_G(\mathbf{R}^d)$ consist of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $f = V_\phi^* F$, for some $F \in L^\infty(\mathbf{R}^d) \cap \mathscr{E}'(\mathbf{R}^d)$ and $F \in \mathscr{E}'(\mathbf{R}^d)$, respectively.

It follows that $f \in \mathcal{S}_C(\mathbf{R}^d)$, if and only if

$$f(x) = (2\pi)^{-\frac{d}{2}} \iint_{\mathbf{R}^{2d}} F(y, \eta) e^{-\frac{1}{2}|x-y|^2} e^{i\langle x, \eta\rangle} \, dy d\eta, \tag{2.31}$$

for some $F \in L^\infty(\mathbf{R}^d) \cap \mathscr{E}'(\mathbf{R}^d)$.

Remark 2.9 By the identity $(V_\phi h, F) = (h, V_\phi^* F)$ and the fact that the map $(f, \phi) \mapsto V_\phi f$ is continuous from $\mathscr{S}(\mathbf{R}^d) \times \mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}(\mathbf{R}^{2d})$, it follows that $f = V_\phi^* F$ is uniquely defined as an element in $\mathscr{S}'(\mathbf{R}^d)$ when $F \in \mathscr{S}'(\mathbf{R}^{2d})$ (cf. [3]). In particular, the space $\mathcal{S}_G(\mathbf{R}^d)$ in Definition 2.8 is well-defined, and it is evident that $\mathcal{S}_C(\mathbf{R}^d) \subseteq \mathcal{S}_G(\mathbf{R}^d)$.

The following is a restatement of [12, Lemma 4.9]. The result is essential when deducing the characterizations of Pilipović spaces in Sect. 4.

Lemma 2.10 *Let $F \in L^\infty(\mathbf{C}^d) \cup \mathscr{E}'(\mathbf{C}^d)$. Then the Bargmann transform of $f = V_\phi^* F$ is given by $\Pi_A F_0$, where*

$$F_0(x + i\xi) = (2\pi^3)^{\frac{d}{4}} F(\sqrt{2}x, -\sqrt{2}\xi) e^{\frac{1}{2}(|x|^2+|\xi|^2)} e^{-i\langle x, \xi\rangle}. \tag{2.32}$$

Moreover, the images of $\mathcal{S}_C(\mathbf{R}^d)$ and $\mathcal{S}_G(\mathbf{R}^d)$ under the Bargmann transform are given by

$$\{\, \Pi_A F \,;\; F \in L^\infty(\mathbf{C}^d) \cap \mathscr{E}'(\mathbf{C}^d) \,\} \quad \text{and} \quad \{\, \Pi_A F \,;\; F \in \mathscr{E}'(\mathbf{C}^d) \,\}, \tag{2.33}$$

respectively.

The next results follow from [12, Theorem 4.10]. The proof is therefore omitted.

Proposition 2.11 *It holds* $\mathcal{S}_C(\mathbf{R}^d) = \mathcal{S}_G(\mathbf{R}^d) = \mathcal{H}_{\flat_1}(\mathbf{R}^d)$.

Due to the image properties for the spaces in Proposition 2.11 under the Bargmann transform, the next result is equivalent with the previous one.

Proposition 2.12 *The sets in* (2.33) *are equal to* $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$.

In the next section we extend Propositions 2.11 and 2.12 by proving that the conclusions in Proposition 2.12 hold for suitable smaller and larger sets than those in (2.33). We also deduce similar identifications for other Pilipović spaces and their Bargmann images.

3 Paley–Wiener properties for Bargmann–Pilipović spaces

In this section we consider spaces of compactly supported functions with interiors in $\mathcal{A}_s(\mathbf{C}^d)$ or in $\mathcal{A}_s'(\mathbf{C}^d)$. We show that the images of such functions under the reproducing kernel Π_A are equal to $\mathcal{A}_s(\mathbf{C}^d)$, for some other choice of $s \le \flat_1$. In the first part we state the main results given in Theorems 3.2–3.4. They are straight-forward consequences of Propositions 3.9, where more detailed information concerning involved constants are given. Thereafter we deduce results which are needed for their proofs. Depending of the choice of s, there are several different situations for characterizing $\mathcal{A}_s(\mathbf{C}^d)$. This gives rise to a quite large flora of main results, where each one takes care of one situation.

First we note that if $A^2_{(\omega)} = \mathcal{A}^2_{[\vartheta]}$, then a split of the variables in the weight ω in $A^2_{(\omega)}$ induce a split of the variables in ϑ in $\mathcal{A}^2_{[\vartheta]}$. (See Lemma 2.7.)

In order to present the main results, it is suitable to make the following definition.

Definition 3.1 Let $t_1, t_2 \in \mathbf{R}^d_+$ be such that $t_1 \le t_2$. Then the function $\chi \in L^\infty(\mathbf{C}^d)$ is called positive, bounded and radial symmetric with respect to t_1 and t_2, if the following conditions are fulfilled:

- $\chi \in L^\infty(\mathbf{C}^d) \cap \mathscr{E}'(\overline{D_{t_2}(0)})$ is non-negative;
- $\chi(z_1, \dots, z_d) = \chi_0(|z_1|, \dots, |z_d|)$ for some function χ_0;
- $\chi \ge c$ on $D_{t_1}(0)$ for some constant $c > 0$.

The set of positive, bounded and radial symmetric functions with respect to t_1 and t_2 is denoted by $\mathcal{R}^\infty_{t_1,t_2}(\mathbf{C}^d)$, and $\mathcal{R}^\infty(\mathbf{C}^d)$ is defined by

$$\mathcal{R}^\infty(\mathbf{C}^d) \equiv \bigcup_{t_1 \le t_2 \in \mathbf{R}^d_+} \mathcal{R}^\infty_{t_1,t_2}(\mathbf{C}^d).$$

3.1 Main results

We begin with characterizing the largest spaces in our investigations, which appears when $s = \flat_1$, and then proceed with spaces of decreasing order. First we recall that elements in $\mathcal{A}_s(\mathbf{C}^d)$ and $\mathcal{A}_{0,s}(\mathbf{C}^d)$ fulfill conditions of the forms

$$|F(z)| \lesssim e^{r_1|z_1|^{\frac{2\sigma}{\sigma+1}}+\cdots+r_d|z_d|^{\frac{2\sigma}{\sigma+1}}} \tag{3.1}$$

when $s = \flat_\sigma$ and

$$|F(z)| \lesssim e^{r_1(\log\langle z_1\rangle)^{\frac{1}{1-2s}}+\cdots+r_d(\log\langle z_d\rangle)^{\frac{1}{1-2s}}} \tag{3.2}$$

when $s \in [0, \frac{1}{2})$.

Theorem 3.2 *Let $F \in A(\mathbf{C}^d)$, $\sigma = 1$ and $s > 1$. Then the following conditions are equivalent:*

1. *$F \in \mathcal{A}_{\flat_1}(\mathbf{C}^d)$;*
2. *(3.1) holds for some $r \in \mathbf{R}_+^d$;*
3. *For some $r_0 \in \mathbf{R}_+^d$ and every $r \in \mathbf{R}_+^d$ with $r_0 \le r$ and every $\chi \in \mathcal{R}_{r_0,r}^\infty(\mathbf{C}^d)$, there exists $F_0 \in A(\overline{D_r(0)})$ such that $F = \Pi_A(F_0 \cdot \chi)$;*
4. *For some $r_0 \in \mathbf{R}_+^d$ and every $r \in \mathbf{R}_+^d$ with $r_0 < r$ and some $\chi \in \mathcal{R}_{r_0,r}^\infty(\mathbf{C}^d)$, there exists $F_0 \in A(\overline{D_r(0)})$ such that $F = \Pi_A(F_0 \cdot \chi)$;*
5. *There exists $F_0 \in \mathscr{E}'(\mathbf{C}^d) \cap L^\infty(\mathbf{C}^d)$ such that $F = \Pi_A F_0$;*
6. *There exists $F_0 \in \mathcal{E}_s'(\mathbf{C}^d)$ such that $F = \Pi_A F_0$.*

Remark 3.3 Since

$$\mathscr{E}'(\mathbf{C}^d) \cap L^\infty(\mathbf{C}^d) \subseteq \mathscr{E}'(\mathbf{C}^d) \subseteq \mathcal{E}_s'(\mathbf{C}^d),$$

Theorem 3.2 still holds true after $\mathcal{E}_s'$ has been replaced by $\mathscr{E}'$ in (6).

Theorem 3.4 *Let $F \in A(\mathbf{C}^d)$, $\sigma = 1$ and $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$. Then the following is true:*

(i) *The following conditions are equivalent:*

1. *$F \in \mathcal{A}_{0,\flat_1}(\mathbf{C}^d)$;*
2. *(3.1) holds for every $r \in \mathbf{R}_+^d$;*
3. *There exists $F_0 \in A(\mathbf{C}^d)$ such that $F = \Pi_A(F_0 \cdot \chi)$;*

(ii) *The map $F \mapsto \Pi_A(F \cdot \chi)$ from $A(\mathbf{C}^d)$ to $\mathcal{A}_{0,\flat_1}(\mathbf{C}^d)$ is a homeomorphism.*

The next result deals with the case when $s = \flat_\sigma$ with $\sigma \in (\frac{1}{2}, 1)$.

Theorem 3.5 *Let $F \in A(\mathbf{C}^d)$, $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$, $\sigma \in (\frac{1}{2}, 1)$ and let*

$$\sigma_0 = \frac{\sigma}{2\sigma - 1}.$$

Then the following is true:

(i) *The following conditions are equivalent:*

(1) $F \in \mathcal{A}_{\flat_\sigma}(\mathbf{C}^d)$ ($F \in \mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d)$);
(2) (3.1) *holds for some (for every)* $r \in \mathbf{R}_+^d$;
(3) *There exists* $F_0 \in \mathcal{A}'_{0,\flat_{\sigma_0}}(\mathbf{C}^d)$ ($F_0 \in \mathcal{A}'_{\flat_{\sigma_0}}(\mathbf{C}^d)$) *such that* $F = \Pi_A(F_0 \cdot \chi)$;

(ii) *The mappings* $F \mapsto \Pi_A(F \cdot \chi)$ *from* $\mathcal{A}'_{0,\flat_{\sigma_0}}(\mathbf{C}^d)$ *to* $\mathcal{A}_{\flat_\sigma}(\mathbf{C}^d)$ *and from* $\mathcal{A}'_{\flat_{\sigma_0}}(\mathbf{C}^d)$ *to* $\mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d)$ *are homeomorphisms.*

The next result deals with the case when $s = \flat_\sigma$ with $\sigma = \frac{1}{2}$.

Theorem 3.6 *Let* $F \in A(\mathbf{C}^d)$, $\sigma = \frac{1}{2}$ *and* $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$. *Then the following is true:*

(i) *The following conditions are equivalent:*

(1) $F \in \mathcal{A}_{\flat_\sigma}(\mathbf{C}^d)$ ($F \in \mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d)$);
(2) (3.1) *holds for some (for every)* $r \in \mathbf{R}_+^d$;
(3) *There exists* $F_0 \in \mathcal{A}'_{0,1/2}(\mathbf{C}^d)$ ($F_0 \in \mathcal{A}_{0,1/2}(\mathbf{C}^d)$) *such that* $F = \Pi_A(F_0 \cdot \chi)$;

(ii) *The mappings* $F \mapsto \Pi_A(F \cdot \chi)$ *from* $\mathcal{A}'_{0,1/2}(\mathbf{C}^d)$ *to* $\mathcal{A}_{\flat_{1/2}}(\mathbf{C}^d)$ *and from* $\mathcal{A}_{0,1/2}(\mathbf{C}^d)$ *to* $\mathcal{A}_{0,\flat_{1/2}}(\mathbf{C}^d)$ *are homeomorphisms.*

The next result deals with the case when $s = \flat_\sigma$ with $\sigma \in (0, \frac{1}{2})$.

Theorem 3.7 *Let* $F \in A(\mathbf{C}^d)$, $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$, $\sigma \in (0, \frac{1}{2})$ *and let*

$$\sigma_0 = \frac{\sigma}{1 - 2\sigma}.$$

Then the following is true:

(i) *The following conditions are equivalent:*

(1) $F \in \mathcal{A}_{\flat_\sigma}(\mathbf{C}^d)$ ($F \in \mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d)$);
(2) (3.1) *holds for some (for every)* $r \in \mathbf{R}_+^d$;
(3) *There exists* $F_0 \in \mathcal{A}_{\flat_{\sigma_0}}(\mathbf{C}^d)$ ($F_0 \in \mathcal{A}_{0,\flat_{\sigma_0}}(\mathbf{C}^d)$) *such that* $F = \Pi_A(F_0 \cdot \chi)$;

(ii) *The mappings* $F \mapsto \Pi_A(F \cdot \chi)$ *from* $\mathcal{A}_{\flat_{\sigma_0}}(\mathbf{C}^d)$ *to* $\mathcal{A}_{\flat_\sigma}(\mathbf{C}^d)$ *and from* $\mathcal{A}_{0,\flat_{\sigma_0}}(\mathbf{C}^d)$ *to* $\mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d)$ *are homeomorphisms.*

In the next result we consider the case when $s \in [0, \frac{1}{2})$ is real.

Theorem 3.8 *Let* $F \in A(\mathbf{C}^d)$, $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$, $s \in [0, \frac{1}{2})$. *Then the following is true:*

(i) *The following conditions are equivalent:*

(1) $F \in \mathcal{A}_s(\mathbf{C}^d)$ ($F \in \mathcal{A}_{0,s}(\mathbf{C}^d)$);
(2) (3.2) *holds for some (for every)* $r \in \mathbf{R}_+^d$;
(3) *There exists* $F_0 \in \mathcal{A}_s(\mathbf{C}^d)$ ($F_0 \in \mathcal{A}_{0,s}(\mathbf{C}^d)$) *such that* $F = \Pi_A(F_0 \cdot \chi)$;

(ii) *The mappings* $F \mapsto \Pi_A(F \cdot \chi)$ *from* $\mathcal{A}_s(\mathbf{C}^d)$ *to* $\mathcal{A}_s(\mathbf{C}^d)$ *and from* $\mathcal{A}_{0,s}(\mathbf{C}^d)$ *to* $\mathcal{A}_{0,s}(\mathbf{C}^d)$ *are homeomorphisms.*

The previous theorems are essentially consequences of Propositions 3.9–3.11, where more detailed information about involved constants are given.

Proposition 3.9 *Let $F \in A(\mathbf{C}^d)$, $\sigma = 1$, $s > 1$ and $r \in \mathbf{R}_+^d$. Then the following conditions are equivalent:*

(1) *For some $r_0 \in \mathbf{R}_+^d$ such that $r_0 < r$, (3.1) holds with r_0 in place of r;*
(2) *For some $t_1 \in \mathbf{R}_+^d$ such that $t_1 < r$, every $t_2 \in \mathbf{R}_+^d$ with $t_1 \le t_2 < r$ and every $\chi \in \mathcal{R}_{t_1,t_2}^\infty(\mathbf{C}^d)$, there exists $F_0 \in A(\overline{D_r(0)})$ such that $F = \Pi_A(F_0 \cdot \chi)$;*
(3) *For some $F_0 \in A(\overline{D_r(0)})$ it holds $F = \Pi_A(F_0 \cdot \chi_{D_r(0)})$;*
(4) *For some $F_0 \in \mathscr{E}'(D_r(0)) \cap L^\infty(\mathbf{C}^d)$ it holds $F = \Pi_A F_0$;*
(5) *For some $r_0 \in \mathbf{R}_+^d$ such that $r_0 < r$, there exists $F_0 \in \mathcal{E}_s'(D_{r_0}(0))$ such that $F = \Pi_A F_0$.*

Theorems 3.5–3.7 essentially follow from the following proposition.

Proposition 3.10 *Let $\tau > \frac{1}{2}$, $r, t_1, t_2 \in \mathbf{R}_+^d$ be such that $t_1 \le t_2$, and let $\chi \in \mathcal{R}_{t_1,t_2}^\infty(\mathbf{C}^d)$. Then the following is true:*

(1) *Let $F \in A(\mathbf{C}^d)$ be such that*

$$|F(z)| \lesssim e^{r_{0,1}|z_1|^{\frac{2}{2\tau+1}} + \cdots + r_{0,d}|z_d|^{\frac{2}{2\tau+1}}} \tag{3.3}$$

holds for some $r_0 \in \mathbf{R}_+^d$ such that $r_0 < r$. Then for some $r_0 \in \mathbf{R}_+^d$ such that $r_0 < r$, there exists $F_0 \in A(\mathbf{C}^d)$ such that $F = \Pi_A(F_0 \cdot \chi)$ and

$$|F_0(z)| \lesssim e^{R_{0,1}|z_1|^{\frac{2}{2\tau-1}} + \cdots + R_{0,d}|z_d|^{\frac{2}{2\tau-1}}}, \tag{3.4}$$

where

$$R_0 = \frac{2\tau - 1}{2}\left(\frac{2r_0}{2\tau + 1}\right)^{\frac{2\tau+1}{2\tau-1}} t_1^{-\frac{4}{2\tau-1}}; \tag{3.5}$$

(2) *Let $r_0, R_0 \in \mathbf{R}_+^d$ be such that $r_0 < r$ and*

$$R_0 = \frac{2\tau - 1}{2}\left(\frac{2r_0}{2\tau + 1}\right)^{\frac{2\tau+1}{2\tau-1}} t_2^{-\frac{4}{2\tau-1}}, \tag{3.6}$$

$F_0 \in A(\mathbf{C}^d)$ be such that (3.4) hold and let $F = \Pi_A(F_0 \cdot \chi)$. Then $F \in A(\mathbf{C}^d)$ and satisfies (3.3) for some $r_0 \in \mathbf{R}_+^d$ such that $r_0 < r$.

Theorem 3.8 follows from the following two propositions, where the first one concerns the case when $s > 0$ and the second one make a more detailed explanation of the case $s = 0$, i.e. the case of analytic polynomials.

Proposition 3.11 *Let $s \in (0, \frac{1}{2})$, $r \in \mathbf{R}_+^d$, and let $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$. Then the following is true:*

(1) *Suppose* $F \in A(\mathbf{C}^d)$ *satisfies*

$$|F(z)| \lesssim e^{r_{0,1}(\log\langle z_1\rangle)^{\frac{1}{1-2s}}+\cdots+r_{0,d}(\log\langle z_d\rangle)^{\frac{1}{1-2s}}} \tag{3.7}$$

for some $r_0 \in \mathbf{R}_+^d$ *such that* $r_0 < r$. *Then there is an* $F_0 \in A(\mathbf{C}^d)$ *such that* $F = \Pi_A(F_0 \cdot \chi)$ *and*

$$|F_0(z)| \lesssim e^{r_{0,1}(\log\langle z_1\rangle)^{\frac{1}{1-2s}}+\cdots+r_{0,d}(\log\langle z_d\rangle)^{\frac{1}{1-2s}}} \tag{3.8}$$

for some $r_0 \in \mathbf{R}_+^d$ *such that* $r_0 < r$;

(2) *Suppose* $F_0 \in A(\mathbf{C}^d)$ *satisfies* (3.8) *for some* $r_0 \in \mathbf{R}_+^d$ *such that* $r_0 < r$, *and let* $F = \Pi_A(F_0 \cdot \chi)$. *Then* $F \in A(\mathbf{C}^d)$ *and satisfies* (3.7) *for some* $r_0 \in \mathbf{R}_+^d$ *such that* $r_0 < r$.

Proposition 3.12 *Let* $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$ *and let* $N \geq 0$ *be an integer. Then the following is true:*

(1) *Suppose* $F \in A(\mathbf{C}^d)$ *is given by*

$$F(z) = \sum_{|\alpha|\leq N} c(F,\alpha)z^\alpha, \tag{3.9}$$

where $\{c(F,\alpha)\}_{|\alpha|\leq N} \subseteq \mathbf{C}$. *Then there is an* $F_0 \in A(\mathbf{C}^d)$ *such that* $F = \Pi_A(F_0 \cdot \chi)$ *and*

$$F_0(z) = \sum_{|\alpha|\leq N} c(F_0,\alpha)z^\alpha, \tag{3.10}$$

where $\{c(F_0,\alpha)\}_{|\alpha|\leq N} \subseteq \mathbf{C}$ *and satisfies* $c(F_0,\alpha) = 0$ *when* $c(F,\alpha) = 0$;

(2) *Suppose* $F_0 \in A(\mathbf{C}^d)$ *satisfies* (3.10) *for some* $\{c(F_0,\alpha)\}_{|\alpha|\leq N} \subseteq \mathbf{C}$, *and let* $F = \Pi_A(F_0 \cdot \chi)$. *Then* $F \in A(\mathbf{C}^d)$ *and satisfies* (3.9) *for some* $\{c(F,\alpha)\}_{|\alpha|\leq N} \subseteq \mathbf{C}$ *such that* $c(F,\alpha) = 0$ *when* $c(F_0,\alpha) = 0$.

3.2 Preparing results and their proofs

For the proofs of Propositions 3.9–3.11 and thereby of Theorems 3.2–3.8 we need some preparatory results. Because the proof of Proposition 3.9 needs some room, we put parts of the statement in the following separate proposition. At the same time we slightly refine some parts concerning the image of compactly supported elements in L^∞ under the map Π_A.

Proposition 3.13 *Let* $s > 1$, $r_0, r \in \mathbf{R}_+^d$ *be such that* $r_0 < r$ *and suppose that either*

$$F_0 \in \mathcal{E}_s'(D_{r_0}(0)), \quad F_0 \in \mathcal{E}_{0,s}'(D_{r_0}(0)), \quad F_0 \in \mathscr{E}'(D_{r_0}(0)) \quad \text{or} \quad F_0 \in L^\infty(\overline{D_r(0)}).$$

Then $F = \Pi_A F_0 \in A(\mathbf{C}^d)$ *and satisfies*

$$|F(z)| \lesssim e^{r_1|z_1|+\cdots+r_d|z_d|}.$$

Proof By the inclusions

$$\mathscr{E} \hookrightarrow \mathcal{E}_{0,s+\varepsilon}' \hookrightarrow \mathcal{E}_s' \hookrightarrow \mathcal{E}_{0,s}'$$

when $\varepsilon > 0$, it suffices to consider the case when $F_0 \in \mathcal{E}_s'(D_{r_0}(0))$ or $F_0 \in L^\infty(\overline{D_r(0)})$ hold.

Let $r_2 = r$. First suppose that $F_0 \in \mathcal{E}_s'(D_{r_0}(0))$ holds, choose $r_1 \in \mathbf{R}_+^d$ such that $r_0 < r_1 < r_2$, $\Psi(y,\eta) = e^{-(|y|^2+|\eta|^2)}$ and let $\Phi_z(y,\eta) = e^{(z,y+i\eta)}$. By identifying $\mathbf{C}^d$ with $\mathbf{R}^{2d}$ and using the fact that $F_0 \in \mathcal{E}_s'(D_{r_0}(0))$ we obtain

$$|\Pi_A F_0(z)| = \pi^{-d}|\langle F_0, \Phi_z\Psi\rangle| \lesssim \sup_{\alpha\in\mathbf{N}^{2d}} \left(\frac{\|D^\alpha(\Phi_z\Psi)\|_{L^\infty(D_{r_1}(0))}}{h_1^{|\alpha|}\alpha!^s} \right) \tag{3.11}$$

for every $h_1 > 0$. We also have $\Psi \in \mathcal{E}_{1/2}(\mathbf{R}^{2d}) \hookrightarrow \mathcal{E}_{0,s}(\mathbf{R}^{2d})$, which implies

$$\|D^\alpha\Psi\|_{L^\infty(D_{r_1}(0))} \lesssim h_2^{|\alpha|}\alpha!^s$$

for every $h_2 > 0$. Furthermore,

$$|D^\alpha\Phi_z(y,\eta)| = |m_\alpha(z)e^{(z,y+i\eta)}| \le |m_\alpha(z)|e^{r_{1,1}|z_1|+\cdots+r_{1,d}|z_d|}, \quad y+i\eta \in D_{r_1}(0),$$

where

$$m_\alpha(z) = \prod_{j=1}^d z_j^{\alpha_j+\alpha_{d+j}}, \quad z \in \mathbf{C}^d,\ \alpha \in \mathbf{N}^{2d}.$$

By choosing $h_1 = 4$ and $h_2 = 1$ above, and letting $y + i\eta \in D_{r_1}(0)$, Leibnitz rule gives

$$\begin{aligned} &4^{-|\alpha|}\alpha!^{-s}|D^\alpha(\Phi_z\Psi)(y,\eta)| \\ &\quad \le 4^{-|\alpha|}\alpha!^{-s}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}|D^\gamma\Phi_z(y,\eta)|\,|D^{\alpha-\gamma}\Psi(y,\eta)| \\ &\quad \lesssim 4^{-|\alpha|}\alpha!^{-s}\left(\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}|m_\gamma(z)|(\alpha-\gamma)!^s\right)e^{r_{1,1}|z_1|+\cdots+r_{1,d}|z_d|} \\ &\quad \lesssim 4^{-|\alpha|}\left(\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}|m_\gamma(z)|\gamma!^{-s}\right)e^{r_{1,1}|z_1|+\cdots+r_{1,d}|z_d|} \\ &\quad \le \sup_{\gamma\le\alpha}\left(\prod_{j=1}^d \frac{|z_j|^{\gamma_j}}{\gamma_j!^s}\cdot\frac{|z_j|^{\gamma_{d+j}}}{\gamma_{d+j}!^s}\right)e^{r_{1,1}|z_1|+\cdots+r_{1,d}|z_d|}. \end{aligned} \tag{3.12}$$

In the last inequality we have used that the number of terms in the sums are bounded by $2^{|\alpha|}$, and that $\binom{n}{k} \le 2^k$ when n, k are non-negative integers such that $k \le n$.

By combining (3.12) with the estimate

$$\frac{|z_j|^{\gamma_j}}{\gamma_j!^s} = \left(\frac{(|z_j|^{\frac{1}{s}})^{\gamma_j}}{\gamma_j!} \right)^s \le e^{s|z_j|^{\frac{1}{s}}},$$

we get

$$\sup_{\alpha \in \mathbf{N}^{2d}} \left(\frac{\|D^\alpha(\Phi_z \Psi)\|_{L^\infty(D_{r_1}(0))}}{h_1^{|\alpha|} \alpha!^s} \right) \lesssim e^{2s(|z_1|^{\frac{1}{s}} + \cdots + |z_d|^{\frac{1}{s}})} e^{r_{1,1}|z_1| + \cdots + r_{1,d}|z_d|}$$
$$\lesssim e^{r_{2,1}|z_1| + \cdots + r_{2,d}|z_d|}.$$

In the last inequality we have used the fact that $r_1 < r_2$ and $s > 1$. From the latter estimate and (3.11) we obtain

$$|\Pi_A F_0(z)| \lesssim e^{r_{2,1}|z_1| + \cdots + r_{2,d}|z_d|},$$

and the result follows when $F_0 \in \mathcal{E}_s'(D_{r_0}(0))$.

Suppose instead that $F_0 \in L^\infty(Q)$ holds, where $Q = \overline{D_{r_2}(0)} \subseteq \mathbf{C}^d$, and let $Q_j = \overline{D_{r_{2,j}}(0)} \subseteq \mathbf{C}$. Then

$$|\Pi_A F_0(z)| \lesssim \int_Q |F_0(z)||e^{(z,w)}| \, d\lambda(w)$$
$$\le \|F_0\|_{L^\infty} \prod_{j=1}^d \left(\int_{Q_j} e^{|z_j||w_j|} \, d\lambda(w_j) \right) \lesssim e^{r_{2,1}|z_1| + \cdots + r_{2,d}|z_d|},$$

and the result follows in this case as well. □

In the next lemma we give options on compactly supported functions which are mapped on the basic monomials, e_α by the operator Π_A.

Lemma 3.14 *Let $t_1, t_2 \in \mathbf{R}_+^d$ be such that $t_1 \le t_2$, $\chi \in \mathcal{R}_{t_1,t_2}^\infty(\mathbf{C}^d)$, and let χ_0 be such that $\chi_0(|z_1|, \ldots, |z_d|) = \chi(z_1, \ldots, z_d)$. If*

$$F_{\alpha,\chi}(z) = \varsigma_\alpha z^\alpha \chi(z),$$

with

$$\varsigma_\alpha = 2^{-d} \alpha!^{\frac{1}{2}} \left(\int_{\Delta_{t_2}} \chi_0(u) e^{-|u|^2} u^{2\alpha} u_1 \cdots u_d \, du \right)^{-1}, \quad \Delta_t = \{ u \in \mathbf{R}_+^d \, ; \, u \le t \}, \tag{3.13}$$

then the following is true:

(1) $\Pi_A F_{\alpha,\chi} = e_\alpha$;
(2) *for some constant* $C > 0$ *which only depends on* $\|\chi\|_{L^\infty}$, c *in Definition* 3.1 *and the dimension* d, *it holds*

$$C^{-1}\left(\prod_{j=1}^{d} t_{2,j}^{-2}(\alpha_j+1)\right) t_2^{-2\alpha}\alpha!^{\frac{1}{2}} \le \varsigma_\alpha \le Ce^{|t_2|^2}\left(\prod_{j=1}^{d} t_{1,j}^{-2}(\alpha_j+1)\right) t_1^{-2\alpha}\alpha!^{\frac{1}{2}}. \tag{3.14}$$

Proof By using polar coordinates in each complex variable when integrating we get

$$\begin{aligned}(\Pi_A F_{\alpha,\chi})(z) &= \pi^{-d}\varsigma_\alpha \int_{\mathbf{C}^d} w^\alpha \chi(w) e^{(z,w)-|w|^2}\,d\lambda(w)\\ &= \pi^{-d}\varsigma_\alpha \int_{\Delta_{t_2}} I_\alpha(u,z) u^\alpha \chi_0(u) e^{-|u|^2} u_1\cdots u_d\,du,\end{aligned} \tag{3.15}$$

where

$$I_\alpha(u,z) = \int_{[0,2\pi)^d} e^{i\langle\alpha,\theta\rangle}\left(\prod_{j=1}^{d} e^{z_j u_j e^{-i\theta_j}}\right) d\theta = \prod_{j=1}^{d} I_{\alpha_j}(u_j,z_j) \tag{3.16}$$

with

$$I_{\alpha_j}(u_j,z_j) = \int_0^{2\pi} e^{i\alpha_j\theta_j} e^{z_j u_j e^{-i\theta_j}}\,d\theta_j.$$

By Taylor expansions we get

$$\begin{aligned}I_{\alpha_j}(u_j,z_j) &= \int_0^{2\pi} e^{i\alpha_j\theta_j}\left(\sum_{k=0}^{\infty}\frac{z_j^k u_j^k e^{-ik\theta_j}}{k!}\right) d\theta_j\\ &= \sum_{k=0}^{\infty}\left(\left(\int_0^{2\pi} e^{i(\alpha_j-k)\theta_j}\,d\theta_j\right)\frac{z_j^k u_j^k}{k!}\right) = \frac{2\pi z_j^{\alpha_j} u_j^{\alpha_j}}{\alpha_j!}\end{aligned}$$

By inserting this into (3.15) and (3.16) we get

$$\begin{aligned}(\Pi_A F_{\alpha,\chi})(z) &= \pi^{-d}\varsigma_\alpha \int_{\Delta_{t_2}} (2\pi)^d \frac{u^\alpha z^\alpha}{\alpha!} u^\alpha \chi_0(u) e^{-|u|^2} u_1\cdots u_d\,du\\ &= \left(2^d \varsigma_\alpha \alpha!^{-\frac{1}{2}} \int_{\Delta_{t_2}} u^{2\alpha}\chi_0(u) e^{-|u|^2} u_1\cdots u_d\,du\right) e_\alpha(z) = e_\alpha(z)\end{aligned}$$

and (1) follows.

Since χ is non-negative and fullfills $\chi \geq c$ on $D_{t_1}(0)$ we get

$$\varsigma_\alpha \lesssim e^{|t_2|^2}\alpha!^{\frac{1}{2}}\left(\int_{\Delta_{t_1}} u^{2\alpha}u_1\cdots u_2\,du\right)^{-1}$$
$$= e^{|t_2|^2}\alpha!^{\frac{1}{2}}\prod_{j=1}^{d}\left(\frac{t_{1,j}^{2\alpha_j+2}}{2\alpha_j+2}\right)^{-1} \asymp e^{|t_2|^2}\alpha!^{\frac{1}{2}}\left(\prod_{j=1}^{d}(t_{1,j}^{-2}(\alpha_j+1))\right)t_1^{-2\alpha}$$

which gives the right inequality in (3.14). By the support properties of χ we also have

$$\varsigma_\alpha \gtrsim \alpha!^{\frac{1}{2}}\left(\int_{\Delta_{t_2}} t^{2\alpha}u_1\cdots u_2\,du\right)^{-1},$$

and the left inequality in (3.14) follows by similar arguments, and (2) follows. □

The next lemma shows that we may estimate entire functions by different Lebesgue norms. We omit the proof, since the result follows from [11, Theorem 3.2].

Lemma 3.15 *Suppose* $s, \tau \in \mathbf{R}$ *and* $r, r_0 \in \mathbf{R}_+^d$ *are such that*

$$s < \frac{1}{2}, \quad \tau > -\frac{1}{2} \quad \textit{and} \quad r_0 < r.$$

Let $p, q \in [1, \infty]$, $F \in A(\mathbf{C}^d)$ *and set*

$$M_{1,r}(z) = r_1|z_1|^{\frac{2}{2\tau+1}} + \cdots + r_d|z_d|^{\frac{2}{2\tau+1}}$$

and

$$M_{2,r}(z) = r_1(\log\langle z_1\rangle)^{\frac{1}{1-2s}} + \cdots + r_d(\log\langle z_d\rangle)^{\frac{1}{1-2s}}.$$

Then

$$\|F \cdot e^{-M_{j,r}}\|_{L^p(\mathbf{C}^d)} \lesssim \|F \cdot e^{-M_{j,r_0}}\|_{L^q(\mathbf{C}^d)}.$$

The next lemma relates Lebesgue estimates of entire functions with estimates on corresponding Taylor coefficients. Here we let the Gamma function on $\mathbf{R}_+^d$ be defined by

$$\Gamma_d(x_1, \ldots, x_d) = \prod_{j=1}^{d}\Gamma(x_j),$$

where Γ is the Gamma function on $\mathbf{R}_+$.

Lemma 3.16 *Let* $\tau > -\frac{1}{2}$*,* $r \in \mathbf{R}_+^d$*,* $M_{1,r}$ *be the same as in Lemma 3.15,* $\omega(z) = e^{\frac{1}{2}|z|^2 - M_{1,r}(z)}$*,* $z \in \mathbf{C}^d$*, and let* $\alpha_0 = (1, \dots, 1) \in \mathbf{N}^d$*. Also let*

$$\vartheta_r(\alpha) = \left((2\tau + 1)(2r)^{-(2\tau+1)(\alpha+\alpha_0)} \left(\frac{\Gamma\left((2\tau + 1)(\alpha + \alpha_0)\right)}{\alpha!} \right) \right)^{\frac{1}{2}}.$$

If $F \in A(\mathbf{C}^d)$ *is given by (2.13), then*

$$\|F \cdot e^{-M_{1,r}}\|_{L^2(\mathbf{C}^d)} = \pi^{\frac{d}{2}} \left(\sum_{\alpha \in \mathbf{N}^d} |c(F, \alpha)\vartheta_r(\alpha)|^2 \right)^{\frac{1}{2}},$$

and $\mathcal{A}^2_{[\vartheta_r]}(\mathbf{C}^d) = A^2_{(\omega)}(\mathbf{C}^d)$ *with equality in norms.*

Proof Since

$$e^{-M_{1,r}(z)} = \prod_{j=1}^{d} e^{-r_j |z_j|^{\frac{2}{2\tau+1}}},$$

Lemma 2.7 shows that we may assume that $d = 1$, giving that $r = r_1$ and $\alpha_0 = 1$.

In view of Theorem 2.6 we have $\mathcal{A}^2_{[\vartheta]}(\mathbf{C}^d) = A^2_{(\omega_1)}(\mathbf{C}^d)$ with equality in norms, when

$$\vartheta(\alpha) = \left(\frac{1}{\alpha!} \int_0^\infty e^{-2rt^{\frac{1}{2\tau+1}}} t^\alpha \, dt \right)^{\frac{1}{2}}.$$

By $u = 2rt^{\frac{1}{2\tau+1}}$ as new variables of integration we obtain

$$\begin{aligned} \vartheta(\alpha) &= \left((2\tau + 1)(2r)^{-(2\tau+1)(\alpha+1)} \frac{1}{\alpha!} \int_0^\infty e^{-u} u^{\alpha(2\tau+1)+2\tau} \, du \right)^{\frac{1}{2}} \\ &= \left((2\tau + 1)(2r)^{-(2\tau+1)(\alpha+1)} \left(\frac{\Gamma\left((2\tau + 1)(\alpha + 1)\right)}{\alpha!} \right) \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that $\theta = \theta_{1,r}$, and the result follows. □

We also need the following version of Stirling's formula.

Lemma 3.17 *Let* $\alpha \geq 0$ *be an integer and let* $\tau > -\frac{1}{2}$*. Then*

$$\frac{\Gamma\left((2\tau + 1)(\alpha + 1)\right)}{\alpha!} \asymp (2\tau + 1)^{(2\tau+1)\cdot\alpha} (\alpha + 1)^\tau \cdot \alpha!^{2\tau}. \tag{3.17}$$

Lemma 3.17 follows by repeated applications of Stirling's formula and the standard limit

$$\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x$$

for every $x \in \mathbf{R}$. In order to be self-contained we present the arguments.

Proof The result is obviously true for $\alpha = 0$. For $\alpha \geq 1$ we have $\alpha^\tau \asymp (\alpha+1)^\tau$. A combination of the the latter relations and Stirling's formula gives

$$\begin{aligned}\frac{\Gamma\left((2\tau+1)(\alpha+1)\right)}{\alpha!} &\asymp \frac{((2\tau+1)\alpha+2\tau)^{(2\tau+1)\cdot\alpha+2\tau+\frac{1}{2}}}{e^{\alpha+2\tau(\alpha+1)}}\cdot\frac{e^\alpha}{\alpha^{\alpha+\frac{1}{2}}}\\ &= \frac{(2\tau+1)^{(2\tau+1)\cdot\alpha+2\tau+\frac{1}{2}}\left(1+\frac{2\tau}{(2\tau+1)\alpha}\right)^{(2\tau+1)\cdot\alpha+2\tau+\frac{1}{2}}\alpha^{2\tau(\alpha+1)}}{e^{2\tau(\alpha+1)}}\\ &\asymp (2\tau+1)^{(2\tau+1)\cdot\alpha}\frac{\alpha^{2\tau(\alpha+1)}}{e^{2\tau\alpha}} \asymp (2\tau+1)^{(2\tau+1)\cdot\alpha}(\alpha+1)^\tau\alpha!^{2\tau},\end{aligned} \tag{3.18}$$

and the result follows. □

Proposition 3.9 essentially follows from the following lemma.

Lemma 3.18 *Let $\tau > -\frac{1}{2}$ and $r_0, r \in \mathbf{R}_+^d$ be such that $r_0 < r$. Then the following is true:*

(1) *If*

$$|c(F,\alpha)| \lesssim \left(\frac{2r_0}{2\tau+1}\right)^{\frac{2\tau+1}{2}\cdot\alpha}\alpha!^{-\tau}, \tag{3.19}$$

then

$$|F(z)| \lesssim e^{r_1|z_1|^{\frac{2}{2\tau+1}}+\cdots+r_d|z_d|^{\frac{2}{2\tau+1}}}; \tag{3.20}$$

(2) *If*

$$|F(z)| \lesssim e^{r_{0,1}|z_1|^{\frac{2}{2\tau+1}}+\cdots+r_{0,d}|z_d|^{\frac{2}{2\tau+1}}} \tag{3.21}$$

then

$$|c(F,\alpha)| \lesssim \left(\frac{2r}{2\tau+1}\right)^{\frac{2\tau+1}{2}\cdot\alpha}\alpha!^{-\tau} \tag{3.22}$$

Proof Let ϑ_r be the same as in Lemma 3.16 and $r_1 \in \mathbf{R}_+^d$ be such that $r_0 < r_1 < r$. First we prove (1). Suppose that (3.19) holds and let $\alpha_0 = (1,\ldots,1) \in \mathbf{N}^d$. Also let $M_{1,r}$ be the same as in Lemma 3.15. Then Lemmas 3.15 and 3.16 give

$$\|F\cdot e^{-M_{1,r}}\|_{L^\infty(\mathbf{C}^d)} \lesssim \|F\cdot e^{-M_{1,r_1}}\|_{L^2(\mathbf{C}^d)} \asymp \left(\sum_{\alpha\in\mathbf{N}^d}\left|c(F,\alpha)\vartheta_{r_1}(\alpha)\right|^2\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&= \left(\sum_{\alpha \in \mathbf{N}^d} \left| c(F,\alpha)\vartheta_{r_0}(\alpha) \left(\frac{r_0}{r_1} \right)^{(\alpha+\alpha_0)\frac{2\tau+1}{2}} \right|^2 \right)^{\frac{1}{2}} \\
&\lesssim \sup_{\alpha \in \mathbf{N}^d} \left| c(F,\alpha)\vartheta_{r_0}(\alpha) \prod_{j=1}^{d} (\alpha_j+1)^{-\frac{\tau}{2}} \right| \\
&\asymp \sup_{\alpha \in \mathbf{N}^d} \left(|c(F,\alpha)| \left(\frac{2\tau+1}{2r_0} \right)^{\frac{2\tau+1}{2}\cdot\alpha} \alpha!^{\tau} \right).
\end{aligned}
$$

Here the second inequality follows from the fact that

$$
\sum_{\alpha \in \mathbf{N}^d} \left(\left(\prod_{j=1}^{d} (\alpha_j+1)^{\frac{\tau}{2}} \right) \left(\frac{r_0}{r_1} \right)^{(2\tau+1)\cdot(\alpha+\alpha_0)} \right)
$$

is convergent since $r_0 < r_1$, and the fifth relation follows from Lemma 3.17. This implies that (3.20) holds and (1) follows.

Next assume that (3.21) holds. By Lemma 3.15 we get

$$
\begin{aligned}
\|F\cdot e^{-M_{1,r_0}}\|_{L^\infty(\mathbf{C}^d)} &\gtrsim \|F\cdot e^{-M_{1,r_1}}\|_{L^2(\mathbf{C}^d)} \\
&\asymp \left(\sum_{\alpha \in \mathbf{N}^d} \left| c(F,\alpha)\vartheta_{r_1}(\alpha) \right|^2 \right)^{\frac{1}{2}} \geq \sup_{\alpha \in \mathbf{N}^d} \left(|c(F,\alpha)|\, \vartheta_{r_1}(\alpha) \right) \\
&\gtrsim \sup_{\alpha \in \mathbf{N}^d} \left(|c(F,\alpha)| \left(\frac{2\tau+1}{2r_1} \right)^{\frac{2\tau+1}{2}\alpha} \prod_{j=1}^{d} (\alpha_j+1)^{\frac{\tau}{2}} \alpha!^{\tau} \right) \\
&\gtrsim \sup_{\alpha \in \mathbf{N}^d} \left(|c(F,\alpha)| \left(\frac{2\tau+1}{2r} \right)^{\frac{2\tau+1}{2}\alpha} \alpha!^{\tau} \right),
\end{aligned}
$$

where the third inequality follows from Lemma 3.17. This gives (2). □

Proposition 3.11 mainly follows from the following result.

Lemma 3.19 *Let $r, r_0 \in \mathbf{R}_+^d$ be such that $r_0 < r$, $s \in (0, \frac{1}{2})$ and let $F \in A(\mathbf{C}^d)$. Then the following is true:*

(1) *if (3.7) holds, then*

$$
|c(F,\alpha)| \lesssim e^{-(R_1|\alpha_1|^{\frac{1}{2s}} + \cdots + R_d|\alpha_d|^{\frac{1}{2s}})}, \quad R = s\left(\frac{1-2s}{r} \right)^{\frac{1-2s}{2s}}; \tag{3.23}
$$

(2) *if* $R_0 \in \mathbf{R}_+^d$ *is given by*

$$|c(F,\alpha)| \lesssim e^{-(R_{0,1}|\alpha_1|^{\frac{1}{2s}}+\cdots+R_{0,d}|\alpha_d|^{\frac{1}{2s}})}, \quad R_0 = s\left(\frac{1-2s}{r_0}\right)^{\frac{1-2s}{2s}}, \tag{3.24}$$

then (3.7) *holds with* r *in place of* r_0.

Proof Let $r_1, r_2, r_3 \in \mathbf{R}_+^d$ be such that $r_1 < r < r_2 < r_3$. By Lemma 2.7 we may assume that $d = 1$, and by Lemma 3.15 the result follows if we prove

$$\begin{aligned}\sup_{\alpha\in\mathbf{N}}\left(|c(F,\alpha)e^{R_3\cdot\alpha^{\frac{1}{2s}}}|^2\right) &\lesssim \int_{\mathbf{C}} |F(z)e^{-r(\log\langle z\rangle)^{\frac{1}{1-2s}}}|^2\,d\lambda(z)\\ &\lesssim \sup_{\alpha\in\mathbf{N}}\left(|c(F,\alpha)e^{R_1\cdot\alpha^{\frac{1}{2s}}}|^2\right),\end{aligned} \tag{3.25}$$

where $R_1, R_2, R_3 \in \mathbf{R}_+$ satisfy

$$R_j = \left(\frac{1-2s}{r_j}\right)^{\frac{1-2s}{2s}}, \quad j = 1, 2, 3.$$

By Theorem 2.6 we have

$$\int_{\mathbf{C}} |F(z)e^{-r(\log\langle z\rangle)^{\frac{1}{1-2s}}}|^2\,d\lambda(z) \asymp \sum_{\alpha\in\mathbf{N}} |c(F,\alpha)\vartheta_r(\alpha)|^2,$$

where

$$\vartheta_r(\alpha) = \left(\frac{\pi}{2\alpha!}\int_0^\infty e^{-r(\log\langle t\rangle)^{\frac{1}{1-2s}}}\,t^\alpha\,dt\right)^{\frac{1}{2}}.$$

Let

$$\theta = \frac{1}{1-2s} > 1 \quad \text{and} \quad g_{r,\alpha}(t) = e^{-r(\log t)^\theta}\,t^\alpha.$$

In order to prove (3.25), we need to show that

$$e^{R_2\alpha^{\frac{1}{2s}}} \lesssim \vartheta_r(\alpha) \lesssim e^{R_1\alpha^{\frac{1}{2s}}}, \tag{3.26}$$

which shall be reached by modifying the proof of (15) in [4].

We have

$$\vartheta_r(\alpha)^2 \lesssim \int_e^\infty e^{(r_1-r)(\log t)^\theta}\,g_{r_1,\alpha}(t)\,dt$$

$$\lesssim \sup_{t\geq e}(g_{r_1,\alpha}(t)) \int_e^\infty e^{(r_1-r)(\log t)^\theta}\, dt \asymp \sup_{t\geq e}(g_{r_1,\alpha}(t)).$$

By straight-forward computations it follows that $g_{r,\alpha}(t)$ attains its global maximum for

$$t_{r,\alpha} = \exp\left(\left(\frac{\alpha}{\theta r}\right)^{\frac{1-2s}{2s}}\right), \tag{3.27}$$

and that

$$g_{r_1,\alpha}(t_{r_1,\alpha}) = \exp\left(r_1^{1-\frac{1}{2s}}\theta^{-\frac{1}{2s}}(\theta-1)\alpha^{\frac{1}{2s}}\right) = e^{2R_1\cdot\alpha^{\frac{1}{2s}}},$$

and the second inequality in (3.26) follows.

In order to prove the first inequality in (3.26), we claim that for some c which is independent of α we have

$$\left(1-\frac{c}{t_{r_2,\alpha}}\right)^\alpha \gtrsim e^{(r-r_2)(\log t_{r_2,\alpha})^\theta}. \tag{3.28}$$

In fact, by (3.27) it follows that

$$\lim_{\alpha\to\infty}\alpha^{-1}t_{r_2,\alpha} = \infty.$$

This together with the fact that $r < r_2$ give

$$\lim_{\alpha\to\infty}\left(1-\frac{c}{t_{r_2,\alpha}}\right)^\alpha = \lim_{\alpha\to\infty}\left(\left(1-\frac{c}{t_{r_2,\alpha}}\right)^{t_{r_2,\alpha}}\right)^{\frac{\alpha}{t_{r_2,\alpha}}} = (e^{-c})^0 = 1$$

and

$$\lim_{\alpha\to\infty} e^{(r-r_2)((1-2s)\alpha/r_2)^{\frac{1}{2s}}} = 0,$$

and (3.28) follows.

By (3.28) and the fact that $\frac{1}{2s} > 1$ we get

$$\begin{aligned}\vartheta_r(\alpha)^2 &\gtrsim \frac{1}{\alpha!}\int_{t_{r_2,\alpha}-c}^{t_{r_2,\alpha}} e^{-r(\log t)^\theta}t^\alpha\, dt \gtrsim \frac{1}{\alpha!}e^{-r(\log t_{r_2,\alpha})^\theta}(t_{r_2,\alpha}-c)^\alpha \\ &\gtrsim \frac{1}{\alpha!}e^{-r_2(\log t_{r_2,\alpha})^\theta}t_{r_2,\alpha}^\alpha = \frac{1}{\alpha!}e^{2R_2\cdot\alpha^{\frac{1}{2s}}} \gtrsim e^{2R_3\cdot\alpha^{\frac{1}{2s}}}.\end{aligned}$$

This gives the result. □

3.3 Proofs of main results

Next we prove Proposition 3.9 and thereby Theorem 3.2.

Proof of Proposition 3.9 It is clear that (2) ⇒ (3) ⇒ (4) ⇒ (5). By Proposition 3.13 it follows that (5) implies (1). We need to prove that (1) implies (2).

Suppose (1) holds and let $r_4 = r$. Choose $r_1, r_2, r_3 \in \mathbf{R}_+^d$ such that $r_0 < r_1 < r_2 \le r_3 < r_4$ and $r_1 r_4 < r_2^2$, $\chi \in \mathcal{R}_{r_2,r_3}^\infty(\mathbf{C}^d)$, and let $F_{\alpha,\chi}$ be as in Lemma 3.14 with r_2 and r_3 in place of r_1 and r_2. If $\tau = 1$, then Lemma 3.18 (2) gives

$$|c(F,\alpha)| \lesssim r_1^\alpha \alpha!^{-\frac{1}{2}}. \tag{3.29}$$

Let

$$F_0(z) = \sum_{\alpha \in \mathbf{N}^d} c(F,\alpha)\varsigma_\alpha z^\alpha. \tag{3.30}$$

We claim that the series in (3.30) is uniformly convergent with respect to z in $\overline{D_R(0)}$.

In fact, if $|z_j| \le r_{4,j}$, then (3.14) gives

$$|c(F,\alpha)\varsigma_\alpha z^\alpha| \lesssim \langle\alpha\rangle^d r_2^{-2\alpha} r_1^\alpha r_4^\alpha = \langle\alpha\rangle^d \rho^\alpha,$$

where $\rho \in \mathbf{R}_+^d$ satisfies $\rho_j = \frac{r_{1,j} r_{4,j}}{r_{2,j}^2} < 1$. Since $\sum_{\alpha \in \mathbf{N}^d} \langle\alpha\rangle^d \rho^\alpha$ is convergent, Weierstrass theorem shows that (3.30) is uniformly convergent and defines an analytic function in $\overline{D_{r_{4,j}}(0)}$. Hence, $F_0 \in A(\overline{D_{r_{4,j}}(0)})$. Furthermore, by the support of χ we get

$$\begin{aligned} \Pi_A(F_0 \cdot \chi) &= \Pi_A\left(\sum_{\alpha \in \mathbf{N}^d} c(F,\alpha)\varsigma_\alpha z^\alpha \cdot \chi\right) \\ &= \sum_{\alpha \in \mathbf{N}^d} c(F,\alpha)\varsigma_\alpha \Pi_A(z^\alpha \cdot \chi) = \sum_{\alpha \in \mathbf{N}^d} c(F,\alpha) e_\alpha = F. \end{aligned} \tag{3.31}$$

Hence (2) holds, and the proof is complete. □

For future references we observe that if ς_α and χ are the same as in Lemma 3.14, then (3.30) shows that the relationship between $c(F,\alpha)$ and $c(F_0,\alpha)$ is given by

$$c(F_0,\alpha) = c(F,\alpha)\varsigma_\alpha \alpha!^{\frac{1}{2}}. \tag{3.32}$$

Proof of Theorem 3.4 The equivalence between (1) and (2) is clear. It is also obvious that (3) implies (2) in view of Proposition 3.9. We shall prove the equivalence between (1) and (3).

Suppose (1) holds. Then (3.29) holds for every $r_1 \in \mathbf{R}_+^d$. Let $r, R \in \mathbf{R}_+^d$ be chosen such that $\chi \in \mathcal{R}_{r,R}^\infty(\mathbf{C}^d)$, $F_{\alpha,\chi}$ be as in Lemma 3.14 and let F_0 be given by (3.30). By (3.29) we have

$$|c(F,\alpha)\varsigma_\alpha| \lesssim \langle\alpha\rangle^d r^{-2\alpha} r_1^\alpha$$

for every $r_1 \in \mathbf{R}_+^d$, giving that

$$|c(F,\alpha)\varsigma_\alpha| \lesssim r_0^\alpha$$

for every $r_0 \in \mathbf{R}_+^d$. This implies that the series in (3.30) is locally uniformly convergent with respect to z and defines an entire function on $\mathbf{C}^d$. Hence $F_0 \in A(\mathbf{C}^d)$. Moreover, by (3.31) it follows that $\Pi_A(F_0\chi) = F$, and we have proved that (1) implies (3).

Next suppose that (3) holds. Then

$$|c(F_0,\alpha)| \lesssim r^\alpha \alpha!^{\frac{1}{2}}$$

for every $r \in \mathbf{R}_+^d$. By (3.14) and (3.32) we get

$$|c(F,\alpha)| = |c(F_0,\alpha)|\varsigma_\alpha^{-1}\alpha!^{-\frac{1}{2}} \lesssim r^\alpha \varsigma_\alpha^{-1} \lesssim r^\alpha R^{2\alpha}\alpha!^{-\frac{1}{2}}.$$

Since $r \in \mathbf{R}_+^d$ can be chosen arbitrarily small we get

$$|c(F,\alpha)| \lesssim r_0^\alpha \alpha!^{-\frac{1}{2}}$$

for every $r_0 \in \mathbf{R}_+^d$. This implies that $F \in \mathcal{A}_{0,\flat_1}(\mathbf{C}^d)$. That is, we have proved that (3) implies (1), and the result follows. □

Next we prove Proposition 3.10.

Proof of Proposition 3.10 Suppose (3.3) holds, and let $r_1, r_2 \in \mathbf{R}_+^d$ be such that $r_0 < r_1 < r_2 < r$. Then Lemma 3.18 gives

$$|c(F,\alpha)| \lesssim \left(\frac{2r_1}{2\tau+1}\right)^{\frac{2\tau+1}{2}\cdot\alpha} \alpha!^{-\tau}.$$

By Lemma 3.14 and (3.32) we get

$$\begin{aligned}|c(F_0,\alpha)| &\lesssim \left(\frac{2r_1}{2\tau+1}\right)^{\frac{2\tau+1}{2}\cdot\alpha} \alpha!^{-\tau}\langle\alpha\rangle^d\alpha! t_1^{-2\alpha}\\ &\lesssim \left(\frac{2r_2}{2\tau+1}\right)^{\frac{2\tau+1}{2}\cdot\alpha} \alpha!^{-(\tau-1)}t_1^{-2\alpha} = \left(\frac{2R_2}{2\tau-1}\right)^{\frac{2\tau-1}{2}\cdot\alpha} \alpha!^{-(\tau-1)},\end{aligned}$$

when

$$R_2 = \frac{2\tau-1}{2}\left(\frac{2r_2}{2\tau+1}\right)^{\frac{2\tau+1}{2\tau-1}} t_1^{-\frac{4}{2\tau-1}}.$$

This proves (1).

Suppose instead that $r_0, R_0, r_1, r_2 \in \mathbf{R}_+^d$ are such that (3.6) hold and $r_0 < r_1 < r_2 < r$, and that $F_0 \in A(\mathbf{C}^d)$ satisfies (3.4). Also let $F = \Pi_A(F_0 \cdot \chi)$. Then

$$\left(\frac{2R_0}{2\tau - 1}\right)^{\frac{2\tau-1}{2}} < \left(\frac{2r_1}{2\tau + 1}\right)^{\frac{2\tau+1}{2}} t_2^{-2}.$$

By combining the latter estimate with Lemma 3.18 we get

$$|c(F_0, \alpha)| \lesssim \left(\frac{2r_1}{2\tau + 1}\right)^{\frac{2\tau+1}{2}\cdot\alpha} t_2^{-2\alpha} \alpha!^{1-\tau}$$

Hence Lemma 3.14 and (3.32) give

$$|c(F, \alpha)| \lesssim \left(\frac{2r_1}{2\tau + 1}\right)^{\frac{2\tau+1}{2}\cdot\alpha} \alpha!^{-\tau}$$

and Lemma 3.18 again implies that (3.3) holds with r_2 in place of r_0. This gives the result. □

Proof of Theorems 3.5–3.7 First suppose $\sigma \in (\frac{1}{2}, 1)$, and let $\sigma_0 = \frac{\sigma}{2\sigma-1}$ and $\tau = \frac{1}{2\sigma}$. Then

$$\frac{2\sigma}{\sigma + 1} = \frac{2}{2\tau + 1} \quad \text{and} \quad \frac{2\sigma_0}{\sigma_0 - 1} = \frac{2}{2\tau - 1}.$$

Theorem 3.5 now follows from these observations and Proposition 3.10 in the case $\tau \in (\frac{1}{2}, 1)$.

In the same way, Theorem 3.6 follows by choosing $\tau = 1$ in Proposition 3.10.

Finally, suppose $\sigma \in (0, \frac{1}{2})$, and let $\sigma_0 = \frac{\sigma}{1-2\sigma}$ and $\tau = \frac{1}{2\sigma}$. Then

$$\frac{2\sigma}{\sigma + 1} = \frac{2}{2\tau + 1} \quad \text{and} \quad \frac{2\sigma_0}{\sigma_0 + 1} = \frac{2}{2\tau - 1},$$

and Theorem 3.7 follows from these observations and Proposition 3.10 in the case $\tau > 1$. □

Next we prove Propositions 3.11 and 3.12 and thereby Theorem 3.8.

Proof of Propositions 3.11 and 3.12 Let $r, r_j, R_j \in \mathbf{R}_+^d$, $j = 0, 1, 2, 3$, be such that

$$r_0 < r_1 < r_2 < r_3 < r \quad \text{and} \quad R_j = s\left(\frac{1 - 2s}{r_j}\right)^{\frac{1-2s}{2s}}.$$

First suppose that $F \in A(\mathbf{C}^d)$ satisfies (3.7) and let F_0 be the formal power series expansion with coefficients given by (3.32). Then $F = \Pi_A(F_0 \cdot \chi)$.

By Lemma 3.19 we get

$$|c(F,\alpha)| \lesssim e^{-(R_{1,1}|\alpha_1|^{\frac{1}{2s}}+\cdots+R_{1,d}|\alpha_d|^{\frac{1}{2s}})}.$$

Hence Lemma 3.14 and (3.32) give

$$\begin{aligned}|c(F_0,\alpha)| &\lesssim \alpha! t_1^{-2\alpha} e^{-(R_{1,1}|\alpha_1|^{\frac{1}{2s}}+\cdots+R_{1,d}|\alpha_d|^{\frac{1}{2s}})}\\ &\lesssim e^{-(R_{2,1}|\alpha_1|^{\frac{1}{2s}}+\cdots+R_{2,d}|\alpha_d|^{\frac{1}{2s}})}.\end{aligned}$$

In the last inequality we have used the fact that $s < \frac{1}{2}$, which implies that $R_2 < R_1$ and

$$t_1^{-2\alpha}\alpha! \lesssim e^{(R_{1,1}-R_{2,1})|\alpha_1|^{\frac{1}{2s}}+\cdots+(R_{1,d}-R_{2,d})|\alpha_d|^{\frac{1}{2s}}}.$$

By applying Lemma 3.19 again it follows that F_0 satisfies (3.8) with r_3 in place of r_0. This gives (1).

Suppose instead that $F_0 \in A(\mathbf{C}^d)$ satisfies (3.8) and let $F = \Pi_A(F_0 \cdot \chi)$. Then Lemma 3.19 gives

$$|c(F_0,\alpha)| \lesssim e^{-(R_{1,1}|\alpha_1|^{\frac{1}{2s}}+\cdots+R_{1,d}|\alpha_d|^{\frac{1}{2s}})},$$

and it follows from Lemma 3.14 and (3.32) that

$$\begin{aligned}|c(F,\alpha)| &\lesssim \alpha!^{-1} t_2^{2\alpha} e^{-(R_{1,1}|\alpha_1|^{\frac{1}{2s}}+\cdots+R_{1,d}|\alpha_d|^{\frac{1}{2s}})}\\ &\lesssim e^{-(R_{1,1}|\alpha_1|^{\frac{1}{2s}}+\cdots+R_{1,d}|\alpha_d|^{\frac{1}{2s}})}.\end{aligned}$$

By applying Lemma 3.19 again we deduce (3.7) with r_2 in place of r_0, and Proposition 3.11 follows.

Proposition 3.12 is a straight-forward consequence of (3.32). The details are left for the reader. □

4 Characterizations of Pilipović spaces

In this section we combine Lemma 2.10 with Theorems 3.2–3.8 to get characterizations of Pilipović spaces.

We begin with the following characterization of $\mathcal{H}_{\flat_1}$. The result is a straight-forward combination of Lemma 2.10 and Theorem 3.2. The details are left for the reader.

Proposition 4.1 *Let ϕ be as in (2.7), $r \in \mathbf{R}_+^d$, χ_r be the characteristic function for $\overline{D_r(0)}$ and let $s > 0$. Then the following conditions are equivalent:*

(1) $f \in \mathcal{H}_{\flat_1}(\mathbf{R}^d)$;

(2) $f = V_\phi^* F$ *for some* $F \in \mathcal{E}_s'(\mathbf{R}^{2d})$*;*
(3) $f = V_\phi^* F$ *for some* $F \in \mathscr{E}'(\mathbf{R}^{2d}) \cap L^\infty(\mathbf{R}^{2d})$*;*
(4) $f = V_\phi^* F$ *for some* $F \in \mathscr{E}'(\mathbf{R}^{2d}) \cap L^\infty(\mathbf{R}^{2d})$ *which satisfies*

$$F(x,\xi) = F_0(x - i\xi)e^{-i\langle x,\xi\rangle}\chi(x,\xi) \tag{4.1}$$

for some $r \in \mathbf{R}_+^d$*,* $\chi = \chi_r$ *and* $F_0 \in A(\overline{D_r(0)})$.

Remark 4.2 It is clear that χ in Proposition 4.1 can be chosen as any $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$ with suitable support properties.

Remark 4.3 In (2.32) there is a factor $e^{\frac{1}{2}(|x|^2+|\xi|^2)}$ which is absent in (4.1). We notice that this factor is not needed in (4.1) because $\mathcal{R}_{t_1,t_2}^\infty(\mathbf{C}^d)$ is invariant under multiplications of such functions.

The next results follow from Lemma 2.10 and Theorems 3.4–3.8. The details are left for the reader.

Proposition 4.4 *Let* ϕ *be as in* (2.7) *and* $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$*. Then the following conditions are equivalent:*

(1) $f \in \mathcal{H}_{0,\flat_1}(\mathbf{R}^d)$*;*
(2) $f = V_\phi^* F$ *for some* $F \in \mathscr{E}'(\mathbf{R}^{2d}) \cap L^\infty(\mathbf{R}^{2d})$ *which satisfies* (4.1) *for some* $F_0 \in A(\mathbf{C}^d)$.

Proposition 4.5 *Let* ϕ *be as in* (2.7)*,* $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$*,* $\sigma \in (\frac{1}{2}, 1)$ *and let*

$$\sigma_0 = \frac{\sigma}{2\sigma - 1}.$$

Then the following conditions are equivalent:

(1) $f \in \mathcal{H}_{\flat_\sigma}(\mathbf{R}^d)$ $(f \in \mathcal{H}_{0,\flat_\sigma}(\mathbf{R}^d))$*;*
(2) $f = V_\phi^* F$ *for some* $F \in \mathscr{E}'(\mathbf{R}^{2d}) \cap L^\infty(\mathbf{R}^{2d})$ *which satisfies* (4.1) *for some* $F_0 \in \mathcal{A}_{0,\flat_{\sigma_0}}'(\mathbf{C}^d)$ $(F_0 \in \mathcal{A}_{\flat_{\sigma_0}}'(\mathbf{C}^d))$.

Proposition 4.6 *Let* ϕ *be as in* (2.7)*,* $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$ *and let* $\sigma = \frac{1}{2}$*. Then the following conditions are equivalent:*

(1) $f \in \mathcal{H}_{\flat_\sigma}(\mathbf{R}^d)$ $(f \in \mathcal{H}_{0,\flat_\sigma}(\mathbf{R}^d))$*;*
(2) $f = V_\phi^* F$ *for some* $F \in \mathscr{E}'(\mathbf{R}^{2d}) \cap L^\infty(\mathbf{R}^{2d})$ *which satisfies* (4.1) *for some* $F_0 \in \mathcal{A}_{0,1/2}'(\mathbf{C}^d)$ $(F_0 \in \mathcal{A}_{0,1/2}(\mathbf{C}^d))$.

Proposition 4.7 *Let ϕ be as in (2.7), $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$, $\sigma \in (0, \frac{1}{2})$ and let*

$$\sigma_0 = \frac{\sigma}{1-2\sigma}.$$

Then the following conditions are equivalent:

(1) $f \in \mathcal{H}_{\flat_\sigma}(\mathbf{R}^d)$ ($f \in \mathcal{H}_{0,\flat_\sigma}(\mathbf{R}^d)$);
(2) $f = V_\phi^* F$ *for some* $F \in \mathscr{E}'(\mathbf{R}^{2d}) \cap L^\infty(\mathbf{R}^{2d})$ *which satisfies* (4.1) *for some* $F_0 \in \mathcal{A}_{\flat_{\sigma_0}}(\mathbf{C}^d)$ ($F_0 \in \mathcal{A}_{0,\flat_{\sigma_0}}(\mathbf{C}^d)$).

Proposition 4.8 *Let ϕ be as in (2.7), $\chi \in \mathcal{R}^\infty(\mathbf{C}^d)$ and let $s \in (0, \frac{1}{2})$. Then the following conditions are equivalent:*

(1) $f \in \mathcal{H}_s(\mathbf{R}^d)$ ($f \in \mathcal{H}_s(\mathbf{R}^d)$);
(2) $f = V_\phi^* F$ *for some* $F \in \mathscr{E}'(\mathbf{R}^{2d}) \cap L^\infty(\mathbf{R}^{2d})$ *which satisfies* (4.1) *for some* $F_0 \in \mathcal{A}_s(\mathbf{C}^d)$ ($F_0 \in \mathcal{A}_{0,s}(\mathbf{C}^d)$).

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