On asymptotic behavior of fractional Cauchy transform

Igor Chyzykov^{1,2} · Galyna Beregova³

Received: 8 October 2018 / Accepted: 24 February 2019 / Published online: 6 June 2019
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Abstract

We study non-regularity of growth of the fractional Cauchy transform

$$f(z) = \int_{-\pi}^{\pi} \frac{d\psi(t)}{(1 - ze^{-it})^{\alpha}}, \quad \alpha > 0, \psi \in BV[-\pi, \pi],$$

in terms of the modulus of continuity of the function ψ . Sharp estimates of the lower logarithmic order of f are found. In the case $\alpha \in (0, 1)$ the estimates are of different form than that for the logarithmic order.

Keywords Fractional Cauchy transform · Modulus of continuity · Lower logarithmic order · Lipschitz class · Maximum modulus

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc, $\alpha > 0$; $(1 - w)^{\alpha}$, $w \in \mathbb{D}$, be the principle branch of the power function, i.e. $(1 - w)^{\alpha}|_{w=0} = 1$.

Our main focus is to study of the following Cauchy type integral

$$f(z) = \int_{-\pi}^{\pi} \frac{d\psi(t)}{(1 - ze^{-it})^{\alpha}}, \quad (1)$$

✉ Igor Chyzykov
chyzykov@matman.uwm.edu.pl

Galyna Beregova
gberegova@yahoo.com

¹ School of Mathematics Science, Guizhou Normal University, No. 116 Baoshan Road (N), Guiyang 550001, Guizhou, China

² Present Address: Faculty of Mathematics and Computer Science, Warmia and Mazury University of Olsztyn, Soneczna 54, 10710 Olsztyn, Poland

³ Department of Computational Mathematics and Programming, Institute of Applied Mathematics and Fundamental Sciences, National University Lviv Polytechnic, 5 Mytropolyt Andrei str., Buiding 4, Lviv, Ukraine

where ψ is a complex-valued function of bounded variation, i.e., $\psi \in BV[-\pi, \pi]$, or a fractional Cauchy transform of the measure ψ^* on $\partial\mathbb{D}$ associated with ψ . The family of all functions of the form (1) is denoted by $\mathcal{F}_\alpha$. A study of $\mathcal{F}_\alpha$ can be found in the book [11], here we mention only few properties. $\mathcal{F}_\alpha$ is a Banach space with respect to the norm $\|f\|_{\mathcal{F}_\alpha} = \inf_{\psi^*} \|\psi^*\|$, where $\|\psi^*\|$ denotes the total variation of ψ^* . A straightforward consequence of the definition is that for $f \in \mathcal{F}_\alpha$ one has

$$M(r, f) \leq \frac{\|\psi^*\|}{(1-r)^\alpha}, \quad (2)$$

where $M(r, f) = \max\{|f(re^{i\theta})| : |z| = r\}$.

Many relations with Hardy, Besov, and Dirichlet-type spaces are obtained in the mentioned monograph.

Theorem A *If $0 < \alpha \leq 1$, then $\mathcal{F}_\alpha \subset H^p$ for $0 < p < 1/\alpha$. If $0 < p \leq 1$ then $H^p \subset \mathcal{F}_{1/p}$, where H^p is the Hardy space of analytic functions in $\mathbb{D}$ satisfying*

$$\sup_{0 < r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Theorem B *If $0 < \alpha < \beta$, then $\mathcal{B}_\alpha \subset \mathcal{F}_\alpha \subset \mathcal{B}_\beta$, where $\mathcal{B}_\alpha$ is the Besov space of analytic functions in $\mathbb{D}$ satisfying*

$$\int_{\mathbb{D}} |f'(z)|(1-|z|)^{\alpha-1} dm(z) < \infty,$$

where m is the planar Lebesgue measure.

Functions from $\mathcal{F}_\alpha$ appear frequently in representation theorems for different classes of functions defined on the unit disc $\mathbb{D}$. In particular, Djrbashian introduced [5] classes A_α , $\alpha > -1$, of analytic functions in $\mathbb{D}$. For $\alpha > 0$, an analytic function f belongs to the class A_α , $\alpha > 0$, if and only if

$$\sup_{0 < r < 1} \int_0^{2\pi} \left(\int_0^r (r-t)^{\alpha-1} \log |f(te^{i\varphi})| dt \right)^+ d\varphi < +\infty.$$

Note that the union of $\bigcup_{\alpha > 0} A_\alpha$ contains the class of analytic functions f in $\mathbb{D}$ of finite order of growth, that is $\log M(r, f) = O(\log \frac{1}{1-r})$ ($r \rightarrow 1-$). Due to results of Djrbashian [5, Chap. IX] functions $f \in \mathcal{F}_{\alpha+1}$ appear in a parametric representation of the class A_α , $\alpha > -1$.

Radial and non-tangential limits of $f \in \mathcal{F}_\alpha$ were investigated in many papers, e.g., Hallenbeck and MacGregor [8,9], and Sheremeta [13], see also [11, Chap. 10; 12]. It turns out that the estimate (2) can be improved in terms of the modulus of continuity for ψ . Let

$$\omega(\delta, \psi) = \sup\{|\psi(x) - \psi(y)| : |x - y| < \delta, x, y \in [-\pi, \pi]\}$$

be the modulus of continuity of ψ .

Denote by Λ_γ [15] the class of functions ψ such that $\omega(\delta, \psi) = O(\delta^\gamma)$, $\gamma \in (0, 1]$. It comes from Hardy and Littlewood [10], see also [6, 14], that there is a strong interplay between smoothness of a measure (or associated function ψ of bounded variation) and the growth of integral (1).

The next result in this spirit is a consequence of Theorem 3.4 [4] (see also Theorem 3.1 [1]).

Theorem C *Let f be an analytic function in $\mathbb{D}$ of the form (1). Let $\alpha > \gamma$ and $0 < \gamma \leq 1$. Then $\psi \in \Lambda_\gamma$, if and only if*

$$M(r, f) = O((1-r)^{\gamma-\alpha}), \quad r \rightarrow 1-.$$

Remark 1 More general results using the concept of a proximate order are obtained in [4]. There are counterparts for harmonic functions represented by Poisson-type integrals as well, see [1, 3].

Remark 2 Growth of p th means of functions $f \in \mathcal{F}_\alpha$ is described in [2].

A common feature of the above mentioned results is that the growth of f depends on which Hölder class Λ_γ the modulus of continuity of ψ belongs to. Let

$$p[f] := \limsup_{r \rightarrow 1-} \frac{\log M(r, f)}{\log \frac{1}{1-r}}, \quad \lambda[f] := \liminf_{r \rightarrow 1-} \frac{\log M(r, f)}{\log \frac{1}{1-r}} \quad (3)$$

denote the logarithmic order and the lower logarithmic order, respectively. It follows from Theorem C that for an arbitrary $f \in \mathcal{F}_\alpha$ and appropriate α, γ we have $p[f] \leq \alpha - \gamma$ if and only if $\psi \in \Lambda_\gamma$. The following question appears naturally. Does a smoothness of the modulus of continuity $\omega(\delta, \psi)$ on a sequence (δ_n) of values δ tending to zero imply an upper estimate for the lower logarithmic order similar to that for the logarithmic order?

We shall see that the answer is in the positive provided that $\alpha > 1$, but in general, the situation is more complicated.

Theorem 3 *Let $\alpha > 0$, $0 \leq \gamma < \lambda < 1$.*

Suppose that $\omega(\delta_n, \psi) = O(\delta_n^\lambda)$, $\delta_n \rightarrow 0+$, while $\omega(\delta, \psi) = O(\delta^\gamma)$, $\delta \rightarrow 0+$, and that f is of the form (1).

- (i) *If $0 < \alpha < \gamma$, then $M(r, f) = O(1)$.*
- (ii) *If $0 < \alpha = \gamma$, then $M(r, f) = O(\log \frac{1}{1-r})$.*
- (iii) *If $\gamma < \alpha$, then*

$$M(r_n, f_\alpha) = O\left(\left(\frac{1}{1-r_n}\right)^\Delta\right), \quad n \rightarrow \infty,$$

on a sequence $r_n \rightarrow 1 - (n \rightarrow \infty)$, where

$$\Delta = \begin{cases} \frac{\alpha(\alpha-\gamma)(1-\lambda)}{\alpha(1-\lambda)+\lambda-\gamma}, & \alpha < 1 \\ \alpha - \lambda, & \alpha > 1, \end{cases} \quad (4)$$

and, for $\alpha = 1$

$$M(r_n, f_\alpha) = O\left(\left(\frac{1}{1-r_n}\right)^{1-\kappa_2} \log \frac{1}{1-r_n}\right), \quad n \rightarrow \infty.$$

Remark 4 The quantity Δ in (4) is continuous at 1 as a function of α . Moreover, $\Delta \rightarrow \alpha - \lambda$ as $\gamma \rightarrow \lambda$.

Remark 5 Any modulus of continuity ω satisfies [7, Chap. 3]

$$\frac{\omega(t_2)}{t_2} \leq \frac{2\omega(t_1)}{t_1}, \quad 0 < t_1 < t_2. \quad (5)$$

If $\omega(\delta_n, \psi) \leq C\delta_n$, $\delta_n \rightarrow 0+$, i.e., $\lambda = 1$, then

$$\omega(\delta, \psi) \leq 2C\delta, \quad \delta > \delta_n,$$

so $\psi \in \Lambda_1$, and $\gamma = \lambda = 1$. This is why we exclude this case in the assumptions of Theorem 3.

Corollary 1 Let $\alpha > 0$. Suppose that for arbitrary $\varepsilon > 0$ there exists a sequence $\delta_n \rightarrow 0+$ ($n \rightarrow \infty$) such that $\omega(\delta_n, \psi) = O(\delta_n^{\lambda-\varepsilon})$ ($n \rightarrow \infty$), while $\psi \in \Lambda_\gamma$, $0 \leq \gamma < \lambda < 1$, $\gamma < \alpha$. Then

$$\lambda[f] \leq \begin{cases} \frac{\alpha(\alpha-\gamma)(1-\lambda)}{\alpha(1-\lambda)+\lambda-\gamma}, & \alpha < 1 \\ \alpha - \lambda, & \alpha \geq 1. \end{cases} \quad (6)$$

Proof of Theorem 3 Let φ be arbitrary on $[-\pi, \pi]$. We extend ψ on $\mathbb{R}$ by the formula $\psi(t + 2\pi) - \psi(t) = \psi(\pi) - \psi(-\pi)$. Since the kernel of the integral (1) is a 2π -periodic function in t , integrating by parts gives us

$$\begin{aligned} f_\alpha(re^{i\varphi}) &= \int_{-\pi+\varphi}^{\pi+\varphi} \frac{d(\psi(t) - \psi(\varphi))}{(1 - re^{i(\varphi-t)})^\alpha} \\ &= \frac{\psi(\pi) - \psi(-\pi)}{(1+r)^\alpha} + i r \alpha \int_{-\pi}^{\pi} \frac{e^{-i\tau}(\psi(\tau + \varphi) - \psi(\varphi)) d\tau}{(1 - re^{-i\tau})^{\alpha+1}}. \end{aligned}$$

Therefore, uniformly in φ , $\omega(\delta) := \omega(\delta; \psi)$

$$|f_\alpha(re^{i\varphi})| \leq 2\alpha \int_0^\pi \frac{\omega(x) dx}{|1 - re^{ix}|^{\alpha+1}} + O(1) \leq 2\alpha \int_0^1 \frac{\omega(x) dx}{|1 - re^{ix}|^{\alpha+1}} + O(1). \quad (7)$$

We start with the simple cases (i) and (ii). Suppose that $0 < \alpha \leq \gamma$. Then $\omega(\delta) = O(\delta^\gamma)$, and standard estimates yield

$$\int_0^1 \frac{\omega(x) dx}{|1 - re^{ix}|^{\alpha+1}} \leq \int_0^{1-r} \frac{Cx^\gamma dx}{(1-r)^{\alpha+1}} + \int_{1-r}^1 \frac{dx}{Cx^{\alpha+1-\gamma}},$$

where C is a positive constant. Hence, the statements (i) and (ii) follow.

Let now $\alpha > \gamma \geq 0$. Consider two cases. First, we suppose that $\alpha < 1$. We then choose

$$\tilde{\delta}_n := \delta_n^{\frac{\alpha(1-\lambda)+\lambda-\gamma}{\alpha(1-\gamma)}} = \delta_n^{1+\frac{(\lambda-\gamma)(1-\alpha)}{1-\gamma}}.$$

Note that $\tilde{\delta}_n \leq \delta_n$, because $\alpha \leq 1$.

In what follows the inequality $g_1(n) \lesssim g_2(n)$ means that there exists a constant C independent of n such that $g_1(n) \leq Cg_2(n)$ ($n \rightarrow \infty$).

We put $r_n = 1 - \tilde{\delta}_n$. Since a modulus of continuity is nondecreasing, we have

$$\int_0^{\tilde{\delta}_n} \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \leq \omega(\delta_n) \int_0^{\tilde{\delta}_n} \frac{dx}{(1 - r_n)^{\alpha+1}} \lesssim \frac{\delta_n^\lambda}{\tilde{\delta}_n^\alpha}. \quad (8)$$

Then

$$\int_{\tilde{\delta}_n}^{\delta_n} \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \lesssim \delta_n^\lambda \int_{\tilde{\delta}_n}^{\delta_n} \frac{dx}{x^{\alpha+1}} \lesssim \frac{\delta_n^\lambda}{\tilde{\delta}_n^\alpha}. \quad (9)$$

Taking into account (5), and setting $\hat{\delta}_n = \delta_n^{\frac{1-\lambda}{1-\gamma}}$, we obtain

$$\int_{\delta_n}^{\hat{\delta}_n} \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \lesssim \int_{\delta_n}^{\hat{\delta}_n} \frac{\delta_n^\lambda x}{\delta_n x^{\alpha+1}} dx \lesssim \frac{\hat{\delta}_n^{1-\alpha}}{\delta_n^{1-\lambda}} = \frac{1}{\delta_n^{\frac{(1-\lambda)(\alpha-\gamma)}{1-\gamma}}}. \quad (10)$$

Similarly

$$\int_{\hat{\delta}_n}^1 \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \lesssim \int_{\hat{\delta}_n}^1 \frac{x^\gamma dx}{x^{\alpha+1}} \lesssim \frac{1}{\hat{\delta}_n^{\alpha-\gamma}} = \frac{1}{\delta_n^{\frac{(1-\lambda)(\alpha-\gamma)}{1-\gamma}}}. \quad (11)$$

Finally, taking into account that $\tilde{\delta}_n \lesssim \delta_n$, from the above estimates we deduce

$$\begin{aligned} \int_0^1 \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} &\lesssim \frac{\delta_n^\lambda}{\tilde{\delta}_n^\alpha} + \frac{1}{\delta_n^{\frac{(1-\lambda)(\alpha-\gamma)}{1-\gamma}}} \leq \frac{\tilde{\delta}_n^{\frac{\lambda\alpha(1-\gamma)}{\alpha(1-\lambda)+\lambda-\gamma}}}{\tilde{\delta}_n^\alpha} + \frac{1}{\delta_n^{\frac{\alpha(1-\gamma)}{\alpha(1-\lambda)+\lambda-\gamma} \frac{(1-\lambda)(\alpha-\gamma)}{1-\gamma}}} \\ &\leq \frac{1}{\delta_n^{\frac{\alpha(\alpha-\gamma)(1-\lambda)}{\alpha(1-\lambda)+\lambda-\gamma}}} + \frac{1}{\tilde{\delta}_n^{\frac{\alpha(\alpha-\gamma)(1-\lambda)}{\alpha(1-\lambda)+\lambda-\gamma}}} \lesssim \frac{1}{\tilde{\delta}_n^{\frac{\alpha(\alpha-\gamma)(1-\lambda)}{\alpha(1-\lambda)+\lambda-\gamma}}}. \end{aligned} \quad (12)$$

This finishes the proof in the first case.

Let now $\alpha > 1$. We put $r_n := 1 - \delta_n$ in this case. Arguing similarly as in the first case, we obtain

$$\int_0^{\delta_n} \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \lesssim \delta_n^\lambda \int_0^{\delta_n} \frac{dx}{(1 - r_n)^{\alpha+1}} \lesssim \frac{1}{\delta_n^{\alpha-\lambda}}. \quad (13)$$

Then

$$\int_{\delta_n}^{\hat{\delta}_n} \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \lesssim \int_{\delta_n}^{\hat{\delta}_n} \frac{\delta_n^\lambda x dx}{\delta_n x^{\alpha+1}} \lesssim \frac{1}{\delta_n^{\alpha-\lambda}}. \quad (14)$$

Similarly, as in the first case,

$$\int_{\hat{\delta}_n}^1 \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \lesssim \int_{\hat{\delta}_n}^1 \frac{x^\gamma dx}{x^{\alpha+1}} \lesssim \frac{1}{\delta_n^{\alpha-\gamma}}. \quad (15)$$

Thus, using the definition of $\hat{\delta}_n$ and the latter estimates we deduce

$$\int_0^1 \frac{\omega(x) dx}{|1 - r_n e^{ix}|^{\alpha+1}} \lesssim \frac{1}{\delta_n^{\alpha-\lambda}} + \frac{1}{\delta_n^{\alpha-\gamma}} = \frac{1}{\delta_n^{\alpha-\lambda}} + \frac{1}{\delta_n^{\frac{(1-\lambda)(\alpha-\gamma)}{1-\gamma}}} \lesssim \frac{1}{\delta_n^{\alpha-\lambda}},$$

because the inequality $\frac{(1-\lambda)(\alpha-\gamma)}{1-\gamma} < \alpha - \lambda$ is equivalent to $\alpha > 1$ provided that $\alpha > \gamma$ and $\gamma < 1$.

In the case $\alpha = 1$, all the estimates for the case $\alpha > 1$ except (14) remain true. Instead of (14) we have

$$\int_{\delta_n}^{\hat{\delta}_n} \frac{\omega(x) dx}{|1 - r_n e^{ix}|^2} \lesssim \int_{\delta_n}^{\hat{\delta}_n} \frac{\delta_n^\lambda x dx}{\delta_n x^2} \lesssim \frac{1}{\delta_n^{1-\lambda}} \log \frac{1}{\delta_n}. \quad (16)$$

The theorem is proved. $\square$

2 Example of a function of non-regular growth

The following theorem shows the sharpness of the corollary.

Theorem 6 *Let $0 < \varkappa_1 < \varkappa_2 < 1$, $\varkappa_1 < \alpha$, and suppose that α is not an odd number. There exists an analytic function f_α in $\mathbb{D}$ of the form (1) such that $\omega(\delta_n, \psi) = O(\delta_n^{\varkappa_2})$, $\delta_n \rightarrow 0+$, while $\psi \in \Lambda_{\varkappa_1}$, $p[f_\alpha] = \alpha - \varkappa_1$, and*

$$\lambda[f_\alpha] = \begin{cases} \frac{\alpha(\alpha-\varkappa_1)(1-\varkappa_2)}{\alpha(1-\varkappa_2)+\varkappa_2-\varkappa_1}, & \alpha < 1 \\ \alpha - \varkappa_2, & \alpha > 1. \end{cases} \quad (17)$$

Proof Given two numbers $\varkappa_1, \varkappa_2, 0 < \varkappa_1 < \varkappa_2 < 1$ we define a nondecreasing function $\psi: [0, 1/2] \rightarrow \mathbb{R}$ recursively. Let $\psi(1/2) = 2^{-\varkappa_1}, t_1 = \frac{1}{2}$. Set

$$t'_n = (t_n)^{\frac{1-\varkappa_1}{1-\varkappa_2}}, \quad t_{n+1} = (t'_n)^{\frac{\varkappa_2}{\varkappa_1}}, \quad n \in \mathbb{N}. \quad (18)$$

One can write these sequences explicitly: $t_n = 2^{-\beta^{n-1}}$, where $\beta = \frac{1-\varkappa_1}{1-\varkappa_2} \frac{\varkappa_2}{\varkappa_1}$, and $t'_n = 2^{-\frac{1-\varkappa_1}{1-\varkappa_2} \beta^{n-1}}, n \in \mathbb{N}$. It is clear that $t_{n+1} < t'_n < t_n, n \in \mathbb{N}$, and $t_{n+1} = o(t'_n), t'_n = o(t_n) (n \rightarrow \infty)$.

We define the function

$$\psi(t) = \begin{cases} (t'_n)^{\varkappa_2-1} t, & t'_n \leq t \leq t_n; \\ (t'_n)^{\varkappa_2}, & t_{n+1} \leq t \leq t'_n. \end{cases} \quad (19)$$

It follows from the definition of the sequences (t_n) and (t'_n) that ψ is a nondecreasing and continuous function on $[0, \frac{1}{2}]$. Moreover, $t^{\varkappa_2} \leq \psi(t) \leq t^{\varkappa_1}, t \in [0, \frac{1}{2}]$, and $\psi(t_n) = t_n^{\varkappa_1}, \psi(t'_n) = (t'_n)^{\varkappa_2}, n \in \mathbb{N}$.

It follows directly from the definition of ψ that $\psi(t)/t$ is nonincreasing. By Dzyadyk and Shevchuk [7, Chap. 3] it implies semiadditivity of ψ , thus ψ is a modulus of continuity.

Define

$$f_\alpha(z) = \int_0^{1/2} \frac{d\psi(t)}{(1 - ze^{-it})^\alpha}.$$

We then have

$$f_\alpha(r) := \sum_{n=1}^{\infty} \int_{t'_n}^{t_n} \frac{(t'_n)^{\varkappa_2-1} dt}{(1 - re^{-it})^\alpha}, \quad r \in [0, 1).$$

Since ψ is a modulus of continuity, $\omega(\delta, \psi) = \psi(\delta) = O(\delta^{\varkappa_1})$, and, by construction we have $\sup\{\gamma : \psi \in \Lambda_\gamma\} = \varkappa_1$. Therefore, by Theorem C, we have that $p[f_\alpha] \leq \alpha - \varkappa_1$. In fact $|f_\alpha(z)| = O((1 - |z|)^{\varkappa_1-\alpha}) (|z| \rightarrow 1-)$.

Let $\alpha > 1$. It follows from the construction of ψ that $\liminf_{\delta \rightarrow 0+} \frac{\omega(\delta, \psi)}{\delta^{\varkappa_2}} = 1$. Hence, by Theorem 3, it follows that $\lambda[f_\alpha] \leq \alpha - \varkappa_2$. We prove that the converse inequality is true as well.

Denote

$$I_n(r) = (t'_n)^{\varkappa_2-1} \int_{t'_n}^{t_n} \frac{\cos(\alpha \tau(t)) dt}{|1 - re^{-it}|^\alpha},$$

where

$$\tau = \tau(t) = \arg \frac{1}{1 - re^{-it}}, \quad \tan \tau = \frac{r \sin t}{1 - r \cos t}. \quad (20)$$

We are going to calculate asymptotics for I_n for an appropriate choice of r . Let $A_n = \log \frac{1}{t'_n} \rightarrow +\infty$ ($n \rightarrow +\infty$). If $1 - r \leq t'_n/A_n$, then uniformly in r and $t \in [t'_n, t_n]$,

$$1 - re^{-it} = 1 - r + r(1 - e^{-it}) \sim it, \quad n \rightarrow \infty.$$

Remark 7 In what follows all asymptotic relations as $n \rightarrow \infty$ that depend on parameters r, t , etc. hold uniformly in these parameters at their specified ranges. For simplicity, this fact will no longer be emphasized.

We have $\tau(t) \sim \frac{\pi}{2}$, $\cos \alpha \tau(t) \sim \cos \frac{\alpha \pi}{2}$ as $t \in [t'_n, t_n]$, $n \rightarrow +\infty$. Therefore,

$$I_n(r) \sim \frac{\cos \frac{\alpha \pi}{2}}{(t'_n)^{1-\kappa_2}} \int_{t'_n}^{t_n} \frac{dt}{t^\alpha} = \frac{C_0(\alpha) + o(1)}{(t'_n)^{\alpha-\kappa_2}}, \quad n \rightarrow +\infty, \quad 1 - r \leq t'_n/A_n. \quad (21)$$

where $C_0(\alpha) = \frac{\cos \frac{\alpha \pi}{2}}{\alpha-1}$.

If $1 - r \geq A_n t_n$, $t \in [t'_n, t_n]$, then $\cos \alpha \tau(t) \sim 1$, and $1 - re^{-it} \sim 1 - r$ ($n \rightarrow \infty$), so we get

$$I_n(r) \sim \frac{(t'_n)^{\kappa_2-1} t_n}{(1-r)^\alpha} = \frac{(t_n)^{\kappa_1}}{(1-r)^\alpha}, \quad n \rightarrow +\infty. \quad (22)$$

Finally, let $1 - r \in [t'_n/A_n, t_n/A_n]$. We split I_n into three integrals in this case, namely,

$$I_n = \left(\int_{t'_n}^{(1-r)/A_n} + \int_{(1-r)/A_n}^{(1-r)A_n} + \int_{(1-r)A_n}^{t_n} \right) \frac{(t'_n)^{\kappa_2-1} \cos(\alpha \tau(t)) dt}{|1 - re^{-it}|^\alpha} =: J_{n1} + J_{n2} + J_{n3}.$$

We begin with J_{n2} . Note that, $\tau(t) = \arctan \frac{r \sin t}{1-r \cos t} \sim \arctan \frac{t}{1-r}$ if $t \in [(1-r)/A_n, (1-r)A_n]$ ($n \rightarrow \infty$). Making the substitution $u = \frac{t}{1-r}$, we obtain

$$\begin{aligned} J_{n2} &\sim (t'_n)^{\kappa_2-1} \int_{A_n^{-1}}^{A_n} \frac{\cos(\alpha \arctan u)(1-r) du}{((1-r)^2 + 4 \sin^2 \frac{u(1-r)}{2})^{\frac{\alpha}{2}}} \\ &\sim \frac{(t'_n)^{\kappa_2-1}}{(1-r)^{\alpha-1}} \int_{A_n^{-1}}^{A_n} \frac{\cos(\alpha \arctan u) du}{(u^2 + 1)^{\frac{\alpha}{2}}} \sim \frac{C_1(\alpha)}{(t'_n)^{1-\kappa_2} (1-r)^{\alpha-1}}, \quad n \rightarrow \infty, \end{aligned} \quad (23)$$

where $C_1(\alpha) = \int_0^\infty \frac{\cos(\alpha \arctan u) du}{(u^2+1)^{\frac{\alpha}{2}}}$. Similar arguments give

$$J_{n1} \leq (t'_n)^{\kappa_2-1} \int_0^{1/A_n} \frac{dt}{(1-r)^{\alpha-1}} = o\left(\frac{1}{(t'_n)^{1-\kappa_2} (1-r)^{\alpha-1}}\right), \quad (24)$$

$$J_{n3} \leq (t'_n)^{\kappa_2-1} \int_{(1-r)A_n}^{t_n} \frac{dt}{t^\alpha} \lesssim \frac{(t'_n)^{\kappa_2-1}}{(A_n(1-r))^{\alpha-1}} = o\left(\frac{1}{(t'_n)^{1-\kappa_2} (1-r)^{\alpha-1}}\right) \quad (25)$$

Combining (23)–(25), we get

$$I_n(r) \sim \frac{C_1(\alpha)}{(t'_n)^{1-\varkappa_2}(1-r)^{\alpha-1}}, \quad 1-r \in [t'_n A_n, t_n/A_n], \quad n \rightarrow \infty. \quad (26)$$

We proceed to estimate the growth of $|f_\alpha(r)|$ from below.

Let now $1-r \in [t_{k+1} A_k, \frac{t'_k}{A_k}]$. Then, using (21) for $n \leq k$, and (22), for $n \geq k+1$, we obtain for some $k_0 \in \mathbb{N}$ (see Remark 7)

$$\begin{aligned} \Re f_\alpha(r) &= \left(\sum_{n=1}^{k_0} + \sum_{n=k_0+1}^k + \sum_{n=k+1}^{\infty} \right) I_n \\ &= O(1) + \sum_{n=k_0+1}^k \frac{C_0(\alpha) + o(1)}{(t'_n)^{\alpha-\varkappa_2}} + \sum_{n=k+1}^{\infty} (1+o(1)) \frac{(t_n)^{\varkappa_1}}{(1-r)^\alpha} \\ &= O\left(\frac{1}{(t'_k)^{\alpha-\varkappa_2}}\right) + (1+o(1)) \frac{(t_{k+1})^{\varkappa_1}}{(1-r)^\alpha} \\ &= O\left(\frac{1}{(t'_k)^{\alpha-\varkappa_2}}\right) + (1+o(1)) \frac{(t'_k)^{\varkappa_2}}{(1-r)^\alpha} \sim \frac{(t'_k)^{\varkappa_2}}{(1-r)^\alpha} \\ &\geq \frac{A_k^{\varkappa_2}}{(1-r)^{\alpha-\varkappa_2}}, \quad k \rightarrow +\infty. \end{aligned} \quad (27)$$

Further, let $1-r \in [A_k t'_k, t_k/A_k]$. In this case, using (21) for $n < k$, (22) for $n \geq k+1$, and (26) for $n = k$, definitions of (t_n) and (t'_n) , we deduce ($k_1 \in \mathbb{N}$)

$$\begin{aligned} \Re f_\alpha(r) &= \left(\sum_{n=1}^{k_1} + \sum_{n=k_1+1}^{k-1} + \sum_{n=k+1}^{\infty} \right) I_n + I_k \\ &= O(1) + \sum_{n=k_1+1}^{k-1} \frac{C_0(\alpha) + o(1)}{(t'_n)^{\alpha-\varkappa_2}} + \sum_{n=k+1}^{\infty} \frac{(1+o(1))(t_n)^{\varkappa_1}}{(1-r)^\alpha} \\ &\quad + \frac{C_1(\alpha) + o(1)}{(t'_k)^{1-\varkappa_2}(1-r)^{\alpha-1}} \\ &= O\left(\frac{1}{(t'_{k-1})^{\alpha-\varkappa_2}}\right) + (1+o(1)) \frac{(t_{k+1})^{\varkappa_1}}{(1-r)^\alpha} + \frac{C_1(\alpha) + o(1)}{(t'_k)^{1-\varkappa_2}(1-r)^{\alpha-1}} \\ &= O\left(\frac{1}{(t_k)^{\frac{\varkappa_1}{\varkappa_2}(\alpha-\varkappa_2)}}\right) + (1+o(1)) \frac{(t'_k)^{\varkappa_2}}{(1-r)^\alpha} + \frac{C_1(\alpha) + o(1)}{(t'_k)^{1-\varkappa_2}(1-r)^{\alpha-1}} \\ &= O\left(\frac{1}{(t'_k)^{\frac{\varkappa_1(1-\varkappa_2)}{\varkappa_2(1-\varkappa_1)}(\alpha-\varkappa_2)}}\right) + \frac{C_1(\alpha) + o(1)}{(t'_k)^{1-\varkappa_2}(1-r)^{\alpha-1}}, \quad k \rightarrow +\infty. \end{aligned} \quad (28)$$

We show that the second addend in the last correlation dominates, i.e.

$$\frac{1}{(t'_k)^{\frac{\varkappa_1(1-\varkappa_2)}{\varkappa_2(1-\varkappa_1)}(\alpha-\varkappa_2)}} = o\left(\frac{1}{(t'_k)^{1-\varkappa_2}(1-r)^{\alpha-1}}\right), \quad 1-r \in [A_k t'_k, t_k/A_k], k \rightarrow \infty,$$

or

$$(1-r)^{\alpha-1} = o\left((t'_k)^{(1-\varkappa_2)\left(\frac{\varkappa_1(\alpha-\varkappa_2)}{\varkappa_2(1-\varkappa_1)}-1\right)}\right), \quad 1-r \in [A_k t'_k, t_k/A_k], k \rightarrow \infty.$$

Since $\alpha > 1$ and $1-r \leq \frac{t_k}{A_k} = \frac{(t'_k)^{\frac{1-\varkappa_2}{1-\varkappa_1}}}{A_k}$, it is sufficient to check if $\frac{(\alpha-1)(1-\varkappa_2)}{1-\varkappa_1} > (1-\varkappa_2)\left(\frac{\varkappa_1(\alpha-\varkappa_2)}{\varkappa_2(1-\varkappa_1)}-1\right)$. This is equivalent to $\varkappa_2 > \varkappa_1$, which holds by assumption. Hence, (28) can be rewritten in the form

$$\begin{aligned} |\Re f_\alpha(r)| &\sim \frac{|C_1(\alpha)|}{(t'_k)^{1-\varkappa_2}(1-r)^{\alpha-1}} \\ &\geq \frac{|C_1(\alpha)|}{\left(\frac{1-r}{A_k}\right)^{1-\varkappa_2}(1-r)^{\alpha-1}} \\ &= \frac{|C_1(\alpha)|A_k^{1-\varkappa_1}}{(1-r)^{\alpha-\varkappa_2}}, \quad 1-r \in \left[A_k t'_k, \frac{t_k}{A_k}\right], k \rightarrow +\infty. \end{aligned} \quad (29)$$

Therefore, taking into account (27) and (29), we obtain

$$(\alpha - \varkappa_2) \log \frac{1}{1-r} \lesssim \log M(r, f) \lesssim (\alpha - \varkappa_1) \log \frac{1}{1-r} \quad (30)$$

as $r \in \bigcup_{k=1}^{\infty} \left([A_k t_{k+1}, \frac{t'_k}{A_k}] \cup [A_k t'_k, \frac{t_k}{A_k}]\right)$. Since for $r \in [t_{k+1}, t_k]$

$$\log\left(\frac{A_k^2}{1-r}\right)^{\alpha-\varkappa_2} \leq \log\left(\frac{1}{1-r}\right)^{\alpha-\varkappa_2} + 2(\alpha-\varkappa_2) \log A_k \lesssim \log\left(\frac{1}{1-r}\right)^{\alpha-\varkappa_2},$$

(30) holds as $r \rightarrow 1-$ without exceptional sets.

This finishes the proof in the case $\alpha > 1$.

Let now $0 < \alpha < 1$. We use the same function f_α . Arguing similarly, we deduce

$$I_n(r) \sim \frac{\cos \frac{\alpha\pi}{2}}{(t'_n)^{1-\varkappa_2}} \int_{t'_n}^{t_n} \frac{dt}{t^\alpha} \sim \frac{C_2(\alpha)t_n^{1-\alpha}}{(t'_n)^{1-\varkappa_2}}, \quad n \rightarrow +\infty, 1-r \leq t'_n/A_n, \quad (31)$$

where $C_2(\alpha) = |C_0(\alpha)| = \frac{\cos \frac{\alpha\pi}{2}}{1-\alpha}$. Correlation (22) remains true. For J_{nm} , $m \in \{1, 2, 3\}$ we get

$$J_{n2} \sim \frac{(t'_n)^{\varkappa_2-1}}{(1-r)^{\alpha-1}} \int_{A_n^{-1}}^{A_n} \frac{\cos(\alpha \arctan u) du}{(u^2+1)^{\frac{\alpha}{2}}} \sim \frac{C_2(\alpha)A_n^{1-\alpha}(1-r)^{1-\alpha}}{(t'_n)^{1-\varkappa_2}}, \quad (32)$$

while J_{n1} allows the same estimate as for $\alpha > 1$,

$$J_{n3} \sim \cos \frac{\alpha\pi}{2} (t'_n)^{\varkappa_2-1} \int_{(1-r)A_n}^{t_n} \frac{dt}{t^\alpha} \sim \frac{C_2(\alpha)}{(t'_n)^{1-\varkappa_2}} (t_n^{1-\alpha} - (A_n(1-r))^{1-\alpha}).$$

Therefore

$$I_n = J_{n1} + J_{n2} + J_{n3} \sim C_2(\alpha) \frac{t_n^{1-\alpha}}{(t'_n)^{1-\varkappa_2}}, \quad 1-r \in [t'_n A_n, t_n/A_n], \quad n \rightarrow \infty. \quad (33)$$

For $1-r \in [t_{k+1} A_k, \frac{t'_k}{A_k}]$, as in the case $\alpha > 1$, using (31) for $n \leq k$, and (22), for $n \geq k+1$, we get

$$\begin{aligned} \Re f_\alpha(r) &= O(1) + \sum_{n=k_0+1}^k \frac{(C_2(\alpha) + o(1))t_n^{1-\alpha}}{(t'_n)^{1-\varkappa_2}} + \sum_{n=k+1}^{\infty} (1+o(1)) \frac{(t_n)^{\varkappa_1}}{(1-r)^\alpha} \\ &= \frac{(C_2(\alpha) + o(1))t_k^{1-\alpha}}{(t'_k)^{1-\varkappa_2}} + (1+o(1)) \frac{(t_{k+1})^{\varkappa_1}}{(1-r)^\alpha} \\ &= \frac{C_2(\alpha) + o(1)}{(t'_k)^{(1-\varkappa_2)\frac{\alpha-\varkappa_1}{1-\varkappa_1}}} + (1+o(1)) \frac{(t'_k)^{\varkappa_2}}{(1-r)^\alpha}, \quad k \rightarrow +\infty. \end{aligned}$$

We write $\tilde{t}_n := (t'_n)^{1+\frac{(\varkappa_2-\varkappa_1)(1-\alpha)}{\alpha(1-\varkappa_1)}}$. We have $t_{n+1} < \tilde{t}_n < t'_n$, because $\varkappa_1 < \alpha < 1$. Consider two subcases. If $t_{k+1} A_k \leq 1-r \leq \tilde{t}_k$, then

$$\begin{aligned} \Re f_\alpha(r) &\geq (1+o(1)) \frac{(t'_k)^{\varkappa_2}}{(1-r)^\alpha} \sim \frac{(\tilde{t}_k)^{\varkappa_2 \frac{\alpha(1-\varkappa_1)}{\alpha(1-\varkappa_2)+\varkappa_2-\varkappa_1}}}{(1-r)^\alpha} \\ &= \frac{1}{(1-r)^{\alpha(1-\frac{\varkappa_2(1-\varkappa_1)}{\alpha(1-\varkappa_2)+\varkappa_2-\varkappa_1})}} = \frac{1}{(1-r)^{\frac{\alpha(\alpha-\varkappa_1)(1-\varkappa_2)}{\alpha(1-\varkappa_2)+\varkappa_2-\varkappa_1}}} \end{aligned}$$

as required. If $\tilde{t}_k \leq 1-r \leq \frac{t'_k}{A_k}$, then

$$\Re f_\alpha(r) \geq \frac{C_2(\alpha) + o(1)}{(t'_k)^{(1-\varkappa_2)\frac{\alpha-\varkappa_1}{1-\varkappa_1}}} \sim \frac{C_2(\alpha)}{\tilde{t}_k^{\frac{\alpha(1-\varkappa_2)(\alpha-\varkappa_1)}{\alpha(1-\varkappa_2)+\varkappa_2-\varkappa_1}}} \geq \frac{C_2(\alpha)}{(1-r)^{\frac{\alpha(1-\varkappa_2)(\alpha-\varkappa_1)}{\alpha(1-\varkappa_2)+\varkappa_2-\varkappa_1}}}.$$

Thus, in both subcases we have the required lower estimate.

Finally, let $1-r \in [t'_k A_k, \frac{t'_k}{A_k}]$. Then using (31) for $n < k$, (22) for $n \geq k+1$, and (33) for $n = k$, and the definitions of (t_n) and (t'_n) , we deduce

$$\Re f_\alpha(r) = O(1) + \sum_{n=k_0+1}^k \frac{(C_2(\alpha) + o(1))t_n^{1-\alpha}}{(t'_n)^{1-\varkappa_2}} + \sum_{n=k+1}^{\infty} (1+o(1)) \frac{(t_n)^{\varkappa_1}}{(1-r)^\alpha}$$

$$\geq \frac{(C_2(\alpha) + o(1))t_k^{1-\alpha}}{(t'_k)^{1-\alpha_2}} \sim \frac{C_2(\alpha)}{(t'_k)^{(1-\alpha_2)\frac{\alpha-\alpha_1}{1-\alpha_1}}} \geq \frac{C_2(\alpha)}{\left(\frac{1-r}{A_k}\right)^{(1-\alpha_2)\frac{\alpha-\alpha_1}{1-\alpha_1}}} \quad k \rightarrow +\infty.$$

It is easy to check that $(1 - \alpha_2)\frac{\alpha-\alpha_1}{1-\alpha_1} > \frac{\alpha(1-\alpha_2)(\alpha-\alpha_1)}{\alpha(1-\alpha_2)+\alpha_2-\alpha_1}$, because $\alpha < 1$.

We have proved that

$$\frac{\alpha(1-\alpha_2)(\alpha-\alpha_1)}{\alpha(1-\alpha_2)+\alpha_2-\alpha_1} \log \frac{1}{1-r} \lesssim \log M(r, f) \lesssim (\alpha - \alpha_1) \log \frac{1}{1-r} \quad (34)$$

as $r \in \bigcup_{k=1}^{\infty} \left([A_k t_{k+1}, \frac{t'_k}{A_k}] \cup [A_k t'_k, \frac{t_k}{A_k}] \right)$ ($k \rightarrow +\infty$). As in the case $\alpha > 1$ we deduce that (34) holds as $r \rightarrow 1-$. $\square$

Conflict of interest Igor Chyzykhov and Galyna Beregova declare that they have no conflict of interest.

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