Analysis and Mathematical Physics (2019) 9:2099–2108
https://doi.org/10.1007/s13324-019-00301-1

Decomposition matrices for the square lattices of the Lie groups $SU(2) \times SU(2)$

M. Bodner[1] · Z. Grabowiecka[2,4] · J. Patera[2,4] · M. Szajewska[3]

Received: 11 October 2017 / Revised: 9 April 2019 / Accepted: 12 April 2019 / Published online: 26 April 2019

Abstract

A method for the decomposition of data functions sampled on a finite fragment of rectangular lattice is described. The symmetry of a square lattice in a 2-dimensional real Euclidean space is either given by the semisimple Lie group $SU(2) \times SU(2)$ or equivalently by the Lie algebra $A_1 \times A_1$, or by the simple Lie group $O(5)$ or its Lie algebra called C_2 or equivalently B_2. In this paper we consider the first of these possibilities which is applied to data which is given in 2 orthogonal directions—hence the method is a concatenation of two 1-dimensional cases. The asymmetry we underline here is a different density of discrete data points in the two orthogonal directions which cannot be studied with the simple Lie group symmetry.

Keywords $SU(2) \times SU(2)$ group · Decomposition matrix · Fourier transform

1 Introduction

The method of decomposition matrices enables multidimensional digital data processing through calculation of the Fourier expansion of digital data, with potentially significantly faster processing and the use of more ideal symmetry lattices for sampling data for specific applications. The method for triangular lattices based on the semisimple Lie group of $SU(3)$ and $G(2)$ has already beed developed [2,3]. Since most of digital data is given on rectangular lattices, in this paper we describe the method for the first case of rectangular lattices, namely those based on the semisimple

✉ M. Szajewska
m.szajewska@math.uwb.edu.pl

1 MIND Research Institute, 111 Academy Dr., Irvine, CA 92617, USA

2 Centre de recherches mathématiques, Université de Montréal, C. P. 6128, Centre-ville, Montreal, QC H3C 3J7, Canada

3 Institute of Mathematics, University of Bialystok, 1M Ciołkowskiego, 15-245 Białystok, Poland

4 Département de mathématiques et de statistique, Université de Montréal, C. P. 6128, Centre-ville, Montreal, QC H3C 3J7, Canada

Lie group $SU(2) \times SU(2)$. The other possibility, that we will focus on in the future is the case when the symmetry is given by Lie group $Sp(4)$ or $O(5)$. While triangular lattices may be more optimal for specific applications, typically digital data is analyzed using rectangular or square lattices. There is a wide range of examples and applications including digital images and video processing, and time series involving time-frequency signal analysis and processing [7].

The square lattice in 2 dimensional real Euclidean space displays two symmetries. The first of these is the symmetry denoted by the semisimple Lie group $SU(2)\times SU(2)$, or equivalently denoted by the Lie algebra $A_1 \times A_1$. The second case is the symmetry derived from the simple Lie group $O(5)$, or from its Lie algebra. In this paper we consider the first of these symmetries because it can conveniently treat data, which are given on rectangular fragments of the square lattice. In contrast, the symmetry of the simple Lie group would not allow this since there, the density of lattice points have to be the same in both directions. The second symmetry is particularly suitable when the data is given on a triangle which is half of a square. Of particular interest here are the cases when the structure of each rectangle is far from that of the square. We can take such a structure rescaled to the square, and then the density of points will be much smaller in one than in the others. In the $A_1 \times A_1$ case an orthogonal system of functions is still allowed on such an asymmetric lattice which can be used in the Fourier decomposition.

2 Weyl groups of A_1 and $A_1 \times A_1$

The 1-dimensional case is well known in the literature, its special functions are the Chebyshev polynomials or the trigonometric functions sin and cos [6]. We examine that case here in order to fix the notation and terminology used in this paper, as well as to display the correspondence with the content of subsequent sections.

The fundamental region F in 1D is a closed segment $[0, 1] = [0, \omega]$. We can fix the density of the points by choosing a positive integer M. The intersection of the segment of some density specified by M and the basic tile F is denoted by F_M. The number of equidistant points in F_M is equal to $M + 1$. Moreover, any choice of nonnegative integers s_0 and s_1, such that $M = s_0+s_1$ is used to find points of F_M. Namely, $x \in F_M$ is written as

$$x = \frac{s_1}{s_0 + s_1}\omega \tag{1}$$

The symmetry group in this case is the Weyl group of A_1 consisting of two elements that are 1 and $r(x) = -x$, with $r^2 = 1$.

In the orthogonal direction the fundamental region is still a segment between 0 and 1, but the number of points is given by $M' + 1$, where M' is a positive number of our choice. Subsequently we are interested mainly in those cases where M is much smaller than M'. The reflection symmetry transforms x to $-x$ and in the other direction y to $-y$, hence the reflection symmetry group is of order 4, containing the elements $1, r_x, r_y, r_{xy} = r_{yx}$. We say that the reflection symmetry group is the Weyl group of $A_1 \times A_1$ or equivalently of $SU(2) \times SU(2)$.

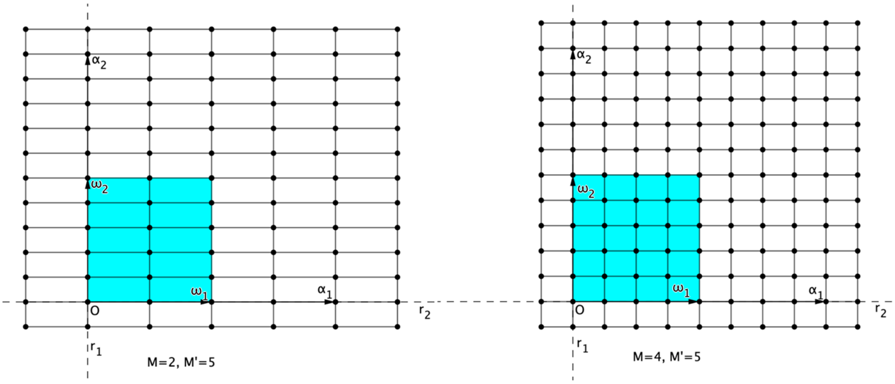

Fig. 1 The fragments of the $A_1 \times A_1$ lattice for $M = 2$, $M' = 5$ and $M = 4$, $M' = 5$

3 The data points in $A_1 \times A_1$

The fundamental region F in 2D is a cartesian product $[0, 1] \times [0, 1]$. The vertices of F are the points $(0, 0)$, $(1, 0) = \omega_1$, $(0, 1) = \omega_2$ and $(1, 1) = \omega_1 + \omega_2$. As in the 1D case, we can fix the density of the points in F by choosing positive integers M and M'. The set of such points, including the points on the boundary of F we denote by $F_{M,M'}$. If we set $M = s_0 + s_1$ and $M' = s'_0 + s'_1$, $s_i, s'_i \in \mathbb{Z}^{\geq 0}$, then a point $(x, y) \in F_{M,M'}$ is written as

$$(x, y) = \left(\frac{s_1}{s_0 + s_1}\omega_1, \frac{s'_1}{s'_0 + s'_1}\omega_2\right) \tag{2}$$

The number of points on $F_{M,M'}$ is $(M + 1)(M' + 1)$. The reflection symmetries are acting on the points in the following way $r_1(x, y) = (-x, y)$ and $r_2(x, y) = (x, -y)$ and $r_1r_2(x, y) = (-x, -y)$. Moreover $r_1^2 = 1$, $r_2^2 = 1$ and $r_1r_2 = r_2r_1$.

Example 1 Let us take $M = 4$, $M' = 5$. The fundamental region $F_{4,4}$ consists of 30 points. Note that $F_{4,5}$ contains all the points of $F_{2,5}$ which has 18 points (see Fig. 1).

4 The special functions of $A_1 \times A_1$

The characters of irreducible representations of Lie group are the special functions that have all properties that we need, except, that they are more complicated than their alternatives that we are going to use here, namely the C- and S-functions.

It is convenient to use the ω-basis which in the orthonormal basis $\{e_1, e_2\}$ has the form

$$\omega_i = \frac{1}{\sqrt{2}}e_i \qquad i = 1, 2.$$

There are four families of special functions, namely C-, S-, CS- and SC-functions that are orthogonal on $F_{M,M'}$.

$$CC_{a,b}(x,y) = e^{2\pi i\langle(a,b),(x,y)\rangle} + e^{2\pi i\langle(-a,b),(x,y)\rangle} + e^{2\pi i\langle(a,-b),(x,y)\rangle} + e^{2\pi i\langle(-a,-b),(x,y)\rangle} \quad (3)$$

$$SS_{a,b}(x,y) = e^{2\pi i\langle(a,b),(x,y)\rangle} - e^{2\pi i\langle(-a,b),(x,y)\rangle} - e^{2\pi i\langle(a,-b),(x,y)\rangle} + e^{2\pi i\langle(-a,-b),(x,y)\rangle} \quad (4)$$

$$CS_{a,b}(x,y) = e^{2\pi i\langle(a,b),(x,y)\rangle} + e^{2\pi i\langle(-a,b),(x,y)\rangle} - e^{2\pi i\langle(a,-b),(x,y)\rangle} + e^{2\pi i\langle(-a,-b),(x,y)\rangle} \quad (5)$$

$$SC_{a,b}(x,y) = e^{2\pi i\langle(a,b),(x,y)\rangle} - e^{2\pi i\langle(-a,b),(x,y)\rangle} + e^{2\pi i\langle(a,-b),(x,y)\rangle} + e^{2\pi i\langle(-a,-b),(x,y)\rangle}, \quad (6)$$

where the inner product is $\langle(a,b),(x,y)\rangle = \frac{ax}{2} + \frac{by}{2}$ and $(x,y) \in F_{M,M'}$. If either x or y are equal to 0, the sums have only two terms. If x and y are both equal to 0, then $C_{a,b}(0,0) = 4$ and $S_{a,b}(0,0) = CS_{a,b}(0,0) = SC_{a,b}(0,0) = 0$. The functions in the ω-basis are the following:

$$\begin{aligned}
C_{a,b}(x,y) &= C_a(x)C_b(y) = 4\cos(\pi ax)\cos(\pi by),\\
S_{a,b}(x,y) &= S_a(x)S_b(y) = -4\sin(\pi ax)\sin(\pi by),\\
CS_{a,b}(x,y) &= C_a(x)S_b(y) = 4i\cos(\pi ax)\sin(\pi by),\\
SC_{a,b}x,y) &= S_a(x)C_b(y) = 4i\sin(\pi ax)\cos(\pi by),
\end{aligned}$$

where $C_\mu(t) = e^{\pi i\mu t} + e^{-\pi i\mu t}$, $S_\mu(t) = e^{\pi i\mu t} - e^{-\pi i\mu t}$, $\mu \in \mathbb{Z}^{\geq 0}$, $t \in \mathbb{R}$.

All four families of functions are orthogonal on the fundamental region $F_{M,M'}$. They differ mainly by their behaviour at the boundary of the fundamental region so we need to describe four subsets of points, namely $F_{M,M'}$, $\tilde{F}_{M,M'}$, $F^{cs}_{M,M'}$ and $F^{sc}_{M,M'}$ for each of the family of special functions

$$\begin{aligned}
F_{M,M'} &= \left\{\frac{s_1}{M}\omega_1 + \frac{s_1'}{M'}\omega_2 \mid s_0 + s_1 = M, s_0' + s_1' = M', s_0, s_1, s_0', s_1' \in \mathbb{Z}^{\geq 0}\right\},\\
\tilde{F}_{M,M'} &= \left\{\frac{s_1}{M}\omega_1 + \frac{s_1'}{M'}\omega_2 \mid s_0 + s_1 = M, s_0' + s_1' = M', s_0, s_1, s_0', s_1' \in \mathbb{Z}^{>0}\right\},\\
F^{cs}_{M,M'} &= \left\{\frac{s_1}{M}\omega_1 + \frac{s_1'}{M'}\omega_2 \mid s_0 + s_1 = M, s_0' + s_1' = M', s_0, s_1 \in \mathbb{Z}^{\geq 0}, s_0', s_1' \in \mathbb{Z}^{>0}\right\},\\
F^{sc}_{M,M'} &= \left\{\frac{s_1}{M}\omega_1 + \frac{s_1'}{M'}\omega_2 \mid s_0 + s_1 = M, s_0' + s_1' = M', s_0, s_1 \in \mathbb{Z}^{>0}, s_0', s_1' \in \mathbb{Z}^{\geq 0}\right\}.
\end{aligned}$$

Figure 2 shows all subsets for $M = 2$ and $M' = 4$.

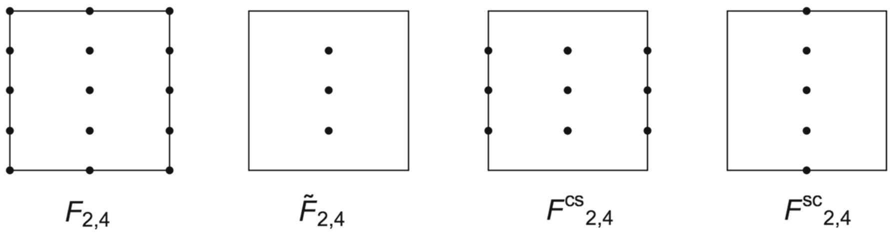

Fig. 2 The set of points on the fundamental region F for $M = 2$, $M' = 4$

The orthogonality relations are

$$\langle C_\lambda(\mathbf{x}), C_{\lambda'}(\mathbf{x})\rangle_{M,M'} = \sum_{\mathbf{x}_i \in F_{M,M'}} \varepsilon_{\mathbf{x}_i} C_\lambda(\mathbf{x_i})\overline{C_{\lambda'}}(\mathbf{x_i}) = 16MM'|\mathrm{Stab}(\lambda)|\delta_{\lambda,\lambda'},$$

$$\lambda \in \Lambda_{M,M'} = \left\{\frac{t_1}{M}\omega_1 + \frac{t_1'}{M'}\omega_2 \mid t_0 + t_1 = M, t_0' + t_1' = M', t_0, t_1, t_0', t_1' \in \mathbb{Z}^{\geq 0}\right\},$$

$$\langle S_\lambda(\mathbf{x}), S_{\lambda'}(\mathbf{x})\rangle_{M,M'} = \sum_{\mathbf{x}_i \in \tilde{F}_{M,M'}} S_\lambda(\mathbf{x_i})\overline{S_{\lambda'}}(\mathbf{x_i}) = 16MM'\delta_{\lambda,\lambda'},$$

$$\lambda \in \tilde{\Lambda}_{M,M'} = \left\{\frac{t_1}{M}\omega_1 + \frac{t_1'}{M'}\omega_2 \mid t_0 + t_1 = M, t_0' + t_1' = M', t_0, t_1, t_0', t_1' \in \mathbb{Z}^{>0}\right\},$$

$$\langle CS_\lambda(\mathbf{x}), CS_{\lambda'}(\mathbf{x})\rangle_{M,M'} = \sum_{\mathbf{x}_i \in F^{cs}_{M,M'}} \varepsilon_{\mathbf{x}_i} C_\lambda(\mathbf{x_i})\overline{C_{\lambda'}}(\mathbf{x_i}) = 16MM'|\mathrm{Stab}(\lambda)|\delta_{\lambda,\lambda'},$$

$$\lambda \in \Lambda^{cs}_{M,M'} = \left\{\frac{t_1}{M}\omega_1 + \frac{t_1'}{M'}\omega_2 \mid t_0 + t_1 = M, t_0' + t_1' = M', t_0, t_1 \in \mathbb{Z}^{\geq 0}, t_0', t_1' \in \mathbb{Z}^{>0}\right\},$$

$$\langle SC_\lambda(\mathbf{x}), SC_{\lambda'}(\mathbf{x})\rangle_{M,M'} = \sum_{\mathbf{x}_i \in F^{sc}_{M,M'}} \varepsilon_{\mathbf{x}_i} C_\lambda(\mathbf{x_i})\overline{C_{\lambda'}}(\mathbf{x_i}) = 16MM'|\mathrm{Stab}(\lambda)|\delta_{\lambda,\lambda'}$$

$$\lambda \in \Lambda^{sc}_{M,M'} = \left\{\frac{t_1}{M}\omega_1 + \frac{t_1'}{M'}\omega_2 \mid t_0 + t_1 = M, t_0' + t_1' = M', t_0, t_1 \in \mathbb{Z}^{>0}, t_0', t_1' \in \mathbb{Z}^{\geq 0}\right\}.$$

The values for orders of orbit of $\varepsilon_{\mathbf{x}_i}$ and orders of stabilizer $\mathrm{Stab}(\lambda)$ of λ are given in Table 1

5 Splitting data into congruence classes

The congruence classes were introduced for the first time in [4] as a generalization of the notion of the triality of $SU(3)$ multiplets in particle physics. Let $X = \frac{s_1}{s_0+s_1}\omega_1 + \frac{s_1'}{s_0'+s_1'}\omega_2$ be an element of $F_{M,M'}$. The points $X \in F_{M,M'}$ can be

Table 1 Orders of orbits of $\varepsilon_{\mathbf{x}_i}$ and stabilizers Stab(λ) for $A_1 \times A_1$

	$\varepsilon_{\mathbf{x}_i}$		$\lvert\mathrm{Stab}(\lambda)\rvert$
$[s_0, s_1, s_0', s_1']$	4	$[t_0, t_1, t_0', t_1']$	1
$[0, s_1, s_0', s_1']$	2	$[0, t_1, t_0', t_1']$	2
$[s_0, 0, s_0', s_1']$	2	$[t_0, 0, t_0', t_1']$	2
$[s_0, s_1, 0, s_1']$	2	$[t_0, t_1, 0, t_1']$	2
$[s_0, s_1, s_0', 0]$	2	$[t_0, t_1, t_0', 0]$	2
$[s_0, 0, 0, s_1']$	1	$[t_0, 0, 0, t_1']$	4
$[0, s_1, 0, s_1']$	1	$[0, t_1, 0, t_1']$	4
$[s_0, 0, s_0', 0]$	1	$[t_0, 0, t_0', 0]$	4
$[0, s_1, s_0', 0]$	1	$[0, t_1, t_0', 0]$	4

There is an assumption that $s_0, s_1, s_0', s_1' \in \mathbb{Z}^{>0}$ and $t_0, t_1, t_0', t_1' \in \mathbb{Z}^{>0}$

split into four congruence classes denoted here by K_{ij}, where $i, j = \{0, 1\}$. The general rule is

$$s_1 = i \mod 2, \quad s_1' = j \mod 2.$$

There are two ways to split any set of $A_1 \times A_1$ data into 4 congruence classes on $F_{M,M'}$. The first way is rather straightforward: one defines the congruence classes of points of $F_{M,M'}$ and considers separately the data function on the points of each congruence class. In this way the entire problem is split into 4 smaller problems to be solved. The advantage of such a splitting is the fact that the special functions of a given congruence class are pairwise orthogonal among themselves. In addition they are orthogonal to special functions of other congruence classes. Another way to split a set of $A_1 \times A_1$ data was described in [5], which showed how any data function $f(x, y)$, $0 \le x, y \le 1$ on $F_{M,M'}$ can be written as a sum of 4 components, each belonging to one congruence class only (see equations (36) and (37) in [5]).

$$f(x, y) = f_{00}(x, y) + f_{10}(x, y) + f_{01}(x, y) + f_{11}(x, y), \tag{7}$$

where

$$\begin{aligned}
f_{00}(x, y) &= \tfrac{1}{4}\{f(x, y) + f(x+1, y) + f(x, y+1) + f(x+1, y+1)\},\\
f_{10}(x, y) &= \tfrac{1}{4}\{f(x, y) - f(x+1, y) + f(x, y+1) - f(x+1, y+1)\},\\
f_{01}(x, y) &= \tfrac{1}{4}\{f(x, y) + f(x+1, y) - f(x, y+1) - f(x+1, y+1)\},\\
f_{11}(x, y) &= \tfrac{1}{4}\{f(x, y) - f(x+1, y) - f(x, y+1) + f(x+1, y+1)\}.
\end{aligned} \tag{8}$$

The four data components are defined by means of center elements of the Lie group $SU(2) \times SU(2)$. The components from each congruence class are defined by means of the whole data function with shifting of its arguments using the center element of the Lie group.

Both methods can be generalized to any other semisimple Lie group with the exception of G_2, F_4 and E_8 because they have only trivial center. Having the data split into the congruence classes, we can describe the Fourier series for each congruence class using the special functions orthogonal on that class.

6 Fourier decomposition of functions on $F_{M,M'}$

Suppose we have any data $f(x, y)$ given on the points of $F_{M,M'}$. Using the Fourier transform on $F_{M,M'}$ one can express a data function $f(x, y)$ sampled on the points of $F_{M,M'}$ as the finite series of $(M+1)(M'+1)$ coefficients d_λ multiplied by special functions Φ_λ

$$f(x, y) = \sum_\lambda d_\lambda \Phi_\lambda(x, y)\,, \qquad (x, y) \in F_{M,M'} \quad \lambda \in \Lambda_{M,M'}, \tag{9}$$

where Φ_λ can be any of four families of special functions defined in Sect. 4. Our task is to calculate the coefficients d_λ.

Orthogonality of special functions allows one to invert the formula (9) and get the coefficients d_λ of the expansion as a product of the decomposition matrix multiplied by the column of values of the data sampled on the points of $F_{M,M'}$

$$d_\lambda = \sum_{x_i \in F_{M,M'}} D^{[M,M']}_{(\lambda)(x_i)} f(x_i).$$

The decomposition matrix has the following form

$$D^{[M,M']} = \left(D^{[M,M']}_{(\lambda)(x_i)} \right) = \left(\frac{\varepsilon_{x_i} \overline{\Phi_\lambda(x_i)}}{16MM'|Stab(\lambda)|} \right).$$

The crucial property of the decomposition matrix is its independence of the data given on $F_{M,M'}$. Such a matrix can be calculated once and used again and again with different sets of data as long as the data analyzed is on $F_{M,M'}$.

Example 2 The decomposition matrix for $M = 2$, $M' = 3$ is the following

$$\begin{pmatrix}
-\frac{1}{96} & \frac{1}{48} & -\frac{1}{48} & \frac{1}{96} & \frac{1}{48} & -\frac{1}{24} & \frac{1}{24} & -\frac{1}{48} & -\frac{1}{96} & \frac{1}{48} & -\frac{1}{48} & \frac{1}{96} \\
\frac{1}{48} & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} & -\frac{1}{24} & \frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & \frac{1}{48} & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} \\
-\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} & \frac{1}{48} & \frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & -\frac{1}{24} & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} & \frac{1}{48} \\
\frac{1}{96} & \frac{1}{48} & \frac{1}{48} & \frac{1}{96} & -\frac{1}{48} & -\frac{1}{24} & -\frac{1}{24} & -\frac{1}{48} & \frac{1}{96} & \frac{1}{48} & \frac{1}{48} & \frac{1}{96} \\
\frac{1}{48} & -\frac{1}{24} & \frac{1}{24} & -\frac{1}{48} & 0 & 0 & 0 & 0 & -\frac{1}{48} & \frac{1}{24} & -\frac{1}{24} & \frac{1}{48} \\
-\frac{1}{24} & \frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & 0 & 0 & 0 & 0 & \frac{1}{24} & -\frac{1}{24} & -\frac{1}{24} & \frac{1}{24} \\
\frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & -\frac{1}{24} & 0 & 0 & 0 & 0 & -\frac{1}{24} & -\frac{1}{24} & \frac{1}{24} & \frac{1}{24} \\
-\frac{1}{48} & -\frac{1}{24} & -\frac{1}{24} & -\frac{1}{48} & 0 & 0 & 0 & 0 & \frac{1}{48} & \frac{1}{24} & \frac{1}{24} & \frac{1}{48} \\
-\frac{1}{96} & \frac{1}{48} & -\frac{1}{48} & \frac{1}{96} & -\frac{1}{48} & \frac{1}{24} & -\frac{1}{24} & \frac{1}{48} & -\frac{1}{96} & \frac{1}{48} & -\frac{1}{48} & \frac{1}{96} \\
\frac{1}{48} & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} & \frac{1}{24} & -\frac{1}{24} & -\frac{1}{24} & \frac{1}{24} & \frac{1}{48} & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} \\
-\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} & \frac{1}{48} & -\frac{1}{24} & -\frac{1}{24} & \frac{1}{24} & \frac{1}{24} & -\frac{1}{48} & -\frac{1}{48} & \frac{1}{48} & \frac{1}{48} \\
\frac{1}{96} & \frac{1}{48} & \frac{1}{48} & \frac{1}{96} & \frac{1}{48} & \frac{1}{24} & \frac{1}{24} & \frac{1}{48} & \frac{1}{96} & \frac{1}{48} & \frac{1}{48} & \frac{1}{96}
\end{pmatrix}$$

Example 3 Consider the $M = 2$ and $M' = 4$ case. The idea of this example is to show the possible enlargement of the decomposition matrix, without the need to recalculate all of the matrix elements from the beginning. First we calculate the decomposition matrix for $M = M' = 2$, then we enlarge the matrix to $M = 2$, $M' = 4$.

Proposition 1 *Let's denote by* $D^{[M,M']}_{(\lambda)(x)}$ *the elements labeled by* $(\lambda)(x)$ *of the decomposition matrix* $D^{[M,M']}$ *which corresponds to any integer numbers* M *and* M'*. The elements of a matrix* $D^{[M,M']}$ *which belong also to the larger matrix* $D^{[\tilde{M},\tilde{M}']}$*,* $M \le \tilde{M}$*,* $M' \le \tilde{M}'$ *can be found by the formula*

$$D^{[\tilde{M},\tilde{M}']}_{(\lambda)(x)} = \frac{MM'}{\tilde{M}\tilde{M}'} \frac{|\,\mathrm{Stab}^{[M,M']}(\lambda)|}{|\,\mathrm{Stab}^{[\tilde{M},\tilde{M}']}(\lambda)|} D^{[M,M']}_{(\lambda)(x)}. \tag{10}$$

For $M = M' = 2$

$\lambda\backslash x$	$(1,1)$	$\left(1,\frac{1}{2}\right)$	$(1,0)$	$\left(\frac{1}{2},1\right)$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{2},0\right)$	$(0,1)$	$\left(0,\frac{1}{2}\right)$	$(0,0)$
$(2,2)$	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{16}$	$-\frac{1}{32}$	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{64}$
$(2,1)$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{16}$	0	$-\frac{1}{16}$	$-\frac{1}{32}$	0	$\frac{1}{32}$
$(2,0)$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{64}$	$-\frac{1}{32}$	$-\frac{1}{16}$	$-\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{64}$
$(1,2)$	$-\frac{1}{32}$	$\frac{1}{16}$	$-\frac{1}{32}$	0	0	0	$\frac{1}{32}$	$-\frac{1}{16}$	$\frac{1}{32}$
$(1,1)$	$\frac{1}{16}$	0	$-\frac{1}{16}$	0	0	0	$-\frac{1}{16}$	0	$\frac{1}{16}$
$(1,0)$	$-\frac{1}{32}$	$-\frac{1}{16}$	$-\frac{1}{32}$	0	0	0	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{32}$
$(0,2)$	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{32}$	$-\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{64}$
$(0,1)$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$-\frac{1}{16}$	0	$\frac{1}{16}$	$-\frac{1}{32}$	0	$\frac{1}{32}$
$(0,0)$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{64}$

For $M = 2, M' = 4$

$\lambda\backslash x$	$(1,1)$	$\left(1,\frac{3}{4}\right)$	$\left(1,\frac{1}{2}\right)$	$\left(1,\frac{1}{4}\right)$	$(1,0)$	$\left(\frac{1}{2},1\right)$	$\left(\frac{1}{2},\frac{3}{4}\right)$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{2},\frac{1}{4}\right)$	$\left(\frac{1}{2},0\right)$	$(0,1)$	$\left(0,\frac{3}{4}\right)$	$\left(0,\frac{1}{2}\right)$	$\left(0,\frac{1}{4}\right)$	$(0,0)$
$(2,4)$	$\frac{1}{128}$	$-\frac{1}{64}$	$\frac{1}{64}$	$-\frac{1}{64}$	$\frac{1}{128}$	$-\frac{1}{64}$	$\frac{1}{32}$	$-\frac{1}{32}$	$\frac{1}{32}$	$-\frac{1}{64}$	$\frac{1}{128}$	$-\frac{1}{64}$	$\frac{1}{64}$	$-\frac{1}{64}$	$\frac{1}{128}$
$(2,3)$	$-\frac{1}{64}$	$\frac{1}{32\sqrt{2}}$	0	$-\frac{1}{32\sqrt{2}}$	$\frac{1}{64}$	$\frac{1}{32}$	$-\frac{1}{16\sqrt{2}}$	0	$\frac{1}{16\sqrt{2}}$	$-\frac{1}{32}$	$-\frac{1}{64}$	$\frac{1}{32\sqrt{2}}$	0	$-\frac{1}{32\sqrt{2}}$	$\frac{1}{64}$
$(2,2)$	$\mathbf{\frac{1}{64}}$	0	$-\mathbf{\frac{1}{32}}$	0	$\mathbf{\frac{1}{64}}$	$-\mathbf{\frac{1}{32}}$	0	$\mathbf{\frac{1}{16}}$	0	$-\mathbf{\frac{1}{32}}$	$\mathbf{\frac{1}{64}}$	0	$-\mathbf{\frac{1}{32}}$	0	$\mathbf{\frac{1}{64}}$
$(2,1)$	$-\mathbf{\frac{1}{64}}$	$-\frac{1}{32\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{32\sqrt{2}}$	$\mathbf{\frac{1}{64}}$	$\mathbf{\frac{1}{32}}$	$\frac{1}{16\sqrt{2}}$	$\mathbf{0}$	$-\frac{1}{16\sqrt{2}}$	$-\mathbf{\frac{1}{32}}$	$-\mathbf{\frac{1}{64}}$	$-\frac{1}{32\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{32\sqrt{2}}$	$\mathbf{\frac{1}{64}}$
$(2,0)$	$\mathbf{\frac{1}{128}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{64}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{128}}$	$-\mathbf{\frac{1}{64}}$	$-\frac{1}{32}$	$-\mathbf{\frac{1}{32}}$	$-\frac{1}{32}$	$-\mathbf{\frac{1}{64}}$	$\mathbf{\frac{1}{128}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{64}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{128}}$
$(1,4)$	$-\frac{1}{64}$	$\frac{1}{32}$	$-\frac{1}{32}$	$\frac{1}{32}$	$-\frac{1}{64}$	0	0	0	0	0	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{32}$	$-\frac{1}{32}$	$\frac{1}{64}$
$(1,3)$	$\frac{1}{32}$	$-\frac{1}{16\sqrt{2}}$	0	$\frac{1}{16\sqrt{2}}$	$-\frac{1}{32}$	0	0	0	0	0	$-\frac{1}{32}$	$\frac{1}{16\sqrt{2}}$	0	$-\frac{1}{16\sqrt{2}}$	$\frac{1}{32}$
$(1,2)$	$-\mathbf{\frac{1}{32}}$	0	$\mathbf{\frac{1}{16}}$	0	$-\mathbf{\frac{1}{32}}$	$\mathbf{0}$	0	$\mathbf{0}$	0	$\mathbf{0}$	$\mathbf{\frac{1}{32}}$	0	$-\mathbf{\frac{1}{16}}$	0	$\mathbf{\frac{1}{32}}$
$(1,1)$	$\mathbf{\frac{1}{32}}$	$\frac{1}{16\sqrt{2}}$	$\mathbf{0}$	$-\frac{1}{16\sqrt{2}}$	$-\mathbf{\frac{1}{32}}$	$\mathbf{0}$	0	$\mathbf{0}$	0	$\mathbf{0}$	$-\mathbf{\frac{1}{32}}$	$-\frac{1}{16\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{16\sqrt{2}}$	$\mathbf{\frac{1}{32}}$
$(1,0)$	$-\mathbf{\frac{1}{64}}$	$-\frac{1}{32}$	$-\mathbf{\frac{1}{32}}$	$-\frac{1}{32}$	$-\mathbf{\frac{1}{64}}$	$\mathbf{0}$	0	$\mathbf{0}$	0	$\mathbf{0}$	$\mathbf{\frac{1}{64}}$	$\frac{1}{32}$	$\mathbf{\frac{1}{32}}$	$\frac{1}{32}$	$\mathbf{\frac{1}{64}}$
$(0,4)$	$\frac{1}{128}$	$-\frac{1}{64}$	$\frac{1}{64}$	$-\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{32}$	$-\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$-\frac{1}{64}$	$\frac{1}{64}$	$-\frac{1}{64}$	$\frac{1}{128}$
$(0,3)$	$-\frac{1}{64}$	$\frac{1}{32\sqrt{2}}$	0	$-\frac{1}{32\sqrt{2}}$	$\frac{1}{64}$	$-\frac{1}{32}$	$\frac{1}{16\sqrt{2}}$	0	$-\frac{1}{16\sqrt{2}}$	$\frac{1}{32}$	$-\frac{1}{64}$	$\frac{1}{32\sqrt{2}}$	0	$-\frac{1}{32\sqrt{2}}$	$\frac{1}{64}$
$(0,2)$	$\mathbf{\frac{1}{64}}$	0	$-\mathbf{\frac{1}{32}}$	0	$\mathbf{\frac{1}{64}}$	$\mathbf{\frac{1}{32}}$	0	$-\mathbf{\frac{1}{16}}$	0	$\mathbf{\frac{1}{32}}$	$\mathbf{\frac{1}{64}}$	0	$-\mathbf{\frac{1}{32}}$	0	$\mathbf{\frac{1}{64}}$
$(0,1)$	$-\mathbf{\frac{1}{64}}$	$-\frac{1}{32\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{32\sqrt{2}}$	$\mathbf{\frac{1}{64}}$	$-\mathbf{\frac{1}{32}}$	$-\frac{1}{16\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{16\sqrt{2}}$	$\mathbf{\frac{1}{32}}$	$-\mathbf{\frac{1}{64}}$	$-\frac{1}{32\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{32\sqrt{2}}$	$\mathbf{\frac{1}{64}}$
$(0,0)$	$\mathbf{\frac{1}{128}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{64}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{128}}$	$\mathbf{\frac{1}{64}}$	$\frac{1}{32}$	$\mathbf{\frac{1}{32}}$	$\frac{1}{32}$	$\mathbf{\frac{1}{64}}$	$\mathbf{\frac{1}{128}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{64}}$	$\frac{1}{64}$	$\mathbf{\frac{1}{128}}$

7 Concluding remarks

The discrete transform in any dimension provides the values of the data at the discrete points by the corresponding values at the points of the special functions in the Fourier series. However, if one chooses to replace the discrete values of the special functions in a Fourier series, by their continuous values, one gets what has been called a continuous approximation of the discrete values [1].

The two methods for splitting data described in Sect. 5 have yet to be compared in practical important large data sets.

The Fourier transform on a square lattice based on C_2 symmetry is quite different than the one studied here. One of the practical restrictions is that the distances between square lattice points are the same in the same directions. In our case we are aiming for the situation where along the one A_1 direction the density of points is relatively small, and along the other A_1 direction the density of points is much larger. This is the freedom that using the simple Lie group doesn't have.

In practical applications for the $A_1 \times A_1$ asymmetric cases the decomposition matrix may be very large. It may be useful to develop it as separate computer software.

The method which we apply here, and in the previous papers of the series [2,3], can be generalized to any dimension in principle. In 3D we would have the lattice of A_3, B_3 and C_3, as well as the four lattices based on semisimple Lie algebras, namely $A_1 \times A_1 \times A_1$, $A_1 \times A_2$, $A_1 \times C_2$ and $A_1 \times G_2$. We will consider those cases elsewhere.

Acknowledgements Authors are grateful for the partial support of the work by the Natural Sciences and Engineering Research Council of Canada, No. RGPIN-2016-04199 and by MIND Research Institute of Irvine, California.

Complience with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Atoyan, A., Patera, J.: Properties of continuous Fourier extension of the discrete cosine transform and its multidimensional generalization. J. Math. Phys. **45**, 2468–2491 (2004)
2. Bodner, M., Patera, J., Szajewska, M.: Decomposition matrices for the special case of data on the triangular lattice of $SU(3)$. Appl. Comput. Harmon. Anal. **43**(2), 346–353 (2017). https://doi.org/10.1016/j.acha.2016.07.001
3. Bodner, M., Patera, J., Szajewska, M.: Decomposition matrices for the special case of data on the triangular lattice of $G(2)$. Appl. Comput. Harmon. Anal. **45**(1), 233–238 (2018). https://doi.org/10.1016/j.acha.2017.02.003
4. Lemire, F.W., Patera, J.: Congruence number, a generalization of $SU(3)$ triality. J. Math. Phys. **21**, 2026 (1980). https://doi.org/10.1063/1.524711
5. Moody, R.V., Patera, J.: Orthogonality within the families of C-, S-, and E- functions of any compact semisimple Lie group. SIGMA (Symmetry, Integrability and Geometry: Methods and Applications), vol. 2, paper 076, p. 14 (2006). arXiv:math-ph/0611020
6. Rivlin, T.J.: Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory. Wiley, New York (1990)
7. Sandryaila, A., Moura, J.M.F.: Big data analysis with signal processing graphs. IEEE Signal Process. Mag. **80**, 80–90 (2014)