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Some properties of reachable sets for control affine systems

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Abstract Let $\dot{q} = X_0 + \sum_{j=1}^{k} u_j X_j$ be a control affine system on a manifold M, let C be a convex compact subset of $\mathbb{R}^k$, $\dim C > 0$, let q_0 be a fixed point of M, and let U be a neighbourhood of q_0. We consider three reachable sets from q_0 for our system which are generated by square integrable controls with values in C, riC—the relative interior of C, and rbC—the relative boundary of C, respectively, with contraints on a state variable q of the form $q \in U$. Among other things, we investigate the relation between closures, interiors and boundaries of the three reachable sets. We also show how methods of the sub-Lorentzian geometry can serve as an auxiliary tool in the study of control affine systems.

Keywords Reachable sets · Affine-control systems · Geometric optimality · Sub-Lorentzian geometry

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1 Introduction and statement of the results

Consider a control affine system

$$\dot{q} = X_0(q) + \sum_{j=1}^{k} u_j X_j(q), \tag{1.1}$$

where $X_0, \ldots, X_k$ are smooth linearly independent vector fields defined on a smooth connected manifold M, and $(u_1, \ldots, u_k) \in C$, where C is a subset of $\mathbb{R}^k$. If $q_0 \in M$ is a fixed point and U is a fixed neighbourhood of q_0, denote by $\mathcal{A}(q_0, U; C)$ the set of endpoints of all trajectories of (1.1) that start from q_0, are generated by square integrable controls $u : [0, T] \longrightarrow C$, and are contained in U; here $T > 0$ is not fixed. The set $\mathcal{A}(q_0, U; C)$ is called the *reachable set from* q_0 for (1.1) corresponding to the set C of control parameters. Trajectories of (1.1) that are generated by L^2 controls with values in C will be sometimes referred to as *admissible trajectories*.

The aim of this paper is to prove Theorem 1.1 which we are going to formulate below. Let S be a set and $\{Y_s : s \in S\}$ be a family of smooth vector fields on M. Denote by $Lie\{Y_s : s \in S\}$ the Lie algebra generated by this family. For a $q \in M$ let us also set $Lie_q\{Y_s : s \in S\} = \{Z(q) : Z \in Lie\{Y_s : s \in S\}\} \subset T_q M$. Suppose now that $C \subset \mathbb{R}^k$ is a compact and convex set of dimension greater than zero. Thus C is of the form $C = \overline{C_0}$, where $C_0 = riC$ and ri stands for the interior relative to the affine hull $Aff(C)$ (the so-called *relative interior*—c.f. [12]). We will also use the notation rbC meaning the boundary of C relative to $Aff(C)$.

Theorem 1.1 *Let C be a compact and convex subset of $\mathbb{R}^k$, $\dim C > 0$, and let $q_0 \in M$ be a fixed point. Suppose that the vector fields in (1.1) satisfy*

$$Lie_{q_0}\left\{ X_0 + \sum_{j=1}^{k} u_j X_j : u \in rbC \right\} = T_{q_0} M. \tag{1.2}$$

Then there exists a fundamental system $\mathcal{O}$ of neighbourhoods of q_0 such that for every $U \in \mathcal{O}$ the following equalities hold true

$$\mathcal{A}(q_0, U; C) = cl_U(int\mathcal{A}(q_0, U; riC)) = cl_U(int\mathcal{A}(q_0, U; rbC)), \tag{1.3}$$

$$int\mathcal{A}(q_0, U; C) = int\mathcal{A}(q_0, U; riC) = int\mathcal{A}(q_0, U; rbC), \tag{1.4}$$

$$\tilde{\partial}\mathcal{A}(q_0, U; C) = \tilde{\partial}\mathcal{A}(q_0, U; riC) = \tilde{\partial}\mathcal{A}(q_0, U; rbC); \tag{1.5}$$

here cl_U (resp. $\tilde{\partial}$) is the closure (resp. boundary) with respect to U.

Remark 1.1 It is useful for further use to notice that the condition (1.2) implies the relation $Lie_{q_0}\{X_0 + \sum_{j=1}^{k} u_j X_j : u \in C\} = T_{q_0} M$ which in turn implies

$Lie_{q_0}\{X_0 + \sum_{j=1}^{k} u_j X_j \ :\ u \in riC\} = T_{q_0}M$. Clearly, $Lie_{q_0}\{X_0, X_1, ..., X_k\} = T_{q_0}M$ can be deduced from (1.2) too.

Keeping the notation from Theorem 1.1 one can also prove, under weaker assumptions, the following

Proposition 1.1 *Suppose that $X_0, \dots, X_k$ are linearly independent. Then $\mathcal{A}(q_0, U; C) = cl_U \mathcal{A}(q_0, U; riC) = cl_U \mathcal{A}(q_0, U; rbC)$ for every $U \in \mathcal{O}$. In particular, the set $\mathcal{A}(q_0, U; C)$ is closed with respect to U. If, moreover, $k \geq 2$ and the distribution $Span\{X_0, \dots, X_k\}$ is generic, then for every open U (not necessarily in $\mathcal{O}$) the set $\mathcal{A}(q_0, U; riC)$ is open, and $\mathcal{A}(q_0, U; C) = \mathcal{A}(q_0, U; rbC)$.*

This moment is suitable for recalling some known facts on reachable sets that can be found in the literature devoted to mathematical control theory (see [1] or eg. [3] and its reference section). First of all, reachable sets for families of smooth vector fields are well described. Such sets, in our situation, correspond to reachable sets for (1.1) generated by piecewise constant controls with values, respectively in C, riC, rbC, and they satisfy, for instance, Krener's theorem (see [10]) which is used later in this paper. It is also known that the formulas (1.4), (1.5) are true with respect to piecewise constant controls (see [9]). Reachable sets generated by piecewise constant controls may differ from ones generated by L^2 controls, however, both have the same closures.

Now let us focus on $\mathcal{A}(q_0, U; C)$. As one can see, in order to prove (1.3) in Theorem 1.1 we need to know that $\mathcal{A}(q_0, U; C)$ is closed relative to U. One could try to use the following compactness results. Define $\mathcal{A}(q_0, U; C; T)$ (resp. $\mathcal{A}(q_0, U; C; \leq T)$) to be the set of endpoints of all trajectories of (1.1) that start from q_0, are generated by square integrable controls $u : [0, T] \longrightarrow C$ (resp. $u : [0, t] \longrightarrow C$ with $t \leq T$), and are contained in U. Following eg. [3] we know that $\mathcal{A}(q_0, U; C; T)$ is compact, provided that graphs of all trajectories of (1.1) are contained in a compact set $K \subset \mathbb{R} \times U$. This assumption, however, is not satisfied in our case, i.e. for $T = T_0$ with T_0 determined as in Proposition 2.1 and Corollary 2.1 below. One could also try to apply the fact that by [9] $\mathcal{A}(q_0, U; M; \leq T)$ is compact, provided that for every $u \in L^2([0, T], C)$ the trajectory of (1.1) starting from q_0 and generated by u is defined on the whole $[0, T]$. Again this assumption, with $T = T_0$ and M replaced by U, is not true in our case. Moreover, $\mathcal{A}(q_0, U; C; \leq T) \neq \mathcal{A}(q_0, M; C; \leq T) \cap U$. All this permits us to conclude that closeness of $\mathcal{A}(q_0, U; C)$ requires a separate proof. At the end recall (see again [3]) that $cl_U \mathcal{A}(q_0, U; C) = cl_U \mathcal{A}(q_0, U; riC) = cl_U \mathcal{A}(q_0, U; rbC)$, the fact which will be used later. Summing up all what we have said, there is a subtal difference between our results and those that can be found in the literature.

Remark at the end of this section that a very interesting situation arises when $C = \{u \in \mathbb{R}^k : |u| \leq 1\}$. Then, as it was noticed in [7], the system (1.1) becomes part of the sub-Lorentzian geometry. In particular, reachable sets from q_0 for (1.1), for this specific C, coincide with the corresponding reachable sets $J^+(q_0, U)$, $I^+(q_0, U)$, $N^+(q_0, U)$ for the sub-Lorentzian structure (H, g) defined as follows: $H = Span\{X_0, \dots, X_k\}$, $g(X_0, X_0) = -1$, $g(X_j, X_j) = +1$, $j = 1, \dots, k$, $g(X_\alpha, X_\beta) = 0$, $\alpha, \beta = 0, \dots, k$, $\alpha \neq \beta$, and X_0 is a time orientation. The reasoning is as follows. Obviously, every trajectory of (1.1) is a nonspacelike future directed curve with respect to (H, g). Conversely, suppose that $\gamma : [0, T] \longrightarrow U$ is nonspacelike future directed, i.e. $\frac{d}{dt}\gamma(t) = \sum_{a=0}^{k} u_\alpha(t)X_\alpha(\gamma(t))$, $-u_0^2(t) + \sum_{j=1}^{k} u_j^2(t) \le 0$. Without loss of generality one can assume that $u_0(t) > 0$ a.e. Now suitable reparameterization $t = t(s)$ gives $\frac{d}{ds}(\gamma(t(s)) = X_0(\gamma(t(s))) + \sum_{j=1}^{k} \frac{u_j(t(s))}{u_0(t(s))} X_j(\gamma(t(s)))$. This implies $\mathcal{A}(q_0, U; C) = J^+(q_0, U)$ and similarly for the other reachable sets.

Note that (1.3), (1.4), (1.5) have been already obtained in [7] in the sub-Lorentzian setting. Thus the present paper can be viewed as a generalization of [7].

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. By the way we will state some partial results. Below we will need some notions and facts from the sub-Lorentzian geometry, such as a sub-Lorentzian metric on a manifold, a time orientation of a sub-Lorentzian metric, and also the sub-Lorentzian distance from a given point q_0, normal neighbourhoods of q_0, the horizontal gradient of a smooth function, and the time-like reachable set from q_0, each induced by a given sub-Lorentzian metric. For all these notions, their properties and correpending results the reader is referred to [5], [6], [7].

Throughout this section C will be a compact convex subset of $\mathbb{R}^k$, $\dim C > 0$. At the begining let us assume that the fields $X_0, \ldots, X_k$ in (1.1) are linearly independent. Unless otherwise stated all controls are supposed to be square integrable with values in C.

Our first aim is to determine a fundamental system of neighbourhoods of a given point $q_0 \in M$ from the hypothesis of Theorem 1.1 To this end let $\hat{U}$ be a neighbourhood of q_0. One can find a function $\varphi \in C^\infty(\hat{U})$ such that $X_0(\varphi)(q_0) = 1$, $X_j(\varphi)(q_0) = 0$, $j = 1, \ldots, k$. Of course such a φ is not uniquely determined.

From now on we suppose that $\hat{U}$ is bounded and small enough to have

$$\inf\left\{X_0(\varphi)(q) : q \in \hat{U}\right\} > kc \sup\left\{|X_j(\varphi)(q)| : q \in \hat{U}, j = 1, \ldots, k\right\}, \quad (2.1)$$

where $c = \max\left\{|u_j| : j = 1, \ldots, k, (u_1, \ldots, u_k) \in C\right\}$. We define a sub-Lorentzian time-oriented metric on $\hat{U}$ in the following manner: $H = Span\{X_0, \ldots, X_k\}$ and

$$\begin{aligned} &g(X_0, X_0) = -kc^2 - \varepsilon, \quad g(X_j, X_j) = +1, \quad j = 1, \ldots, k, \\ &g(X_\alpha, X_\beta) = 0, \quad \alpha \ne \beta, \quad \alpha, \beta = 0, \ldots, k, \end{aligned} \quad (2.2)$$

where $\varepsilon > 0$, and X_0 is a time orientation. In other words the family

$$\frac{1}{\sqrt{kc^2 + \varepsilon}} X_0, X_1, \ldots, X_k$$

is an orthonormal frame for g. Let $\gamma : [0, T] \longrightarrow \hat{U}$, $\gamma(0) = q_0$, be an admissible trajectory generated by a control $(u_1, \ldots, u_k) : [0, T] \longrightarrow C$. Clearly, by (2.2), $g(\dot{\gamma}(t), \dot{\gamma}(t)) = -kc^2 - \varepsilon + \sum_{j=1}^{k} u_j^2(t) \le -kc^2 - \varepsilon + kc^2 = -\varepsilon < 0$ for almost every $t \in [0, T]$. Thus trajectories of (1.1) which are contained in $\hat{U}$ are timelike future directed curves with respect to the structure (H, g). By the way we obtain $\mathcal{A}(q_0, \hat{U}; C) \subset I^+(q_0, \hat{U})$.

Now, after shrinking $\hat{U}$, we suppose that $\hat{U}$ is contained in a normal, relative to (H, g), neighbourhood of a point q_0. For such a $\hat{U}$ we can prove the following proposition.

Proposition 2.1 *There exists a constant $T_0 > 0$ with the following property: if $\gamma : [0, T] \longrightarrow \hat{U}$, $\gamma(0) = q_0$, is an arbitrary trajectory of (1.1) then $T \leq T_0$.*

Proof Denote by $f[V]$ the sub-Lorentzian distance from q_0 relative to V, where V is a normal neighbourhood of q_0 containing $\hat{U}$. It is known [5] that $f[V]$ is bounded on V by a positive constant, say, A. Therefore if $\gamma : [0, T] \longrightarrow \hat{U}$ is a trajectory of (1.1) starting from $\gamma(0) = q_0$ then $L(\gamma) \leq f[V](\gamma(T)) \leq A$, where $L(\gamma)$ is the sub-Lorentzian length of γ. However,

$$L(\gamma) = \int_0^T \sqrt{-g(\dot{\gamma}(t), \dot{\gamma}(t))}dt \geq T\sqrt{\varepsilon},$$

so it is enough to define $T_0 = \frac{A}{\sqrt{\varepsilon}}$ and the proof is over. □

Note that A increases along with ε, so substituting for ε bigger and bigger numbers does not cause T_0 to become smaller. Using Proposition 2.1 we immediately obtain

Corollary 2.1 *For any sufficiently small neighbourhood $\hat{U}$ of q_0 there exist a number $T > 0$ such that*

$$\mathcal{A}(q_0, \hat{U}; C) = \mathcal{A}(q_0, \hat{U}; C; \leq T).$$

Proceeding further let us remark that if $\gamma : [0, T] \longrightarrow \hat{U}$ is a trajectory of (1.1), the function $t \longrightarrow \varphi(\gamma(t))$ is increasing. Indeed, by (2.1) we have

$$\frac{d}{dt}(\varphi(\gamma(t)) = X_0(\varphi)(\gamma(t)) + \sum_{j=1}^{k} u_j(t) X_j(\varphi)(\gamma(t)) > 0.$$

In the sub-Lorentzian language, this last remark can be explained by observing that the horizontal gradient $\nabla_H \varphi$ is timelike past directed, so φ must increase along timelike future directed curves. Now, let U be a neighbourhood of q_0 such that U is starshaped about q_0, $U \subset \hat{U}$, and $\mathcal{A}(q_0, \hat{U}; C) \cap \partial U = \{\varphi = const\}$ for some $const > \varphi(q_0)$. Obviously, for any $\hat{U}$ one can construct a U with above properties, hence sets of type U form a fundamental system of neighbourhoods of q_0 which we will denote by $\mathcal{O}$. We shall see below that after slight modification this system of neighbourhoods of q_0 is sufficient to prove Theorem 1.1

In the sequel we assume that U is a fixed element of the family $\mathcal{O}$. Let us note an easy.

Corollary 2.2 *Let $\gamma : [0, T] \longrightarrow M$ be a trajectory of (1.1) with $\gamma(0) \in U$. If $\gamma(t_0) \in \partial U$ for a $t_0 \in (0, T)$ then there exists an $\varepsilon > 0$ such that $\gamma(t) \notin \overline{U}$ for every $t \in (t_0, t_0 + \varepsilon)$.*

We will need a notion of convergence which is suitable for admissible trajectories. The most useful is a concept of C^0 convergence which is extensively used eg. in the Lorentzian geometry. More precisely let $\eta_\nu : [a_\nu, b_\nu] \longrightarrow M$, $\nu = 1, 2, \ldots,$ and $\eta : [a, b] \longrightarrow M$ be curves in M. We say that a sequence $\{\eta_\nu\}$*tends to* η *in the* C^0*topology on curves* if $\eta_\nu(a_\nu) \longrightarrow \eta(a)$, $\eta_\nu(b_\nu) \longrightarrow \eta(b)$ as $\nu \longrightarrow \infty$ and for every neighbourhood V of the set $\eta([a, b])$ there exists a number $\Lambda > 0$ such that $\eta_\nu([a_\nu, b_\nu]) \subset V$ for every $\nu > \Lambda$.

Before proving the main result concerning sequences of admissible trajectories we will prove a simple lemma. Remark that if $v \in L^2([a, b], \mathbb{R}^k)$ and $F \subset \mathbb{R}^k$ then the notation $v([a, b]) \subset F$ means that $v(t) \in F$ for almost every $t \in [a, b]$. Here and below we make no distinction in notation between a sequence and its subsequences.

Lemma 2.1 *Let $u^{(\nu)}, u \in L^2([a, b], \mathbb{R}^k)$, $\nu = 1, 2, \ldots,$ be such that $u^{(\nu)} \longrightarrow u$ weakly in L^2. Suppose moreover that $u^{(\nu)}([a, b]) \subset C$ for every ν. Then also $u([a, b]) \subset C$.*

Proof Let $W = \left\{v \in L^2([a, b], \mathbb{R}^k) : \ v([a, b]) \subset C\right\}$. Since C is convex, W is a convex subset of $L^2([a, b], \mathbb{R}^k)$. It is known [8] that convex sets are weakly closed if and only if they are closed, therefore it is enough to observe that W is a closed subset of $L^2([a, b], \mathbb{R}^k)$ in L^2 topology. To this end take a sequence $v^{(\nu)}$ of elements of W which is L^2-convergent to a $v \in L^2([a, b], \mathbb{R}^k)$. Then, after passing to a subsequence, $v^{(\nu)} \longrightarrow v$ a.e. on $[a, b]$. Since C is closed, it follows that $v([a, b]) \subset C$. □

It will be convenient to introduce the following notation. Let $h \in L^2([0, T], \mathbb{R}^{k+1})$; denote by Φ_t^h the flow of non-autonomous vector field $Z(t, q) = \sum_{\alpha=0}^k h_\alpha(t) X_\alpha(q)$ computed starting from $t = 0$. In other words the curve $[0, T] \ni t \longrightarrow \Phi_t^h(q_0)$ is the solution to the Cauchy problem

$$\dot{q} = Z(t, q), \quad q(0) = q_0.$$

For instance, if $u : [0, T] \longrightarrow C$ is a control then $t \longrightarrow \Phi_t^{(1,u)}(q_0)$ is the corresponding trajectory of (1.1) initiating at q_0.

The proposition below, which is a little bit too strong for our purposes, is of its own interest.

Proposition 2.2 *Let $\gamma_\nu : [0, l_\nu] \longrightarrow U$ be a sequence of trajectories of (1.1) such that $\gamma_\nu(0) = q_0$, $\nu = 1, 2, \ldots,$ and $\gamma_\nu(l_\nu) \longrightarrow q$ as $\nu \longrightarrow \infty$ with $q \in U$. Then, after passing to a subsequence, $\gamma_\nu \longrightarrow \gamma$ in the C^0 topology on curves, where $\gamma : [0, l] \longrightarrow U$ is a trajectory of (1.1), $\gamma(0) = q_0$, $\gamma(l) = q$.*

Proof We know by Proposition 2.1 that there exists a constant $T_0 > 0$ such that $l_\nu \leq T_0$. Thus, passing to a subsequence if necessary, we may assume that $\{l_\nu\}$ converges to a number l, and either $l_\nu \nearrow l$ or $l_\nu \searrow l$.

Suppose that $l_\nu \nearrow l$. Let $\gamma_\nu(t) = \Phi_t^{(1,u^{(\nu)})}(q_0)$, $0 \leq t \leq l_\nu$, where $u^{(\nu)}$ is a control generating γ_ν. For every ν sufficiently large, $u^{(\nu)}$ can be extended to a control

$\hat{u}^{(\nu)} : [0, l] \longrightarrow C$ in such a way that $\hat{\gamma}_\nu(t) = \Phi_t^{(1,\hat{u}^{(\nu)})}(q_0)$ does not leave U for $t \in [0, l]$. Obviously $\left\| \hat{u}^{(\nu)} \right\|_{L^2} \le c\sqrt{kl}$ for every ν, so again passing to a subsequence we may suppose that there exists a $u \in L^2([0, l], \mathbb{R}^k)$ such that $(1, \hat{u}^{(\nu)}) \longrightarrow (1, u)$ weakly in L^2. Now if $\gamma(t) = \Phi_t^{(1,u)}(q_0)$, $0 \le t \le l$, then $\hat{\gamma}_\nu \rightrightarrows \gamma$ on $[0, l]$ as $\nu \longrightarrow \infty$ by properties of flows of type $\Phi_t^h(\cdot)$ (see [2]). Since $u([0, l]) \subset C$ by Lemma 2.1, γ is a trajectory of (1.1) and, by Corollary 2.2, γ does not leave U. Further $\hat{\gamma}_\nu$ tends to γ continuously [11], i.e. $\gamma_\nu(l_\nu) = \hat{\gamma}_\nu(l_\nu) \longrightarrow \gamma(l)$, $\nu \longrightarrow \infty$. Now it is clear that $\gamma_\nu \longrightarrow \gamma$ in the C^0 topology on curves.

Suppose now that $l_\nu \searrow l$. Take a small $\varepsilon > 0$. For every ν sufficiently large $l_\nu \le l+\varepsilon$. If ε is small enough (and ν's are sufficiently large), every $u^{(\nu)}$ can be extended to a control $\hat{u}^{(\nu)} : [0, l+\varepsilon] \longrightarrow C$ in such a way that $\hat{\gamma}_\nu(t) = \Phi_t^{(1,\hat{u}^{(\nu)})}(q_0)$ does not leave U for $t \in [0, l+\varepsilon]$. Analogously as in the first case there exists a $\hat{u} \in L^2([0, l+\varepsilon], \mathbb{R}^k)$ such that, possibly after passing to a subsequence, $(1, \hat{u}^{(\nu)}) \longrightarrow (1, \hat{u})$ weakly in L^2. Now, if $\hat{\gamma}(t) = \Phi_t^{(1,\hat{u})}(q_0)$, $0 \le t \le l + \varepsilon$, then

$$\hat{\gamma}_\nu \rightrightarrows \hat{\gamma} \quad \text{on} \quad [0, l + \varepsilon] \tag{2.3}$$

and, since this convergence is continuous, $\hat{\gamma}_\nu(l_\nu) \longrightarrow \hat{\gamma}(l) = q$. Also, again by Lemma 2.1 and Corollary 2.2, $\hat{\gamma}([0, l + \varepsilon]) \subset U$.

Let $\gamma = \hat{\gamma}_{|[0,l]}$. To end the proof we must show that $\gamma_\nu \longrightarrow \gamma$ in the C^0 topology on curves. Suppose the converse. Then one can find an open set $V \supset \gamma([0, l])$ such that, again after passing to a subsequence, for every ν there exists a $t_\nu \in [0, l_\nu]$ with $\gamma_\nu(t_\nu) \notin V$. Take an open $\hat{V}$ such that $V \subset \hat{V}$ and $\hat{\gamma}([0, l + \varepsilon]) \subset \hat{V}$. By (2.3), again passing to a subsequence if necessary, we have $\hat{\gamma}_\nu([0, l_\nu]) \subset \hat{V}$ for every ν. Thus $\gamma_\nu(t_\nu) \in \hat{V} \backslash V$ for every ν. Passing to a subsequence again we may suppose that there exists a $t \in [0, l]$ such that $t_\nu \longrightarrow t$. By (2.3) $\gamma_\nu(t_\nu) \longrightarrow \gamma(t)$ and since $\hat{V} \backslash V$ is relatively closed in $\hat{V}$ we finally obtain $\gamma(t) \notin V$ which is a contradiction. □

Corollary 2.3 *Suppose that the fields $X_0, \dots, X_k$ in (1.1) are linearly independent. Then the set $\mathcal{A}(q_0, U; C)$ is closed with respect to U.*

Proof Take a sequence $\{q_\nu\} \subset \mathcal{A}(q_0, U; C)$ such that $q_\nu \longrightarrow q \in U$. For every ν there exists a trajectory γ_ν of (1.1) joining q_0 to q_ν. The sequence $\{\gamma_\nu\}$ satisfies the assumptions of Proposition 2.2. Therefore, after passsing to a subsequence, $\gamma_\nu \longrightarrow \gamma$ in the C^0 topology on curves, where γ is a trajectory of (1.1) that joins q_0 to q. In this way $q \in \mathcal{A}(q_0, U; C)$ which ends the proof. □

Now using [3] we have $cl_U \mathcal{A}(q_0, U; C) = cl_U \mathcal{A}(q_0, U; riC) = cl_U \mathcal{A}(q_0, U; rbC)$ which together with Corollary 2.3 gives

Proposition 2.3 $\mathcal{A}(q_0, U; C) = cl_U \mathcal{A}(q_0, U; riC) = cl_U \mathcal{A}(q_0, U; rbC)$ *for every* $U \in \mathcal{O}$.

Now we strengthen our assumption; namely, up to the end of this section we assume that the fields $X_0, \dots, X_k$ in (1.1) are linearly independent and such that $Lie_{q_0}\{X_0 + \sum_{j=1}^k u_j X_j : u \in rbC\} = T_{q_0}M$. This condition is open, so we will assume that all the sets from $\mathcal{O}$ are so small that

$$\forall U \in \mathcal{O} \quad \forall q \in U \quad Lie_q \left\{ X_0 + \sum_{j=1}^{k} u_j X_j \,:\, u \in rbC \right\} = T_q M. \tag{2.4}$$

Recall that U is a fixed element of $\mathcal{O}$. Introduce three other reachable sets in U: $\mathcal{F}_{q_0}(C)$ (resp. $\mathcal{F}_{q_0}(riC)$, $\mathcal{F}_{q_0}(rbC)$) is the reachable set from q_0 for the family of vector fields $\{X_0+\sum_{j=1}^k u_j X_j \,:\, u \in C\}$ (resp. for the family $\{X_0+\sum_{j=1}^k u_j X_j \,:\, u \in riC\}$, $\{X_0 + \sum_{j=1}^k u_j X_j \,:\, u \in rbC\}$). The set $\mathcal{F}_{q_0}(C)$ may be regarded as the set of endpoints of all trajectories of (1.1) that are generated by piecewise constant controls with values in C. Similar remarks apply to $\mathcal{F}_{q_0}(riC)$, $\mathcal{F}_{q_0}(rbC)$. Obviously $\mathcal{F}_{q_0}(C) \subset \mathcal{A}(q_0, U; C)$, $\mathcal{F}_{q_0}(riC) \subset \mathcal{A}(q_0, U; riC)$, $\mathcal{F}_{q_0}(rbC) \subset \mathcal{A}(q_0, U; rbC)$.

Using again [3] (and Proposition 2.3) we have

$$cl_U \mathcal{F}_{q_0}(riC) = cl_U \mathcal{F}_{q_0}(rbC) = \mathcal{A}(q_0, U; C). \tag{2.5}$$

Because of (2.4) (cf. remark 1.1) we can apply Krener's theorem [10] for $\mathcal{F}_{q_0}(rbC)$ and $\mathcal{F}_{q_0}(riC)$ to obtain

$$\mathcal{F}_{q_0}(riC) \subset cl_U(int\mathcal{F}_{q_0}(riC)), \quad \mathcal{F}_{q_0}(rbC) \subset cl_U(int\mathcal{F}_{q_0}(rbC)). \tag{2.6}$$

Now (2.5), (2.6) give (1.3).

Next we will prove the first equality in (1.4). Clearly $int\mathcal{A}(q_0, U; riC) \subset int\mathcal{A}(q_0, U; C)$, so the reverse inclusion must be proven. Let $q \in int\mathcal{A}(q_0, U; C)$. There exists an open set V such that $q \in V \subset \mathcal{A}(q_0, U; C)$. Denote by $\mathcal{F}_q^-(riC)$ the reachable set from q in U for the family of vector fields $\{-X_0 - \sum_{j=1}^k u_j X_j \,:\, u \in riC\}$. Again by Krener's theorem $int\mathcal{F}_q^-(riC) \neq \varnothing$, thus $int\mathcal{F}_q^-(riC) \cap V \neq \varnothing$. By the first equality in (1.3), $int\mathcal{A}(q_0, U; riC)$ is dense in $\mathcal{A}(q_0, U; C)$, so there is a point q_1 such that $q_1 \in int\mathcal{A}(q_0, U; riC) \cap int\mathcal{F}_q^-(riC) \cap V$. It follows that there exist admissible trajectories γ_1, γ_2 contained in U such that γ_1 joins q_0 to q_1, γ_2 joins q_1 to q and both are generated by controls with values in riC. Also $\gamma_1 \cup \gamma_2$ is contained in $int\mathcal{A}(q_0, U; riC)$ by a well-kown fact saying that any admissible trajectory that starts from the interior of the reachable set cannot leave this interior. Summing up $q \in int\mathcal{A}(q_0, U; riC)$.

The equality $int\mathcal{A}(q_0, U; C) = int\mathcal{A}(q_0, U; rbC)$ is proved analogously. Finally, observe that (1.5) follows directly from (1.3) and (1.4). Thus the proof of Theorem 1.1 is over.

3 Proof of Proposition 1.1

By Corollary 2.3 we need to prove only the second part. We start from the notion of so-called *Goh curves*. Let H be a rank m vector distribution on an n-dimensional manifold M. An absolutely continuous curve $\gamma : [a, b] \longrightarrow M$ is called *horizontal* if $\dot{\gamma}(t) \in H_{\gamma(t)}$ for almost every $t \in [a, b]$. Let $H^\perp$ denote the set of all such covectors $\lambda \in T^*M$ that $\langle \lambda, \xi \rangle = 0$ for every $\xi \in H_{\pi(\lambda)}$, $\pi : T^*M \longrightarrow M$ being the canonical projection. $H^\perp$ is, in a natural way, a subbundle of rank $2n - m$ of the cotangent bundle T^*M, and is called the *annihilator* of H. An absolutely continuous curve $\lambda : [a, b] \longrightarrow T^*M$ is called an *abnormal biextremal* if $\lambda([a, b]) \subset H^{\perp}$, λ never intersects the zero section, and moreover $\Omega_{\lambda(t)}(\dot{\lambda}(t), \zeta) = 0$ for almost every $t \in [a, b]$ and every $\zeta \in T_{\lambda(t)}H^{\perp}$; here by Ω we denote the restriction to $H^{\perp}$ of the standard symplectic form on T^*M. Now a horizontal curve $\gamma : [a, b] \longrightarrow M$ is said to be *abnormal* if there exists an abnormal biextremal $\lambda : [a, b] \longrightarrow T^*M$ such that $\gamma = \pi \circ \lambda$. Finally, an abnormal curve $\gamma : [a, b] \longrightarrow M$ is called a *Goh curve*, if there exists such an abnormal biextremal $\lambda : [a, b] \longrightarrow T^*M$ projecting onto γ, that for every $t \in [a, b]$ and for every smooth local sections v, w of H defined in a neighbourhood, say G, of $\gamma(t)$, we have $\langle \lambda(t), [v, w](\gamma(t)) \rangle = 0$ for each t such that $\gamma(t) \in G$. According to [4] Goh curves do not exist for generic distributions of rank $m \geq 3$.

Now we pass to the proof of the second part of Proposition 1.1 Fix a $U \in \mathcal{O}$. By virtue of the above remark it is enough to show that every admissible trajectory $\gamma : [0, T] \longrightarrow U$ which is generated by a control $u : [0, T] \longrightarrow riC$, and is such that $\gamma([0, T]) \subset \tilde{\partial}\mathcal{A}(q_0, U; C)$ is a Goh curve. Admissible curves that start from q_0 and are contained in $\tilde{\partial}\mathcal{A}(q_0, U; C)$ are called *geometrically optimal* curves. Clearly, γ is geometrically optimal, and moreover, in view of our assumption imposed on u, one can suppose that it is geometrically optimal with respect to the set of control parameters being equal to riC. Now we are going to write down necessary conditions for geometric optimality following from the Pontriagin maximum principle—see e.g. [1] (we agreed to use L^2 controls, however, C is bounded so our controls are in fact L^{∞} and Pontriagin maximum principle is applicable).

To be more precise, for each $\alpha = 0, \ldots, k$ denote by h_α the function $h_\alpha : T^*U \longrightarrow \mathbb{R}$ defined by $h_\alpha(q, p) = \langle p, X_\alpha(q) \rangle$. For every $u \in C$ let us also set $h_u = h_0 + \sum_{j=1}^{k} u_j h_j$. Now the maximum principle asserts that if a curve $\gamma : [0, T] \longrightarrow U$, generated by $u : [0, T] \longrightarrow riC$, is geometrically optimal, then there exists an absolutely continuous curve $\lambda : [0, T] \longrightarrow T^*U$, $\lambda(t) \in T^*_{\gamma(t)}U \backslash \{0\}$ for every $t \in [0, T]$, such that (i) $\dot{\lambda}(t) = \overrightarrow{h_0}(\lambda(t)) + \sum_{j=1}^{k} u_j(t)\overrightarrow{h_j}(\lambda(t))$ a.e. on $[0, T]$; (ii) $h_{u(t)}(\lambda(t)) = \max_{v \in riC} h_v(\lambda(t))$ a.e. on $[0, T]$; (iii) $h_{u(t)}(\lambda(t)) = 0$ on $[0, T]$. By $\overrightarrow{h_\alpha}$ we denoted the Hamiltonian vector field corresponding to h_α.

Now (ii) implies that $h_j(\lambda(t)) = 0$ for $j = 1, \ldots, k$ and every $t \in [0, T]$. This, together with (iii), gives $h_0(\lambda(t)) = 0$ for every $t \in [0, T]$ which means $\lambda([0, T]) \subset H^{\perp}$, where $H = Span\{X_0, \ldots, X_k\}$. Moreover, by definition of $\overrightarrow{h_\alpha}$'s,

$$\Omega(\zeta, \dot{\lambda}(t)) = d_{\lambda(t)}h_0(\zeta) + \sum_{j=1}^{k} u_j(t) d_{\lambda(t)}h_j(\zeta)$$

which implies $\Omega(\zeta, \dot{\lambda}(t)) = 0$ for every $\zeta \in T_{\lambda(t)}H^{\perp}$. Thus we conclude that γ is an abnormal curve.

Next, using [1], γ satisfies the Goh condition, which in our case reads

$$\big\langle \lambda(t), [X_i, X_j](\gamma(t)) \big\rangle = 0, \quad i, j = 1, \ldots, k, t \in [0, T]. \tag{3.1}$$

Differentiating the equation $\langle\lambda(t), X_j(\gamma(t))\rangle = 0$, $j = 1, \ldots, k$, with respect to t on the interval $[0, T]$ one obtains

$$0 = \frac{d}{dt}\langle\lambda(t), X_j(\gamma(t))\rangle = \langle\lambda(t), [X_j, X_0](\gamma(t))\rangle + \sum_{i=1}^{k} u_i(t)\langle\lambda(t), [X_j, X_i](\gamma(t))\rangle$$

which, using (3.1), gives

$$\langle\lambda(t), [X_\alpha, X_\beta](\gamma(t))\rangle = 0, \quad \alpha, \beta = 0, 1, \ldots, k, t \in [0, T].$$

The proof of Proposition 1.1 is finished.

It is easy to give an example of an affine control system where $\mathcal{A}(q_0, U; riC)$ is not open. To this end take $a, b \in \mathbb{R}$ such that $a < 0 < b$, and consider a system

$$\dot{q} = \frac{\partial}{\partial x} + \frac{1}{2}y^{2k}\frac{\partial}{\partial z} + u\left(\frac{\partial}{\partial y} - \frac{1}{2}xy^{2k-1}\frac{\partial}{\partial z}\right),$$

where k is a positive integer and $u \in C = [a, b]$. Using the same argument as in [7] one can see that the curve $\gamma(t) = (t, 0, 0)$ (which is generated by the null control $u(t) = 0$) is the only, up to a change of parameter, admissible curve joining $(0, 0, 0)$ to a point $(0, 0, T)$ for any $T > 0$. Now (1.4) yields that $\gamma([0, T]) \subset \tilde{\partial}\mathcal{A}(0, \mathbb{R}^3; C)$, and what follows $\mathcal{A}(0, \mathbb{R}^3; riC)$ is not open.

4 Remarks

First of all let us note that throughout the paper we do not assume completeness of admissible trajectories for positive times.

Now we make some comments on a system $\mathcal{O}$ of neighbourhoods of q_0. It is intuitively obvious that (1.3) does not hold for every neighbourhood of q_0. To construct an example of such a "bad" neighbourhood let us consider the control affine system determined by the Heisenberg sub-Lorentzian metric on $\mathbb{R}^3$ (see [6]). In this case, as it was explained in the introduction, $C = [-1, 1]$. Take a number $\delta > 0$ and define a convex neighbourhood of the origin

$$U_\delta = \left\{(x, y, z) : -\tfrac{1}{2}x - \delta < y < -\tfrac{1}{2}x + \delta, \tfrac{1}{2}x - \delta < y < \tfrac{1}{2}x + \delta, -\delta^2 < z < \delta^2\right\}.$$

Let δ_n be an arbitrary sequence of real numbers such that $0 < \delta_n \nearrow \frac{4}{3}\delta$. Now, by [6], it is clear that $q_n = (\delta_n, 0, \frac{1}{4}\delta_n^2) \in \tilde{\partial}\mathcal{A}(0, U_\delta; [-1, 1])$ while $q = \lim q_n \notin \mathcal{A}(0, U_\delta; [-1, 1])$, hence $\mathcal{A}(0, U_\delta; [-1, 1])$ is not closed with respect to U_δ for every $\delta > 0$, and consequently (1.3) is no longer true for a U_δ's. At the same time, let us note that $\mathcal{A}(0, \mathbb{R}^3; [-1, 1]; \leq T) \cap U_\delta$ is closed with respect to U_δ for every $T > 0$, as it follows from theorems mentioned in the introduction.

In order to prove Theorem 1.1, it is in fact sufficient to consider small enough normal neighbourhoods of q_0 but such an approach is not very constructive and requires entering into the sub-Lorentzian geometry details. From practical reasons it is simpler to construct the elements of $\mathcal{O}$ as presented in sect. 2, i.e. for a point q_0 find a function φ, next take a sufficiently small starshaped neighbourhood of q_0 so that (2.1) holds, and finally modify the boundary ∂U in such a way that $\mathcal{A}(q_0, \hat{U}; C) \cap \partial U$ is a level set of φ. Let us emphasize that we do not need to know the set $\mathcal{A}(q_0, \hat{U}; C)$ - for small U's it is easy to estimate the region in which $\mathcal{A}(q_0, \hat{U}; C)$ is contained.

Next remark concerns assumptions imposed on the distribution $H = Span\{X_0, \ldots, X_k\}$. We saw in remark 1.1 that under assumptions of Theorem 1.1 H is bracket generating. Now we will explain what can be said about (1.3), (1.4) and (1.5) if H is not bracket generating. Our reasoning is based on [13]. So let $\hat{H}$ be the smallest integrable distribution containing H; since H is not bracket generating, $\hat{H}$ is not equal to the whole TM. Our M is foliated by leaves of $\hat{H}$, i.e. M is the union of integral submanifolds of $\hat{H}$. Let L be the (unique) leaf that passes through q_0. Obviously, reachable sets from q_0 for the system (1.1) are now contained in L. Note here that, in general, L is an immersed submanifold, so the appropriate topology to consider is the manifold topology of L (which may be stronger than the topology induced by M). Now (1.3), (1.4) and (1.5) remain valid if we modify assumptions of Theorem 1.1 as follows. At first, $\mathcal{O}$ is the fundamental system of neighbourhoods of q_0 on L. Secondly, we should replace $T_{q_0}M$ by $T_{q_0}L$ in the right hand side of (1.2). Finally, the operations cl_U and $\tilde{\partial}$ must be taken with respect to the manifold topology of L.

At the end let us emphasize the application of the sub-Lorentzian geometry methods to the study of control affine systems. Sub-Lorentzian metrics can play similar auxiliary role in the investigation of control affine systems as, for instance, Riemannian metrics do in many problems in differential topology.

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