Bipolar-valued hesitant fuzzy graph and its application

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Abstract

In the real-world scenario, one has to find a dominating person in a social network, conferences, meetings or any group discussion. The fuzzy graph (network) is one of the most powerful tools to find the strongest influential person in a network. This paper aims to develop a concept of fuzzy graphs (FGs) in the setup of bipolar-valued hesitant fuzzy sets (BVHFs). The concept of bipolar-valued hesitant fuzzy graph (BVHFG) is different from the concept of bipolar fuzzy graph (BFG). BVHFG is the generalization of hesitant fuzzy graph (HFG), which not only considers the satisfaction degree of units in a network but also considers the satisfaction degree to some implicit counter property of units with several bipolar fuzzy values. We first introduce the definition of BVHFG, represented by another class of imprecise membership grades that refers to BVHF membership grades. We shall subsequently see the scope of BVHF membership grades in BVHFG is greater than the scope of bipolar-valued membership grades in BFG. In addition, we also discuss the basic operations and functional properties of BVHFGs. Finally, we propose a numerical method to find the most dominating person using our proposed work. As the proposed method of ranking considers the degree of hesitation as well as bipolarity, this method has the edge over earlier work. To establish the importance of our method, we also find domination degrees for HFG and BVHFG using the same example and show that there is a significant change in the ranking of dominating persons.

Keywords Bipolar-valued hesitant fuzzy sets · Bipolar-valued hesitant fuzzy graph · Directed bipolar-valued hesitant fuzzy graph · Domination degree

1 Introduction

Fuzzy set theory introduced by Klir and Yuan (1996) can be seen as a body of concepts that deal with the kinds of uncertainties which arise when the class boundary is not properly defined. A fuzzy set is basically represented by its membership function which is a generalization of the characteristic function. After the development of fuzzy sets, it has become the most popular research field in various disciplines. Several generalizations of fuzzy sets have been introduced in

the field of research, for example, Zhang (1994) presented bipolar-valued fuzzy sets, as an extension of fuzzy sets whose membership lies within $[-1, 1]$. In the bipolar-valued fuzzy set, membership degree 0 indicates that the element has no relevance to the given condition, membership degree $(0, 1]$ indicates the satisfaction degree of the element corresponding to some given properties, membership degree $[-1, 0)$ indicates the degree of satisfaction of an element for some inherent inverse properties of the given condition. In this generalization, we assume that it is dealing with the degree of satisfaction of the elements that satisfy the corresponding property and some inherent counter property of the given condition, but there is no consideration for hesitation on this satisfaction degree. This generalization is still not more-convenient with real-world problems. Therefore, Mandal and Ranadive (2019) propose the concept of bipolar-valued hesitant fuzzy set, and it deals the situation in bipolar membership degree with hesitation index.

Graph theory originated in 1736 for Euler's solution to the Königsberg bridge puzzle, provides a powerful tool for solving problems in various disciplines such as computer

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analysis, data structure, mathematical modeling, economics, engineering, social sciences, and statistics. In many cases, some aspects of the graph-theory problems are uncertain, for example, ranking of attributes, group decision-making problems, consistency of preference relation problems, etc. In order to solve such problems, Rosenfeld gave the concept of fuzzy graph, inspired by Kauffman's (1973) fundamental ideas. Currently, a lot of work is being done in the research field using fuzzy graphs, such as, link prediction in social networks (Mahapatra et al. 2019, 2020), generalized neutrosophic planar graphs (Mahapatra et al. 2021), coloring of covid-19 affected regions (Mahapatra et al. 2021, 2020), radio fuzzy graphs (Mahapatra et al. 2019), and packaged food smart traceability and communication (Mahapatra et al. 2020). The concept of fuzzy homomorphism, fuzzy isomorphism, fuzzy co-weak isomorphism was given by Bhutani (1989). After the concept of fuzzy graphs, this area has been variously generalized. Based on the basic idea of Zhang (1994) bipolar fuzzy set theory, Akram (2011, 2013, 2016), Akram and Dudek (2012) proposed the notion of bipolar fuzzy graphs to deal with the bipolarity of real-world problems and gave some basic operations such as Cartesian product, composition and union of two bipolar fuzzy graphs. The book by Akram, Sarwar and Dudek (2021) entitled *Graphs for the Analysis of Bipolar Fuzzy Information* is an excellent source for research in fuzzy graphs. The features of bipolar fuzzy graph are very useful in the application of decision-making problems (Akram and Waseem 2018). More information on BFGs, irregular BFGs, and bipolar fuzzy hypergraphs can be found in Samanta and Pal (2012a, (b); Samanta et al. (2014); Rashmanlou et al. (2015); Akram and Akmal (2016); Alghamdi et al. (2018). In fuzzy graph theory, each vertex is characterized by membership degrees only, but this theory is not more convenient for modelling some real-life problems. Therefore, Karaaslan (2019) provided the definition of hesitant fuzzy graph whose basic idea was introduced by Torra (2010).

In this paper, we establish the concept of BVHFG with a different approach from BFG. Here we formulate the many real-world problems in a network with several bipolar-valued hesitant membership grades, more reliable with the decision process and many other problems. In this sequence, we also introduce some basic operations such as Cartesian product, strong product and union of two BVHFGs, we also investigate the concept of homomorphism, isomorphism, weak isomorphism, co-weak isomorphism of these graphs. In the final section, we illustrate by mean of an example, finding the most dominating person in the meeting and compare the usage of BVHFG and HFG.

The content of the paper is organized as follows. Section two reviews some basic knowledge on graphs, fuzzy

graphs and BVHFSs. Section three refers to the definition and example of BVHFGs, then some common operational laws along with mapping relationships of BVHFGs and propositions are explored. Section four develops a two-stage algorithmic rule by virtue of BVHFGs, and then, a numerical instance concerning finding the most influential person is given. In section five, we tend to study the comparative analysis between BVHFG and HFG to demonstrate the relevancy and effectiveness of the conferred notion of BVHFG. At last, Section six concludes the total paper and points out the significance of BVHFG.

2 Preliminaries

In this segment, we cover some basic definitions based on graphs, digraphs, fuzzy graphs and bipolar-valued hesitant fuzzy sets (BVHFSs).

2.1 Basic definitions on graphs and fuzzy graphs

Definition 1 (Bollobás 2013) A graph is an ordered pair $G = (V, E)$, where V is the set of vertices and E is the set of edges of G formed by a pair of vertices $E \subset \{(x, y) | x, y \in V\}$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, then the following operations are defined as:

- Cartesian product of two graphs G_1 and G_2 denoted by $G_1 \times G_2 = (V_1 \times V_2, E_1 \times E_2)$ is defined as $V_1 \times V_2 = \{(v_1, v_2) | v_1 \in V_1 \text{ and } v_2 \in V_2\}$ and $E_1 \times E_2 = \{(a, a_2)(a, b_2) | a \in V_1, a_2 b_2 \in E_2\} \cup \{(a_1, c)(b_1, c) | a_1 b_1 \in E_1, c \in V_2\}$.
- Strong product of two graphs G_1 and G_2 denoted by $G_1 \otimes G_2 = (V_1 \otimes V_2, E_1 \otimes E_2)$ is defined as $V_1 \times V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$ and $E_1 \times E_2 = \{(a, a_2)(a, b_2) | a \in V_1, a_2 b_2 \in E_2\} \cup \{(a_1, c)(b_1, c) | a_1 b_1 \in E_1, c \in V_2\} \cup \{(a_1, a_2)(b_1, b_2) | a_1 b_1 \in E_1, a_2 b_2 \in E_2\}$.

Definition 2 A graph $G = (V, E)$ whose edges have the orientations and arrows on edges are used to represent the directions is called directed graph (digraph), denoted by $\vec{G}_d = (V, \vec{E})$.

Definition 3 (Rosenfeld 1975) A fuzzy graph $G^* = (V, \mu, \nu)$ over the graph $G = (V, E)$ is characterized by membership functions $\mu : V \rightarrow [0, 1]$ and $\nu : V \times V \rightarrow [0, 1]$ such that

$$\nu(v_1 v_2) \leq \mu(v_1) \wedge \mu(v_2) \quad \forall v_1, v_2 \in V,$$

where $\mu(v_1)$ and $\nu(v_1 v_2)$ represent the membership value of the vertex v_1 and edge $v_1 v_2$ in G^* , respectively.

Definition 4 (Mordeson and Nair 2012) Let $G^* = (V, \mu, \nu)$ be a fuzzy graph, then complimentary fuzzy graph $\bar{G}^*$ of a fuzzy graph G^* has the same vertices as G^* , two vertices are adjacent in $\bar{G}^*$ if and only if they are not adjacent in G^* .

Definition 5 (Poulik and Ghorai 2020) Let $G^* = (V, \mu, \nu)$ be a fuzzy graph, then degree of vertex v_1 in a fuzzy graph G^* is defined as $deg(v_1) = \sum_{v_1 \neq v \in V} \nu(v_1 v)$.

2.2 Bipolar-valued hesitant fuzzy set

Definition 6 (Mandal and Ranadive 2019). Let X be frame of reference. A bipolar-valued hesitant fuzzy set (BVHFS) A on X is defined as:

$$A = \{ \langle x, A(x) \rangle \mid x \in X \},$$

where $A(x)$ is a set of some values in $[0, 1] \times [-1, 0]$. For convenience, we call $A(x)$ a bipolar-valued hesitant fuzzy element (BVHFE) expressed as:

$$A(x) = \{ a_x \mid a_x \in [0, 1] \times [-1, 0] \},$$

here $a_x = (a_x^P, a_x^N)$ is a bipolar-valued fuzzy number (BVFN) such that $a_x^P \in [0, 1]$ and $a_x^N \in [-1, 0]$.

Definition 7 (Mandal and Ranadive 2019) Let X be universe of discourse and for $x \in X$ let $A(x), A_1(x), A_2(x)$ be the BVHFE's, then

- $A_1(x) \cup A_2(x) = \{ (\max(a_{1x}^P, a_{2x}^P), \min(a_{1x}^N, a_{2x}^N)) \mid a_{1x} \in A_1(x), a_{2x} \in A_2(x) \};$
- $A_1(x) \cap A_2(x) = \{ (\min(a_{1x}^P, a_{2x}^P), \max(a_{1x}^N, a_{2x}^N)) \mid a_{1x} \in A_1(x), a_{2x} \in A_2(x) \};$
- $A(x)^c = \{ (1 - a_x^P, -1 - a_x^N) \mid a_x \in A(x) \}.$

Definition 8 (Mandal and Ranadive 2019) Let $a_x = (a_x^P, a_x^N) \in A(x)$ be a BVFN, and the value of score $s(a_x)$ is defined as:

$$s(a_x) = \frac{1}{2}(a_x^P - a_x^N),$$

which is the mean of the satisfaction degree corresponding to some given property of attribute and satisfaction degree to some implicit counter property of attribute of an element.

Definition 9 (Mandal and Ranadive 2019) Let $A(x)$ be a BVHFE, and the score function $s(A(x))$ is defined as:

$$s(A(x)) = \frac{1}{l(A(x))} \sum_{a_x \in A(x)} s(a_x),$$

here $l(A(x))$ denote the number of bipolar values in $A(x)$, and a_x is the element in $A(x)$, taken as the form of BVFN.

Inspired by the basic idea of Deepak and John (2014), we define score-based intersection and union of two BVHFSs as follows:

Definition 10 Let A and B be two BVHFSs over X . Then, score-based intersection and union of two BVHFE's $A(x)$ and $B(x)$ are represented by $A(x) \tilde{\wedge} B(x)$ and $A(x) \tilde{\vee} B(x)$, respectively, which is characterized by

$$A(x) \tilde{\wedge} B(x) = \begin{cases} A(x), & \text{if } s(A(x)) < s(B(x)), \\ B(x), & \text{if } s(B(x)) < s(A(x)), \\ A(x) \text{ or } B(x), & \text{if } s(A(x)) = s(B(x)), \end{cases}$$

and

$$A(x) \tilde{\vee} B(x) = \begin{cases} A(x), & \text{if } s(A(x)) > s(B(x)), \\ B(x), & \text{if } s(B(x)) > s(A(x)), \\ A(x) \text{ or } B(x), & \text{if } s(A(x)) = s(B(x)). \end{cases}$$

The following is a trivial consequence of our definition.

Proposition 1 Let A and B be two BVHFSs over the non-empty set X . Then, $s(A(x) \tilde{\wedge} B(x)) = s(A(x)) \wedge s(B(x))$ and $s(A(x) \tilde{\vee} B(x)) = s(A(x)) \vee s(B(x))$.

2.3 Bipolar-valued hesitant fuzzy relation

Definition 11 Let A and B be two BVHFSs over the non-empty set X . Then, the score-based Cartesian product of two BVHFSs A and B is represented by $A \tilde{\times} B$ and characterized by

$$A \tilde{\times} B = \{ \langle (x, y), (A \tilde{\times} B)(x, y) \rangle \mid (x, y) \in X \times X \}, \\ = \{ \langle (x, y), A(x) \tilde{\times} B(y) \rangle \mid (x, y) \in X \times X \}.$$

Definition 12 Let X be a non-empty set. Let A and B be two BVHFSs on X , for $x, y \in X$, let $A(x, y) : X \times X \rightarrow P([0, 1] \times [-1, 0])$ be bipolar-valued hesitant fuzzy relation on X , and then, we call A is score-based bipolar-valued hesitant fuzzy relation on B if $s(A(x, y)) \leq s(B(x)) \wedge s(B(y))$ for all $x, y \in X$.

3 Bipolar-valued hesitant fuzzy graph

In this section, we define the notion of bipolar-valued hesitant fuzzy graph (BVHFG) using the score function, which is the mean of satisfaction degree of an element corresponding to some given attribute property and some implicit counter property of attribute. In addition, we give some examples, basic operations and define the isomorphisms on BVHFGs.

Definition 13 Let $G = (V, E)$ be a graph. A bipolar-valued hesitant fuzzy graph (BVHFG) with V as a reference set is a pair $G^* = (A, B)$ where A and B are BVHFSs in V and $\tilde{V}^2$ respectively, which is characterized by membership functions $A : V \rightarrow P([0, 1] \times [-1, 0])$ and $B : \tilde{V}^2 \rightarrow P([0, 1] \times [-1, 0])$ with the condition

$$s(B(xy)) \leq s(A(x)) \wedge s(A(y)) \quad \forall xy \in \tilde{V}^2$$

$$\text{and } s(B(xy)) = 0 \quad \forall xy \in (\tilde{V}^2 - E),$$

here $B(xy)$ and $A(x)$ are BVHFEs defined as $B(xy) = \{(b_{xy}^P, b_{xy}^N) \mid (b_{xy}^P, b_{xy}^N) \in [0, 1] \times [-1, 0]\}$ and $A(x) = \{(a_x^P, a_x^N) \mid (a_x^P, a_x^N) \in [0, 1] \times [-1, 0]\}$.

Remark 1

- If we neglect the negative membership degrees of vertices in a network, which represents the satisfaction degrees to some implicit counter property of attribute, then BVHFG reduces to HFG (Karaaslan 2019).
- The concept of bipolar-valued hesitant fuzzy graph differs from the concept of bipolar fuzzy graph in its approach to the definition of 13.
- From definition 13, we can easily understand that the class of imprecise membership grades in BVHFG is greater than the class of imprecise membership grades in BFG. Membership grades of bipolar fuzzy graphs also satisfy the condition of BVHFG, but the converse is not always true. Here in Fig. 1 all membership grades satisfy the condition of BVHFG as in definition 13. However, for the edge between vertex A and vertex B , the positive satisfaction degree $0.5 \not\leq \min(0.4, 0.8)$ and negative satisfaction degree $-0.4 \not\geq \max(-0.6, -0.3)$, i.e., it does not satisfy the condition of BFG. Hence, the class of membership grades of BVHFG is larger than the class of membership grades in BFG.

Example 1 Suppose there are six participants in the debate competition program and board members select the one participant as a winner according to the four functional properties, that is, influence power, conversation style, knowledge

Fig. 1 BVHFG Vs BFG

of the given topic, speech ingenuity. Instead of providing a traditional decision matrix, board members comprehensively evaluate the correlation of four properties between six participants. Let V be a set of six participants $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1 v_2, v_1 v_4, v_2 v_3, v_2 v_5, v_3 v_4, v_4 v_5, v_5 v_6, v_6 v_1\}$ be the correlation of four properties among participants, then BVHFG and BVHFSs A and B over V and $\tilde{V}^2$ are given in Fig. 2 and Table 1, respectively:

Definition 14 Let $G^* = (A, B)$ be a BVHFG over the graph $G = (V, E)$, score-based degree of a vertex $v_1 \in V$ in BVHFG is denoted by $\mathfrak{D}(v_1)$ and defined as $\mathfrak{D}(v_1) = \sum_{v \neq v_1 \in V} s(B(v_1 v))$.

Example 2 For Example 1, we calculate the score-based degree of each vertex in BVHFG and find that $\mathfrak{D}(v_1) = 1.200$, $\mathfrak{D}(v_2) = 1.233$, $\mathfrak{D}(v_3) = 0.425$, $\mathfrak{D}(v_4) = 1.275$, $\mathfrak{D}(v_5) = 1.35$, $\mathfrak{D}(v_6) = 0.892$.

Definition 15 Let $G_1^* = (A_1, B_1)$ and $G_2^* = (A_2, B_2)$ be two BVHFGs over the graph $G = (V, E)$ then we say that G_1^* is score-based BVHF-subgraph of G_2^* if it satisfies the conditions

$$s(A_1(x)) \leq s(A_2(x)), \quad s(B_1(xy)) \leq s(B_2(xy))$$

$$\forall x \in V, \quad \forall xy \in \tilde{V}^2.$$

3.1 Basic operations on BVHFGs

Let $G_1^* = (A_1, B_1)$ and $G_2^* = (A_2, B_2)$ be two BVHFGs over the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. We now give some operations and related results for BVHFGs:

1. Cartesian product: Cartesian product of two BVHFGs denoted by $G_1^* \tilde{\times} G_2^* = (A_1 \tilde{\times} A_2, B_1 \tilde{\times} B_2)$ is defined as:

Fig. 2 Bipolar-valued hesitant fuzzy graph (BVHFG)

Table 1 Bipolar-valued hesitant fuzzy table

	V_1	V_2	V_3	V_4
A	$\{(0.2, -0.7), (0.4, -0.6)\}$	$\{(0.5, -0.3), (0.7, -0.4), (0.8, -0.1)\}$	$\{(0.3, -0.8), (0.5, -0.6)\}$	$\{(0.2, -0.9)\}$
Score	0.475	0.467	0.55	0.55
	V_5	V_6		
	$\{(0.1, -0.8), (0.3, -0.7)\}$	$\{(0.6, -0.7), (0.4, -0.8)\}$		
	0.475	0.625		
	$V_1 V_2$	$V_1 V_4$	$V_2 V_3$	$V_2 V_5$
B	$\{(0.4, -0.3), (0.6, -0.2)\}$	$\{(0.2, -0.5), (0.6, -0.3)\}$	$\{(0.3, -0.5), (0.2, -0.7)\}$	$\{(0.4, -0.5), (0.6, -0.3), (0.1, -0.7)\}$
Score	0.375	0.400	0.425	0.433
	$V_3 V_4$	$V_4 V_5$	$V_5 V_6$	$V_6 V_1$
	$\{(0.2, -0.6), (0.4, -0.5)\}$	$\{(0.1, -0.7), (0.2, -0.8)\}$	$\{(0.5, -0.6), (0.3, -0.7), (0.2, -0.5)\}$	$\{(0.3, -0.5), (0.2, -0.7)\}$
	0.425	0.450	0.467	0.425

- $(A_1 \tilde{\times} A_2)(x_1, x_2) = A_1(x_1) \tilde{\wedge} A_2(x_2) \forall (x_1, x_2) \in V_1 \times V_2$.
- $(B_1 \tilde{\times} B_2)((x, x_2)(x, y_2)) = A_1(x) \tilde{\wedge} B_2(x_2 y_2) \forall x \in V_1, x_2 y_2 \in E_2$.
- $(B_1 \tilde{\times} B_2)((x_1, z)(y_1, z)) = B_1(x_1 y_1) \tilde{\wedge} A_2(z) \forall x_1 y_1 \in E_1, z \in V_2$.

Proposition 2 Let G_1^* and G_2^* be two BVHFGs, then $G_1^* \tilde{\times} G_2^*$ is a BVHFG.

Proof For any $x \in V_1$ and $x_2 y_2 \in E_2$ we have,

$$\begin{aligned}
 & s((B_1 \tilde{\times} B_2)((x, x_2)(x, y_2))) \\
 &= s(A_1(x) \tilde{\wedge} B_2(x_2 y_2)) \\
 &= s(A_1(x)) \wedge s(B_2(x_2 y_2)) \\
 &\leq s(A_1(x)) \wedge (s(A_2(x_2)) \wedge s(A_2(y_2))) \\
 &= (s(A_1(x)) \wedge s(A_2(x_2))) \wedge (s(A_1(x)) \wedge s(A_2(y_2))) \\
 &= s(A_1(x) \tilde{\wedge} A_2(x_2)) \wedge s(A_1(x) \tilde{\wedge} A_2(y_2)) \\
 &= s((A_1 \tilde{\times} A_2)(x, x_2)) \wedge s((A_1 \tilde{\times} A_2)(x, y_2)).
 \end{aligned}$$

For any $x_1 y_1 \in E_1$ and $z \in V_2$ we have,

$$\begin{aligned}
 & s((B_1 \tilde{\times} B_2)((x_1, z)(y_1, z))) \\
 &= s(B_1(x_1 y_1) \tilde{\wedge} A_2(z)) \\
 &= s(B_1(x_1 y_1)) \wedge s(A_2(z)) \\
 &\leq (s(A_1(x_1)) \wedge s(A_1(y_1))) \wedge s(A_2(z)) \\
 &= (s(A_1(x_1)) \wedge s(A_2(z))) \wedge (s(A_1(y_1)) \wedge s(A_2(z))) \\
 &= s(A_1(x_1) \tilde{\wedge} A_2(z)) \wedge s(A_1(y_1) \tilde{\wedge} A_2(z)) \\
 &= s(A_1 \tilde{\times} A_2)(x_1, z) \wedge s(A_1 \tilde{\times} A_2)(y_1, z).
 \end{aligned}$$

2. Strong product: Strong product of two BVHFGs denoted by $G_1^* \otimes G_2^* = (A_1 \otimes A_2, B_1 \otimes B_2)$ is defined as:

- $(A_1 \tilde{\otimes} A_2)(x_1, x_2) = A_1(x_1) \tilde{\wedge} A_2(x_2) \forall (x_1, x_2) \in V_1 \times V_2$.
- $(B_1 \tilde{\otimes} B_2)((x, x_2)(x, y_2)) = A_1(x) \tilde{\wedge} B_2(x_2 y_2) \forall x \in V_1, x_2 y_2 \in E_2$.
- $(B_1 \tilde{\otimes} B_2)((x_1, z)(y_1, z)) = B_1(x_1 y_1) \tilde{\wedge} A_2(z) \forall x_1 y_1 \in E_1, z \in V_2$.
- $(B_1 \tilde{\otimes} B_2)((x_1, x_2)(y_1, y_2)) = B_1(x_1 y_1) \tilde{\wedge} B_2(x_2 y_2) \forall x_1 y_1 \in E_1, x_2 y_2 \in E_2$.

Proposition 3 Let G_1^* and G_2^* be two BVHFGs, then $G_1^* \tilde{\otimes} G_2^*$ is a BVHFG. □

Proof The proof of this proposition is similar as that of proposition 2 and hence, is omitted.

3. Union: Union of two BVHFGs denoted by $G_1^* \cup G_2^* = (A_1 \cup A_2, B_1 \cup B_2)$ s.t. $s(A_1(x)) = 0$ if $x \notin V_1$ and $s(A_2(x)) = 0$ if $x \notin V_2$ is defined as:

- $(A_1 \cup A_2)(x) = A_1(x) \tilde{\vee} A_2(x) \forall x \in V_1 \cup V_2$.
- $(B_1 \cup B_2)(xy) = B_1(xy) \tilde{\vee} B_2(xy) \forall xy \in E_1 \cup E_2$.

Proposition 4 Let G_1^* and G_2^* be two BVHFGs, then $G_1^* \cup G_2^*$ is a BVHFG. □

Proof For any $xy \in E_1 \cup E_2$ we have

$$\begin{aligned}
 & s((B_1 \cup B_2)(xy)) \\
 &= s(B_1(xy) \tilde{\vee} B_2(xy)) \\
 &= s(B_1(xy)) \vee s(B_2(xy)) \\
 &\leq (s(A_1(x)) \wedge s(A_1(y))) \vee (s(A_2(x)) \wedge s(A_2(y))) \\
 &= (s(A_1(x)) \vee s(A_2(x))) \wedge (s(A_1(y)) \vee s(A_2(y))) \\
 &= s(A_1(x) \tilde{\vee} A_2(x)) \wedge s(A_1(y) \tilde{\vee} A_2(y)) \\
 &= S((A_1 \cup A_2)(x)) \wedge s((A_1 \cup A_2)(y)).
 \end{aligned}$$

3.2 Isomorphism between bipolar-valued hesitant fuzzy graphs

Definition 16 Let G_1^* and G_2^* be the BVHFGs. A homomorphism $f : G_1^* \rightarrow G_2^*$ is a mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) \leq s(A_2(f(x_1))) \forall x_1 \in V_1$.
- $s(B_1(x_1y_1)) \leq s(B_2(f(x_1)f(y_1))) \forall x_1y_1 \in \widetilde{V}_1^2$.

Definition 17 Let G_1^* and G_2^* be the BVHFGs. An isomorphism $f : G_1^* \rightarrow G_2^*$ is a bijective mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) = s(A_2(f(x_1))) \forall x_1 \in V_1$.
- $s(B_1(x_1y_1)) = s(B_2(f(x_1)f(y_1))) \forall x_1y_1 \in \widetilde{V}_1^2$.

Definition 18 Let G_1^* and G_2^* be the BVHFGs. Then, a weak isomorphism $f : G_1^* \rightarrow G_2^*$ is a bijective mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) = s(A_2(f(x_1))) \forall x_1 \in V_1$.
- $s(B_1(x_1y_1)) \leq s(B_2(f(x_1)f(y_1))) \forall x_1y_1 \in \widetilde{V}_1^2$.

Definition 19 Let G_1^* and G_2^* be the BVHFGs. A co-weak isomorphism $f : G_1^* \rightarrow G_2^*$ is a bijective mapping $f : V_1 \rightarrow V_2$ satisfying the following conditions:

- $s(A_1(x_1)) \leq s(A_2(f(x_1))) \forall x_1 \in V_1$.
- $s(B_1(x_1y_1)) = s(B_2(f(x_1)f(y_1))) \forall x_1y_1 \in \widetilde{V}_1^2$.

Remark 2

- If $G_1^* = G_2^* = G^*$, then the homomorphism f over itself is called an endomorphism. An isomorphism f over G is called an automorphism.
- Let G^* be a BVHFG of a graph G . Let $Aut(G^*)$ be the set of all bipolar-valued hesitant automorphism of G^* . Let $e : G^* \rightarrow G^*$ be a map defined by $e(x) = x \forall x \in V$. Clearly $e \in Aut(G)$.

$$s(A_1(m_1)) = s(A_2(h_1(m_1))) = s(A_2(m_2)) = s(A_3(h_2(m_2))) = s(A_3(h_2(h_1(m_1))));$$

$$= s(A_3(h_2oh_1(m_1))) \forall m_1 \in V_1.$$

- If $G_1^* = G_2^*$, then the weak and co-weak isomorphism actually become isomorphic.
- If $f : V_1 \rightarrow V_2$ is a bijective map, then $f^{-1} : V_2 \rightarrow V_1$ is also a bijective map.

Proposition 5 If G_1^*, G_2^*, G_3^* are BVHFGs, then the isomorphism between these graphs is an equivalence relation.

Proof For reflexivity we can use identity mapping between BVHFGs, and it is trivial. For symmetry, we assume a function $h : V_1 \rightarrow V_2$ is an isomorphism on G_1^* onto G_2^* such that $h(m_1) = m_2 \forall m_1 \in V_1$ with conditions

$$s(A_1(m_1)) = s(A_2(h(m_1))), s(B_1(m_1n_1)) = s(B_2(h(m_1)h(n_1))) \forall m_1 \in V_1, \forall m_1n_1 \in \widetilde{V}_1^2; \quad (1)$$

further, since h is isomorphism, we have $h^{-1}(m_2) = m_1 \forall m_2 \in V_2$, satisfies condition (1), we have

$$s(A_1(h^{-1}(m_2))) = s(A_2(m_2)), s(B_1(h^{-1}(m_2)h^{-1}(n_2))) = s(B_2(m_2n_2)) \forall m_2 \in V_2, \forall m_2n_2 \in \widetilde{V}_2^2.$$

Thus, a mapping $h^{-1} : V_2 \rightarrow V_1$ is an isomorphism from G_2 onto G_1 . For transitivity, we assume $h_1 : V_1 \rightarrow V_2$ such that $h_1(m_1) = m_2 \forall m_1 \in V_1$ and $h_2 : V_2 \rightarrow V_3$ such that $h_2(m_2) = m_3 \forall m_2 \in V_2$ are isomorphisms between G_1 onto G_2 and G_2 onto G_3 , respectively. Further $h_2oh_1 : V_1 \rightarrow V_3$ is a composition of h_1 and h_2 such that $(h_2oh_1)(m_1) = h_2(h_1(m_1)) \forall m_1 \in V_1$.

Since the map $h_1 : V_1 \rightarrow V_2$ is an isomorphism, we have

$$s(A_1(m_1)) = s(A_2(h_1(m_1))) = s(A_2(m_2)) \forall m_1 \in V_1; \quad (2)$$

$$s(B_1(m_1n_1)) = s(B_2(h_1(m_1)h_1(n_1))) = s(B_2(m_2n_2)) \forall m_1n_1 \in \widetilde{V}_1^2. \quad (3)$$

Again since the map $h_2 : V_2 \rightarrow V_3$ is an isomorphism, we have

$$s(A_2(m_2)) = s(A_3(h_2(m_2))) = s(A_3(m_3)) \forall m_2 \in V_2; \quad (4)$$

$$s(B_2(m_2n_2)) = s(B_3(h_2(m_2)h_2(n_2))) = s(B_3(m_3n_3)) \forall m_2n_2 \in \widetilde{V}_2^2. \quad (5)$$

From expressions (2) and (4), we have

From expressions (3) and (5), we have

$$\begin{aligned} s(B_1(m_1n_1)) &= s(B_2(h_1(m_1)h_1(n_1))) \\ &= s(B_2(m_2n_2)) = s(B_3(h_2(m_2)h_2(n_2))); \\ &= s(B_3(h_2(h_1(m_1))(h_2(h_1(n_1))))); \\ &= s(B_3((h_2oh_1)(m_1)(h_2oh_1)(n_1))) \forall m_1n_1 \in \tilde{V}_1^2. \end{aligned}$$

Hence h_2oh_1 is an isomorphism between G_1^* and G_3^* .

□

Proposition 6 If G_1^* , G_2^* , G_3^* are BVHFGs, then the weak isomorphism between these graphs is a partial order relation.

Proof Reflexivity is trivial. For anti-symmetry we assume a function $h : V_1 \rightarrow V_2$ is a weak isomorphism on G_1^* onto G_2^* such that $h_1(m_1) = m_2 \forall m_1 \in V_1$ with conditions

$$\begin{aligned} s(A_1(m_1)) &= s(A_2(h_1(m_1))), \\ s(B_1(m_1n_1)) &\leq s(B_2(h_1(m_1)h_1(n_1))) \\ \forall m_1 \in V_1, \forall m_1n_1 &\in \tilde{V}_1^2; \end{aligned} \quad (6)$$

suppose $h_2 : V_2 \rightarrow V_1$ is a weak isomorphism between G_1^* and G_2^* such that $h_2(a_2) = a_1 \forall a_2 \in V_2$ with condition

$$\begin{aligned} s(A_2(a_2)) &= s(A_1(h_2(a_2))), s(B_2(a_2b_2)) \\ &\leq s(B_1(h_2(a_2)h_2(b_2))) \forall a_2 \in V_2, \\ \forall a_2b_2 &\in \tilde{V}_1^2. \end{aligned} \quad (7)$$

From expressions (6) and (7), we conclude that both these inequalities hold if and only if the BVHFGs have the same number of edges and corresponding edges have the same weights, which shows that G_1^* and G_2^* are similar. Furthermore, we can show that the transitivity among the graphs G_1^* , G_2^* and G_3^* is the same as previous proposition. □

Proposition 7 If $G^* = (A, B)$ is a BVHFG and $Aut(G^*)$ is the set of all automorphisms of G^* . Then, $(Aut(G^*), o)$ forms a group.

Proof For any $\phi, \psi \in Aut(G^*)$ and $x, y \in V$. Then, we have

$$\begin{aligned} s(B((\phi\psi)(x)(\phi\psi)(y))) \\ &= s(B(\phi(\psi(x))(\phi(\psi(y)))) \\ &= s(B(\psi(x)\psi(y))) \\ &= s(B(xy)), \\ s(A((\phi\psi)(x))) &= s(A(\phi(\psi(x)))) \\ &= s(A(\psi(x))) \\ &= s(A(x)), \end{aligned}$$

clearly $\phi\psi \in Aut(G^*)$. Also $Aut(G^*)$ satisfies associative law under the mapping composition, let $I : G^* \rightarrow G^*$ be an identity mapping such that $\phi o I = I o \phi = \phi \forall \phi \in Aut(G^*)$, also for each $\phi \in Aut(G^*)$ there exists $\phi^{-1} \in G^*$ such

that $s(A(\phi^{-1}(x))) = s(A(\phi(\phi^{-1}(x)))) = s(A((x)))$, $s(B(\phi^{-1}(x)\phi^{-1}(y))) = s(B(\phi(\phi^{-1}(x))\phi(\phi^{-1}(y)))) = s(B(xy))$. Hence, $(Aut(G^*), o)$ forms a group. □

4 Directed-BVHFGs and their application in decision making

In this section, we present the definition of directed-BVHFGs and develop a method for finding in-degrees and out-degrees of vertices in directed-BVHFGs with the help of bipolar-valued hesitant fuzzy weighted averaging (BVHFWA) operator. Along with this, we have given a numerical example to find the domination degree of people in a meeting based on directed graph.

Definition 20 A directed-BVHFG $\vec{G}_d^* = (A, \vec{B})$ of the graph $G = (V, E)$ is given by the pair of membership functions $A : V \rightarrow P([0, 1] \times [-1, 0])$ and $\vec{B} : \vec{V}^2 \rightarrow P([0, 1] \times [-1, 0])$ with condition

$$\begin{aligned} s(\vec{B}(xy)) &\leq s(A(x)) \wedge s(A(y)) \forall xy \in \vec{V}^2 \\ \text{and } s(\vec{B}(xy)) &= 0 \forall xy \in (\vec{V}^2 - E). \end{aligned}$$

Definition 21 Let X be universe of discourse and $\{A_i(x) | x \in X, i = 1, 2 \dots n\}$ be a collection of BVHFE's and $w = (w_1, w_2 \dots w_n)^T$ be the weight vector of $A_i(x) = (a_{ix}^P, a_{ix}^N) (i = 1, 2 \dots n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Then, bipolar-valued hesitant fuzzy weighted averaging (BVHFWA) operator is a mapping $BVHFWA : A^n \rightarrow A$, where

$$\begin{aligned} BVHFWA(A_1(x), A_2(x) \dots A_n(x)) &= \oplus_{i=1}^n (w_i \tilde{A}_i(x)) \\ &= \left\{ \left(1 - \prod_{i=1}^n (1 - a_{ix}^P)^{w_i}, -1 + \prod_{i=1}^n (1 + a_{ix}^N)^{w_i} \right) \right\} \\ a_{1x} &\in A_1(x), a_{2x} \in A_2(x), \dots a_{nx} \in A_n(x), \end{aligned} \quad (8)$$

here if $w = \left(\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n} \right)^T$, then BVHFWA operator will become the bipolar-valued hesitant fuzzy averaging (BVHFA) operator. Equation (8) can be written as:

$$\begin{aligned} BVHFA(A_1(x), A_2(x) \dots A_n(x)) &= \oplus_{i=1}^n \left(\frac{1}{n} A_i(x) \right) \\ &= \left\{ \left(1 - \prod_{i=1}^n (1 - a_{ix}^P)^{\frac{1}{n}}, -1 + \prod_{i=1}^n (1 + a_{ix}^N)^{\frac{1}{n}} \right) \right\} \\ a_{1x} &\in A_1(x), a_{2x} \in A_2(x), \dots a_{nx} \in A_n(x). \end{aligned} \quad (9)$$

Definition 22 Let $G_d^* = (A, B)$ be a directed-BVHFG over the graph $G = (V, E)$ and $v_r, r = 1, 2, \dots n$ be adjacent BVHF-vertices of $v_k \in V$. With the help of Eq. (9), we

define out-degree and in-degree of a vertex v_k denoted by $O_d(v_k)$ and $I_d(v_k)$, respectively, and defined :

$$O_d(v_k) = \left\{ \left(1 - \prod_{i=1}^n (1 - b_{v_k, v_i}^P), \quad -1 + \prod_{i=1}^n (1 + b_{v_k, v_i}^N) \right) \mid v_i v_k \in B \right\}, \quad (10)$$

$$I_d(v_k) = \left\{ \left(1 - \prod_{i=1}^n (1 - b_{v_i, v_k}^P), \quad -1 + \prod_{i=1}^n (1 + b_{v_i, v_k}^N) \right) \mid v_i v_k \in B \right\}. \quad (11)$$

After finding the in-degree and out-degree of each vertex, we represent its score value by $s(O_d(v))$ and $s(I_d(v))$, respectively, and calculate it from definition 9.

To find the domination degree of each vertex v_k , we use $s(O_d(v_k)) - s(I_d(v_k))$ where $s(O_d(v_k))$ and $s(I_d(v_k))$ represent the score value of out-degree and in-degree of vertex v_k , respectively, and denoted by $\omega(v_k)$.

Step 1: Let $A = \{ \langle p_1, \{(0.40, -0.80), (0.50, -0.30), (0.90, -0.20)\} \rangle, \langle p_2, \{(0.30, -0.50), (0.60, -0.90)\} \rangle, \langle p_3, \{(0.10, -0.60), (0.50, -0.20), (0.60, -0.40), (0.70, -0.40)\} \rangle, \langle p_4, \{(0.35, -0.60), (0.40, -0.80)\} \rangle, \langle p_5, \{(0.45, -0.30), (0.80, -0.10), (0.75, -0.30)\} \rangle, \langle p_6, \{(0.60, -0.20), (0.90, -0.50)\} \rangle, \langle p_7, \{(0.20, -0.70), (0.55, -0.80), (0.80, -0.40)\} \rangle \}$ be the BVHFS on the set V as in Table 2, which demonstrates the potential power of persons which are present in business meeting.

Step 2: Let $B = \{ \langle P_1 P_3, \{(.3, -.7), (.4, -.3), (.5, -.2)\} \rangle, \langle P_1 P_6, \{(.5, -.6), (.6, -.3)\} \rangle, \langle P_1 P_7, \{(.2, -.5), (.3, -.3), (.7, -.4)\} \rangle, \langle P_2 P_1, \{(.1, -.2), (.5, -.4), (.7, -.3)\} \rangle, \langle P_2 P_3, \{(.2, -.2), (.3, -.1), (.4, -.4), (.5, -.5)\} \rangle, \langle P_3 P_7, \{(.45, -.8)\} \rangle, \langle P_3 P_4, \{(.1, -.5), (.5, -.9)\} \rangle, \langle P_4 P_7, \{(.1, -.3), (.2, -.5), (.3, -.2)\} \rangle, \langle P_5 P_4, \{(.1, -.1), (.4, -.2), (.6, -.3)\} \rangle, \langle P_5 P_2, \{(.4, -.4), (.5, -.2)\} \rangle, \langle P_6 P_3, \{(.2, -.1), (.3, -.2), (.5, -.3), (.7, -.1)\} \rangle, \langle P_6 P_4, \{(.1, -.1), (.2, -.5)\} \rangle, \langle P_6 P_7, \{(.2, -.2), (.6, -.3), (.7, -.1)\} \rangle, \langle P_6 P_2, \{(.2, -.3), (.5, -.2)\} \rangle, \langle P_7 P_5 \}$

Algorithm 1: Based on directed BVHFGs

Input: A directed bipolar-valued hesitant fuzzy graph which describes the correlation among elements.

Output: Most dominating element.

- 1: Input the BVHFS defined on the set V .
 - 2: Input the edge set represents correlation among elements.
 - 3: Each edge represented by BVHF membership degrees with the condition $s(B(xy)) \leq s(A(x)) \wedge s(A(y))$.
 - 4: By using the Eqn. (10) and Eqn. (11) calculate the in-degree and out-degree of each element.
 - 5: Find the score value of in degree and out degree of each element.
 - 6: Calculate the domination degree of elements.
 - 7: Ranking of elements according to their domination degree.
 - 8: Pick the most dominating element.
-

4.1 Numerical example

We cannot measure the impact power of person's properly, so we are always hesitant to evaluate the impact power of person's. Apart from this, if we do not know about a person properly, then we can also have a negative impact on him. In this part, we present a directed bipolar-valued hesitant fuzzy graph for such case.

Let us consider the directed-BVHFG of potential power of seven person's $P = \{p_1, p_2 \dots p_7\}$ in a business meeting. The potential power of person's determined in positive and negative membership degrees, positive hesitant membership degree denotes satisfaction degree of person corresponding to potential power, while negative hesitant membership degree denotes satisfaction degree of person to some implicit counter property of potential power.

, $\{(.3, -.2), (.4, -.3), (.5, -.1), (.8, -.1)\} \rangle, \}$ be the set of bipolar-valued directed hesitant edges as in Table 3, which demonstrates the impact power of one person onto another person in a meeting as in Fig. 3.

Step 3: With the help of Eqs. (10) and (11), we find the out-degree and in-degree of each person as in Table 4.

Step 4: Now we calculate the score value of in-degree and out-degree of each person by with the help of definition 8 and definition 9 as in Table 5.

Step 5: Finally, we calculate the domination degree using (Karaaslan 2019) of each person in the meeting as in Table 6.

Step 6: The ranking of each person present in the meeting based on their domination degree is as follows:

$$P_2 > P_7 > P_1 > P_4 > P_3 > P_6 > P_5,$$

Table 2 BVHF membership of persons potential power

	P_1	P_2
A	$\{(0.40, -0.80), (0.50, -0.30), (0.90, -0.20)\}$	$\{(0.30, -0.50), (0.60, -0.90)\}$
Score	0.516	0.575
	P_3	P_4
	$\{(0.10, -0.60), (0.50, -0.20), (0.60, -0.40), (0.70, -0.40)\}$	$\{(0.35, -0.60), (0.40, -0.80)\}$
	0.437	0.537
	P_5	P_6
	$\{(0.45, -0.30), (0.80, -0.10), (0.75, -0.30)\}$	$\{(0.60, -0.20), (0.90, -0.50)\}$
	0.45	0.550
	P_7	
	$\{(0.20, -0.70), (0.55, -0.80), (0.80, -0.40)\}$	
	0.575	

Table 3 Impact power of one person onto another person

	P_1P_3	P_1P_6
B	$\{(0.30, -0.70), (0.40, -0.30), (0.50, -0.20)\}$	$\{(0.50, -0.60), (0.60, -0.30)\}$
Score	0.40	0.50
	P_3P_7	P_2P_1
	$\{(0.20, -0.50), (0.30, -0.30), (0.70, -0.40)\}$	$\{(0.10, -0.20), (0.50, -0.40), (0.70, -0.30)\}$
	0.40	0.367
P_2P_3	P_5P_7	P_3P_4
$\{(0.20, -0.20), (0.30, -0.10), (0.40, -0.40), (0.50, -0.50)\}$	$\{(0.45, -0.80)\}$	$\{(0.10, -0.50), (0.50, -0.90)\}$
0.365	0.625	0.5
	P_4P_7	P_5P_4
	$\{(0.10, -0.30), (0.20, -0.50), (0.30, -0.20)\}$	$\{(0.10, -0.10), (0.40, -0.20), (0.60, -0.30)\}$
	0.267	0.283
	P_5P_2	P_6P_3
	$\{(0.40, -0.40), (0.50, -0.20)\}$	$\{(0.20, -0.10), (0.30, -0.20), (0.50, -0.30), (0.70, -0.10)\}$
	0.375	0.30
P_6P_4	P_6P_7	P_6P_2
$\{(0.10, -0.10), (0.20, -0.50)\}$	$\{(0.20, -0.20), (0.60, -0.30), (0.70, -0.10)\}$	$\{(0.20, -0.30), (0.50, -0.20)\}$
0.225	0.35	0.30
	P_7P_5	
	$\{(0.30, -0.20), (0.40, -0.30), (0.50, -0.10), (0.80, -0.10)\}$	
	0.337	

it is clear that the most dominating person in the meeting is P_2 .

5 Comparative analysis

In this section, we evaluate the importance of HFG and BVHFG in decision making methods.

Following Karaaslan (2019), the hesitant fuzzy graph (HFG) $\tilde{G} = (\mathcal{A}, \mathcal{G})$ of the graph $G = (V, E)$ is characterized by membership functions $\mathcal{A} : V \rightarrow P([0, 1])$ and $\mathcal{G} : \tilde{V}^2 \rightarrow P([0, 1])$ with the condition

$$\delta(\mathcal{G}(xy)) \leq \delta(\mathcal{A}(x)) \wedge \delta(\mathcal{A}(y)) \quad \forall xy \in \tilde{V}^2$$

$$\text{and } \delta(\mathcal{G}(xy)) = 0 \quad \forall xy \in (\tilde{V}^2 - E).$$

Fig. 3 Directed-BVHFG

where δ is the score valued function given by Xia and Xu (2011) such that $\delta(\mathcal{A}(x)) = \frac{1}{l(\mathcal{A}(x))} \sum_{\gamma \in \mathcal{A}(x)} \gamma$, here $l(\mathcal{A}(x))$ represent the number of elements in $\mathcal{A}(x)$.

In HFG, all the informations is expressed with only positive membership degrees, which represents the satisfaction degree of an element corresponding to the set, and it ignores the degree of satisfaction of the element for some implicit counter property of the set. However, BVHFG simultaneously considers the positive and negative membership satisfaction degree, We will use the following tables for comparative study between BVHFG and HFG. In HFG, hesitant fuzzy table is composed only of the persons satisfaction degree corresponding to the set. Thus, hesitant fuzzy table of potential power of persons will only include the positive satisfaction degrees as in Table 7:

Table 4 Out-degree and in-degree of each person

$O_d(P_1) = \{(0.79, -0.832), (0.8, -0.72), (0.832, -0.79)\}$
$O_d(P_2) = \{(0.865, -0.664), (0.832, -0.784)\}$
$O_d(P_3) = \{(0.45, -0.8), (0.55, -0.95)\}$
$O_d(P_4) = \{(0.496, -0.72)\}$
$O_d(P_5) = \{(0.784, -0.496), (0.7, -0.52)\}$
$O_d(P_6) = \{(0.916, -0.546), (0.28, -0.55), (0.904, -0.496), (0.6, -0.44)\}$
$O_d(P_7) = \{(0.958, -0.546)\}$
$I_d(P_1) = \{(0.865, -0.664)\}$
$I_d(P_2) = \{(0.7, -0.52), (0.6, -0.44)\}$
$I_d(P_3) = \{(0.79, -0.832), (0.832, -0.784), (0.916, -0.546)\}$
$I_d(P_4) = \{(0.55, -0.95), (0.784, -0.496), (0.28, -0.55)\}$
$I_d(P_5) = \{(0.958, -0.664)\}$
$I_d(P_6) = \{(0.8, -0.72)\}$
$I_d(P_7) = \{(0.832, -0.79), (0.45, -0.8), (0.496, -0.72), (0.904, -0.496)\}$

Based on the potential power of individuals, we establish the hesitant degree of impact power of one person to another person; here, the directed edge $P_i P_j$ represents impact power of P_i on P_j as in Table 8

Following Karaaslan (2019), we calculate the out-degree and in-degree of each person, by using the equations

$$O_d(v_k) = \{(1 - \prod_{i=1}^n (1 - a_{v_k, v_i})) | v_k v_i \in B\},$$

$$I_d(v_k) = \{(1 - \prod_{i=1}^n (1 - a_{v_i, v_k})) | v_i v_k \in B\},$$

which is given in Table 9.

Score value of in-degree and out-degree of each person is given in Table 10

Now we evaluate the domination degree of each person as in Table 11.

Through HFG domination degree of each person in this meeting is ranked as follows:

$$P_7 > P_2 > P_4 > P_1 > P_6 > P_5 > P_3,$$

clearly the person P_7 is the most dominating person in the meeting.

When the results of HFG and BVHFG are examined, we realize that the domination degree and ranking of dominating persons change significantly in two cases. In the BVHFG, P_2 is the most dominating person in the meeting, while in HFG the person P_7 is the most dominating person in the meeting. Furthermore, when we examine the ranking of persons in two cases, significant difference between two results is observed.

The main reason for this difference is the capability of BVHFG, and it is simultaneously considering the positive and negative membership degree with no restriction while the HFG considering only positive membership values.

Table 5 Score value of in-degree and out-degree

$$\begin{aligned}
S(O_d(P_1)) &= \frac{1}{3} \times \left\{ \frac{0.79 - (-0.832)}{2} + \frac{0.8 - (-0.72)}{2} + \frac{0.832 - (-0.79)}{2} \right\} = \frac{1}{3} \times 2.382 = 0.794 \\
S(I_d(P_1)) &= 1 \times \left\{ \frac{0.865 - (-0.664)}{2} \right\} = 1 \times 0.764 = 0.764 \\
S(O_d(P_2)) &= \frac{1}{2} \times \left\{ \frac{0.865 - (-0.664)}{2} + \frac{0.832 - (-0.784)}{2} \right\} = \frac{1}{2} \times 1.572 = 0.786 \\
S(I_d(P_2)) &= \frac{1}{2} \times \left\{ \frac{0.7 - (-0.52)}{2} + \frac{0.6 - (-0.44)}{2} \right\} = \frac{1}{2} \times 1.13 = 0.566 \\
S(O_d(P_3)) &= \frac{1}{2} \times \left\{ \frac{0.45 - (-0.8)}{2} + \frac{0.55 - (-0.95)}{2} \right\} = \frac{1}{2} \times 1.375 = 0.687 \\
S(I_d(P_3)) &= \frac{1}{3} \times \left\{ \frac{0.79 - (-0.832)}{2} + \frac{0.832 - (-0.784)}{2} + \frac{0.916 - (-0.546)}{2} \right\} = \frac{1}{3} \times 2.349 = 0.783 \\
S(O_d(P_4)) &= 1 \times \left\{ \frac{0.496 - (-0.72)}{2} \right\} = 1 \times 0.608 = 0.608 \\
S(I_d(P_4)) &= \frac{1}{3} \times \left\{ \frac{0.55 - (-0.95)}{2} + \frac{0.784 - (-0.496)}{2} + \frac{0.28 - (-0.55)}{2} \right\} = \frac{1}{3} \times 1.806 = 0.602 \\
S(O_d(P_5)) &= \frac{1}{2} \times \left\{ \frac{0.784 - (-0.496)}{2} + \frac{0.7 - (-0.52)}{2} \right\} = \frac{1}{2} \times 1.25 = 0.625 \\
S(I_d(P_5)) &= 1 \times \left\{ \frac{0.958 - (-0.664)}{2} \right\} = 1 \times 0.811 = 0.811 \\
S(O_d(P_6)) &= \frac{1}{4} \times \left\{ \frac{0.916 - (-0.546)}{2} + \frac{0.28 - (-0.55)}{2} + \frac{0.904 - (-0.496)}{2} + \frac{0.6 - (-0.44)}{2} \right\} = \frac{1}{4} \times 2.364 = 0.591 \\
S(I_d(P_6)) &= 1 \times \left\{ \frac{0.8 - (-0.72)}{2} \right\} = 1 \times 0.76 = 0.76 \\
S(O_d(P_7)) &= 1 \times \left\{ \frac{0.958 - (-0.546)}{2} \right\} = 1 \times 0.752 = 0.752 \\
S(I_d(P_7)) &= \frac{1}{4} \times \left\{ \frac{0.832 - (-0.79)}{2} + \frac{0.45 - (-0.8)}{2} + \frac{0.496 - (-0.72)}{2} + \frac{0.904 - (-0.496)}{2} \right\} = \frac{1}{4} \times 2.744 = 0.686
\end{aligned}$$

Table 6 Dominant degree of each person

Persons	$I_d - O_o$	Dominant degree
$\omega(P_1)$	0.794–0.764	0.030
$\omega(P_2)$	0.786–0.566	0.220
$\omega(P_3)$	0.687–0.783	-0.096
$\omega(P_4)$	0.608–0.602	0.006
$\omega(P_5)$	0.625–0.811	-0.186
$\omega(P_6)$	0.591–0.760	-0.169
$\omega(P_7)$	0.752–0.686	0.066

Table 7 Hesitant fuzzy membership of each person

Person	Hesitant fuzzy membership	Score value
P_1	{0.4, 0.5, 0.9}	0.6
P_2	{0.3, 0.6}	0.45
P_3	{0.1, 0.5, 0.6, 0.7}	0.457
P_4	{0.35, 0.40}	0.357
P_5	{0.45, 0.8, 0.75}	0.667
P_6	{0.6, 0.9}	0.75
P_7	{0.2, 0.55, 0.8}	0.517

6 Conclusion

In this paper, we have introduced the concept of BVHFG, some operations and propositions related to BVHFGs, such as Cartesian product, strong product, union, homomorphism, and isomorphism between BVHFGs. We have also investigated the concept of directed-BVHFG, which is a more helpful tool to express the various decision making problems. Furthermore, we have proposed a problem to find most dominating person in a meeting through directed-BVHFG. In the process of solving the problem, we construct a way to find the in-degree and out-degree of vertices, and then, we find out the score value of in-degree and out-degree of vertices, and then we calculate domination degree of persons.

In the final section, we point to comparative analyses between BVHFG and HFG. When we compare the outcome of hesitant fuzzy graph and bipolar-valued hesitant fuzzy graph, we notice the valuable difference between these two results. From the numerical illustrations, we conclude that bipolar-valued hesitant fuzzy graph provides more accurate results.

Table 8 Influence power of one person onto another person

Edge	Hesitant fuzzy membership	Score value
P_1P_3	{0.3, 0.4, 0.5}	0.4
P_1P_6	{0.5, 0.6}	0.55
P_1P_7	{0.2, 0.3, 0.7}	0.4
P_2P_1	{0.1, 0.5, 0.7}	0.433
P_2P_3	{0.2, 0.3, 0.4, 0.5}	0.350
P_3P_4	{0.45}	0.450
P_3P_7	{0.1, 0.5}	0.300
P_4P_7	{0.1, 0.2, 0.3}	0.200
P_5P_4	{0.1, 0.4, 0.6}	0.367
P_5P_2	{0.4, 0.5}	0.450
P_6P_3	{0.2, 0.3, 0.5, 0.7}	0.425
P_6P_4	{0.1, 0.2}	0.15
P_6P_2	{0.2, 0.6, 0.7}	0.500
P_6P_7	{0.2, 0.5}	0.350
P_7P_5	{0.3, 0.4, 0.5, 0.8}	0.500

Table 9 Out-degree and in-degree of each person

$O_d(P_1) = \{0.79, 0.8, 0.832\}$	$I_d(P_1) = \{0.865\}$
$O_d(P_2) = \{0.865, 0.832\}$	$I_d(P_2) = \{0.7, 0.6\}$
$O_d(P_3) = \{0.45, 0.55\}$	$I_d(P_3) = \{0.79, 0.832, 0.916\}$
$O_d(P_4) = \{0.496\}$	$I_d(P_4) = \{0.55, 0.784, 0.28\}$
$O_d(P_5) = \{0.784, 0.7\}$	$I_d(P_5) = \{0.958\}$
$O_d(P_6) = \{0.916, 0.28, 0.904, 0.6\}$	$I_d(P_6) = \{0.8\}$
$O_d(P_7) = \{0.958\}$	$I_d(P_7) = \{0.832, 0.45, 0.496, 0.904\}$

Table 10 Score value of out-degree and in-degree

$\delta(O_d(P_1)) = 0.807$	$\delta(I_d(P_1)) = 0.865$
$\delta(O_d(P_2)) = 0.849$	$\delta(I_d(P_2)) = 0.650$
$\delta(O_d(P_3)) = 0.500$	$\delta(I_d(P_3)) = 0.846$
$\delta(O_d(P_4)) = 0.496$	$\delta(I_d(P_4)) = 0.538$
$\delta(O_d(P_5)) = 0.742$	$\delta(I_d(P_5)) = 0.958$
$\delta(O_d(P_6)) = 0.675$	$\delta(I_d(P_6)) = 0.800$
$\delta(O_d(P_7)) = 0.958$	$\delta(I_d(P_7)) = 0.671$

Table 11 Dominant degree of each person

$\omega(P_1)$	0.807–0.865	-0.058
$\omega(P_2)$	0.849–0.650	0.199
$\omega(P_3)$	0.500–0.846	-0.034
$\omega(P_4)$	0.807–0.865	-0.042
$\omega(P_5)$	0.807–0.865	-0.216
$\omega(P_6)$	0.807–0.865	-0.125
$\omega(P_7)$	0.807–0.865	0.287

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