Analytic solution of in-plane vortex–rotor interactions with arbitrary orientation and its impact on rotor trim

Berend G. van der Wall¹ · Lennert B. van der Wall²

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Abstract

The general aerodynamic problem of arbitrarily oriented in-plane vortex-rotor interaction was investigated in the past only by numerical simulation. Just one special case of in-plane vortex-rotor interaction with the vortex axis in flight direction was recently solved analytically. In this article, the analytical solution for arbitrary in-plane vortex orientation and position relative to the rotor is given for the first time. The solution of the integrals involved as derived here encompasses and simplifies the previous derivation of the special case significantly. Results provide the vortex impact on rotor trim (thrust, aerodynamic rolling and pitching moments about the hub) and the rotor controls required to mitigate these disturbances. For the special case with the vortex axis in flight direction, the results are identical to the former solution and results for the other in-plane vortex orientations and positions agree with the numerical results obtained so far.

Keywords Interactional aerodynamics · Vortex–rotor Interaction · Rotor aerodynamics · Rotor trim · Aerodynamic disturbance rejection

Abbreviations

$a_i(r), b_i(r)$	Fourier coefficients of blade element lift caused by rotor controls, $i = 0, 1, 2, \dots$
$A_k(r)$	Fourier coefficients of the vortex kernel function $K(r, \Psi)$, $k = 0, 1, 2, \dots$
A, B	Non-dimensional effective begin and end of rotor blade, referenced to R
c	Airfoil chord, m
$C_{l\alpha}$	Lift curve slope, $C_{l\alpha} = 2\pi$
d_i	Radial integral coefficients, $d_i = (B^i - A^i)/i$, $i = 1, 2, 3, \dots$
dr	Rotor blade element width, m
$K(r, \Psi)$	Kernel function of vortex-induced velocities
L'	Blade element lift per unit span, N/m
M_x, M_y	Aerodynamic rotor rolling and pitching moment about the hub center
N_b	Number of rotor blades

r	Radial blade coordinate, m
r_a	Root cutout of the rotor blade, m
r_c	Vortex core radius, m
R	Rotor radius, m
T	Rotor thrust, N
v_{i0}	Induced velocity due to rotor thrust, m/s
v_{iV}	Vortex induced velocity, m/s
V_T, V_P	Velocities acting at the blade element tangential and perpendicular to the rotor disk, m/s
V_∞	Helicopter flight speed, m/s
x, y, z	Rotor coordinate system, x pos. downstream, y pos. starboard, z pos. up
x_0, y_0	Vortex closest point relative to the hub center, m
x_V, y_V, z_V	Vortex coordinate system
y_{V0}	Vortex distance relative to the rotor center, m
α	Blade element angle of attack, rad
α_S	Rotor disk angle of attack, rad
Γ_V	Vortex circulation strength, m ² /s
Δ	Perturbation of a variable
η	Geometric constant, $\eta = 2y_{V0}r_c$
Θ, Θ_0	Rotor blade pitch angle, collective control angle, rad
Θ_C, Θ_S	Rotor blade lateral and longitudinal cyclic control angles, rad
Θ_t	Linear rotor blade pre-twist angle, rad/R

✉ Berend G. van der Wall
 berend.vanderwall@dlr.de

Lennert B. van der Wall
 lvdw172@gmail.com

¹ German Aerospace Center (DLR), Institute of Flight Systems, Lilienthalplatz 7, 38108 Braunschweig, Germany

² Kleiberweg 4, 38110 Braunschweig, Germany

ϑ	Pitch angle per vortex strength, $\vartheta = \Theta/\lambda_{V0}$
λ_{i0}	Thrust-induced inflow ratio normal to the rotor disk, $\lambda_{i0} = v_{i0}/(\Omega R)$
λ_{iV}	Vortex-induced inflow ratio normal to the rotor disk, $\lambda_{iV} = v_{iV}/(\Omega R)$
λ_{V0}	Non-dimensional vortex strength, $\lambda_{V0} = \Gamma_V/(2\pi\Omega R^2)$
μ	Rotor advance ratio, $\mu = V_\infty \cos\alpha_S/(\Omega R)$
μ_z	Rotor axial inflow ratio, $\mu_z = -V_\infty \sin\alpha_S/(\Omega R)$
ξ	Transform of radial variable, $\xi = r^2 - y_{V0}^2 + r_c^2$
ρ	Air density, kg/m ³
σ	Rotor solidity, $\sigma = N_b c R/(\pi R^2)$
ϕ	Inflow angle, rad, $\phi = \arctan V_p/V_T$
ψ	Rotor blade azimuth, rad, $\psi = \Omega t$
ψ_V	Vortex orientation angle relative to the rotor x -axis, rad
Ψ	Transform of rotor blade azimuth, $\Psi = \psi - \psi_V - \pi/2$
Ω	Rotor rotational speed, rad/s

1 Introduction

Vortex-rotor aerodynamic interaction is a phenomenon that was more intensely investigated by flight testing and by numerical simulation from the mid-1970s to the end of the 1980s with respect to flight mechanics response of helicopters encountering the wake of large and heavy fixed-wing aircraft [1–5]. The subject was taken up again from 2000 on for handling qualities aspects [6–8]. All these were based on numerical simulation with different degrees of simplifications. Computational fluid dynamics were also applied to the fundamental problem addressing the mutual vortex-wake interaction [9]. Recently, a GARTEUR action group (HC-AG23) was investigating aspects of wind turbine blade tip vortices and their impact on helicopter operations in offshore wind farms [10].

The first entirely analytical solution based on blade element theory and steady aerodynamics was given 2017 for the special case of a vortex parallel to the x -axis of the rotor [11, 12]. Despite this being a case of the highest practical relevance—for example when helicopters perform air-refueling with a steady flight behind a tanker aircraft such as sketched in Fig. 1—the general solution of in-plane vortex-rotor interaction with arbitrary position and orientation (including parallelism to the y -axis of the rotor) was still missing and left to numerical simulation. The analytical solution of this general problem is given here for the first time.

Oblique in-plane vortex-rotor interaction occurs when crossing a fixed-wing aircraft wake from the side or when flying in the wake of large wind turbines towards the turbine or away from it in wind direction. Although this kind of interaction will never be stationary and rather a transient

Fig. 1 Helicopter in the wake of a large fixed-wing aircraft

process, the solution of the stationary interaction problem provides insight into the physics. This was solved only numerically in the past [13–15].

A clear differentiation must be made between the “classical” blade-vortex interaction (BVI), where a blade tip vortex generated by any of the rotor blades interacts with any rotor blade at any angle of interaction essentially in-plane of the rotor disk. These vortices have a core radius in the order of 15% of the blade chord, which typically is about 7% rotor radius, thus the vortex core radius is in the order of 1% of the rotor radius. In this article, we focus on blade tip vortices generated by, e.g., large fixed-wing aircraft, whose wing tip chord is about ten times larger than that of the helicopter rotor blade, such that the wing vortex core radius is in the order of 10% of the rotor radius, as used in the results section. While the classical BVI phenomenon requires fully unsteady aerodynamics treatment, the large wave length of the fixed-wing vortex interaction with the rotor blades may still be treated by quasi-steady blade element momentum theory.

2 Problem definition

Consider a rotor with N_b blades of constant chord c and an airfoiled part of it extending from the inner root cutout $r_a \geq 0$ to its radius R in steady forward flight with speed $V_\infty \geq 0$ as shown in Fig. 2a, where a vortex is placed whose closest point with respect to the rotor hub is at $(x_0, y_0) \in \mathbb{R}$ and which has an orientation angle $\psi_V \subseteq [-\pi, \pi]$ relative to the rotor x -axis. Helicopter trim in forward flight requires the rotor slightly tilted forward by the shaft angle of attack $\alpha_S < 0$, depending on flight speed.

Blade element theory computes the blade section lift based on the flow components in-plane and normal to the radial axis of the blade, V_T , and perpendicular to the rotor disk, V_p . The rotor blade rotates in counter-clockwise direction as seen from above at a rotational speed of Ω and its azimuth angle is defined by $\psi = \Omega t$ with t as time in seconds. The tangential velocity consists of the rotational speed at the respective radial station Ωr and the periodic contribution of the speed of flight $V_\infty \cos\alpha_S \sin\psi$, while the

Fig. 2 Rotor disk with in-plane vortex

perpendicular component (positive downwards) includes a constant contribution caused by flight speed and rotor inclination $-V_\infty \sin \alpha_S$, the induced velocity v_{i0} caused by the rotor thrust (also assumed as constant), and the induced velocity caused by the vortex v_{iV} , which is a nonlinear function of both rotor radial coordinate and azimuth position:

$$\begin{aligned} V_T(r, \psi) &= \Omega r + V_\infty \cos \alpha_S \sin \psi \\ V_P(r, \psi) &= -V_\infty \sin \alpha_S + v_{i0} + v_{iV}(r, \psi). \end{aligned} \quad (1)$$

As shown in Fig. 2b, these components generate an inflow angle ϕ that has to be considered for computation of the aerodynamic angle of attack $\alpha = \Theta - \phi$ at the blade element having a local pitch angle Θ relative to the rotor disk, which consists of a linear built-in pre-twist relative to 75% radius $\Theta_t(r - 0.75R)$ and the pilot collective, lateral and longitudinal cyclic control angles $\Theta_0, \Theta_C, \Theta_S$. Applying small angle assumption and with $\phi = \arctan(V_P/V_T) \approx V_P/V_T$ from Fig. 2b the section angle of attack becomes

$$\alpha(r, \psi) = \Theta_0 + \Theta_C \cos \psi + \Theta_S \sin \psi + \Theta_t(r - 0.75R) - \frac{-V_\infty \sin \alpha_S + v_{i0} + v_{iV}(r, \psi)}{\Omega r + V_\infty \cos \alpha_S \sin \psi}. \quad (2)$$

The vortex-induced velocity field can be represented by its circulation strength $\Gamma_V \in \mathbb{R}$ (the sign denotes its sense of rotation), its core radius $r_c \in \mathbb{R}^+$ and the location normal to the vortex axis $y_V \in \mathbb{R}$ in the following manner, where x, y are the coordinates of the blade element of interest, see Fig. 2a. This is a special case of the “Vatistas” vortex swirl velocity model [16] frequently used in the literature.

$$\begin{aligned} v_{iV}(r, \psi) &= -\frac{\Gamma_V}{2\pi} \frac{y_V(r, \psi)}{y_V^2(r, \psi) + r_c^2} \\ y_V(r, \psi) &= (y - y_0) \cos \psi_V - (x - x_0) \sin \psi_V. \\ x &= r \cos \psi \quad y = r \sin \psi \end{aligned} \quad (3)$$

Its maximum induced velocity is obtained at the core radius itself for $y_V = \pm r_c$ with $v_{iV, \max} = \mp \Gamma_V / (4\pi r_c)$ and its velocity profile is sketched in Fig. 3 (note that in [11] the vortex sense of rotation was assumed opposite, which can be represented here by the sign of the circulation Γ_V).

By means of the following substitution, the distance of the blade element can be expressed in terms of its radial position and a transform of the rotor azimuth into an azimuth relative to the vortex y_V -direction Ψ , which is zero when the rotor blade radial direction is oriented normal to the vortex axis. Using the abbreviations $C_V = \cos \psi_V$; $S_V = \sin \psi_V$:

$$\begin{aligned} y_V(r, \psi) &= (r \sin \psi - y_0) \cos \psi_V - (r \cos \psi - x_0) \sin \psi_V \\ &= r \sin \psi \cos \psi_V - r \cos \psi \sin \psi_V + x_0 \sin \psi_V - y_0 \cos \psi_V \\ &= r \sin(\psi - \psi_V - \pi/2 + \pi/2) - y_{V0} \quad \Psi := \psi - \psi_V - \pi/2 \\ &= r \cos \Psi - y_{V0} = y_V(r, \Psi) \quad \text{where } y_{V0} = y_0 C_V - x_0 S_V. \end{aligned} \quad (4)$$

$y_{V0} \in \mathbb{R}$ represents the vortex position relative to the rotor hub center in a coordinate system parallel to the vortex coordinates, i.e., rotated by ψ_V . The sectional lift per unit span L' is computed in blade element theory with the dynamic pressure based on air density ρ and the tangential velocity V_T , the chord length c and the lift curve slope $C_{l\alpha}$ (which is assumed constant), and the angle of attack α . Stall, compressibility and other nonlinearities are neglected here. Based on the local lift, the steady values of rotor thrust T and of the

Fig. 3 Vortex-induced velocity profile

aerodynamic rolling and pitching moments M_x, M_y can be computed by dual integration over the azimuth and the radial coordinate and consideration of the number of blades N_b .

$$L'(r, \psi) = \frac{\rho}{2} V_T^2(r, \psi) c C_{la} \alpha(r, \psi),$$

$$\begin{Bmatrix} T \\ M_x \\ M_y \end{Bmatrix} = N_b \int_{r_a}^R \frac{1}{2\pi} \int_0^{2\pi} L'(r, \psi) \begin{Bmatrix} 1 \\ r \sin \psi \\ -r \cos \psi \end{Bmatrix} d\psi dr. \quad (5)$$

The next step is to introduce dimensionless expressions. All coordinates and lengths are divided by the rotor radius R , all velocities by the rotor tip speed ΩR , lift per unit span by $\rho(\Omega R)^2 \pi R$, thrust by $\rho(\Omega R)^2 \pi R^2$ and the moments by $\rho(\Omega R)^2 \pi R^2 R$. Regarding the velocities this generates the advance ratio μ , the axial inflow ratio μ_z , the induced inflow ratio caused by the rotor thrust λ_{i0} , the vortex-induced inflow ratio λ_{iV} , and the non-dimensional vortex strength λ_{v0} .

$$\mu = \frac{V_\infty \cos \alpha_s}{\Omega R} \quad \mu_z = -\frac{V_\infty \sin \alpha_s}{\Omega R} \quad \lambda_{i0} = \frac{v_{i0}}{\Omega R} \quad \lambda_{v0} = \frac{\Gamma_v}{2\pi \Omega R^2} \quad (6)$$

$$V_T = r + \mu \sin \psi \quad V_P = \mu_z + \lambda_{i0} + \lambda_{iV} \quad \lambda_{iV} = -\lambda_{v0} \frac{y_V(r, \Psi)}{y_V^2(r, \Psi) + r_c^2} = -\lambda_{v0} K(r, \Psi).$$

For convenience, the following symbols $x, y, y_V, r, dr, r_c, V_T, V_P, L', T, M_x, M_y$ are kept the same, but from now on are the dimensionless form. The expression of the dimensionless lift from Eq. (5) consists of two independent contributions: one from rotor control and the general operating condition L'_0 , and the other from the vortex-induced velocities $\Delta L'_V$. Inserting Eq. (2), Eq. (6) and introducing the rotor solidity $\sigma = N_b c R / (\pi R^2)$ as the ratio of total blade area to rotor disk area these expressions are:

$$L'(r, \psi) = \frac{\sigma C_{la}}{2N_b} [L'_0(r, \psi) + \Delta L'_V(r, \psi)]$$

$$L'_0(r, \psi) = [\Theta_0 + \Theta_C \cos \psi + \Theta_S \sin \psi + \Theta_I(r - 0.75)]$$

$$(r + \mu \sin \psi)^2 - (\mu_z + \lambda_{i0})(r + \mu \sin \psi)$$

$$\Delta L'_0(r, \psi) = [\Delta \Theta_0 + \Delta \Theta_C \cos \psi + \Delta \Theta_S \sin \psi](r + \mu \sin \psi)^2$$

$$\Delta L'_V(r, \psi) = \lambda_{v0} K(r, \Psi)(r + \mu \sin \psi). \quad (7)$$

The subject of interest is which rotor controls would be required to eliminate the vortex influence on rotor trim, i.e., $\Delta T = \Delta M_x = \Delta M_y = 0$. Adding additional pilot controls

$\Delta \Theta_0, \Delta \Theta_C, \Delta \Theta_S$, we only need to look at the incremental lift $\Delta L'_0$ as also given in Eq. (7) caused by these. Then the perturbation part of Eq. (5) becomes:

$$\frac{2}{\sigma C_{la}} \begin{Bmatrix} \Delta T \\ \Delta M_x \\ \Delta M_y \end{Bmatrix} \stackrel{!}{=} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \int_A^B \left(\frac{1}{2\pi} \int_0^{2\pi} \Delta L'_0(r, \psi) \begin{Bmatrix} 1 \\ r \sin \psi \\ -r \cos \psi \end{Bmatrix} d\psi \right) dr$$

$$\stackrel{!}{=} - \int_A^B \left(\frac{1}{2\pi} \int_0^{2\pi} \Delta L'_V(r, \psi) \begin{Bmatrix} 1 \\ r \sin \psi \\ -r \cos \psi \end{Bmatrix} d\psi \right) dr. \quad (8)$$

Therein the radial integration may be performed from $A \geq r_a \geq 0$ to $B \leq 1$ to account for tip losses at both ends

of the blade and the constant at the left can be omitted entirely because $\sigma C_{la}/2 \neq 0$. The trivial case is vanishing vortex strength $\lambda_{v0} = 0$ with $\Delta L'_V = 0$ and consequently also requires $\Delta L'_0 = 0$ which is the case for $\Delta \Theta_0 = \Delta \Theta_C = \Delta \Theta_S = 0$. The contributions of the additional pilot controls and of the vortex-induced velocities to thrust, rolling and pitching moment increments can be solved separately due to their linear superposition.

3 Solution of the integrals

3.1 Contribution of control angles

$\Delta L'_0$ can easily be represented as a Fourier series in terms of rotor azimuth and then the integrals can be solved most conveniently. Since the vortex-induced contribution includes the non-dimensional vortex strength λ_{v0} as constant multiplier, it is convenient to define the rotor controls as divided by it: $\Delta \vartheta = \Delta \Theta / \lambda_{v0}$ and Eq. (7) provides

$$\begin{aligned}
\Delta L'_0(r, \psi) &= (\Delta \vartheta_0 + \Delta \vartheta_C \cos \psi + \Delta \vartheta_S \sin \psi) \\
&\quad \left[r^2 + 2r\mu \sin \psi + \frac{\mu^2}{2}(1 - \cos 2\psi) \right] \\
&= \Delta \vartheta_0 \left(r^2 + \frac{\mu^2}{2} \right) + \Delta \vartheta_S r\mu + \\
&\quad \left[\Delta \vartheta_0 2r\mu + \Delta \vartheta_S \left(r^2 + \frac{\mu^2}{2} + \frac{\mu^2}{4} \right) \right] \sin \psi \\
&\quad + \Delta \vartheta_C \left(r^2 + \frac{\mu^2}{2} - \frac{\mu^2}{4} \right) \cos \psi \\
&\quad - \left(\Delta \vartheta_0 \frac{\mu^2}{2} + \Delta \vartheta_S r\mu \right) \cos 2\psi + \Delta \vartheta_C r\mu \sin 2\psi \\
&\quad - \Delta \vartheta_C \frac{\mu^2}{4} \cos 3\psi - \Delta \vartheta_S \frac{\mu^2}{4} \sin 3\psi \\
&= \frac{a_0(r)}{2} + \sum_{i=1}^3 [a_i(r) \cos i\psi + b_i(r) \sin i\psi],
\end{aligned} \tag{9}$$

with the Fourier coefficients $a_i(r), b_i(r)$ of which only the mean and the first harmonic ($i = 0, 1$) are needed for steady thrust and hub aerodynamic moments.

$$\begin{aligned}
a_0(r) &= (2r^2 + \mu^2)\Delta \vartheta_0 + 2r\mu\Delta \vartheta_S \\
a_1(r) &= \left(r^2 + \frac{\mu^2}{4} \right) \Delta \vartheta_C \quad b_1(r) = 2r\mu\Delta \vartheta_0 + \left(r^2 + \frac{3\mu^2}{4} \right) \Delta \vartheta_S.
\end{aligned} \tag{10}$$

Then the left part of Eq. (8) is easily solved:

$$\begin{aligned}
\begin{Bmatrix} \Delta T(\Delta L'_0) \\ \Delta M_x(\Delta L'_0) \\ \Delta M_y(\Delta L'_0) \end{Bmatrix} &= \int_A^B \left(\frac{1}{2\pi} \int_0^{2\pi} \Delta L'_0(r, \psi) \begin{Bmatrix} 1 \\ r \sin \psi \\ -r \cos \psi \end{Bmatrix} d\psi \right) dr \\
&= \frac{1}{2} \int_A^B \begin{Bmatrix} a_0(r) \\ rb_1(r) \\ -ra_1(r) \end{Bmatrix} dr \\
&= \begin{bmatrix} d_3 + \frac{\mu^2}{2}d_1 & \mu d_2 & 0 \\ \mu d_3 & \frac{d_4}{2} + \frac{3\mu^2}{8}d_2 & 0 \\ 0 & 0 & -\frac{d_4}{2} - \frac{\mu^2}{8}d_2 \end{bmatrix} \begin{Bmatrix} \Delta \vartheta_0 \\ \Delta \vartheta_S \\ \Delta \vartheta_C \end{Bmatrix}.
\end{aligned} \tag{11}$$

Therein $d_i = (B^i - A^i)/i, i = 1, 2, 3, 4$ are radial integration constants. In hover, where $\mu = 0$, thrust and hub moments are fully decoupled, but in forward flight due to the difference of dynamic pressure on the advancing and retreating sides, thrust and aerodynamic rolling moment are both influenced by the collective and longitudinal cyclic control angles.

3.2 Contribution of vortex-induced velocities

This part is the challenging one due to the nonlinearities of the vortex-induced velocities field involved. From Eqs. (6) and (7), the right part of Eq. (8) becomes (the transform of $\psi \rightarrow \Psi$ was given in Eq. (4), thus $\psi = \Psi + \psi_V + \pi/2$ and $\sin \psi = C_V \cos \Psi - S_V \sin \Psi$ and $\cos \psi = -S_V \cos \Psi - C_V \sin \Psi$):

$$\begin{aligned}
\Delta L'_V(r, \Psi) &= K(r, \Psi) [r + \mu (C_V \cos \Psi - S_V \sin \Psi)] \\
&\quad \begin{Bmatrix} \Delta T(\Delta L'_V) \\ \Delta M_x(\Delta L'_V) \\ \Delta M_y(\Delta L'_V) \end{Bmatrix} \\
&= \int_A^B \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta L'_V(r, \Psi) \begin{Bmatrix} 1 \\ r(C_V \cos \Psi - S_V \sin \Psi) \\ r(S_V \cos \Psi + C_V \sin \Psi) \end{Bmatrix} d\Psi \right) dr,
\end{aligned} \tag{12}$$

with $K(r, \Psi) = K(r, -\Psi)$ from Eq. (6) as a symmetric kernel function around $\Psi = 0$, because it only includes $\cos \Psi$ functions. The azimuthal integration bounds can be shifted arbitrarily as long as they span over one rotor revolution period of 2π , because the integrand always is periodic within this range. This kernel function may be expressed as a Fourier series with only Cosine terms due to its symmetry,

$$K(r, \Psi) = \frac{y_V(r, \Psi)}{y_V^2(r, \Psi) + r_c^2} = \frac{A_0(r)}{2} + \sum_{k=1}^{\infty} A_k(r) \cos k\Psi, \tag{13}$$

and from all coefficients $A_k(r)$ only those for $k = 0, 1, 2$ are needed in Eq. (12) for the computation of the steady part of rotor thrust and aerodynamic hub moments; their explicit form is given at the end of the Appendix in Eq. (51) and details of the derivation are given in the supplementary material. With the abbreviations $C_V = \cos \psi_V, S_V = \sin \psi_V$ as introduced in Eq. (4) and $C_{2V} = \cos(2\psi_V), S_{2V} = \sin(2\psi_V)$, Eq. (12) becomes

$$\begin{aligned}
\begin{Bmatrix} \Delta T(\Delta L'_V) \\ \Delta M_x(\Delta L'_V) \\ \Delta M_y(\Delta L'_V) \end{Bmatrix} &= \int_A^B \begin{Bmatrix} \frac{1}{2}rA_0(r) + \frac{\mu C_V}{2}A_1(r) \\ \frac{\mu}{4}rA_0(r) + \frac{C_V}{2}r^2A_1(r) + \frac{\mu}{4}C_{2V}rA_2(r) \\ \frac{S_V}{2}r^2A_1(r) + \frac{\mu}{4}S_{2V}rA_2(r) \end{Bmatrix} dr.
\end{aligned} \tag{14}$$

The solution is given in the Appendix with all details of the derivation, which includes a transform of the variable Ψ and a successive development into a Laurent series with partial fraction decomposition, and subsequent integration over the radial

coordinate. The derivation elaborated here and detailed in the Appendix is universal now as it covers the entire range of vortex orientation ψ_V , it is significantly simpler and shorter than its predecessor for only the special case $\psi_V = 0$ given before in [11], and only relies on real analysis. Major improvements include reduction from 4th order to 2nd order polynomials. Long division avoids usage of all but 0th Fourier coefficients. While contour integrals may shorten the derivation of Eq. (28) of the Appendix by about 10%, they do not lead to greater insight. With the following abbreviations

$$G(\sqrt{+}) = \ln \left| 1 + \frac{r_c}{\sqrt{+}} \right| + \frac{\ln \left| \sqrt{+}^2 + y_{V0}^2 \right|}{2}$$

$$\sqrt{\pm} = \sqrt{\frac{1}{2}(\sqrt{\xi^2 + \eta^2} \pm \xi)} \quad \xi = r^2 - y_{V0}^2 + r_c^2 \quad \eta = 2y_{V0}r_c, \quad (15)$$

the vortex-induced contribution to the steady thrust and aerodynamic hub moments in Eq. (12) is

$$\Delta T(\Delta L'_V) = \left[\mu C_V G(\sqrt{+}) + \text{sgn}(y_{V0}) \sqrt{-} \right]_A^B$$

$$\begin{Bmatrix} \Delta M_x(\Delta L'_V) \\ \Delta M_y(\Delta L'_V) \end{Bmatrix} = \begin{bmatrix} C_V & -S_V \\ S_V & C_V \end{bmatrix} \begin{Bmatrix} d_2 \\ 0 \end{Bmatrix} + \mu \left(y_{V0} G(\sqrt{+}) + r_c \arctan \frac{y_{V0}}{\sqrt{+}} \right) \begin{Bmatrix} C_V \\ S_V \end{Bmatrix}$$

$$+ \begin{bmatrix} |y_{V0}| & -r_c \\ -\mu S_V \text{sgn}(y_{V0}) & 0 \end{bmatrix} \begin{Bmatrix} \sqrt{-} \Big|_A^B \\ \sqrt{+} \Big|_A^B \end{Bmatrix}.$$

It is easily verified that for $\psi_V = 0$ (vortex parallel to the rotor x -axis) $\Delta M_y = 0$ and—only in hover when $\mu = 0$ —that for $\psi_V = \pi/2$ (vortex parallel to the rotor y -axis) also results in $\Delta M_x = 0$; while in forward flight always $\Delta M_x \neq 0$.

3.3 Computation of rotor controls

It remains to solve Eq. (8) for the rotor controls that are required to mitigate the vortex influence on rotor trim by means of the results obtained in Eq. (11) and those of Eq. (16):

$$\begin{Bmatrix} \Delta T(\Delta L'_0) \\ \Delta M_x(\Delta L'_0) \\ \Delta M_y(\Delta L'_0) \end{Bmatrix} = \begin{bmatrix} d_3 + \frac{\mu^2}{2} d_1 & \mu d_2 & 0 \\ \mu d_3 & \frac{d_4}{2} + \frac{3\mu^2}{8} d_2 & 0 \\ 0 & 0 & -\frac{d_4}{2} - \frac{\mu^2}{8} d_2 \end{bmatrix} \begin{Bmatrix} \Delta \vartheta_0 \\ \Delta \vartheta_S \\ \Delta \vartheta_C \end{Bmatrix} \stackrel{!}{=} - \begin{Bmatrix} \Delta T(\Delta L'_V) \\ \Delta M_x(\Delta L'_V) \\ \Delta M_y(\Delta L'_V) \end{Bmatrix} \quad (17)$$

equiv. to : $\underline{\underline{A}} \underline{\underline{\Delta \vartheta}} = \underline{\underline{\vec{F}}} \Rightarrow \underline{\underline{\Delta \vartheta}} = \underline{\underline{A}}^{-1} \underline{\underline{\vec{F}}}.$

It is seen that the collective and the longitudinal cyclic control angles are coupled with each other for $\mu > 0$, which is caused by the difference of the dynamic pressure between the advancing and the retreating side. The lateral control

angle is not coupled to any of these because the dynamic pressure in the rear or front position of the rotor blade is independent of the advance ratio. In hover, all three equations can be solved independently, because in this special case the dynamic pressure at a blade element is constant throughout the entire revolution. Eq. (17) is a linear algebraic equation system with the rotor control's system matrix $\underline{\underline{A}}$, the control vector $\underline{\underline{\Delta \vartheta}}$ and the vortex' disturbance vector $\underline{\underline{\vec{F}}}$. It may be solved by system matrix inversion as indicated above or, because the third equation always is independent of the others and can be solved separately, the first two can be evaluated manually and it results in

$$\begin{Bmatrix} \Delta \vartheta_0 \\ \Delta \vartheta_S \end{Bmatrix} = \frac{-1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{Bmatrix} \Delta T(\Delta L'_V) \\ \Delta M_x(\Delta L'_V) \end{Bmatrix}$$

$$\Delta \vartheta_C = \frac{-1}{a_{33}} \Delta M_y(\Delta L'_V). \quad (18)$$

Therein, a_{ij} are the system matrix elements, i denoting the line and j the column index. In hover $a_{12} = a_{21} = 0$.

4 Results

Essential parameters for results presentations are the vortex proximity relative to the hub center y_{V0} and its orientation ψ_V , and the vortex core radius r_c . From the rotor, its operational parameters advance ratio μ and disk angle of

attack α_s could be considered, but the latter usually remains small within $\alpha_s > -10$ deg and therefore it may be neglected because $\cos \alpha_s \approx 1$. Further the radial integration bounds A, B could be considered as parameters, but the majority of

existing rotor blades has a root cutout around $r_a = 0.22$ and adding root losses leads to $A \approx 0.25$. Also, most analysis approximate the aerodynamic blade tip loss caused by rotor blade tip vortices by an effective blade length of $B \approx 0.97$. Therefore, A, B are considered as constants and only the advance ratio remains, which may be investigated from hovering condition with $\mu = 0$ to forward flight at maximum speed with about $\mu = 0.4$ for modern series production helicopters. The errors due to reversed flow in the inner blade portion on the retreating side are considered small because of the rather small radial range exposed to them with a maximum extension of $\mu - A \approx 0.15$ at $\psi = 270$ deg and the very small dynamic pressure within the area covered by reversed flow.

To address representative cases of e.g. the air refueling of helicopters flying at short distance behind a large fixed-wing aircraft, vortex core radii encountered by the helicopter can be estimated to be in the range of $r_c \geq 0.1$ of the rotor radius, e.g., derived from [2–5] or taken from [6]. At a given circulation strength, the peak vortex-induced velocity is largest for the smallest core radius, see Eq. (3) and the text below. For large core radii, for example $r_c = 1$, the peak velocities become rather small and the induced gradients within the rotor disk flatten out, getting closer to a linear distribution across the rotor disk, especially when the vortex including its core radius is outside the rotor radius.

The following figures vary the vortex distance to the rotor center y_{V0} from two rotor radii outside on the one side to two radii outside on the other, $-2 \leq y_{V0} \leq +2$, and the vortex orientation with respect to the rotor x -axis varies from $-\pi \leq \psi_V \leq +\pi$, as illustrated in Fig. 4. In that figure, when $y_{V0} = \pm 1$, the vortex is the tangent to the rotor disk and when $y_{V0} = 0$ it is crossing the rotor center, while its orientation ψ_V remains constant. When $\psi_V = 0$ the vortex is always parallel to the rotor x -axis and $y_{V0} = y$, and for $\psi_V = \pm\pi$, it remains parallel to the rotor x -axis, but the sense of rotation as seen by the rotor blades becomes opposite and $y_{V0} = -y$. In the case of $\psi_V = \pi/2$, the vortex is always parallel to the y -axis of the rotor and $y_{V0} = -x$, and in analogy to the former for $\psi_V = -\pi/2$, it remains parallel to the rotor y -axis, but the sense of rotation as seen by the rotor blades becomes opposite and $y_{V0} = x$.

Results for the rotor controls required to mitigate the vortex impact on rotor trim are shown in Fig. 5 for the two extremes of the vortex parallel to the rotor x - and y -axis, i.e., $\psi_V = -\pi/2, 0, \pi/2, \pi$. Fixed parameters are $A = 0.25, B = 0.97, r_c = 0.1$ and the graphs shown are as in [11] for the purpose of comparability, i.e., here the sign of Γ_V is reversed for direct comparison. The solid line exactly represents the results shown in [11] as a special case for $\psi_V = 0$, all other $\psi_V \neq 0$ are analytically solved here for the first time. In hover, all control angles are uncoupled from each other as outlined before, see Eq. (17), and the collective

control $\Delta\Theta_0$ shown in Fig. 5a is only needed to compensate thrust increase or loss caused by the vortex influence. This is of course independent on the vortex orientation in hover due to the rotational symmetry of the dynamic pressure at the blades and only the vortex distance to the rotor center in whatever direction defines the amount and sign of collective control needed to eliminate the change of thrust caused by the vortex.

A position at $\psi_V = 0, y_{V0} = -1$ is tangent to the retreating side and with the mentioned reversed sense of rotation the rotor experiences the vortex downwash only, thus requires a positive collective control to keep thrust constant. With the vortex axis passing the rotor center, $y_{V0} = 0$, the downwash on the advancing side is as large as the upwash on the retreating side, the thrust is thus not changed and no collective control angle needed. When $y_{V0} = 1$ the vortex axis is tangent to the advancing side, the rotor immersed in the upwash side of the vortex and negative collective control required to eliminate the vortex-induced increase of thrust.

The longitudinal control angle $\Delta\Theta_S$ is shown in Fig. 5b; again, the solid line for $\psi_V = 0$ exactly represents the results shown in [11]. For this orientation, when $y_{V0} = -1$ the rotor experiences a large lateral gradient of vortex-induced velocities with maximum on the retreating side, thus a negative longitudinal control angle is needed to compensate this. A vortex passing the rotor center induces downwash on the advancing and upwash on the retreating side of the disk, thus a positive control angle $\Delta\Theta_S$ is needed to eliminate the vortex-induced aerodynamic moment, and when $y_{V0} = 1$ the vortex is tangent to the advancing side, a large lateral vortex-induced velocity gradient with maximum upwash on the advancing side, thus again a negative control angle required to keep the aerodynamic moment zero.

When the vortex is rotated by $\psi_V = \pm\pi/2$, i.e., it is parallel to the y -axis of the rotor (see Fig. 4b), it does only induce gradients in the rotor's x -direction, but not in its y -direction, and therefore no aerodynamic rolling moment needs to be compensated by longitudinal control inputs, thus $\Delta\Theta_S = 0$ in either case. For $\psi_V = \pi$ the vortex again is parallel to the rotor's x -axis, but its induced velocity field within the rotor is opposite compared to $\psi_V = 0$, therefore, the longitudinal cyclic control is opposite as well.

Results for the lateral cyclic control $\Delta\Theta_C$ are shown in Fig. 5c. This was not shown in [11], because the special case therein was for $\psi_V = 0$, which does only generate induced velocity gradients in the rotor y -direction, but not in x , therefore $\Delta\Theta_C = 0$, and of course as well for $\psi_V = \pi$. When $\psi_V = \pm\pi/2$, the vortex-induced velocity gradients are in the rotor's x -direction and now the lateral cyclic control acts to compensate its influence on the aerodynamic pitching moment in the same manner as the longitudinal control angles shown before.

Fig. 4 Variation of vortex position and orientation

Fig. 5 Rotor controls needed for vortex disturbance rejection in hovering flight ($\mu = 0, \Gamma_V < 0$)

Finally, because so far lateral vortex orientations were only computed by numerical simulation, e.g. [13–15], an example from [15] for $\psi_V = -\pi/2$ is given in Fig. 5d for the collective, longitudinal and lateral control angles. DLR's comprehensive rotor simulation program S4 has been applied to the Bo105 Mach-scaled model rotor with elastic blades and unsteady aerodynamics, for details see [15]. The same range of vortex position y_{V0} was covered for a vortex core radius of $r_c = 0.2$ (lines) and the most characteristic positions $y_{V0} = -1, 0, +1$ repeated with $r_c = 0.1$ (symbols). As can be seen in comparison with the analytical results shown in Fig. 5a–c with $r_c = 0.1$ the agreement of the fully analytic solution with the numerical solution is very good. A minor difference exists with respect to the longitudinal control angle $\Delta\Theta_S$, which is zero in the analytical solution but achieves small values in the numerical simulation. This can be traced back to the rigid blades assumed in the analytical solution and the elastic blades as used in the numerical simulation.

Both collective and longitudinal cyclic control are coupled in forward flight, and the case for an advance ratio of $\mu = 0.3$, representative for cruise flight conditions of today's helicopters, is shown next in Fig. 6, also for direct comparison purposes with the results shown in [11]. A, B, r_c are the same as before. Again, the solid lines in all three graphs are identical to those shown in [11] in the special case of $\psi_V = 0$, all other $\psi_V \neq 0$ are analytically solved here for the first time.

Now the dynamic pressure acting at the rotor blades is significantly differing between advancing and retreating side of the rotor, but not so between the front and rear region of it. Therefore, the lateral cyclic control angle $\Delta\Theta_C$ shown in Fig. 6c remains symmetric as in hover, see Fig. 5c, but with little reduced magnitude due to the term $d_2\mu^2/8$ in Eq. (17).

The asymmetry of dynamic pressure between the advancing and retreating side of the rotor disk results in larger force disturbances on the advancing side of the rotor. Any thrust compensation by the collective control also introduces an aerodynamic rolling moment that requires longitudinal

Fig. 6 Rotor controls needed for vortex disturbance rejection in cruise ($\mu = 0.3, \Gamma_V < 0$)

cyclic control for compensation, while any aerodynamic rolling moment compensation by the longitudinal cyclic control angle also introduces changes of the thrust which again requires some collective control angle for balancing it. Both control angles are thus coupled and cause asymmetries of the controls needed to compensate the vortex impact on rotor trim, as seen in Fig. 6a, b and as expressed by Eq. (18). Changing the vortex orientation from $\psi_V = 0$ (solid line) to $\psi_V = \pi$ (dotted line), the curves in both graphs are mirrored about both the vertical and horizontal axis, because the sense of vortex rotation as seen in the rotor disk appears opposite and because $y_{V0} = y$ for $\psi_V = 0$, but $y_{V0} = -y$ for $\psi_V = \pi$.

The analytic results for the longitudinal control angle in the case of $\psi_V = \pm\pi/2$ are shown here for the first time, so far they have been computed only numerically see [13–15]. An example from [15] is given in in Fig. 6d, again with lines for $r_c = 0.2$ and with symbols for $r_c = 0.1$. The agreement between the analytical solution and the nonlinear numerical simulation is again very good, including the longitudinal control angle. Some asymmetry is present in the numerical simulation result for the lateral control angle, compared to symmetry in the analytic result of Fig. 6c. This again can be traced down to the fully elastic blade formulation in the numerical simulation versus the rigid blade in the analytical simulation.

5 Conclusions

The problem of the in-plane vortex-rotor interaction of arbitrary vortex distance relative to the rotor center and arbitrary orientation with respect to the longitudinal axis is solved here analytically for the first time in closed form by using blade element theory. The solution derived is shorter than the former one obtained for the special case of a vortex parallel to the rotor x -axis ($\psi_V = 0$). The results provide the vortex impact on rotor trim (thrust, aerodynamic rolling and pitching moments about the hub) and the rotor controls required to mitigate these disturbances. The results agree with former analytic solution obtained for the special case and with numerical solutions for all other vortex positions and orientations.

Appendix

The following includes the entire derivation of the analytical solution of the integrals in Eq. (12). We want to solve the following two integrals containing the short-hands $c_\Psi = \cos\Psi$, $s_\Psi = \sin\Psi$; y_{V0} was given in Eq. (4).

$$\begin{aligned}\Delta L'_V(r, \Psi) &= K(r, \Psi) [r + \mu (C_V c_\Psi - S_V s_\Psi)] \\ K(r, \Psi) &= \frac{rc_\Psi - y_{V0}}{(rc_\Psi - y_{V0})^2 + r_c^2} \\ \begin{Bmatrix} \Delta T(\Delta L'_V) \\ \Delta M_x(\Delta L'_V) \\ \Delta M_y(\Delta L'_V) \end{Bmatrix} &= \int_A^B \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta L'_V(r, \Psi) \begin{Bmatrix} 1 \\ r(C_V c_\Psi - S_V s_\Psi) \\ r(S_V c_\Psi + C_V s_\Psi) \end{Bmatrix} d\Psi \right) dr.\end{aligned}\quad (19)$$

The constants within the integrals are subject to the following constraints:

$$r \in [A; B] \subseteq [0; 1], \quad r_c > 0, \quad y_{V0}, \Psi_V \in \mathbb{R}, \quad \mu \in \mathbb{R}^+$$

The cases $r = 0$ or $y_{V0} = 0$ lead to simpler integrals. In the following, we always assume them unequal zero, but the solution we obtain will be valid in those special cases as well.

The first integral ΔT

We start with the inner integral of ΔT over Ψ and notice the integrand is 2π -periodic and has an even kernel $K(r, \Psi)$, see Eq. (19), such that $K(r, \Psi) = K(r, -\Psi)$. Because s_Ψ is odd the second part of the integrand $K(r, \Psi)s_\Psi$ is odd as well! Using King's Property, its integral over the symmetric interval $[-\pi; \pi]$ vanishes:

$$I_T(\psi_V) = \int_{-\pi}^{\pi} K(r, \Psi) (r + \mu C_V c_\Psi) d\Psi = I_T(0) \Big|_{y_0 \rightarrow y_{V0}, \mu \rightarrow \mu C_V}. \quad (20)$$

The solution to the special case $\psi_V = 0$ yields the solution to the general case by a simple substitution. The same is true for the outer integral over r , because we only substituted constants independent of r !

The inner integral of the special case $\psi_V = 0$

Let us compute the inner integral $I_T(0)$. To simplify notation, we define the symbol $X_\Psi = rc_\Psi - y_{V0}$. Then we rewrite the numerator as a polynomial in X_Ψ and use long division:

$$\begin{aligned}I_T(0) &= \int_{-\pi}^{\pi} \frac{X_\Psi}{X_\Psi^2 + r_c^2} \left[r + \frac{\mu}{r} (X_\Psi + y_{V0}) \right] d\Psi \quad K_1(\Psi) := \frac{X_\Psi}{X_\Psi^2 + r_c^2} \\ &= \frac{\mu}{r} \int_{-\pi}^{\pi} 1 + \frac{\left(\frac{r^2}{\mu} + y_{V0}\right) X_\Psi - r_c^2}{X_\Psi^2 + r_c^2} d\Psi \quad K_2(\Psi) := \frac{-r_c}{X_\Psi^2 + r_c^2} \\ &= \frac{\mu}{r} \int_{-\pi}^{\pi} 1 + \left\{ \frac{r^2}{\mu} + y_{V0}, \quad r_c \right\} \bar{K}(\Psi) d\Psi \quad \bar{K}(\Psi) := \begin{Bmatrix} K_1(\Psi) \\ K_2(\Psi) \end{Bmatrix}.\end{aligned}\quad (21)$$

The first part of the integral is simply $2\pi\mu/r$, but the integration of the kernel $\bar{K}(\Psi)$ is more involved. Luckily,

$K_i(\Psi)$ are 2π -periodic C^1 -functions, so they each have a Fourier-Series representation we can easily integrate. Our task has become to find the Fourier-Series of $K_i(\Psi)$. While we could try to compute both series independently, we greatly reduce the workload if we combine both functions into $K(\Psi) := K_1(\Psi) + iK_2(\Psi) =: f(e^{i\Psi})$, $c_\Psi = (e^{i\Psi} + e^{-i\Psi})/2$, $\Psi \in \mathbb{R}$:

$$f(z) := \frac{\left(\frac{r}{2}(z + z^{-1}) - y_{V0}\right) - ir_c}{\left(\frac{r}{2}(z + z^{-1}) - y_{V0}\right)^2 + r_c^2} = \frac{1}{\frac{r}{2}(z + z^{-1}) - y_{V0} + ir_c} = \frac{2}{r} \cdot \frac{z}{z^2 - \frac{2}{r}(y_{V0} - ir_c)z + 1}. \quad (22)$$

If we can find a Laurent-Series representation of $f(z)$ that converges for any $|z| = 1$, we may reorder the Laurent-Series

We see that $\sqrt{+}$ is increasing in $r \geq 0$, so $\sqrt{+} \geq r_c > 0$ and $|z_1| > |z_2|$ —together with $z_1 z_2 = 1$ we get

$$|z_1^{-1}| = |z_2| < 1 < |z_1| \quad (24)$$

With all roots at our disposal, we can compute the partial fraction decomposition of $f(z)$:

$$\begin{aligned} f(z) &= \frac{2}{r} \frac{z}{(z - z_1)(z - z_1^{-1})} \stackrel{\text{PFD}}{=} \frac{-2}{r(z_1 - z_1^{-1})} \left(\frac{-z_1}{z - z_1} + \frac{z_1^{-1}}{z - z_1^{-1}} \right) \quad \left| X := \frac{-2}{r(z_1 - z_1^{-1})} \right. \\ &=: X \left(\frac{1}{1 - z_1^{-1}z} + \frac{z_1^{-1}z^{-1}}{1 - z_1^{-1}z^{-1}} \right) = X \left(\sum_{k=0}^{\infty} z_1^{-k} z^k + \sum_{k=1}^{\infty} z_1^{-k} z^{-k} \right) \quad \left| |z_1^{-1}| \stackrel{!}{<} |z| \stackrel{!}{<} |z_1| \right. \\ &= X \left[1 + \sum_{k=1}^{\infty} z_1^{-k} (z^k + z^{-k}) \right]. \end{aligned} \quad (25)$$

evaluated at $z = e^{i\Psi}$ into the Fourier series representation of $K(r, \Psi)$. To obtain such a Laurent-Series, we compute the partial fraction decomposition (PFD) of $f(z)$ and its denominator roots $z_{1,2}$. From Viète's Formula, we know $z_1 z_2 = 1$, and the quadratic formula yields z_1 in terms of the short-hands $\xi := r^2 - y_{V0}^2 + r_c^2$, $\eta := 2y_{V0}r_c \neq 0$. In the following we choose the sign in the expression for the root z_1 such that z_1 has the larger absolute value of the two roots.

The $\stackrel{!}{<}$ notation is introduced here as a new required condition. From Eq. (24) we see that all $|z| = 1$ lie within a closed subset of the open region of convergence of both geometric series. It follows that the Fourier series $K(\Psi) = f(e^{i\Psi})$, converges uniformly for all $\Psi \in \mathbb{R}$:

$$\begin{aligned} z_1 &:= \frac{1}{r} \left[y_{V0} - ir_c + \operatorname{sgn}(y_{V0}) \sqrt{-\xi - i\eta} \right] \quad \left| \sqrt{\pm} := \sqrt{\frac{1}{2}(\sqrt{\xi^2 + \eta^2} \pm \xi)} \right. \\ &= \frac{1}{r} \left[y_{V0} - ir_c + \operatorname{sgn}(y_{V0}) \left(\sqrt{-} - i \operatorname{sgn}(\eta) \sqrt{+} \right) \right] \quad \left| \operatorname{sgn}(\eta) = \operatorname{sgn}(y_{V0}) \right. \\ &= \frac{1}{r} \left[\operatorname{sgn}(y_{V0}) \left(|y_{V0}| + \sqrt{-} \right) - i \left(r_c + \sqrt{+} \right) \right] \quad \left| \sqrt{+} \cdot \sqrt{-} = |y_{V0}| r_c \right. \\ &= \frac{\sqrt{+} + r_c}{r\sqrt{+}} \left(y_{V0} - i\sqrt{+} \right) \quad \left| z_2 = z_1^{-1} = \frac{\sqrt{+} - r_c}{r\sqrt{+}} \left(y_{V0} + i\sqrt{+} \right) \right. \end{aligned} \quad (23)$$

$$X = \frac{-2}{r(z_1 - z_1^{-1})} = \frac{-2\sqrt{+}}{2y_{v0}r_c - i2\sqrt{+}} = -\frac{\operatorname{sgn}(y_{v0})\sqrt{-} + i\sqrt{+}}{\sqrt{\xi^2 + \eta^2}} \quad \left| \sqrt{+} \cdot \sqrt{-} = |y_{v0}|r_c \right.$$

$$K(\Psi) = f(e^{i\Psi}) = X \left(1 + 2 \sum_{k=1}^{\infty} z_1^{-k} c_{k\Psi} \right) = K_1(\Psi) + iK_2(\Psi) \quad \left| \sqrt{+}^2 + \sqrt{-}^2 = \sqrt{\xi^2 + \eta^2} \right.$$
(26)

Splitting $K(\Psi)$ into its real- and imaginary part, we finally get

yields the other anti-derivative. We add sign $(x)\pi/2$ to the integration constant $\vec{C} \in \mathbb{R}^2$ and get:

$$K_1(\Psi) = \Re(X) + \sum_{k=1}^{\infty} 2\Re(Xz_1^{-k})c_{k\Psi} \quad K_2(\Psi) = \Im(X) + \sum_{k=1}^{\infty} 2\Im(Xz_1^{-k})c_{k\Psi}$$
(27)

As those series converge uniformly, we may interchange summation and integration and use Eq. (26):

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \vec{K}(\Psi) d\Psi = \begin{Bmatrix} \Re(X) \\ \Im(X) \end{Bmatrix} =: -\vec{S}(r)$$

$$\vec{S} := \begin{bmatrix} \operatorname{sgn}(y_{v0})0 \\ 01 \end{bmatrix} \quad \vec{f}(r) := \frac{1}{\sqrt{\xi^2 + \eta^2}} \begin{Bmatrix} \sqrt{-} \\ \sqrt{+} \end{Bmatrix}. \quad (28)$$

With these solutions we can finally complete the inner integration from Eq. (21):

$$I_T(0) = \frac{2\pi\mu}{r} \left[1 - \left\{ \frac{r^2}{\mu} + y_{v0}, r_c \right\} \vec{S}(r) \right]. \quad (29)$$

The outer integral of the special case $\Psi_v = 0$

Now that we finished the inner integral $I_T(0)$ in Eq. (29), we have to compute the outer integral over r :

$$\Delta T(0) = \frac{1}{2\pi} \int_A^B I_T(0) dr = \int_A^B \frac{\mu}{r} \left[1 - \left\{ \frac{r^2}{\mu} + y_{v0}, r_c \right\} \vec{S}(r) \right] dr. \quad (30)$$

The first term yields $\mu \ln|r|$. The only other terms still dependent on r are $\vec{f}(r)r^{\pm 1}$ —our task has become to find their anti-derivatives. Recall Eq. (23), Eq. (28) and the short-hands $\xi = r^2 - y_{v0}^2 + r_c^2$, $\eta = 2y_{v0}r_c$:

$$\vec{f}(r) = \frac{1}{\sqrt{\xi^2 + \eta^2}} \begin{Bmatrix} \sqrt{-} \\ \sqrt{+} \end{Bmatrix} =: \begin{Bmatrix} f_{-}(r) \\ f_{+}(r) \end{Bmatrix} \Rightarrow \frac{d}{dr} \sqrt{\pm} = \pm r f_{\pm}(r). \quad (31)$$

By the above we already solved one anti-derivative. A substitution $u := \sqrt{+}$ together with the identities

$$\sqrt{+} \cdot \sqrt{-} = |y_{v0}|r_c, \quad \sqrt{+}^2 - \sqrt{-}^2 = \xi = r^2 - y_{v0}^2 + r_c^2 \Rightarrow \sqrt{+}^2 r^2 = (\sqrt{+}^2 - r_c^2)(\sqrt{+}^2 + y_{v0}^2)$$

$$x \in \mathbb{R} \setminus \{0\} : \arctan \frac{1}{x} = -\arctan x + \operatorname{sgn}(x)\frac{\pi}{2}, \quad x \in \mathbb{R} \setminus \{\pm 1\} : \operatorname{ar tanh} \frac{1}{x} = \operatorname{ar tanh} x$$
(32)

$$\int \vec{f}(r)r dr = \begin{Bmatrix} -\sqrt{-} \\ \sqrt{+} \end{Bmatrix} + \vec{C} \quad \vec{K} := \frac{1}{y_{v0}^2 + r_c^2} \begin{bmatrix} |y_{v0}| & -r_c \\ r_c & |y_{v0}| \end{bmatrix}$$

$$\int \vec{f}(r)r^{-1} dr = -\vec{K}\vec{F}(r) + \vec{C} \quad \vec{F}(r) := \begin{Bmatrix} \operatorname{ar tanh} \frac{\sqrt{+}}{r_c} \\ \operatorname{arctan} \frac{|y_{v0}|}{\sqrt{+}} \end{Bmatrix}. \quad (33)$$

The solution of the integral and derivation of the result in Eq. (33) is elaborate and includes the substitution $u := \sqrt{+}$ together with the identities given in Eq. (32), which results in functions with elementary integrals that lead to the given solution. Details of the derivation are given in the supplementary material. The function $\operatorname{ar tanh}(\dots)$ denotes the inverse hyperbolic tangent extended beyond $(-1;1)$:

$$\operatorname{ar tanh} : \mathbb{R} \setminus \{\pm 1\} \rightarrow \mathbb{R} \quad \operatorname{ar tanh} x := \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|. \quad (34)$$

We may verify the anti-derivatives by differentiation using the identities Eq. (32). Plucking the anti-derivatives into Eq. (30), we notice the $\operatorname{arctan}(\dots)$ -terms cancel out:

$$\Delta T(0) = \mu \left[\ln|r| + \operatorname{ar tanh} \frac{\sqrt{+}}{r_c} + \operatorname{sgn}(y_{v0}) \frac{\sqrt{-}}{\mu} \right]_A^B. \quad (35)$$

However, the solution has a flaw—it is numerically unstable around $r = 0$, as both $\ln[.]$ and the $\operatorname{ar tanh}[.]$ tend to $\pm\infty$ when $r \rightarrow 0$. As the original integrand was smooth over the entire integration interval, we expect both parts to cancel out, and the remainder being well behaved:

$$\begin{aligned}
\ln |r| + \operatorname{arctanh} \frac{\sqrt{+}}{r_c} &= \frac{1}{2} \ln \left| r^2 \frac{r_c + \sqrt{+}}{r_c - \sqrt{+}} \right| = \frac{1}{2} \ln \left| r^2 \frac{(r_c + \sqrt{+})^2 (\sqrt{+}^2 + y_{V0}^2)}{(\sqrt{+}^2 - r_c^2) (\sqrt{+}^2 + y_{V0}^2)} \right| \\
&= \frac{1}{2} \ln \left| \frac{(r_c + \sqrt{+})^2 (\sqrt{+}^2 + y_{V0}^2)}{\sqrt{+}^2} \right| = \ln \left| 1 + \frac{r_c}{\sqrt{+}} \right| + \frac{1}{2} \ln |\sqrt{+}^2 + y_{V0}^2| \\
&=: G(\sqrt{+})
\end{aligned} \tag{36}$$

We obtain a numerically stable solution entirely dependent on short-hands $\xi = r^2 - y_{V0}^2 + r_c^2$, $\eta = 2y_{V0}r_c$, $\sqrt{\pm}$ from Eq. (23) and $G(x)$ from Eq. (36)

$$\Delta T(\Delta L'_V) = \left[\mu C_V G(\sqrt{+}) + \operatorname{sgn}(y_{V0}) \sqrt{-} \right] \Big|_A^B \tag{37}$$

Remark If we also want to eliminate $\sqrt{-}$, we may use $\sqrt{+} \cdot \sqrt{-} = |y_{V0}|r_c$ from Eq. (32). The solution is also valid for the special case $y_{V0} = 0$ we initially left out.

The second integral $\Delta \bar{M}$

We again start with the inner integral of $\Delta \bar{M}$ over Ψ and simplify it with the same transformations we used for ΔT in the exact same order. We get

$$\bar{I}_M(\psi_V) = \int_{-\pi}^{\pi} K_1(\Psi) \left(r + \mu c_{\Psi+\Psi_V} \right) \left\{ \frac{c_{\Psi+\Psi_V}}{s_{\Psi+\Psi_V}} \right\} d\Psi. \tag{38}$$

$$\begin{aligned}
I_{M_x}(0) &= \int_{-\pi}^{\pi} \frac{X_{\Psi}}{X_{\Psi}^2 + r_c^2} \left[r + \frac{\mu}{r} (X_{\Psi} + y_{V0}) \right] \frac{X_{\Psi} + y_{V0}}{r} d\Psi = \int_{-\pi}^{\pi} \frac{\mu}{r^2} \cdot \frac{X_{\Psi} (X_{\Psi} + y_{V0})^2}{X_{\Psi}^2 + r_c^2} + \frac{X_{\Psi} (X_{\Psi} + y_{V0})}{X_{\Psi}^2 + r_c^2} d\Psi \\
&= \int_{-\pi}^{\pi} \frac{\mu}{r^2} \left[X_{\Psi} + 2y_{V0} + \frac{(y_{V0}^2 - r_c^2) X_{\Psi} - 2y_{V0} r_c^2}{X_{\Psi}^2 + r_c^2} \right] + 1 + \frac{y_{V0} X_{\Psi} - r_c^2}{X_{\Psi}^2 + r_c^2} d\Psi.
\end{aligned} \tag{41}$$

Like before, we sort the integrand into odd and even parts to use King's Property. This time we need to use the addition theorems $c_{x+y} = c_x c_y - s_x s_y$, $s_{x+y} = s_x c_y + c_x s_y$, $x, y \in \mathbb{R}$:

$$\begin{aligned}
(r + \mu c_{\Psi+\Psi_V}) \left\{ \frac{c_{\Psi+\Psi_V}}{s_{\Psi+\Psi_V}} \right\} &= [r + \mu (C_V c_{\Psi} - S_V s_{\Psi})] \left\{ \frac{C_V c_{\Psi} - S_V s_{\Psi}}{S_V c_{\Psi} + C_V s_{\Psi}} \right\} \\
&= \underbrace{\begin{bmatrix} C_V & -S_V \\ S_V & C_V \end{bmatrix}}_{=\operatorname{Rot}_V} \left\{ \begin{bmatrix} r + \mu (C_V c_{\Psi} - S_V s_{\Psi}) \\ (r + \mu C_V c_{\Psi}) s_{\Psi} - \mu S_V s_{\Psi}^2 \end{bmatrix} \right\}.
\end{aligned} \tag{39}$$

The function $K_1(\Psi)$ is even, therefore all terms with the underlined multiplier s_{Ψ} are odd and vanish like before when we use King's Property:

$$\begin{aligned}
\bar{I}_M(\Psi_V) &= \operatorname{Rot}_V \int_{-\pi}^{\pi} K_1(\Psi) \left\{ \frac{(r + \mu C_V c_{\Psi}) c_{\Psi}}{\mu S_V (c_{\Psi}^2 - 1)} \right\} d\Psi \\
&= \operatorname{Rot}_V \left\{ \begin{array}{l} I_{M_x}(0) \Big|_{y_0 \rightarrow y_{V0}, \mu \rightarrow \mu C_V} \\ I_{M_x} \left(\frac{\pi}{2} \right) \Big|_{x_0 \rightarrow -y_{V0}, \mu \rightarrow -\mu S_V} \end{array} \right\}.
\end{aligned} \tag{40}$$

We only need the solutions of the first component I_{M_x} under two special cases $\psi_V \in \{0; \pi/2\}$ to generate the general solution via a simple substitution. Again, the same applies to the outer integral over r !

The inner integral of the special cases $\psi_V \in \{0; \pi/2\}$

We begin with the case $\psi_V = 0$. To simplify notation, we define the symbol $X_{\Psi} := r c_{\Psi} - y_{V0}$. We rewrite the numerator into a polynomial in X_{Ψ} and use long division:

The integration over $X_{\Psi} + 2y_{V0} = r c_{\Psi} + y_{V0}$ yields $2\pi y_{V0}$ and the middle part is simply 2π . The remaining parts consist entirely of $K_i(\Psi)$ from Eq. (21) and can be collected into a constant matrix **A**, making use of Eq. (28):

$$\begin{aligned}
I_{M_x}(0) &= 2\pi \left[1 + \frac{\mu y_{V0}}{r^2} \right] + \underline{\underline{A}} \int_{-\pi}^{\pi} \bar{K}(\Psi) d\Psi = 2\pi \left[1 + \frac{\mu y_{V0}}{r^2} - \underline{\underline{A}} \bar{f}(r) \right] \\
\underline{\underline{A}} &:= \frac{\mu}{r^2} \{ y_{V0}^2 - r_c^2, 2y_{V0} r_c \} + \{ y_{V0}, r_c \} =: \frac{\mu}{r^2} \underline{\underline{A}}_1 + \underline{\underline{A}}_2.
\end{aligned} \tag{42}$$

Now we tackle the other case $\Psi_V = \pi/2$. Like before, we define $X_\Psi = rc_\Psi - y_{V0}$, rewrite the numerator into a polynomial in X_Ψ and use long division:

$$\begin{aligned} I_{M_x}\left(\frac{\pi}{2}\right) &= -\mu \int_{-\pi}^{\pi} \frac{X_\Psi}{X_\Psi^2 + r_c^2} \left[\frac{1}{r^2} (X_\Psi + y_{V0})^2 - 1 \right] d\Psi = -\mu \int_{-\pi}^{\pi} \frac{1}{r^2} \cdot \frac{X_\Psi (X_\Psi + y_{V0})^2}{X_\Psi^2 + r_c^2} - \frac{X_\Psi}{X_\Psi^2 + r_c^2} d\Psi \\ &= -\mu \int_{-\pi}^{\pi} \frac{1}{r^2} \left[X_\Psi + 2y_{V0} + \frac{(y_{V0}^2 - r_c^2)X_\Psi - 2y_{V0}r_c^2}{X_\Psi^2 + r_c^2} \right] - \frac{X_\Psi}{X_\Psi^2 + r_c^2} d\Psi. \end{aligned} \quad (43)$$

The integration over $X_\Psi + 2y_{V0} = rc_\Psi + y_{V0}$ again yields $2\pi y_{V0}$. The remaining parts consist entirely of $K_i(\Psi)$ from Eq. (21) and can be collected into a constant matrix $\mathbf{B}$. We notice its first part equals $\mathbf{A}_1$ and by use of Eq. (28):

$$\begin{aligned} I_{M_x}\left(\frac{\pi}{2}\right) &= -\mu \left[\frac{2\pi y_{V0}}{r^2} + \underline{\underline{B}} \int_{-\pi}^{\pi} \tilde{K}(\Psi) d\Psi \right] = -2\pi\mu \left[\frac{y_{V0}}{r^2} - \underline{\underline{B}} \tilde{f}(r) \right] \\ \underline{\underline{B}} &:= \frac{1}{r^2} \{ y_{V0}^2 - r_c^2, 2y_{V0}r_c \} + \{ -1, 0 \} =: \frac{1}{r^2} \underline{\underline{B}}_1 + \underline{\underline{B}}_2 = \frac{1}{r^2} \underline{\underline{A}}_1 + \underline{\underline{B}}_2. \end{aligned} \quad (44)$$

The outer integral of the special cases $\Psi_V \in \{0; \pi/2\}$

With the results from Eqs. (42) and (44), we may tackle the outer integration over r :

$$\begin{aligned} \Delta M_x(0) &= \frac{1}{2\pi} \int_A^B r I_{M_x}(0) dr = \int_A^B r + \frac{\mu y_{V0}}{r} - \left(\frac{\mu}{r} \underline{\underline{A}}_1 + r \underline{\underline{A}}_2 \right) \underline{\underline{S}} \tilde{f}(r) dr \\ \Delta M_x\left(\frac{\pi}{2}\right) &= \frac{1}{2\pi} \int_A^B r I_{M_x}\left(\frac{\pi}{2}\right) dr = -\mu \int_A^B \frac{y_{V0}}{r} - \left(\frac{1}{r} \underline{\underline{A}}_1 + r \underline{\underline{B}}_2 \right) \underline{\underline{S}} \tilde{f}(r) dr \quad \left| \underline{\underline{B}}_1 = \underline{\underline{A}}_1 \right. \end{aligned} \quad (45)$$

The only challenging integrals are $\tilde{f}(r)r^{\pm 1}$, but we already found their anti-derivatives in Eq. (33). We also note the $\tilde{f}(r)r^{-1}$ parts are identical in both cases. Thus

$$\begin{aligned} \Delta M_x(0) &= \left[\frac{r^2}{2} + \mu y_{V0} \ln |r| + \mu \underline{\underline{A}}_1 \underline{\underline{SK}} \tilde{F}(r) - \underline{\underline{A}}_2 \underline{\underline{S}} \left\{ \frac{-\sqrt{-}}{\sqrt{+}} \right\} \right]_A^B \quad \left| \underline{\underline{A}}_1 \underline{\underline{SK}} = \{ y_{V0}, \operatorname{sgn}(y_{V0}) r_c \} \right. \\ &= \left[\frac{r^2}{2} + \mu y_{V0} \ln |r| + \mu y_{V0} \operatorname{arctanh} \frac{\sqrt{+}}{r_c} + \mu \operatorname{sgn}(y_{V0}) r_c \operatorname{arctan} \frac{|y_{V0}|}{\sqrt{+}} + |y_{V0}| \sqrt{-} - r_c \sqrt{+} \right]_A^B \\ \Delta M_x\left(\frac{\pi}{2}\right) &= -\mu \left[y_{V0} \ln |r| + y_{V0} \operatorname{arctanh} \frac{\sqrt{+}}{r_c} + \operatorname{sgn}(y_{V0}) r_c \operatorname{arctan} \frac{|y_{V0}|}{\sqrt{+}} - \operatorname{sgn}(y_{V0}) \sqrt{-} \right]_A^B. \end{aligned} \quad (46)$$

The $\ln(\dots)$ and $\operatorname{arctanh}(\dots)$ terms are again numerically unstable around $r = 0$, so we apply Eq. (36) to remove that instability. We also transfer $\operatorname{sgn}(y_{V0})$ into $\operatorname{arctan}(\dots)$ and

obtain a numerically stable solution in terms of the short-hands $\xi = r^2 - y_{V0}^2 + r_c^2$, $\eta = 2y_{V0}r_c$, $\sqrt{\pm}$ from Eq. (23) and $G(x)$ from Eq. (36):

$$\begin{Bmatrix} \Delta M_x(0) \\ \Delta M_x\left(\frac{\pi}{2}\right) \end{Bmatrix} = \left[\frac{r^2}{2} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \left[y_{V0}G(\sqrt{+}) + r_c \arctan \frac{y_{V0}}{\sqrt{+}} \right] \begin{Bmatrix} \mu \\ -\mu \end{Bmatrix} + \begin{bmatrix} |y_{V0}| & -r_c \\ \mu \operatorname{sgn}(y_{V0}) & 0 \end{bmatrix} \begin{Bmatrix} \sqrt{-} \\ \sqrt{+} \end{Bmatrix} \right]_A^B. \quad (47)$$

Remark If we also want to eliminate $\sqrt{-}$, we may use $\sqrt{+} \cdot \sqrt{-} = |y_{V0}|r_c$ from Eq. (32). The solution is also valid for the special case $y_{V0} = 0$ we initially left out.

Solution of the general case $\psi_V \in \mathbb{R}$

As a conclusion, we will use the special cases $\psi_V = 0$ and $\psi_V = \pi/2$ from Eq. (37) and Eq. (47) to give the general solution via the transformations in a concise form.

$$\Delta T(\psi_V) = \Delta T(0)|_{y_0 \rightarrow y_{V0}, \mu \rightarrow \mu C_V} \quad \Delta \vec{M}(\psi_V) = \operatorname{Rot}_V \begin{Bmatrix} \Delta M_x(0)|_{y_0 \rightarrow y_{V0}, \mu \rightarrow \mu C_V} \\ \Delta M_x\left(\frac{\pi}{2}\right)|_{x_0 \rightarrow -y_{V0}, \mu \rightarrow -\mu S_V} \end{Bmatrix}. \quad (48)$$

Constants and function definitions:

$$G(\sqrt{+}) = \ln \left| 1 + \frac{r_c}{\sqrt{+}} \right| + \frac{\ln |\sqrt{+}^2 + y_{V0}^2|}{2} \quad y_{V0} = y_0 C_V - x_0 S_V \quad C_V = \cos \psi_V \quad S_V = \sin \psi_V \quad (49)$$

$$\sqrt{\pm} = \sqrt{\frac{1}{2}(\sqrt{\xi^2 + \eta^2} \pm \xi)} \quad \xi = r^2 - y_{V0}^2 + r_c^2 \quad \eta = 2y_{V0}r_c \quad d_2 = \frac{A^2 - B^2}{2}.$$

Solution to the general case:

$$\Delta T(\Delta L'_V) = \left[\mu C_V G(\sqrt{+}) + \operatorname{sgn}(y_{V0}) \sqrt{-} \right]_A^B$$

$$\begin{Bmatrix} \Delta M_x(\Delta L'_V) \\ \Delta M_y(\Delta L'_V) \end{Bmatrix} = \begin{bmatrix} C_V & -S_V \\ S_V & C_V \end{bmatrix} \left[\begin{Bmatrix} d_2 \\ 0 \end{Bmatrix} + \mu \left(y_{V0}G(\sqrt{+}) + r_c \arctan \frac{y_{V0}}{\sqrt{+}} \right) \right]_A^B \begin{Bmatrix} C_V \\ S_V \end{Bmatrix} \quad (50)$$

$$+ \begin{bmatrix} |y_{V0}| & -r_c \\ -\mu S_V \operatorname{sgn}(y_{V0}) & 0 \end{bmatrix} \begin{Bmatrix} \sqrt{-} \\ \sqrt{+} \end{Bmatrix}_A^B.$$

Remark If we also want to eliminate $\sqrt{-}$, we may use $\sqrt{+} \cdot \sqrt{-} = |y_{V0}|r_c$ from Eq. (32). The solution is also valid for the special case $y_{V0} = 0$ we initially left out.

Remark As a reference, here are the first Fourier-Coefficients suggested in Eq. (13), Eq. (14) and the text before: $A_k(r) = 2\Re(Xz_1^{-k})$, $k = 0, 1, 2$ of the kernel $K_1(\Psi)$ in Eq. (21),

wherein $f_+(r)$ and $f_-(r)$ are given in Eq. (31). Details of the derivation are given in the supplementary material.

$$A_0(r) = -2\operatorname{sgn}(y_{V0})f_-(r)$$

$$A_1(r) = \frac{2}{r} [1 - |y_{V0}|f_-(r) - r_c f_+(r)] \quad (51)$$

$$A_2(r) = \frac{2\operatorname{sgn}(y_{V0})}{r^2} [2|y_{V0}| - 2\sqrt{-} - r^2 f_-(r)].$$

The derivation of $A_2(r)$ is non-trivial and elaborate, it only uses expansion and Eq. (32) repeatedly. However, the explicit form of these coefficients is not needed here since the only required result is given in Eq. (50).

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