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Coalition Formation in Games with Externalities

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Abstract

This paper studies an extensive form game of coalition formation with random proposers in games with externalities. It is shown that an agreement will be reached without delay if any set of coalitions profits from merging. Even under this strong condition, the equilibrium coalition structure is not necessarily efficient. There may be multiple equilibria even in the absence of externalities, and symmetric players are not necessarily treated symmetrically in equilibrium. If the grand coalition forms without delay in equilibrium, expected payoffs must be in the core of the characteristic function game that assigns to each coalition its equilibrium payoff. Compared with the rule of order process of Ray and Vohra (Games Econ Behav 26:286–336, 1999), the bargaining procedure with random proposers tends to give a large advantage to the proposer, whereas the bargaining procedure with a rule of order tends to favor the responders. The equilibria of the two procedures cannot be ranked in general in terms of efficiency.

Keywords Coalition formation · Externalities, partition function · Random proposers · Core · Multiple equilibria

JEL Classification C71 · C72 · C78

1 Introduction

In most economic situations, the payoff of a coalition depends on which other coalitions form, that is, there are externalities between coalitions. A function assigning to each coalition a payoff depending on the whole coalition structure is called a *partition function* [71]. In spite of the empirical relevance of externalities, standard cooperative game theory is not based on

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the partition function but on the characteristic function, which assigns a fixed payoff to each coalition regardless of how outsiders are organized. In order to derive a characteristic function from a situation with externalities, players are assumed to have conjectures about how the rest of the players will organize themselves given that a coalition forms. These conjectures are usually pessimistic (players fear the worst) or optimistic (players expect the best). Thus, when a coalition forms, the rest of the players are expected to partition themselves so as to either minimize or maximize the payoff of the coalition, regardless of their own interest.

An alternative to indiscriminated optimism or pessimism is to use an extensive form game of coalition formation in order to allow a coalition to predict the reaction of the outsiders as an equilibrium reaction, so that conjectures are consistent (see [9] and [64]). Ray and Vohra [64] extend the model of coalitional bargaining of Chatterjee et al. [15] to games with externalities. Both models are natural generalizations of the Rubinstein [66] two-player alternating offers bargaining game. A distinctive feature of these models is that players respond to proposals according to a predetermined *rule of order* and the first player to reject a proposal automatically becomes the next proposer. Because the first rejector becomes the next proposer, responders may have substantial bargaining power even if they are in a weak position in terms of the availability of alternative coalitions.

The present paper uses an alternative model of coalitional bargaining that has been studied by Binmore [7] for two players, Baron and Ferejohn [4] for symmetric majority games, and Okada [58] for characteristic function games. The distinctive feature of this model is that a player who rejects an offer does not automatically become the next proposer. Instead, proposers are selected randomly by Nature. By giving less power to the responders, this model incorporates competition: any player may have an opportunity to “step in” with a proposal during the negotiations (even if only after a rejection), and the responders’ bargaining position reflects the availability of alternative coalitions.

As most of the literature on coalitional bargaining, I assume that payoffs are transferable within coalitions (but not between coalitions) and coalitions cannot be enlarged once formed, perhaps because forming a coalition entails sunk costs. Players share a common discount factor and are risk neutral. The solution concept is stationary subgame perfect equilibrium.

The results are as follows:

(1) If any merger of two or more coalitions is (weakly) profitable, agreement is immediate in equilibrium.
(2) Even under this strong condition, the resulting coalition structure is not necessarily efficient.
(3) If the grand coalition forms immediately in equilibrium, expected payoffs must lie in the core of a characteristic function game that assigns to each coalition its equilibrium payoff.
(4) Multiple equilibria with immediate agreement can exist, even in symmetric games without externalities.
(5) Symmetric games may have asymmetric equilibria (with immediate agreement) where some players have a greater expected payoff than others.
(6) Compared with the rule of order procedure studied by Ray and Vohra [64], bargaining with random proposers favors the proposer (who may get a disproportionate share even if players are symmetric), whereas bargaining with a rule of order favors the responders (who may get an equal share even if players are very asymmetric). In general, the two procedures cannot be ranked in terms of efficiency. For a specific class of games (symmetric games without externalities where only one coalition forms), the procedure with random proposers is at least as efficient as the procedure with a rule of order.

2 The Model

2.1 The Partition Function

Let $N = \{1, 2, ..., n\}$ be the set of players. Non-empty subsets of N are called *coalitions*. A *coalition structure* $\pi := \{S_1, ..., S_m\}$ is a partition of N into coalitions, hence it satisfies $S_j \cap S_k = \emptyset$ if $j \neq k$, $\cup_{j=1}^{m} S_j = N$. The set of all coalition structures is denoted by $\Pi(N)$, with typical element π. Analogously, the set of partitions of $T \subset N$ is denoted by $\Pi(T)$, with typical element π_T.

An *embedded coalition* is a pair (S, π) with $S \in \pi$. The set of embedded coalitions is denoted by $E(N)$. A *partition function* φ assigns a real number to each embedded coalition (S, π), thus $\varphi : E(N) \longrightarrow \mathbb{R}$. The value $\varphi(S, \pi)$ represents the payoff of coalition S given that coalition structure π forms.[1] We assume that players can guarantee a nonnegative payoff as singletons ($\varphi(\{i\}, \pi) \geq 0$ for all $i \in N$ and $\pi \in \Pi$ with $\{i\} \in \pi$). We also rule out the trivial case in which all values are zero by assuming $\varphi(N, \{N\}) > 0$. The pair (N, φ) is called a *partition function game*.

Given a coalition structure $\pi = \{S_1, ..., S_m\}$ and a partition function φ, we will denote the m-dimensional vector $(\varphi(S_i, \pi))_{i=1}^{m}$ by $\overline{\varphi}(S_1, ..., S_m)$. It will sometimes be convenient to write the partition function in terms of $\overline{\varphi}$. In order to simplify notation, we will denote $\varphi(N, \{N\})$ by $\varphi(N)$ or $\overline{\varphi}(N)$. We will also write $\varphi(\{i\}, \{\{i\}, \{j, k\}\})$ as $\varphi(i, \{i, jk\})$ and $\overline{\varphi}(\{i\}, \{j, k\})$ as $\overline{\varphi}(i, jk)$.

A partition function (N, φ) is *positive* if

$$\varphi(S, \pi) \geq 0 \text{ for all } (S, \pi), \text{ and } \varphi(S, \pi) > 0 \text{ for all } (S, \pi) \text{ such that } |S| \geq 2.$$

A partition function game (N, φ) is *superadditive* if for all $\pi \in \Pi(N)$, S_i, $S_j \in \pi$, $S_i \neq S_j$ it holds that

$$\varphi(S_i \cup S_j, (\pi \backslash \{S_i, S_j\}) \cup \{S_i \cup S_j\}) \geq \varphi(S_i, \pi) + \varphi(S_j, \pi).$$

Superadditivity means that a merger of any two coalitions keeping the partition of the remaining players unchanged is weakly profitable.

A partition function game (N, φ) is *cohesive* if

$$\varphi(N) \geq \sum_{S \in \pi} \varphi(S, \pi) \text{ for all } \pi \in \Pi(N).$$

Cohesiveness means that total payoffs are maximized when players form the grand coalition. Thus, starting from an arbitrary partition, a merger of all coalitions to form the grand coalition is weakly profitable.

A partition function game (N, φ) is *fully cohesive* if for all $(S, \pi) \in E(N)$ and for all $\pi_S \in \Pi(S)$ it holds that

$$\varphi(S, \pi) \geq \sum_{T \in \pi_S} \varphi(T, (\pi \backslash \{S\}) \cup \pi_S).$$

Full cohesiveness means that any merger of coalitions keeping the partition of the remaining players unchanged is weakly profitable.[2]

[1] Like the vast majority of the literature, we assume that $\varphi(S, \pi)$ is independent of how payoffs are divided within S. This assumption would not be appropriate in the presence of moral hazard (see [26]).

[2] This terminology is taken from [17, 18].

Notice that cohesiveness and superadditivity are independent properties, both of them weaker than full cohesiveness.[3] The following examples illustrate these three properties.

Example 1 A game that is superadditive but not cohesive

$$
\begin{aligned}
&N = \{1, 2, 3\} \\
&\overline{\varphi}(1, 2, 3) = (3, 3, 3) \\
&\overline{\varphi}(ij, k) = (7, 0) \\
&\overline{\varphi}(123) = 8
\end{aligned}
$$

Even though any merger of two coalitions is profitable, total payoffs are maximized when all players remain singletons. The externality that a two-player merger imposes on the outsider is stronger than the internal gain, thus the game is not cohesive.

Example 2 A game that is cohesive but not superadditive

$$
\begin{aligned}
&N = \{1, 2, 3\} \\
&\overline{\varphi}(1, 2, 3) = (1, 1, 1) \\
&\overline{\varphi}(ij, k) = (0, 3) \\
&\overline{\varphi}(N) = 4
\end{aligned}
$$

Total payoffs are maximized when the grand coalition forms, but the merger of two players is not profitable.

Example 3 A game that is cohesive and superadditive but not fully cohesive

$$
\begin{aligned}
&N = \{1, 2, 3, 4\} \\
&\overline{\varphi}(1, 2, 3, 4) = (3, 3, 3, 3) \\
&\overline{\varphi}(ij, k, l) = (7, 0, 0) \\
&\overline{\varphi}(ijk, l) = (8, 0) \\
&\overline{\varphi}(ij, kl) = (2, 2) \\
&\overline{\varphi}(N) = 15
\end{aligned}
$$

The merger of any two coalitions is profitable and the grand coalition achieves the highest payoff, but the merger of three singletons is unprofitable.

Example 4 A fully cohesive game

$$
\begin{aligned}
&N = \{1, 2, 3, 4\} \\
&\overline{\varphi}(1, 2, 3, 4) = (2, 2, 2, 2) \\
&\overline{\varphi}(ij, k, l) = (5, 0, 0) \\
&\overline{\varphi}(ijk, l) = (7, 0) \\
&\overline{\varphi}(ij, kl) = (1, 1) \\
&\overline{\varphi}(N) = 8
\end{aligned}
$$

This example shows that full cohesiveness does not imply that going from a finer to a coarser partition will always increase aggregate payoffs. It also shows that full cohesiveness is compatible with a situation in which coalition formation can only reduce aggregate payoffs.

We will assume that the partition function is positive and fully cohesive for most of the results (these assumptions will be explicitly stated when needed).

[3] For the special case of games in characteristic function form, superadditivity and full cohesiveness are equivalent.

2.2 The Bargaining Procedure

Time is discrete and indexed by $t = 1, 2, ...$ Given the underlying partition function game (N, φ), bargaining proceeds as follows:

- Nature selects a player randomly to be the *proposer* according to the probability vector $\theta := (\theta_i)_{i \in N}$, where $\theta_i \geq 0$ for all $i \in N$ and $\sum_{i \in N} \theta_i = 1$. This probability vector is called a *protocol*.
- The proposer makes a proposal $(S, x^{S\backslash\{i\}})$ where S is a coalition to which i belongs and $x^{S\backslash\{i\}} \in \mathbb{R}_+^{s-1}$ is a vector of payments.
- If $S = \{i\}$, it is understood that i accepts his own proposal. If $|S| > 1$ the rest of players in S (called responders) accept or reject sequentially (the order does not affect the results). If all players in S accept, S is formed and each $j \in S\backslash\{i\}$ receives $x_j^{S\backslash\{i\}}$ from i immediately (we assume that players are not financially constrained).
- After S is formed, bargaining continues between players in $N\backslash S$ with an adjusted protocol (more on this below) provided that $|N\backslash S| > 1$. If $N\backslash S = \{j\}$, it is understood that j has formed a singleton coalition.
- If at least one player rejects, the game proceeds to the next period in which Nature selects a new proposer according to θ.[4]
- Bargaining continues until all players have formed a coalition. Given that coalition structure π forms at time t, coalition S receives a payoff of $\varphi(S, \pi)$. This payoff goes to the player i whose proposal to form S was accepted. Let $\delta \in [0, 1)$ be the discount factor. Evaluated at time 1, i's payoff is $\delta^{t-1}\varphi(S, \pi) - \delta^{r-1} \sum_{j \in S\backslash\{i\}} x_j^{S\backslash\{i\}}$, where $r \leq t$ is the period in which coalition S formed.[5] This payoff may be negative.

We will denote the extensive form game described above by $G(N, \varphi, \theta, \delta)$.

A *reduced game* is a subgame starting immediately after a coalition is formed. The set T of players in the reduced game consists of all players who have not formed a coalition yet, and the partition function they face, $\varphi^{\pi_{N\backslash T}} : E(T) \longrightarrow \mathbb{R}$, is obtained from φ by fixing $\pi_{N\backslash T}$. The protocol $\theta^{\pi_{N\backslash T}}$ is a probability distribution such that $\theta_i^{\pi_{N\backslash T}} \geq 0$ for all $i \in T$ and $\sum_{j \in T} \theta_j^{\pi_{N\backslash T}} = 1$. We will denote the reduced game arising after the partition $\pi_{N\backslash T}$ has formed by $G(T, \varphi^{\pi_{N\backslash T}}, \theta^{\pi_{N\backslash T}}, \delta)$.

A (pure) strategy for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy, prescribes a proposal $(S, x^{S\backslash\{i\}})$ and a response function assigning "yes" or "no" to all possible proposals of the other players.[6]

We will only consider *stationary subgame perfect equilibria (SSPE)*. An SSPE is a subgame perfect equilibrium with the property that the strategies of the players depend only on the set of coalitions that have formed so far, $\pi_{N\backslash T}$, and the current proposal, if any.

Let σ be a combination of stationary strategies (not necessarily an equilibrium). Suppose that no coalitions have formed yet. Thus, we are at the beginning of the game or at a subgame that is equivalent to the beginning of the game. We denote the *expected payoff* of player i given σ by $w_i(\sigma)$. This expectation is computed before Nature draws the proposer. We

[4] Thus a period elapses if a proposal is rejected, but not if a coalition forms. Assuming that some time elapses in both cases would not alter the results in Sects. 3 and 4.

[5] Thus utility is transferable, and proposers pay responders immediately after the proposal to form S is accepted. Assuming that proposers pay responders only after a complete coalition structure is formed would not change the results in Sects. 3 and 4.

[6] Since no time elapses after a coalition is formed there may be several "stages" at time t, each of them with a smaller set of remaining players than the previous one, and each player taking an action in at most one of those stages.

denote by $w_i^j(\sigma)$ the expected payoff for player i given that player j has been selected to be the proposer.

Suppose a proposal has been made to player i. The expected payoff of player i if he rejects a proposal is called the *continuation value* of player i. With stationary strategies, this value is a constant across all subgames in which no coalitions have formed yet. It is also i's expected payoff if somebody else rejects a proposal. Stationarity also implies that i's continuation value equals $\delta w_i(\sigma)$. Continuation values will play a central role in the analysis because in a subgame perfect equilibrium a responder must accept any proposal that gives him more than his continuation value, and reject any proposal that gives him less.[7]

Expected payoffs and continuation values can be defined analogously for a reduced game. We denote player i's expected payoff in the reduced game arising after $\pi_{N\setminus T}$ has formed by $w_i^{\pi_{N\setminus T}}(\sigma)$. We will drop σ from the notation when no confusion can arise.

An equilibrium is *efficient* if it maximizes the aggregate payoffs of the players. Efficiency implies two requirements: immediate agreement and formation of a coalition structure with maximum aggregate payoffs.

3 Main Results

3.1 No Delay

Given a strategy combination, we say that a coalition structure forms without delay if all proposals that are made with positive probability are accepted. Proposition 1 states that a coalition structure will form without delay if the underlying partition function is positive and fully cohesive. This proposition is an extension of theorem 1 in Okada [58] to partition function games. The proof rests on the following straightforward lemmas:

Lemma 1 *Let (N, φ) be positive and fully cohesive. For any stationary strategy combination σ of the game $G(N, \varphi, \theta, \delta)$ it holds that*

(i) $\sum_{i\in N} w_i(\sigma) \leq \varphi(N)$.
(ii) $\sum_{i\in T} w_i^{\pi_{N\setminus T}}(\sigma) \leq \varphi^{\pi_{N\setminus T}}(T)$.

Proof Since the game is positive and fully cohesive, the maximum aggregate payoff is achieved if the grand coalition is formed immediately. Delay of the agreement or formation of subcoalitions can only reduce aggregate payoffs. The same reasoning applies to any reduced game. □

Lemma 2 *Let σ^* be an SSPE of $G(N, \varphi, \theta, \delta)$. At any subgame (on or off the equilibrium path):*

(i) Any proposal $(S, x^{S\setminus\{i\}})$ such that $x_j^{S\setminus\{i\}} > \delta w_j^{\pi_{N\setminus T}}(\sigma^*)$ for all $j \in S\setminus\{i\}$ must be accepted.
(ii) Any proposal $(S, x^{S\setminus\{i\}})$ such that $x_j^{S\setminus\{i\}} < \delta w_j^{\pi_{N\setminus T}}(\sigma^*)$ for some $j \in S\setminus\{i\}$ must be rejected.

[7] Recall that the proposer's offer is not contingent on the final coalition structure. This assumption is without loss of generality. Given a proposal to form S with a contingent payoff division, players can compute their expected payoff from accepting the proposal using the probability of each coalition structure (conditional on the partition so far). The responders are indifferent between a contingent proposal and a proposal that gives then the same expected payoff. Risk neutrality and lack of financial constraints imply that the proposer is also indifferent.

Proof (i) Suppose the proposal has been accepted by all players except the last one. Subgame perfection implies that the last player must accept, and by backwards induction all other players will accept as well. (ii) Otherwise player j is not playing a best response. □

Proposition 1 *Let (N, φ) be positive and fully cohesive. In any SSPE σ^* of the game $G(N, \varphi, \theta, \delta)$, a coalition structure is formed without delay.*

Proof Consider any SSPE σ^*, and let i be the proposer. Suppose σ^* is such that i proposes $(S, x^{S\setminus\{i\}})$ with positive probability and this proposal is rejected with positive probability. We will show that i can do better by making a proposal that will be accepted for sure.

By making proposal $(S, x^{S\setminus\{i\}})$, i receives his continuation value δw_i. This is clear if the proposal is rejected with certainty. If the proposal is accepted with positive probability, it must be the case that all $j \in S\setminus\{i\}$ vote yes with positive probability and at least one $j \in S\setminus\{i\}$ mixes between accepting and rejecting. The proposal must be such that $x_j^{S\setminus\{i\}} \geq \delta w_j$ for all $j \in S\setminus\{i\}$ (otherwise j would be better-off rejecting and receiving δw_j) and $x_j^{S\setminus\{i\}} = \delta w_j$ for any j who is mixing between accepting and rejecting (otherwise j would be better off accepting for sure). It then follows that i's payoff when the proposal is accepted must also be δw_i. If i's payoff was lower, i could do better by offering less to the responders so that the proposal is rejected for sure. If i's payoff was higher, i could do better by offering $\delta w_j + \varepsilon$ to the responders j that are getting exactly δw_j, so that each responder gets more than their continuation value and the proposal is accepted for sure. This would increase i's payoff if ε is small enough.

Lemma 1 together with $\varphi(N) > 0$ and $\delta < 1$ implies $\sum_{j\in N} \delta w_j(\sigma^*) < \varphi(N)$. Player i could propose the grand coalition and allocate to all players, including himself, a payoff above $\delta w_j(\sigma^*)$. Lemma 2 (i) implies that this proposal must be accepted. Since i can profit by deviating from σ^*, making a proposal that is rejected with positive probability cannot be part of an equilibrium. The same reasoning applies to any reduced game, so that the whole coalition structure forms without delay, [8] □

Corollary 1 *Consider a positive and fully cohesive game (N, φ). In any SSPE σ^* of the game $G(N, \varphi, \theta, \delta)$, every player i in N proposes a solution of the following maximization problem*

$$\max_{S, x^{S\setminus\{i\}}} \sum_{\substack{\pi\in\Pi(N)\\ \pi\ni S}} \mu(\pi|(\sigma^*, S))\varphi(S, \pi) - \sum_{j\in S\setminus\{i\}} x_j^{S\setminus\{i\}}$$
$$s.t. \quad i \in S \subseteq N,\ x_j^{S\setminus\{i\}} \geq \delta w_j(\sigma^*) \tag{1}$$

where $\mu(\pi|(\sigma^*, S))$ is the probability that coalition structure π forms given that players follow σ^* and that S is the first coalition to form. Moreover, each responder receives exactly $\delta w_j(\sigma^*)$.

Proof Let $(S, x^{S\setminus\{i\}})$ be a proposal i makes with positive probability according to σ_i^*. It must be the case that $x_j^{S\setminus\{i\}} = \delta w_j(\sigma^*)$: if $x_j^{S\setminus\{i\}} < \delta w_j(\sigma^*)$ for some j the proposal would be rejected, contradicting proposition 1 and if $x_j^{S\setminus\{i\}} > \delta w_j(\sigma^*)$ for some j player i could do

[8] Okada [58] assumes superadditivity and $v(S) \geq 0$ for all S. Proposition 1 replaces superadditivity by full cohesiveness (which is equivalent for characteristic function games). Requiring the partition function to be positive is a stronger requirement than $v(S) \geq 0$ for all S, but it can be relaxed to $\varphi(S, \pi) \geq 0$ for all (S, π) if we assume (as Okada [58]) that the game ends with coalition structure $\{\pi_{N\setminus T}, \langle T\rangle\}$ if the set of remaining players T is such that $\varphi^{\pi_{N\setminus T}}(T) = 0$.

better by cutting j's payoff slightly. Thus, i's payoff is $\sum_{\substack{\pi\in\Pi(N)\\ \pi\ni S}} \mu(\pi|(\sigma^*, S))\varphi(S, \pi) - \sum_{j\in S\setminus\{i\}} \delta w_j(\sigma^*)$.

Now suppose S does not solve the maximization problem. Let T be one of the coalitions that solves the maximization problem. Then player i would be better-off by proposing T and offering each $j \in T\setminus\{i\}$ a payoff slightly above $\delta w_j(\sigma^*)$, and σ^* cannot be an equilibrium. □

Corollary 2 *Consider a positive and fully cohesive game (N, φ). In any SSPE σ^* of the game $G(N, \varphi, \theta, \delta)$*

$$w_i^i(\sigma^*) > w_i(\sigma^*) > \delta w_i(\sigma^*) \geq 0 \textit{ for all } i \textit{ with } \theta_i > 0.$$

Proof Because $\delta < 1$ and $\sum_{i\in N} w_i(\sigma) \leq \varphi(N)$, the proposer can offer each $j \in N\setminus\{i\}$ a payoff between $\delta w_j(\sigma^*)$ and $w_j(\sigma^*)$, and still get more than $w_i(\sigma^*)$. Hence, $w_i^i(\sigma^*) > w_i(\sigma^*)$.

Players can secure a nonnegative payoff by rejecting all proposals (as responders) and forming a singleton coalition (as proposers), thus w_i is nonnegative. Furthermore, player i can get a strictly positive payoff as a proposer by proposing the grand coalition and offering everyone (including i itself) $\delta w_j + \frac{\varphi(N)-\sum_{j\in N}\delta w_j(\sigma)}{n} > 0$. Hence, $w_i(\sigma^*) > 0$.

Since $\delta \in [0, 1)$, it then follows that $w_i(\sigma^*) > \delta w_i(\sigma^*) \geq 0$.[9] □

Delay may arise if the partition function is not positive (Example 5) or not fully cohesive (Example 6). On the other hand, these are sufficient but not necessary conditions: Example 7 shows that there may be equilibria without delay even if the game is neither positive nor fully cohesive.

Example 5 Delay with a partition function that is fully cohesive but not positive.

$N = \{1, 2, 3, 4, 5\}$, $\varphi(S, \pi) = 6$ if $|S| = 3, 4$ or 5 and 0 otherwise. All players have the same proposer probability.

Delay is possible due to the fact that, once a three-player coalition forms, the remaining two players are indifferent between forming a coalition of their own, forming singletons, or creating some delay by making unacceptable proposals.[10] The delay probability in an equilibrium cannot be too high, or it would be profitable to form a larger coalition.

The following example is based on example 3.1 in Ray and Vohra [64]. It illustrates the possibility of delay if the game is not fully cohesive.[11]

Example 6 Delay with a partition function that is not fully cohesive.

We are going to consider a symmetric game with five players. We depart from our usual notation and denote each coalition by its size.

9 If $\theta_i = 0$, it is easy to see that $w_i = 0$. Player i's expected equilibrium payoff is $w_i = p\delta w_i$, where p is the probability that i receives a proposal in equilibrium. Even if $p = 1$, given that $\delta < 1$, the only solution of this equation is $w_i = 0$.

10 This is a peculiarity of 0 payoffs, which are special since players are indifferent between getting 0 now and getting 0 later. With positive payoffs, for example $\varphi(S, \pi) = 6$ if $|S| = 3, 4, 5, \overline{\varphi}(ijk, l, m) = (6, 1, 1)$ and $\overline{\varphi}(ijk, lm) = (6, 2)$, this indifference does not arise, even if there are no (strict) gains from merging. Once a three-player coalition forms, the next proposer will offer δ to the remaining player and keep $2 - \delta$. Since $\delta < 1$, this is strictly higher than what the proposer can get by forming a singleton.

11 The game in Example 6 is not superadditive. For an example of delay in a superadditive but not fully cohesive game see Example 16 in the appendix. Example 16 also illustrates that equilibria with all combinations of properties (immediate agreement or delay, symmetric or asymmetric) can coexist in the same game.

$N = \{1, 2, 3, 4, 5\}, \overline{\varphi}(3, 2) = (9, 8), \overline{\varphi}(2, 2, 1) = (4, 4, 4), \overline{\varphi}(4, 1) = (9, 0), \varphi(S, \pi) = 0$ for all other π and $S \in \pi$.[12] All players have the same proposer probability.

Consider the following strategy combination. Players 1, 2 and 3 propose coalition $\{1, 2, 3\}$ (offering the responders their continuation value), and players 4 and 5 make unacceptable proposals. After coalition $\{1, 2, 3\}$ is formed, players 4 and 5 form coalition $\{4, 5\}$. For large values of δ, the continuation values associated with these strategy combination are close to 3 (for players 1, 2 and 3) and 4 (for players 4 and 5). Players 4 and 5 prefer to wait (and get almost 4) rather than form a three-player coalition and get about 3. Other deviations are not profitable either: a two-player coalition would be followed by a singleton and another two-player coalition, and thus would earn only 4. A singleton would be followed by a four-player coalition and thus would earn 0.

There cannot be a symmetric equilibrium in which $(3, 2)$ forms immediately. This is because then continuation values would be close to $\frac{17}{5} = 3.4$, so that the proposer of a three-player coalition would earn 2.2, which is less than what he can get by waiting. Interestingly, coalition structure $(3, 2)$ can form without delay in an *asymmetric* equilibrium (see Sect. 4.1).

Example 7 Immediate agreement with a partition function that is neither positive, nor superadditive, nor cohesive.

$N = \{1, 2, 3, 4\}, \overline{\varphi}(ij, kl) = (-1, -1), \overline{\varphi}(ijk, l) = (3, 0), \overline{\varphi}(N) = -1$. All other values are zero. All players have the same proposer probability.

It is easy to see that there is an equilibrium in which all players propose three-player coalitions.

3.2 Formation of the Grand Coalition

In this section, we focus on no-delay SSPE in which the grand coalition always forms (we will say that those equilibria exhibit *immediate formation of the grand coalition*). This type of equilibrium is especially relevant if the grand coalition is the only coalition structure that maximizes aggregate payoffs. However, we will provide examples showing that the grand coalition being the only efficient coalition structure is neither necessary nor sufficient for such an equilibrium to exist (unless $\delta = 0$). We will also show that SSPE with immediate formation of the grand coalition can only occur if the core of a certain game is nonempty.

The following lemma will be useful

Lemma 3 *Let (N, φ) be a partition function game. Suppose there is an SSPE of the game $G(N, \varphi, \theta, \delta)$ with immediate formation of the grand coalition. Then the expected payoff for player i equals*

$$w_i = \theta_i \varphi(N). \tag{2}$$

Proof Given that all players propose the grand coalition and that each responder must receive exactly δw_j in equilibrium, the following equation determines w_i

$$w_i = \theta_i \left[\varphi(N) - \delta \sum_{j \in N \setminus \{i\}} w_j \right] + (1 - \theta_i)\delta w_i.$$

[12] This partition function is not positive, but the example can easily be modified replacing 0 values with a small $\varepsilon > 0$.

Re-arranging terms yields

$$w_i = \theta_i \left[\varphi(N) - \delta \sum_{j \in N} w_j \right] + \delta w_i.$$

Using $\sum_{j \in N} w_j = \varphi(N)$, we obtain $w_i = \theta_i \varphi(N)$. □

3.2.1 Two Examples of Inefficiency

It is well-known that the grand coalition need not form even if it is the only efficient coalition structure [15, 58].

Example 8 $N = \{1, 2, 3\}$, $\overline{\varphi}(1, 2, 3) = (0, 0, 0)$, $\overline{\varphi}(ij, k) = (8, 0)$, $\overline{\varphi}(123) = 9$; $\theta_i = \frac{1}{3}$ for $i = 1, 2, 3$.

Suppose all players propose the grand coalition. Expected payoffs equal 3 according to Eq. (2). This is not an equilibrium for high values of δ because player 1 would prefer to propose to just one other player, offering 3δ and keeping $8 - 3\delta > 9 - 6\delta$.

Conversely, the following example shows that the grand coalition can form even if it is not efficient.

Example 9 $N = \{1, 2, 3, 4, 5\}$. Denoting coalitions by their size, $\overline{\varphi}(5) = 10$, $\overline{\varphi}(2, 3) = (12, 1)$, $\overline{\varphi}(2, 1, 1, 1) = (0, 1, 1, 1)$, $\varphi(S, \pi) = 0$ *for all other* π.

Let $\theta_i = \frac{1}{5}$ for all i. If the grand coalition always forms, each player has an expected payoff of 2. The only coalition that could profit from deviating is a coalition of size 2, but this would only be the case if the coalition of size 2 is followed by a coalition of size 3. Since a coalition of size 2 would be followed by three singletons, no player has an incentive to deviate.

3.2.2 Immediate Formation of the Grand Coalition and the Core

Chatterjee et al. [15] show that, in the limit when δ tends to 1, immediate formation of the grand coalition can only occur in equilibrium if the expected payoff vector (given that all players propose the grand coalition) lies in the core of the characteristic function game. Okada [58] shows an analogous result in his model. This implies that no efficient equilibrium exists for high values of δ if the underlying game is strictly superadditive and has an empty core. In this section, we extend this result to partition function games.

Given a partition function, there are several characteristic functions that can be associated to it. Two well-known possibilities are the optimistic characteristic function (see [70]) and the pessimistic characteristic function (see [2]).

Given a partition function (N, φ), the *optimistic characteristic function* v^+ is defined as

$$v^+(S) \equiv \max_{\substack{\pi \in \Pi(N) \\ \pi \ni S}} \varphi(S, \pi) \text{ for all } S \subseteq N.$$

Analogously, the *pessimistic characteristic function* v^- is defined as

$$v^-(S) \equiv \min_{\substack{\pi \in \Pi(N) \\ \pi \ni S}} \varphi(S, \pi) \text{ for all } S \subseteq N.$$

Proposition 2 establishes a necessary condition for immediate formation of the grand coalition for all values of δ: the expected payoff vector found in Lemma 3 must be in the core of a characteristic function game that assigns to each coalition its expected payoff given some *equilibrium* strategies. We will denote this characteristic function by v^*. Note however that v^* depends on the extensive form game and on the equilibrium strategy vector σ^*, so that a complete notation would be $v^*_{G(N,\varphi,\theta,\delta),\sigma^*}$. The game v^* captures the idea of rational expectations of S about its coalitional payoff. If the coalition structure contains random elements (as will be the case in general, since the coalition structure that forms will depend on which players get the initiative to be proposers) the value v^* will be an average of several values; note also that, since in general the equilibrium payoff of S may depend on δ, v^* is defined as the limit when $\delta \to 1$. Clearly, $Core(v^+) \subseteq Core(v^*) \subseteq Core(v^-)$.

Proposition 2 *Let (N, φ) be a partition function game. Suppose there is a sequence $\delta^k \to 1$ and a corresponding sequence of SSPE $\sigma^*(\delta^k)$ of the game $G(N, \varphi, \theta, \delta^k)$ with immediate formation of the grand coalition. Then the expected payoff vector $w = (\theta_i \varphi(N))_{i \in N}$ is in the core of the characteristic function game (N, v^*), where*

$$v^*(S) := \lim_{\delta \to 1} \sum_{\substack{\pi \in \Pi(N) \\ \pi \ni S}} \sum_{t=1}^{\infty} \delta^{t-1} \mu(\pi^t | (\sigma^*(\delta), S)) \varphi(S, \pi).$$

and $\mu(\pi^t | (\sigma^*(\delta), S))$ is the probability that coalition structure π forms at time t given that players follow the strategy combination $\sigma^*(\delta)$ and that S is the first coalition to form.

Proof From Lemma 3, immediate formation of the grand coalition implies that expected payoffs are $w_i = \theta_i \varphi(N)$ for all $i \in N$ regardless of the value of δ.

Suppose i is selected to be the proposer. If he sticks to the prescribed strategy and proposes the grand coalition, he offers δw_j to each $j \in N \backslash \{i\}$ and keeps $\varphi(N) - \sum_{j \in N \backslash \{i\}} \delta w_j$. If instead he proposes coalition $S \subset N$, he offers δw_j to each player j in $S \backslash \{i\}$. The expected payoff for i will then depend on the payoff coalition S gets in the game. Player i will only propose the grand coalition if, for all $S \ni i$

$$\sum_{\substack{\pi \in \Pi(N) \\ \pi \ni S}} \sum_{t=1}^{\infty} \delta^{t-1} \mu(\pi^t | (\sigma^*(\delta), S)) \varphi(S, \pi) - \sum_{j \in S \backslash \{i\}} \delta w_j \leq \varphi(N) - \sum_{j \in N \backslash \{i\}} \delta w_j \tag{3}$$

Since any player may be selected to be the proposer, Condition (3) must be satisfied for all $i \in N$.

In the limit when $\delta \to 1$ the advantage of the proposer disappears and each player i gets w_i regardless of whether he is a proposer or a responder. Taking into account that $w_j = \theta_j \varphi(N)$, Condition (3) becomes

$$v^*(S) \leq \sum_{i \in S} w_i \text{ for all } S \subset N. \tag{4}$$

that is, the vector w must be in the core of the game (N, v^*). □

Corollary 3 *If the core of v^- is empty, no equilibrium with immediate formation of the grand coalition exists.*

Proposition 2 has a clear interpretation: the grand coalition cannot be an equilibrium for all values of δ if some other coalition can form and increase its payoff given the *equilibrium* reaction of $N \backslash S$. The following examples illustrate the difference between v^+, v^- and v^*.

Example 10 $\overline{\varphi}(1, 2, 3) = (0, 0, 0)$, $\overline{\varphi}(12, 3) = (5, 0)$, $\overline{\varphi}(1, 23) = (4, 5)$, $\overline{\varphi}(13, 2) = (5, 0)$, $\overline{\varphi}(123) = 9$; $\theta_i = \frac{1}{3}$ for $i = 1, 2, 3$.

It is easy to see that $v^*(1) = v^+(1) = 4$ (if player 1 forms a singleton, players 2 and 3 will form a coalition, so 1 can count on a payoff of 4) and $v^-(1) = v^*(i) = v^+(i) = v^-(i) = 0$ for $i = 2, 3$. Moreover, since the formation of a two or three-player coalition determines the partition, $v^-(S) = v^*(S) = v^+(S)$ for all other S. If the grand coalition always forms, each player's expected payoff (and actual payoff when $\delta \to 1$) is 3. The payoff vector (3, 3, 3) is not in the core of v^* (even though it is in the core of v^-), so immediate formation of the grand coalition is not an equilibrium for sufficiently high values of δ ($\delta > \frac{5}{6}$). If player 1 gets the initiative, he prefers to form a singleton and obtain 4.

Example 11 $\overline{\varphi}(1, 2, 3) = (4, 0, 0)$, $\overline{\varphi}(ij, k) = (5, 0)$, $\overline{\varphi}(123) = 9$; $\theta_i = \frac{1}{3}$ for $i = 1, 2, 3$.

This example is very similar to the previous one but now $v^*(i) = v^-(i) = 0$ for all i and $v^+(1) = 4$. There is an SSPE in which the grand coalition forms, even though player 1's expected payoff (and actual payoff when $\delta \to 1$) is less than what he would get if all players were alone (i.e., the equilibrium payoff vector is not in the core of v^+). The reason is that player 1 cannot secure 4 for himself: if he decides to stay alone players 2 and 3 will form a coalition and player 1 will get zero. Moreover, no player would profit from proposing a two-player coalition since any two players get $6 > 5$, so the expected payoff vector is in the core of v^*.

There is no straightforward relationship between the partition function being fully cohesive and the core of v^* being nonempty. The game in Example 10 is fully cohesive but the core of v^* is empty. Example 11 could be modified so that it stops being cohesive by letting $\overline{\varphi}(1, 2, 3) = (10, 0, 0)$ without affecting the nonemptiness of the core of v^*.

4 Further Results

The results in this section can be viewed as negative results. As will be shown below, equilibria of symmetric games are not necessarily symmetric, there are multiple equilibria (and this multiplicity survives even if we focus on symmetric equilibria in characteristic function games), and the comparison with the rule of order bargaining procedure in terms of efficiency is ambiguous (except in symmetric games with the one-stage property, where the random proposer bargaining procedure is at least as efficient as the rule of order bargaining procedure).

4.1 Asymmetric No-delay Equilibria in Symmetric Games

For the game with a rule of order, Ray and Vohra [64] show that continuation values must be symmetric in a no-delay equilibrium. The following example shows that asymmetric no-delay equilibria are possible in the game with random proposers.

Consider the partition function in Example 6. $N = \{1, 2, 3, 4, 5\}$, $\overline{\varphi}(3, 2) = (9, 8)$, $\overline{\varphi}(2, 2, 1) = (4, 4, 4)$, $\overline{\varphi}(4, 1) = (9, 0)$, $\varphi(S, \pi) = 0$ for all other π and $S \in \pi$.

Suppose all players propose $\{1, 2, i\}$, except players 1 and 2 who propose $\{1, 2, 3\}$. All responders are offered their continuation value. Assuming $\theta_i = \frac{1}{5}$ for all i, these strategies induce expected payoffs with $w_1 = w_2$ and $w_4 = w_5$. Taking into account that, after a three-player coalition forms, each of the two remaining players gets 4 on average, expected payoffs are given by

$$w_1 = \frac{1}{5}(9 - \delta w_1 - \delta w_3) + \frac{4}{5}\delta w_1$$
$$w_3 = \frac{1}{5}(9 - 2\delta w_1) + \frac{2}{5}\delta w_3 + \frac{2}{5}4$$
$$w_4 = \frac{1}{5}(9 - 2\delta w_1) + \frac{4}{5}4$$

The solution to this system of equations is $w_1 = \frac{5(9-7\delta)}{4\delta^2-25\delta+25}$, $w_3 = \frac{85-69\delta}{4\delta^2-25\delta+25}$, $w_4 = \frac{34\delta^2-143\delta+125}{4\delta^2-25\delta+25}$. For large values of δ, continuation values are close to 2.5 for players 1 and 2, and 4 for players 3, 4 and 5. It can be checked that each proposer gets at least δw_i, thus no player would rather make an unacceptable proposal. Also, since $w_1 \leq w_3 \leq w_4$, proposers' choice of coalition partners is optimal.

Continuation values in the game with random proposers are influenced by the likelihood of getting proposals. In this example, players who receive proposals more often have lower continuation values (since they are more likely to end up in a three-player coalition, which is less profitable than a two-player coalition), and they get more proposals because of their low continuation values.

Externalities play an important role in this example. Players with high continuation values tend to be excluded from coalitions and, in the absence of externalities, this makes it harder for them to have a high continuation value and tends to equalize expected payoffs in equilibrium.[13] However, with externalities, being less likely to receive proposals is not necessarily a bad thing (it may lead to being in a better coalition later on) and may result in a high continuation value.

4.2 Multiplicity of Symmetric No-delay Equilibria

For symmetric games with a rule of order, Ray and Vohra [64] show that, if δ is sufficiently large, all no-delay equilibria lead to the same coalition structure in terms of the size of the coalitions that form. In this coalition structure, proposers form a coalition with highest per capita payoff (given the anticipated reaction of outsiders).[14] In contrast, under the random proposer bargaining procedure, there can be multiple no-delay equilibria, and this is true even in games without externalities.[15]

Example 12 $N = \{1, 2, 3, 4, 5\}$. Consider the following characteristic function game, where we denote a coalition by the number of players: $v(1) = 0$, $v(2) = 4$, $v(3) = 9$, $v(4) = 11.5$, $v(5) = 13$. Note that this is a superadditive game. Suppose $\theta_i = \frac{1}{5}$ for all i and $\delta \geq \frac{25}{26}$. There are two (symmetric) SSPE of the game with random proposers, leading to partitions [3, 2] and [4, 1] respectively.

In the first SSPE, each player proposes to two other players at random; once a three-player coalition has formed, the remaining two players form a coalition as well. Expected payoffs

[13] For weighted majority games, players of the same type must have the same expected payoff if they are treated symmetrically by the protocol; see lemma 2 in Montero [54].

[14] There may still be multiple equilibria, but other equilibria would have delay (see their example 3.4).

[15] To the best of my knowledge this has not been noted before for $\delta < 1$. Banks and Duggan ([3], example 4) have an example of a game with random proposers and multiple equilibria, but their example does not have transferable utility. Other papers establish uniqueness for certain classes of games (see [24, 25, 55, 61, 72, 73]). Ray and Vohra ([65], p. 294) fail to refer to any example of multiple equilibria with random proposers in their survey.

computed at the start of the game are thus $\frac{v(3)+v(2)}{5} = 2.6$, and continuation values are 2.6δ. Forming a three-player coalition gives the proposer $v(3) - 2 \times 2.6\delta$. If δ is large enough, it does not pay to propose a four-player coalition (one gains 2.5 in terms of coalitional value but has to pay 2.6δ to an additional coalition partner) or a five-player coalition (one gains 4 in coalitional value but has to pay 2.6δ each to two other players). This equilibrium exists for $\delta \geq \frac{25}{26}$.

In the second SSPE, each player proposes to three other players at random, and the remaining player automatically becomes a singleton. Expected payoffs are $\frac{v(4)}{5} = 2.3$. It does not pay to form a smaller coalition (one loses 2.5 in value and saves only 2.3δ in payments to the other players) and, if players are sufficiently patient, it does not pay to form the grand coalition (one gains 1.5 but has to pay 2.3δ). This equilibrium exists for $\delta \geq \frac{15}{23}$.

4.3 Random Proposers Versus Rule of Order

The essential difference between random proposers and rule-of-order bargaining is not whether the first proposer is selected deterministically or at random, but how a proposer is selected after a rejection. With random proposers, continuation values equal *expected payoffs* multiplied by δ. In contrast, with a rule of order, continuation values equal *the payoff of being proposer* multiplied by δ. Because a player gets his maximum payoff when he is the proposer, continuation values may be "too high" with a rule of order, leading to delay even in positive and fully cohesive games (see [15] and [58]). Also, since a player can retain the initiative by rejecting a proposal regardless of the underlying situation, the bargaining procedure with a rule of order leads to little competition between the players and a tendency to equal division, even in very asymmetric situations such as one seller facing multiple buyers (see example 2 and footnote 12 in [15]). On the other hand, it can be argued that the advantage of the proposer in the game with random proposers is too high: it does not necessarily disappear as δ tends to 1, unlike in the game with a rule of order. In general, the two bargaining procedures cannot be ranked in terms of efficiency, as the examples below illustrate. We also show that the random proposers procedure is at least as efficient for symmetric characteristic function games with the one-stage property (Proposition 3).

Example 13 *(Greater efficiency with random proposers)* Consider the characteristic function game with $N = \{1, 2, 3, 4, 5\}$, $v(S) = 18$ for $|S| = 4$, $v(S) = 14$ for $|S| = 3$ and 0 otherwise.

Recall that Ray and Vohra [64] show that, for symmetric games and provided that the equilibrium exhibits no delay, players form the coalition that maximizes the expected per capita payoff given the reaction of outsiders. Since coalitions of size 3 have the maximum per capita payoff in this example, the total equilibrium payoff is 14.

Consider the game with random proposers with $\theta_i = \frac{1}{5}$ for all i. If players always proposed a three-player coalition, each player's expected payoff would be $\frac{14}{5}$, and the payoff of the proposer would be $14 - 2\delta\frac{14}{5}$. However, he would get an even higher payoff by proposing a coalition of four $(18 - 3\delta\frac{14}{5})$. In equilibrium, only coalitions of four players form. Expected payoffs are then $\frac{18}{5} = 3.6$, so that a deviation to proposing a coalition of three is not profitable. The total equilibrium payoff is $18 > 14$.

The reason why larger coalitions with lower per capita payoffs may form is that a player who rejects a proposal will be left out with positive probability, and this negatively affects his continuation value. Since the responders are paid less than an equal share of the value of the coalition that forms, forming large coalitions may be profitable. This example points to a trade-off between efficiency and equitable distribution, since payoff division within the four-player coalition remains unequal even as $\delta \to 1$.[16]

Proposition 3 generalizes Example 13 to symmetric characteristic function games with the one-stage property,[17] that is, to games where only one coalition with $v(S) > 0$ can be formed.

Proposition 3 *Let (N, v) be a symmetric characteristic function game such that $v(S) > 0$ implies $v(T) = 0$ for all $T \subseteq N \backslash S$. The equilibrium of the game with random proposers and $\theta_i = \frac{1}{n}$ for all i is at least as efficient as the equilibrium of the game with a rule of order.*

Proof Let k be the largest coalition size among those with the highest per capita payoff. A coalition of size k always forms with a rule of order. We now show that a coalition with lower total payoff cannot form with positive probability in the game with random proposers.

For games with the one-stage property, symmetry of the game and of the protocol imply that all players have the same expected payoff w.[18] Denote the value of a coalition of size m by $v(m)$, and the probability that a coalition of size m is formed in this hypothetical equilibrium by λ_m; then $w = \frac{\sum_{m=1}^{n} \lambda_m v(m)}{n} \le \frac{v(k)}{k}$. If $v(k) = 0$, no coalition with positive value can form and total equilibrium payoffs are 0 for both bargaining procedures. If $v(k) > 0$, it follows that $\delta w < \frac{v(k)}{k}$.

A coalition of cardinality l with $v(l) < v(k)$ cannot form in equilibrium in the game with random proposers. The reasons are obvious for $l > k$. For $l < k$, the proposer would always want to enlarge the coalition to k players, since doing so would increase the value of the coalition by at least $(k - l)\frac{v(k)}{k}$, while he would only have to pay $(k - l)\delta w$. □

Proposition 3 requires the game to be symmetric. If the game is not symmetric, it is possible for the equilibrium of the rule of order to be more efficient, as the following example shows.

Example 14 *(Greater efficiency with a rule of order; characteristic function game is not symmetric)* $N = \{1, 2, 3, 4, 5\}$; $v(1, i) = 1 + \alpha$ for $i = \{2, 3, 4, 5\}$; $v(2, 3, 4, 5) = 1$, $v(N) = 1 + \alpha$, where $\alpha > 0$.

This game is similar to an apex game (see [20]), except that the coalition of the apex game (player 1) with a minor player has a greater payoff than the coalition $N \backslash \{1\}$ of the minor players. Let $\delta \to 1$. With a rule of order, the apex player and one minor player form a coalition and divide the payoff equally. With random proposers and $\theta_i = \frac{1}{5}$ for all i, all SSPE are in mixed strategies, with minor players randomizing between proposing $\{1, i\}$ and $N \backslash \{1\}$ (analogously to [54]).[19] Because the minor player coalition forms with positive probability for any $\alpha > 0$, there is inefficiency.

Proposition 3 also requires the game to have the one-stage property, that is, once a coalition forms no other coalition is profitable. If several disjoint coalitions are profitable, the equilibrium with a rule of order may be more efficient, as the following example shows.

Example 15 *(Greater efficiency with a rule of order; characteristic function game does not have the one-stage property)* $N = \{1, 2, 3, 4, 5\}$. We denote a coalition by its size. $v(1) = 0$, $v(2) = 5$, $v(3) = 14$, $v(4) = 18$, $v(5) = 19$.

16 If instead payoff division was fixed as in Bloch [9], there would be little difference between the two protocols. Since the proposer gets the same payoff as the rest of players in a coalition, he would choose the coalition of maximum per capita payoff given the reaction of outsiders in both cases. If payoff division is fixed but not egalitarian, the two bargaining procedures may make very different predictions (see [57]).

17 This terminology comes from Selten [69].

18 These games are covered by the uniqueness result of Yan [73].

19 There are multiple SSPE in terms of strategies, but they all lead to the same payoffs.

In the game with a rule of order, a coalition of size 3 will form (since it has the highest per capita value), followed by a coalition of size 2. Total payoffs are then 19. However, in the game with random proposers and $\theta_i = \frac{1}{5}$ for all i, a four-player coalition is formed with probability 1, so that total payoffs are only 18.

Kawamori ([40], theorem 2) also discusses the effect of rejectors becoming the next proposer on efficiency in characteristic function games. He focuses exclusively on whether immediate formation of the grand coalition occurs, rather than on expected equilibrium payoffs more generally. An implication of his theorem is that, if the grand coalition forms immediately in the game with a rule of order, then it must also form immediately in the game with random proposals with $\theta_i = \frac{1}{n}$. Proposition 3 strengthens this result for the special class of symmetric games with the one-stage property and shows that, even if the grand coalition does not form, total payoffs will be at least as high in the game with random proposers. Outside this class of games, Example 15 shows that replacing $\theta_i = \frac{1}{n}$ with a rejector-proposer protocol can help increase total payoffs, though it can never help with immediate formation of the grand coalition.

5 Related Literature

The reader is referred to Bloch [10] and Ray and Vohra [65] for more extensive surveys.

The early literature on coalition formation in games with externalities assumes that players have either optimistic or pessimistic conjectures about the reaction of outsiders. Some of these papers are based on an underlying normal-form game, and players who form a coalition can coordinate their actions in the normal-form game and transfer payoffs between themselves; the reaction of outsiders then refers to the strategies of players outside the coalition. Others, like the present paper, take the partition function as a primitive,[20] and the reaction of outsiders then refers to the way players outside the coalition partition themselves.

Shenoy [70] considers optimistic conjectures in partition function games. In his dominance concept, a partition is dominated if there is a coalition that would be better off in another partition; there is no consideration of the incentives of outsiders to form the latter partition. In a common pool game, Funaki and Yamato [27] construct a partition function under the assumption that coalitions play Nash against each other and consider both optimistic and pessimistic conjectures about the partition of outsiders. They find that the core is always nonempty under pessimistic conjectures but may be empty if conjectures are optimistic. Abe and Funaki [1] present conditions for the nonemptiness of the core for general partition function games under each of the two conjectures.

Aumann and Peleg [2] define the α-core and the β-core, both of which assume pessimistic conjectures in a normal-form game (outsiders coordinate and play a strategy that minimizes the payoff of the coalition). Under these extremely pessimistic conjectures, Meinhardt [47] shows that common pool games are convex.

An alternative to optimism or pessimism is to assume that coalitions always expect outsiders to form a coalition of their own, or to remain singletons, regardless of their own interest. Both possibilities are considered by Hart and Kurz [34] in their simultaneous games of coalition formation (see also [19, 74, 75]) and by Rajan [62] in the context of coalition formation in oligopoly. Chander and Tulkens [14] define the γ-characteristic function and the associated γ-core; these concepts are based on the assumption that players outside the coalition remain

[20] In applications, the partition function is often derived from a normal form game under the assumption that coalitions act as single players and play a Nash equilibrium against each other; see Ray and Vohra [65].

singletons. They show the nonemptiness of this core in an economy with environmental externalities. The δ -characteristic function and the associated δ-core are defined analogously, with players in $N\backslash S$ forming a coalition of their own; this is the assumption in Maskin [50].

Modifications of the optimistic and pessimistic cores have been studied by Koczy [43, 45] and Huang and Sjostrom [37, 38]. These concepts are recursive, that is, they assume that if a coalition forms the remaining players will choose an allocation in the core of the reduced game.

Hafalir [33] introduces a "rational expectations" core. This concept assumes that outsiders organize themselves in a partition that maximizes their total payoff; the same assumption is in Borm et al. [13]. The idea of the v^* characteristic function is similar since S moves first and players in $N\backslash S$ react rationally. However, since there are no side payments between coalitions, it is not necessarily the case that $N\backslash S$ would partition itself so as to maximize its total payoff in an SSPE as we have seen; all we know is that $N\backslash S$ will partition itself according to an SSPE of the reduced game played by players in $N\backslash S$.

Okada [60] takes a normal form game as a starting point. A proposal to form a coalition includes a correlated strategy for the normal form game. He shows that in a totally efficient equilibrium (i.e., one in which the grand coalition forms not only in the main game but also in all reduced games) equilibrium payoffs belong to what he calls the Nash core. In this core concept, after S sets a correlated strategy $N\backslash S$ chooses the Nash bargaining solution of the reduced game. The core of the game v^* is similar to the Nash core in that coalition S anticipates an equilibrium reaction of $N\backslash S$. However, there are some differences. Since the proposal to form S includes a correlated strategy, the strategies played in the normal form game (and subsequent payoffs) may depend on the order in which coalitions formed, while in a partition function game this is not the case. Proposition 2 only requires that the grand coalition forms in the overall game, while Theorem 2 in Okada [60] requires that it forms at all subgames. The price to pay is potential multiplicity of equilibria. There may be two equilibria, σ and σ', such that the grand coalition forms in equilibrium σ (and then Proposition 2 implies that payoffs are in the core of the corresponding v^*_σ) but a different coalition structure forms in σ'.

Bloch and van den Nouweland [12] approach the question of constructing a characteristic function from a partition function axiomatically, and discuss the properties of several expectation formation rules, including (among others) optimism, pessimism, external players forming singletons and external players forming a complementary coalition. Among other results, they show that what they call exogenous rules (including assuming that players in $N\backslash S$ stay together or split into singletons, as in the γ and δ characteristic functions) satisfy a notion of consistency. The game v^* can be interpreted as yet another expectation formation rule, with the caveat that v^* depends on θ and, if there are multiple equilibria, on σ^*, so that different equilibria of the game may be associated to different expectations for the coalition S. In cases where v^* is invariant to the choice of θ and σ^* (as is the case in Example 10), we can talk about an equilibrium characteristic function v^*, and emptiness of its core implies that the grand coalition cannot form immediately in any SSPE.

The present paper belongs to the class of models that combine an extensive form game with a partition function, so that the reaction of outsiders is an equilibrium reaction. Building on Rubinstein [66] and Chatterjee et al. [15], Ray and Vohra [64] assume that players respond to proposals according to a predetermined *rule of order* and the first player to reject a proposal automatically becomes the next proposer. Bloch [9] differs from Ray and Vohra [64] in that payoff division is fixed; in particular, players are assumed to divide payoffs equally in symmetric games. The assumption of exogenous payoff division is common in applications (e.g. [5, 48, 67]). Kóczy [44] finds a connection between the equilibria of a modification of

Bloch's [9] game and the recursive pessimistic core. De Clippel and Serrano [21] resembles Bloch [9] in that payoff division is fixed (based on de Clippel and Serrano [22]) and moves are sequential, but allows several coalitions to form in parallel.

I use an alternative model of bargaining that was introduced by Binmore [7] for two players and extended by Okada [58] to characteristic function games; I extend it to partition function games. The distinctive feature of this model is that a player who rejects an offer does not automatically become the next proposer. Instead, proposers are selected randomly by Nature. A consequence is that a player's continuation value is tied to the player's ex ante expected payoff, while in a rule of order game it is tied to the player's payoff as a proposer. These differences translate into different results in terms of immediate agreement, and uniqueness, symmetry and efficiency of the equilibrium as we have seen.

As most of the literature on coalitional bargaining, I assume that payoffs are transferable within coalitions (but not between coalitions) and coalitions cannot be enlarged once formed, perhaps because forming a coalition entails sunk costs. At the other extreme, if renegotiation is costless it will eventually lead to an efficient outcome in cohesive games (see Seidmann and Winter [68] and Okada [59] for characteristic function games, and Gomes [28] and Hyndman and Ray [39] for partition function games). In Bloch and Gomes [11], renegotiation is possible in principle but coalitions can take irreversible actions that commit them to not be enlarged; this possibility leads to inefficiency. There are also models where coalitions can break up [31].

The possibility of renegotiation does not always increase efficiency. Gomes ([29], Example 4) has an example with three players where each of the players prefers that the other two form a coalition. This leads to a war of attrition and multiple equilibria, all with delay. In the present model there would be immediate agreement because the first proposer would form a singleton (and would be committed to remain this way since coalitions cannot be enlarged).

I have assumed that players share a common discount factor. An alternative source of friction studied in the literature is the risk of breakdown (see [8]). In Montero [53] I consider the possibility of random termination of the negotiations after a rejection, in which case all players who have not formed a coalition yet become singletons. The results are qualitatively similar, though the actual equilibria can be different. Kawamori and Miyakawa [41, 42] study a variant of the model of Hart and Mas-Colell [35] where only the rejector becomes a singleton with some probability.[21]

There are several other papers involving extensive form games and a partition function. Cornet [17] studies a three-player model based on Binmore [6], where players make a payoff demand for themselves rather than propose a coalition. McQuillin ([52], theorem 3) provides support for the expectation that external players form the complementary coalition. He shows that in a simplified version of the bilateral coalescence model of Gul [32] applied to partition function games, the only values that turn out to be relevant in the limit as players become arbitrarily patient are the values of the partition $\{S, N\backslash S\}$. There are also bargaining mechanisms in the literature that always achieve efficiency, either conditional on the grand coalition being efficient [49] or more generally [33]. The aim of these mechanisms is to implement a particular extension of the Shapley value to partition function games.

Finally, a different strand of the literature consists of models that impose less structure about the timing of moves, based on farsighted concepts such as equilibrium binding agreements [23, 63] the largest consistent set [16, 46, 51], and the farsightedly stable set [36].

[21] Recently there has been renewed interest in models where negotiations to form a coalition can break down, resulting in bargaining returning to the starting point (see [30, 56]). These models have not been extended to partition function games.

6 Concluding Remarks

As in all noncooperative bargaining games, the results are dependent on the specific assumptions of the model. Perhaps the most controversial assumption is that coalitions cannot be enlarged, while in some economic applications such as mergers or international trade agreements the membership of the coalition can be gradually expanded. The model is best suited to situations where forming a coalition involves what Seidmann and Winter [68] call "irreversible actions" as would be the case with agreements to standardize equipment (e.g. railway gauges).

Section 3 of this paper extends the results of the random proposer bargaining model of Okada [58] from characteristic function games to partition function games. A key feature of random proposer bargaining is that continuation values are closely connected to expected payoffs. This guarantees immediate agreement for fully cohesive games, though the final outcome is not necessarily efficient. If the grand coalition is the only efficient coalition structure, efficiency can only be achieved if the core of the characteristic function game that takes the equilibrium reaction of the remaining players into account is nonempty.

Section 4 of this paper shows that random proposer bargaining games may have multiple equilibria, and this multiplicity is not necessarily connected with externalities or with delay as we have seen in Sect. 4.2. Compared with the rule of order bargaining procedure, coalition payoffs are divided equally in the limit with a rule of order (provided players are symmetric and agreement is immediate), whereas the game with random proposers typically yields a proposer advantage. While this may increase efficiency in some games as Proposition 3 shows, the two procedures cannot be ranked in terms of efficiency in general.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

7 Appendix: Delay with a Superadditive but not Fully Cohesive Partition Function

Example 16 $N = \{1, 2, 3, 4, 5, 6, 7\}$, $\theta_i = \frac{1}{7}$ for all i. Coalitions are denoted by their size.

$\overline{\varphi}(3, 3, 1) = (6, 6, 4)$
$\overline{\varphi}(4, 2, 1) = (0, 1, 1)$
$\overline{\varphi}(4, 3) = (10, 2)$
$\overline{\varphi}(5, 2) = (1, 1)$
$\overline{\varphi}(6, 1) = (12, 0)$

$\overline{\varphi}(7) = 12$

All other payoffs are 0. Note that this partition function is superadditive.[22]

There is an equilibrium with delay for $\delta \geq \frac{28}{29}$. This example is based in the following idea: a coalition structure of type (3, 3, 1) is going to arise in this equilibrium. In order for this to be the case, coalitions have to form in a given order: the singleton coalition is the most attractive, but it cannot form first because then a coalition of size 6 would follow and the singleton would have a payoff of 0. Thus, a coalition of size 3 must form first, followed by the singleton. On the other hand, the coalition of size 3 cannot form immediately in a symmetric equilibrium, because then expected equilibrium payoffs would be $\frac{6+6+4}{7} = \frac{16}{7}$ for each player, and the proposer would get approximately $6 - 2 \times \frac{16}{7} < \frac{16}{7}$ and would be better off waiting for someone else to form a coalition. Thus, in order to construct an equilibrium with coalition structure (3, 3, 1), we must give up on symmetry, immediate agreement or both.

We now construct an asymmetric equilibrium in which some players form a coalition of size 3 and others wait, leading to delay.[23] In this equilibrium, players 1, 2 and 3 propose coalition $\{1, 2, 3\}$ if selected to be proposers, and players 4, 5, 6 and 7 make an unacceptable proposal. After coalition $\{1, 2, 3\}$ has formed, the next proposer forms a singleton, and the remaining three players form a three-player coalition. Off the equilibrium path, a singleton is followed by a six-player coalition and a coalition of four players is followed by a singleton and a two-player coalition.

In order to show that these strategies constitute an equilibrium, we calculate the continuation values induced by the strategies at every subgame and check that proposers cannot do better by proposing a different coalition or by waiting (i.e., by making unacceptable proposals).

Step 1. A coalition of size 3 is indeed followed by a singleton and by another coalition of size 3. Therefore, $v^*(3) = 6$.

Starting at the end of the game, suppose a three-player coalition has formed, followed by a singleton. It is optimal for the remaining players to form a three-player coalition, since all other coalitions have a value of 0.

Going back to the previous stage, suppose a three-player coalition has formed. Given the strategies, each remaining player has a continuation value of $\frac{10\delta}{4} = 2.5\delta$. Will the next proposer form a singleton? Since a singleton will be followed by another three-player coalition, its payoff is 4. A two-player coalition would lead to a negative payoff for the proposer. A three-player coalition would lead to a payoff of $6 - 5\delta$. A four-player coalition would lead to $10 - 7.5\delta$. If $\delta \geq \frac{4}{5}$, forming a singleton is optimal.

Step 2. A singleton is followed by a six-player coalition, thus $v^*(1) = 0$.

If players form the six player coalition, each player's continuation value is 2δ, and given this value it is optimal for the proposer to propose a six-player coalition (rather than for example a three-player coalition) regardless of the value of δ. Thus, a singleton can expect a payoff of 0.

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[22] The game in example 16 is not cohesive, but we can obtain delay in a cohesive game using this example as a subgame: consider a symmetric game with 15 players such that coalitions of size 8 earn a payoff of 800, the value of the grand coalition is 816, and given that a coalition of 8 players forms the payoffs for the remaining players are as in example 16. The game can also be made positive by replacing the 0 payoffs by small positive numbers.

[23] Thus delay occurs with positive probability but not for sure. With stationary strategies, delay with probability 1 means perpetual disagreement and this can never happen in equilibrium if $\varphi(N) > 0$.

Step 3. Analogously it can be shown that, after a coalition of 4 players forms, followed by a singleton, it is optimal for a coalition of two players to form. Anticipating that a two-player coalition will follow, it is optimal to form the singleton if $\delta \geq \frac{3}{4}$. Hence, $v^*(4) = 0$ for $\delta \geq \frac{3}{4}$. Similarly, a coalition of 2 or 5 players can get only 1.
We have shown that the strategies described are optimal in subgames where one coalition has already formed. The characteristic function associated to the strategy we have constructed is $v^*(1) = 0$, $v^*(2) = 1$, $v^*(3) = 6$, $v^*(4) = 0$, $v^*(5) = 1$, $v^*(6) = v^*(7) = 12$. It remains to show that the strategies are optimal also at the beginning of the game.

Step 4. At the beginning of the game, it is optimal for players 1, 2 and 3 to form $\{1, 2, 3\}$, and for the remaining players to wait.
First we compute the expected payoff of players 1, 2 and 3 (denoted by w_l) and that of players 4, 5, 6 and 7 (denoted by w_h). Let i be the selected proposer. With probability $\frac{3}{7}$, $i \in \{1, 2, 3\}$ and expected payoffs are 2 for the first three players and 2.5 for the last four. With probability $\frac{4}{7}$, $i \in \{1, 2, 3, 4\}$ and each player's payoff is the continuation value, thus

$$w_l = \frac{3}{7}2 + \frac{4}{7}\delta w_l$$
$$w_h = \frac{3}{7}2.5 + \frac{4}{7}\delta w_h$$

The solution is $w_l = \frac{6}{7-4\delta}$ and $w_h = \frac{15}{2(7-4\delta)}$. Continuation values are then $\frac{6\delta}{7-4\delta}$ and $\frac{15\delta}{2(7-4\delta)}$ respectively.

Consider now the situation of the proposer. We have established above that the payoff of a singleton is 0. As for a two-player coalition, it can earn a payoff of at most 1 according to the partition function. If a player is going to form a three-player coalition, he will propose to two of the players with a low continuation value. The total payoff for the coalition will be 6 as established in step 1, and thus the payoff for the proposer will be $6 - 2\delta w_l = \frac{6(7-6\delta)}{7-4\delta}$. A coalition of size 4 or 5 would get a payoff of 0 or 1. A coalition of size 6 gets 12, and, given that the proposer includes players with the lowest continuation values, the payoff for the proposer is then $12 - 3\delta w_l - 2\delta w_h = \frac{3(28-27\delta)}{7-4\delta}$ if $i \in \{4, 5, 6, 7\}$ or $12 - 2\delta w_l - 3\delta w_h = \frac{3(56-55\delta)}{2(7-4\delta)}$ if $i \in \{1, 2, 3\}$. The grand coalition is dominated by a coalition of size 6. Finally, the proposer can make an unacceptable proposal and obtain his continuation value.

If the proposer $i \in \{1, 2, 3\}$, the best of these options is to form coalition $\{1, 2, 3\}$ provided that $\delta \geq \frac{28}{31}$. If $i \in \{4, 5, 6, 7\}$, the best option is to wait and get the continuation value w_h provided that $\delta \geq \frac{28}{29}$.

There is also a symmetric equilibrium with delay that results in the same type of partition. In this equilibrium, each of the players randomizes between making an acceptable proposal to form a coalition of size 3 and making an unacceptable proposal. Once a coalition forms, strategies in the reduced game are as in the previous equilibrium, so all the v^* values are as before.

In order for this to be an equilibrium, players must be indifferent between forming a three-player coalition and creating delay, hence $6 - 2\delta w = \delta w$, which implies $w = \frac{2}{\delta}$; the continuation value is then $\delta w = 2$. We now construct an equilibrium that results in $w = \frac{2}{\delta}$ at the start of the game. Suppose each player proposes the three–player coalition with probability λ and makes an unacceptable proposal with probability $1 - \lambda$. Expected payoffs w must solve the following equation. With probability $\frac{1}{7}$, the player is selected as a proposer and gets $6 - 2\delta w$ with probability λ and δw with probability $1 - \lambda$. With probability $\frac{6}{7}$, another player is selected. It may then be the case that this player forms a three-player coalition involving i (this happens with probability $\lambda\frac{2}{6}$), a three-player coalition not involving i (in which case the rest of the coalition structure forms without delay and i expects $\frac{10}{4}=2.5$), or creates delay (in which case i gets δw). Thus $w=\frac{1}{7}(\lambda(6-2\delta w)+(1-\lambda)\delta w)+\frac{6}{7}(\lambda\frac{2}{6}\delta w+\lambda\frac{4}{6}2.5+(1-\lambda)\delta w)$. Using $w=\frac{2}{\delta}$ we can solve for the equilibrium value of λ, which is $\frac{7(1-\delta)}{\delta}$. This equilibrium exists if $\delta \geq \frac{7}{8}$ (otherwise λ would be above 1; note also that total expected payoffs $\frac{14}{\delta}$ cannot be above 16). Creating delay or proposing a three-player coalition is optimal; the only other alternative is to form a coalition of size 6, but this leads to the same payoff, $12-5\delta\frac{2}{\delta}=2$.

There is also an asymmetric no-delay equilibrium, similar to the example in Sect. 4.1. All players propose coalition $\{1,2,i\}$, except 1 and 2, who propose $\{1,2,3\}$. Unlike the others, this equilibrium is efficient since it involves immediate agreement and a coalition structure with the highest possible total payoff. Expected payoffs are $w_1=w_2=\frac{14(3-2\delta)}{8\delta^2-49\delta+49}$, $w_3=\frac{4(28-23d)}{8\delta^2-49\delta+49}$, $w_4=w_5=w_6=w_7=\frac{32\delta^2-159\delta+147}{8\delta^2-49\delta+49}$. As $\delta \to 1$, these values converge to 1.75 for the first two players and 2.5 for everyone else. This is an equilibrium since $w_2 \leq w_3 \leq w_4$, implying that players are proposing to the cheapest available coalition partners. Furthermore, none of the players can profit from forming an alternative coalition (such as a coalition of size 6) or creating delay. This equilibrium exists for $\delta \geq \frac{4}{5}$ (see step 1 above).

Finally, there is a symmetric no-delay equilibrium with a different coalition structure. A coalition of size 6 forms immediately, each player's expected payoff is $w=\frac{12}{7}$ and, since this payoff is quite small (about 1.71), the proposer gets a high payoff (about 3.43) and has no incentive to propose a smaller coalition or cause delay.

The example also shows that all combinations of equilibrium properties (symmetric, asymmetric, immediate agreement and delay) can occur for the same game if δ is sufficiently large.

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