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Constructing Patterns of (Many) ESSs Under Support Size Control

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Abstract

As is well known, equilibrium analysis of evolutionary partnership games can be done by studying a so-called standard quadratic optimization problem, where a possibly indefinite quadratic form is maximized over the standard (probability) simplex. Despite the mathematical simplicity of this model, the nonconvex instances in this problem class allow for remarkably rich patterns of coexisting (strict) local solutions, which correspond to evolutionarily stable states (ESSs) in the game; seen from a dynamic perspective, ESSs form the asymptotically stable fixed points under the continuous-time replicator dynamics. In this study, we develop perturbation methods to enrich existing ESS patterns by a new technique, continuing the research strategy started by Chris Cannings and coworkers in the last quarter of the past century.

Keywords Local solutions · Quadratic optimization · Evolutionary stability · Global optimization

1 Introduction

1.1 Motivation

We consider the *Standard Quadratic Optimization Problem (StQP)* given by

$$\max_{\mathbf{x}\in\Delta^n} \mathbf{x}^\top \mathsf{A}\mathbf{x} \tag{1}$$

where $\mathsf{A} \in \mathcal{S}^n$ ($\mathcal{S}^n$ denoting symmetric $n \times n$-matrices) and Δ^n is the standard-simplex

$$\Delta^n = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1,\ x_i \geq 0 \quad \text{for all } i \in N \right\},$$

where $N = \{1, \ldots, n\}$, also denoted as $[1\!:\!n]$.

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Despite its mathematical simplicity, above model serves in different domains of applied mathematics: apart from straightforward applications like in portfolio optimization [15], it can be used in mathematical biology [12,13] as well as in Evolutionary Game Theory and Game Dynamics [1,11,17,20]. For a more detailed account on the interrelation between these domains, we refer to [3,4] and references therein. Here, we want to focus on perturbation approaches to generate richer ESSs patterns from existing ones, building upon the seminal work of Chris Cannings and coworkers [6–9,18,19], as well as upon the more recent study [5].

1.2 Notation and Preliminaries

Given any set $S \subseteq \mathbb{R}^n$, we denote by conv S the convex hull of S, and the linear hull is denoted by span S, while closure and interior of S are denoted by cl S and S°, respectively. For a (convex) set C, the set of its extreme points is denoted by Ext C. Later, we will use the relation Ext conv $S \subseteq S$ holding for any set $S \subseteq \mathbb{R}^n$. For matrices $\{\mathsf{A}, \mathsf{B}\} \subset \mathcal{S}^n$ we write $\mathsf{A} \prec \mathsf{B}$ or $\mathsf{B} \succ \mathsf{A}$ if $\mathsf{B} - \mathsf{A}$ is positive-definite, and likewise $\mathsf{A} \preceq \mathsf{B}$ or $\mathsf{B} \succeq \mathsf{A}$ if $\mathsf{B} - \mathsf{A}$ is positive-semidefinite (in particular, if either of them coincides with the zero matrix denoted by O in this paper). Treating $\{\mathsf{A}, \mathsf{B}\}$ as points in a Euclidean space, we consider the *Frobenius* inner product $\mathsf{A} \bullet \mathsf{B} := \mathrm{trace}(\mathsf{AB})$ and the norm $\|\mathsf{A}\| = \sqrt{\mathsf{A} \bullet \mathsf{A}}$; as usual, we abbreviate $\mathsf{A} \perp \mathsf{B}$ for $\mathsf{A} \bullet \mathsf{B} = 0$. Additional usual notation involving (sometimes also non-square or non-symmetric) matrices is $\ker \mathsf{A} := \{\mathbf{v} : \mathsf{A}\mathbf{v} = \mathbf{o}\}$ and $\mathrm{colspace}(\mathsf{A}) := \left\{\mathsf{A}\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\right\}$. Returning to $\mathsf{A} \in \mathcal{S}^n$, by

$$\mathrm{SOL}(\mathsf{A}) := \left\{\bar{\mathbf{x}} \in \Delta^n : \bar{\mathbf{x}}^\top \mathsf{A}\bar{\mathbf{x}} = \max_{\mathbf{x} \in \Delta^n} \mathbf{x}^\top \mathsf{A}\mathbf{x}\right\} = \mathrm{Argmax}\left\{\mathbf{x}^\top \mathsf{A}\mathbf{x} : \mathbf{x} \in \Delta^n\right\}$$

we denote the set of all global solutions to (1). For $n \geq 2$ and $\mathsf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathcal{S}^n$ define

$$\mathcal{B}(\mathsf{A}) := (\mathbf{e}\,\mathbf{a}_n^\top + \mathbf{a}_n\,\mathbf{e}^\top - \mathsf{A} - a_{nn}\mathbf{e}\,\mathbf{e}^\top)_{[1:n-1]\times[1:n-1]} = -(\mathsf{I}_{n-1} \;\; -\mathbf{e})\mathsf{A}\begin{pmatrix}\mathsf{I}_{n-1}\\ -\mathbf{e}^\top\end{pmatrix},$$

where $\mathsf{I}_k \in \mathcal{S}^k$ is the identity matrix, and $\mathbf{e}$ denotes the all ones vector of appropriate dimension. We note

$$\mathcal{B}(\mathsf{A} + \lambda\mathbf{e}\mathbf{e}^\top) = \mathcal{B}(\mathsf{A}) \quad \text{for all } \lambda \in \mathbb{R} \tag{2}$$

and

$$\bar{\mathbf{v}}^\top \mathcal{B}(\mathsf{A})\bar{\mathbf{v}} = -\mathbf{v}^\top \mathsf{A}\mathbf{v} \quad \text{whenever} \quad \mathbf{v} \perp \mathbf{e} \text{ and } \bar{\mathbf{v}} = \mathbf{v}_{[1:n-1]}\,. \tag{3}$$

Clearly we have rank $\mathsf{A} \geq$ rank $\mathcal{B}(\mathsf{A})$. Furthermore, for $\mathbf{x} \in \Delta^n$, let

$$I(\mathbf{x}) = \{i \in N : x_i > 0\}$$

be the *support* of $\mathbf{x}$, and $\Delta_I := \{\mathbf{x} \in \Delta^n : I(\mathbf{x}) \subseteq I\}$. The *extended support* with respect to A is given by

$$J_{\mathsf{A}}(\mathbf{x}) = \left\{i \in N : [\mathsf{A}\mathbf{x}]_i = \mathbf{x}^\top \mathsf{A}\mathbf{x}\right\}\,.$$

We will use some more convenient notation (again, we refer to [3] for background on the interplay of optimization and game equilibrium notions): given $\mathsf{A} \in \mathcal{S}^n$, we denote by

$$\mathrm{ESS}(\mathsf{A}) := \left\{\mathbf{x} \in \Delta^n : \mathbf{x} \text{ is an ESS for } \mathsf{A}\right\}$$

the set of all strict local maximizers of (1), which is always finite but may be empty; by

$$\mathrm{NSS}(\mathsf{A}) := \left\{\mathbf{x} \in \Delta^n : \mathbf{x} \text{ is an NSS for } \mathsf{A}\right\}$$

the set of all local maximizers of (1), and by

$$\mathrm{NES}(\mathsf{A}) := \left\{ \mathbf{x} \in \Delta^n : \mathbf{x} \text{ is an NES for } \mathsf{A},\ \text{ i.e., } \mathsf{A}\mathbf{x} \le (\mathbf{x}^\top \mathsf{A}\mathbf{x})\,\mathbf{e} \right\}$$

the set of all KKT points of (1). The latter two sets are never empty; however, they may be infinite. Given $\mathsf{A} \in \mathcal{S}^n$ with finite $\mathrm{NES}(\mathsf{A})$, the procedures `FINDEQ` and `CHECKSTAB` discussed, e.g., in [2,5], find all points in $\mathrm{NES}(\mathsf{A})$ and all members of $\mathrm{ESS}(\mathsf{A})$, respectively. Obviously,

$$\mathrm{SOL}(\mathsf{A}) \subseteq \mathrm{ESS}(\mathsf{A}) \cup \mathrm{SOL}(\mathsf{A}) \subseteq \mathrm{NSS}(\mathsf{A}) \subseteq \mathrm{NES}(\mathsf{A}) \tag{4}$$

with the equality $\mathrm{SOL}(\mathsf{A}) = \mathrm{NSS}(\mathsf{A}) = \mathrm{NES}(\mathsf{A})$ if A is negative-semidefinite. For indefinite A these sets may differ. We furthermore denote

$$\mathrm{pattern}(\mathsf{A}) := \{ I(\mathbf{x}) : \mathbf{x} \in \mathrm{ESS}(\mathsf{A}) \}$$

and a subset thereof, containing only the supports of the *quasistrict* members of $\mathrm{ESS}(\mathsf{A})$,

$$\mathrm{qpattern}(\mathsf{A}) := \{ I(\mathbf{x}) : \mathbf{x} \in \mathrm{ESS}(\mathsf{A}),\, J_{\mathsf{A}}(\mathbf{x}) = I(\mathbf{x}) \}\ .$$

Note that any $\mathbf{p} \in \mathrm{NES}(\mathsf{A})$ satisfies $I(\mathbf{p}) \subseteq J_{\mathsf{A}}(\mathbf{p})$, and quasistrictness of $\mathbf{p}$ means that these two index sets coincide.

A further set of interest is

$$\begin{aligned} \mathcal{E}(\mathsf{A}) := & \left\{ \mathbf{x} \in \Delta^n : \mathsf{A}\mathbf{x} \le (\mathbf{x}^\top \mathsf{A}\mathbf{x})\,\mathbf{e} \text{ and } |I(\mathbf{x})| > 1 \Rightarrow \mathcal{B}(\mathsf{A}_{I(\mathbf{x})\times I(\mathbf{x})}) \succ \mathsf{O} \right\} \\ & \cup \{ \mathbf{e}_i : \mathbf{e}_i \in \mathrm{NES}(\mathsf{A}) \}\ . \end{aligned}$$

The significance of the latter two sets is the following: We try to find a perturbation $\widetilde{\mathsf{A}}$ of a matrix A such that $\mathrm{ESS}(\widetilde{\mathsf{A}})$ contains perturbations of some members of $\mathcal{E}(\mathsf{A})$. These will be quasistrict ESSs of $\widetilde{\mathsf{A}}$, with supports not contained in $\mathrm{pattern}(\mathsf{A})$, and the enrichment of the pattern can be expressed as $\mathrm{qpattern}(\mathsf{A})$ being a strict subset of $\mathrm{qpattern}(\widetilde{\mathsf{A}})$.

For the readers' convenience, we provide a short glossary of the sets used in our analysis, all refer to a fixed instance of (1) given by a symmetric matrix $\mathsf{A} \in \mathcal{S}^n$:

$\mathrm{SOL}(\mathsf{A})$	all global solutions (maximizers) to (1);
$\mathrm{NSS}(\mathsf{A})$	all local solutions to (1) $\equiv$ all neutrally stable states of evolutionary game based on A;
$\mathrm{ESS}(\mathsf{A})$	all strict local solutions to (1) $\equiv$ all evolutionarily stable states of evol. game based on A;
$\mathrm{NES}(\mathsf{A})$	all KKT/first-order stationary points (1) $\equiv$ all Nash equilibrium states of game based on A;
$\mathcal{E}(\mathsf{A})$	all $\mathbf{p} \in \mathrm{NES}(\mathsf{A})$ which either are pure or have a strictly concave objective on their face $\Delta_{I(\mathbf{p})}$;
$\mathrm{pattern}(\mathsf{A})$	ESS pattern = system of supports of all ESSs of evolutionary game based on A;
$\mathrm{qpattern}(\mathsf{A})$	quasistrict ESS pattern = system of supports of all quasistrict ESSs of evol. game based on A.

We collect some basic properties of the set $\mathcal{E}(\mathsf{A})$ in the following

Proposition 1 **(a)** *$\mathcal{E}(A)$ is nonempty and finite; to be more precise, we have*

$$\mathbf{p} \in \mathcal{E}(A) \implies \mathrm{NES}(A) \cap \Delta_{I(\mathbf{p})} = \{\mathbf{p}\} \tag{5}$$

and

$$\emptyset \neq \mathit{Ext\,conv}\,\mathrm{SOL}(A) \subseteq \mathit{Ext\,conv}\,\mathrm{NSS}(A) \subseteq \mathcal{E}(A) \subseteq \mathit{Ext\,conv}\,\mathrm{NES}(A)\,, \tag{6}$$

and all set inclusions above are strict in general.

(b) *Further we have* $\mathrm{ESS}(A) \subseteq \mathcal{E}(A) \subseteq \mathrm{NES}(A)$*, but in general* $\mathcal{E}(A)$ *and* $\mathrm{NSS}(A)$ *are incomparable.*

(c) *However, any* $\mathbf{p} \in \mathcal{E}(A)$ *with* $I(\mathbf{p}) = J_A(\mathbf{p})$ *satisfies already* $\mathbf{p} \in \mathrm{ESS}(A)$*. In other words, any* $\mathbf{p} \in \mathcal{E}(A) \setminus \mathrm{ESS}(A)$ *is a not quasistrict NES. Yet* $\mathbf{p}$ *being a not quasistrict NES does not imply* $\mathbf{p} \in \mathcal{E}(A)$*.*

(d) *In case that* $A = -FF^\top$ *for some* $F \in \mathbb{R}^{n\times k}$ *we have* $\mathcal{E}(A) = \mathit{Ext}(\mathrm{SOL}(A))$*. Moreover, we have* $\mathcal{E}(A) = \mathit{Ext}(\Delta^n \cap \ker F^\top)$*, provided the latter set is not empty, in which case we have* $J_A(\mathbf{p}) = [1\!:\!n]$ *for any* $\mathbf{p} \in \mathcal{E}(A)$*.*

Proof Denote by $f(\mathbf{x}) := \frac{1}{2}\mathbf{x}^\top \mathsf{A}\mathbf{x}$ with directional derivative $\partial_\mathbf{v} f(\mathbf{x}) = \mathbf{v}^\top \mathsf{A}\mathbf{x}$.

(a) Implication (5) comes from $f(\mathbf{x}) < f(\mathbf{p})$ for all $\mathbf{x} \in \Delta_{I(\mathbf{p})} \setminus \{\mathbf{p}\}$ due to (3) using $\mathbf{v} = \mathbf{x} - \mathbf{p} \perp \mathbf{e}$ and $\mathbf{v}^\top \mathsf{A}\mathbf{p} = 0$. Further, the inequality $0 > f(\mathbf{x}) - f(\mathbf{p}) = \frac{1}{2}(\mathbf{x}-\mathbf{p})^\top \mathsf{A}\mathbf{x}$ excludes the equilibrium condition $(\mathsf{A}\mathbf{x}) \leq (\mathbf{x}^\top \mathsf{A}\mathbf{x})\mathbf{e}$ at $\mathbf{x}$. Let us turn towards the set relations in (6); we deal with them from left to right;

(i) $\mathrm{SOL}(\mathsf{A})$ is nonempty and closed, so $\mathrm{conv}\,\mathrm{SOL}(\mathsf{A}) \neq \emptyset$ is compact, thus $\mathrm{Ext\,conv}\,\mathrm{SOL}(\mathsf{A}) \neq \emptyset$.

(ii) Let $\mathbf{p} \in \mathrm{Ext\,conv}\,\mathrm{SOL}(\mathsf{A})$. Then $\mathbf{p} \in \mathrm{SOL}(\mathsf{A}) \subseteq \mathrm{NSS}(\mathsf{A})$, and extremality in this set is obviously settled in case $|I(\mathbf{p})| = 1$. Turning to the case $|I(\mathbf{p})| > 1$, we note extremality of $\mathbf{p}$ in $\mathrm{conv}\,\mathrm{SOL}(\mathsf{A})$ implies

$$\mathbf{p} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \text{ with } 0 < \lambda < 1, \{\mathbf{x},\mathbf{y}\} \subseteq \mathrm{SOL}(\mathsf{A}) \implies \mathbf{x} = \mathbf{y} = \mathbf{p}\,.$$

Hence we deduce $\Delta_{I(\mathbf{p})} \cap \mathrm{SOL}(\mathsf{A}) = \{\mathbf{p}\}$. But this in turn even implies $\Delta_{I(\mathbf{p})} \cap \mathrm{NES}(\mathsf{A}) = \{\mathbf{p}\}$ because $f(\mathbf{x}) < f(\mathbf{p})$ must hold for any $\mathbf{x} \in \Delta_{I(\mathbf{p})} \setminus \{\mathbf{p}\}$, and therefore the directional derivative $(\mathbf{p}-\mathbf{x})^\top \mathsf{A}\mathbf{x} = \partial_{\mathbf{p}-\mathbf{x}} f(\mathbf{x}) > 0$, meaning $\mathbf{x} \notin \mathrm{NES}(\mathsf{A})$. Obviously then also $\Delta_{I(\mathbf{p})} \cap \mathrm{Ext\,conv}\,\mathrm{NSS}(\mathsf{A}) = \{\mathbf{p}\}$ holds. If now $\{\mathbf{x},\mathbf{y}\} \subseteq \mathrm{conv}\,\mathrm{NSS}(\mathsf{A}) \setminus \{\mathbf{p}\}$ and $0 < \lambda < 1$ such that $\mathbf{p} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, then $\{\mathbf{x},\mathbf{y}\} \subseteq \Delta_{I(\mathbf{p})}$, implying $\mathbf{x} = \mathbf{y} = \mathbf{p}$, so $\mathbf{p}$ is extremal in $\mathrm{conv}\,\mathrm{NSS}(\mathsf{A})$. For $\mathsf{A} = \mathrm{Diag}\,(1,2)$ we have $\mathrm{SOL}(\mathsf{A}) = \{\mathbf{e}_2\}$ but also $\mathbf{e}_1 \in \mathrm{Ext\,conv}\,\mathrm{NSS}(\mathsf{A})$.

(iii) Let $\mathbf{p} \in \mathrm{Ext\,conv}\,\mathrm{NSS}(\mathsf{A}) \subseteq \mathrm{NSS}(\mathsf{A}) \subseteq \mathrm{NES}(\mathsf{A})$, so the case $|I(\mathbf{p})| = 1$ is settled. In case that $|I(\mathbf{p})| > 1$ we claim that $f(\mathbf{x}) < f(\mathbf{p})$ for all $\mathbf{x} \in \Delta_{I(\mathbf{p})} \setminus \{\mathbf{p}\}$. Otherwise, since f is quadratic, there would be a whole segment (the intersection of a straight line with $\Delta_{I(\mathbf{p})}$) of points along which f is constantly equalling $f(\mathbf{p})$, because $\partial_{\mathbf{x}-\mathbf{p}} f(\mathbf{p}) = 0$ for all $\mathbf{x} \in \Delta_{I(\mathbf{p})}$. Due to extremality of $\mathbf{p}$, this segment cannot contain two members of $\mathrm{NSS}(\mathsf{A})$ on either side of $\mathbf{p}$. So there must be points $\mathbf{p}_k$ on this segment arbitrarily close to $\mathbf{p}$, say $\|\mathbf{p}-\mathbf{p}_k\| < \frac{1}{k}$, that are not in $\mathrm{NSS}(\mathsf{A})$, thus there are points $\mathbf{q}_k \in \Delta^n$ with $\|\mathbf{p}_k - \mathbf{q}_k\| < \frac{1}{k}$ and $f(\mathbf{q}_k) > f(\mathbf{p}_k) = f(\mathbf{p})$. These points $\mathbf{q}_k$ get arbitrarily close to $\mathbf{p}$, meaning that $\mathbf{p}$ is not in $\mathrm{NSS}(\mathsf{A})$, a contradiction. For $\mathsf{A} = \mathrm{Diag}\,(0,1)$ we have $\mathbf{e}_1 \in \mathcal{E}(\mathsf{A}) \setminus \mathrm{NSS}(\mathsf{A}) = \mathcal{E}(\mathsf{A}) \setminus \mathrm{ESS}(\mathsf{A})$.

(iv) Let $\mathbf{p} \in \mathcal{E}(\mathsf{A}) \subseteq \mathrm{NES}(\mathsf{A}) \subseteq \mathrm{conv}\,\mathrm{NES}(\mathsf{A})$. Since $\mathbf{e}_i$ are the extreme points of Δ^n we are done in case $|I(\mathbf{p})| = 1$. Moreover, if $|I(\mathbf{p})| > 1$, we have $f(\mathbf{x}) < f(\mathbf{p})$ for all $\mathbf{x} \in \Delta_{I(\mathbf{p})} \setminus \{\mathbf{p}\}$ as observed when proving (5) above, and, arguing as in (ii), $\Delta_{I(\mathbf{p})} \cap \mathrm{conv}\,\mathrm{NES}(\mathsf{A}) = \{\mathbf{p}\} = \Delta_{I(\mathbf{p})} \cap \mathrm{NES}(\mathsf{A})$. As $\mathbf{p} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ with $0 < \lambda < 1$ and $\{\mathbf{x},\mathbf{y}\} \subseteq \mathrm{conv}\,\mathrm{NES}(\mathsf{A})$ requires $\{\mathbf{x},\mathbf{y}\} \subseteq \Delta_{I(\mathbf{p})}$, we obtain $\mathbf{x} = \mathbf{y} = \mathbf{p}$, i.e. extremality of $\mathbf{p}$ in conv NES(A). For

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{p} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{q} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

we have $\mathrm{NES}(\mathsf{A}) = \{\mathbf{q}\} \cup \mathrm{conv}\ \{\mathbf{e}_1, \mathbf{p}\}$, but $\mathbf{p} \notin \mathcal{E}(\mathsf{A})$ and $\mathbf{p}$ is not quasistrict.

(b) The leftmost inclusion follows similarly as in the proof of (5), while the rightmost is a consequence of (6) and of the generally valid relation $\mathrm{Ext\,conv}\ S \subseteq S$ which we used already in our proofs of (ai) and (aii). Examples of $\mathbf{p} \in \mathcal{E}(\mathsf{A}) \setminus \mathrm{NSS}(\mathsf{A})$ or $\mathbf{q} \in \mathrm{NSS}(\mathsf{A}) \setminus \mathcal{E}(\mathsf{A})$ are $\mathbf{p} = \mathbf{e}_1$ for $\mathsf{A} = \mathrm{Diag}\ (0, 1)$ – see the proof of (aiii) — and $\mathbf{q} = [\frac{1}{2}, \frac{1}{2}]^\top$ for $\mathsf{A} = \mathsf{O}$, respectively.
(c) If $J_{\mathsf{A}}(\mathbf{p}) = I(\mathbf{p})$ and $\mathbf{p} \in \mathcal{E}(\mathsf{A})$, then any $\mathbf{x} \in \Delta^n$ with $\mathbf{x}^\top \mathsf{A}\mathbf{p} = \mathbf{p}^\top \mathsf{A}\mathbf{p}$ must lie in $\Delta_{I(\mathbf{p})}$, therefore as in the proof of (aiv) we have $\mathbf{p}^\top \mathsf{A}\mathbf{x} = \mathbf{x}^\top \mathsf{A}\mathbf{p} > \mathbf{x}^\top \mathsf{A}\mathbf{x}$ unless $\mathbf{x} = \mathbf{p}$, which shows $\mathbf{p} \in \mathrm{ESS}(\mathsf{A})$. The example $\mathbf{p}$ given in the proof of (aiv) proves the last assertion.
(d) is obvious from the fact that SOL(A) is convex itself, and that all critical points of a smooth concave maximization problem are global solutions, so $\mathrm{SOL}(\mathsf{A}) = \mathrm{NSS}(\mathsf{A}) = \mathrm{NES}(\mathsf{A})$. □

2 Main Results

Our aim is to increase the number of ESSs of a matrix A by perturbing it in a way such that members of $\mathcal{E}(\mathsf{A}) \setminus \mathrm{ESS}(\mathsf{A})$ get perturbed into (quasistrict) ESSs of the perturbed matrix. To this end, we consider perturbations of A of the form $\widetilde{\mathsf{A}} = \mathsf{A} + \varepsilon\mathsf{B}$, with ε small. We seek simple sufficient conditions on B that lead to successful perturbations in the sense that $|\mathrm{qpattern}(\widetilde{\mathsf{A}})| > |\mathrm{qpattern}(\mathsf{A})|$. We further aim at results telling us that we have found a matrix with the largest number of quasistrict ESSs of prescribed support size.

The plan of this section is as follows: In Sect. 2.1 (Lemma 2), we keep both A and $\mathbf{x} \in \mathcal{E}(\mathsf{A})$ fixed and derive a sufficient condition on B guaranteeing $I(\mathbf{x}) \in \mathrm{qpattern}(\widetilde{\mathsf{A}})$ for $\varepsilon > 0$ small enough. This is accomplished by studying the first-order expansion in ε of the first-order optimality condition for the StQP with matrix $\widetilde{\mathsf{A}}$. In Sect. 2.2 (Lemma 3), we prove, loosely speaking, that if close to A there are sufficiently many matrices $\overline{\mathsf{A}}$ satisfying $I(\mathbf{x}) \in \mathrm{qpattern}(\overline{\mathsf{A}})$, then there will be B satisfying the sufficient condition of Lemma 2, such that also $I(\mathbf{x}) \in \mathrm{qpattern}(\widetilde{\mathsf{A}})$ for $\varepsilon > 0$ small enough. Further results (Theorems 8 and 9) explore ways to restrict A and B to smaller sets of matrices, varying either A or B while not losing any generality in $\mathrm{qpattern}(\widetilde{\mathsf{A}})$. Negative-semidefinite matrices A are particularly interesting starting points for perturbation, as they have large sets $\mathcal{E}(\mathsf{A})$. Proposition 6 and Corollary 10 deal with those. In Sect. 2.3, we introduce $n \times n$ cyclically symmetric matrices, forming a vector space $\mathcal{C}^n$ of dimension $\lceil \frac{n-1}{2} \rceil$. With those, a further simplification allows to restrict our perturbation procedure to a finite set of candidate matrices A, with B ranging in a certain linear subspace of $\mathcal{C}^n$ depending on A. In Sect. 2.4, the preceding results are applied to the setting of negative-semidefinite candidate matrices $\mathsf{A} \in \mathcal{C}^n$, where the task is to find a maximal subset $\mathcal{P} \subseteq \mathcal{E}(\mathsf{A})$, such that there is $\mathsf{B} \in \mathcal{C}^n$ satisfying the sufficient condition of Lemma 2 for any $\mathbf{x} \in \mathcal{P}$. Examples 12 and 13 for $n \in \{6, 7\}$ explicate the procedure. In Sect. 2.5, we report on our results for orders $n \in [4:23]$. The subsections are interspersed with several examples illustrating our results, some also show limitations.

2.1 First-Order Conditions

The following lemma deals with perturbations of the form $\mathsf{A}+\varepsilon\mathsf{B}$ and fathoms what can be deduced from a first-order expansion of the resulting FINDEQ inequalities.

Lemma 2 *Let $A \in \mathcal{S}^n$, $\mathbf{x} \in \Delta^n$, $I := I(\mathbf{x})$, and $J := J_A(\mathbf{x}) \neq I$. Assume $(A\mathbf{x})_i = v$ for $i \in J$ and $(A\mathbf{x})_i < v$ for $i \notin J$, as well as $\mathcal{B}(A_{I\times I}) \succ O$, in case that $|I| \geq 2$. Thus $\mathbf{x} \in \mathcal{E}(A)$. Let $A_1 := A_{I\times I}$, $A_2 := A_{(J\setminus I)\times I}$, and similarly define submatrices B_1 and B_2 of $B \in \mathcal{S}^n$. Further let $\bar{A}_1 := \begin{pmatrix} A_1 & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}$ and $\bar{B}_1 := \begin{pmatrix} B_1 & \mathbf{o} \\ \mathbf{o}^\top & 0 \end{pmatrix}$, where $\mathbf{e}$ is the all ones and $\mathbf{o}$ the zero vector of suitable dimension. We abbreviate the relation $\mathbf{b}-\mathbf{a} \in \operatorname{int}\mathbb{R}^d_+$ by $\mathbf{a} < \mathbf{b}$.*

a) If B satisfies

$$\left[(B_2 \mid \mathbf{o}) - (A_2 \mid \mathbf{e})\bar{A}_1^{-1}\bar{B}_1\right]\begin{pmatrix} \mathbf{x}_I \\ -v \end{pmatrix} < \mathbf{o}\,, \tag{7}$$

then for $\varepsilon > 0$ small enough, $A+\varepsilon B$ will have an ESS $\mathbf{x}_\varepsilon$ such that $I(\mathbf{x}_\varepsilon) = J_{A+\varepsilon B}(\mathbf{x}_\varepsilon) = I$, and hence we have $I \in qpattern(A+\varepsilon B)$.

b) If B violates $\left[(B_2 \mid \mathbf{o}) - (A_2 \mid \mathbf{e})\bar{A}_1^{-1}\bar{B}_1\right]\begin{pmatrix} \mathbf{x}_I \\ -v \end{pmatrix} \leq \mathbf{o}$, then for $\varepsilon > 0$ small enough, $A+\varepsilon B$ will not have an NES $\mathbf{x}_\varepsilon$ such that $I(\mathbf{x}_\varepsilon) = I$.

Proof From $\mathcal{B}(\mathsf{A}_1) \succ \mathsf{O}$ we deduce that $\bar{\mathsf{A}}_1$ is invertible,[1] so we have $\begin{pmatrix} \mathbf{x}_I \\ -v \end{pmatrix} = \bar{\mathsf{A}}_1^{-1}\begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix}$. For $\varepsilon > 0$ small enough, $\bar{\mathsf{A}}_1+\varepsilon\bar{\mathsf{B}}_1$ will therefore be invertible as well, and $\begin{pmatrix} (\mathbf{x}_\varepsilon)_I \\ -v_\varepsilon \end{pmatrix} := (\bar{\mathsf{A}}_1+\varepsilon\bar{\mathsf{B}}_1)^{-1}\begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix}$ with $(\mathbf{x}_\varepsilon)_{N\setminus I} := \mathbf{o}$ gives rise to a candidate $\mathbf{x}_\varepsilon$ for an ESS of $\mathsf{A}+\varepsilon\mathsf{B}$, satisfying $I(\mathbf{x}_\varepsilon) = I$. By continuity, for $\varepsilon > 0$ small enough we will have $\mathcal{B}(\mathsf{A}_1+\varepsilon\mathsf{B}_1) \succ \mathsf{O}$ and $\big((\mathsf{A}+\varepsilon\mathsf{B})\mathbf{x}_\varepsilon\big)_i < v_\varepsilon$ for $i \notin J$. Showing $\big((\mathsf{A}+\varepsilon\mathsf{B})\mathbf{x}_\varepsilon\big)_i < v_\varepsilon$ for $i \in J\setminus I$ is the only remaining task in the proof of *a)*. The latter inequality can be restated as $[(\mathsf{A}_2 \mid \mathbf{e}) + \varepsilon(\mathsf{B}_2 \mid \mathbf{o})]\begin{pmatrix} (\mathbf{x}_\varepsilon)_I \\ -v_\varepsilon \end{pmatrix} < \mathbf{o}$. Using now $(\bar{\mathsf{A}}_1+\varepsilon\bar{\mathsf{B}}_1)^{-1} = \bar{\mathsf{A}}_1^{-1} - \varepsilon\bar{\mathsf{A}}_1^{-1}\bar{\mathsf{B}}_1\bar{\mathsf{A}}_1^{-1} + \mathcal{O}(\varepsilon^2)$ we obtain $\begin{pmatrix} (\mathbf{x}_\varepsilon)_I \\ -v_\varepsilon \end{pmatrix} = \begin{pmatrix} \mathbf{x}_I \\ -v \end{pmatrix} - \varepsilon\bar{\mathsf{A}}_1^{-1}\bar{\mathsf{B}}_1\begin{pmatrix} \mathbf{x}_I \\ -v \end{pmatrix} + \mathcal{O}(\varepsilon^2)$, so that indeed we have, using $(\mathsf{A}_2 \mid \mathbf{e})\begin{pmatrix} \mathbf{x}_I \\ -v \end{pmatrix} = \mathbf{o}$,

$$[(\mathsf{A}_2 \mid \mathbf{e}) + \varepsilon(\mathsf{B}_2 \mid \mathbf{o})]\begin{pmatrix} (\mathbf{x}_\varepsilon)_I \\ -v_\varepsilon \end{pmatrix} = \varepsilon\left[(\mathsf{B}_2 \mid \mathbf{o}) - (\mathsf{A}_2 \mid \mathbf{e})\bar{\mathsf{A}}_1^{-1}\bar{\mathsf{B}}_1\right]\begin{pmatrix} \mathbf{x}_I \\ -v \end{pmatrix} + \mathcal{O}(\varepsilon^2) < \mathbf{o}, \tag{8}$$

for $\varepsilon > 0$ small enough, by our assumption (7). Turning to *b)*, by the assumption made there, at least one component of the L.H.S. of (8) will become positive for $\varepsilon > 0$ small enough, showing $\mathbf{x}_\varepsilon \notin \mathrm{NES}(\mathsf{A}+\varepsilon\mathsf{B})$. □

Remark 1 For A and $\mathbf{x}$ fixed, also I, J and v are fixed, so the matrices B satisfying (7) form the interior of a polyhedral cone, and thus a convex cone.

[1] Assume that $\bar{\mathsf{A}}_1$ is not invertible. Then there is a vector $\mathbf{z}$ and a scalar u such that $\bar{\mathsf{A}}_1\begin{pmatrix} \mathbf{z} \\ u \end{pmatrix} = \begin{pmatrix} \mathsf{A}_1\mathbf{z}+u\mathbf{e} \\ \mathbf{e}^\top\mathbf{z} \end{pmatrix} = \mathbf{o}$, while $\begin{pmatrix} \mathbf{z} \\ u \end{pmatrix} \neq \mathbf{o}$. This implies $\mathbf{z} \neq \mathbf{o}$, $\mathbf{e}^\top\mathbf{z} = 0$, and $\mathbf{z}^\top\mathsf{A}_1\mathbf{z} = 0$. Denote $m := |I|$ and $\bar{\mathbf{z}} := \mathbf{z}_{[1:m-1]}$ and observe that $\bar{\mathbf{z}} \neq \mathbf{o}$. Furthermore, denoting columns and elements of A_1 by $\mathbf{a}_i$ resp. a_{ij}, and using $\mathcal{B}(\mathsf{A}_1) \succ \mathsf{O}$, we derive $0 < \bar{\mathbf{z}}^\top\mathcal{B}(\mathsf{A}_1)\bar{\mathbf{z}} = \mathbf{z}^\top(\mathbf{e}\,\mathbf{a}_m^\top + \mathbf{a}_m\,\mathbf{e}^\top - \mathsf{A}_1 - a_{mm}\mathbf{e}\,\mathbf{e}^\top)\mathbf{z} = -\mathbf{z}^\top\mathsf{A}_1\mathbf{z} = 0$, a contradiction.

2.2 Perturbation

If there are (in a certain sense) enough arbitrarily small perturbations $\bar{\mathsf{A}}$ of A that transform $\mathbf{x} \in \mathcal{E}(\mathsf{A}) \setminus \mathrm{ESS}(\mathsf{A})$ into $\bar{\mathbf{x}} \in \mathrm{ESS}(\bar{\mathsf{A}})$, with $I(\mathbf{x}) = I(\bar{\mathbf{x}}) = J_{\bar{\mathsf{A}}}(\bar{\mathbf{x}})$, then there will be a matrix B that satisfies (7), as the following lemma claims.

Lemma 3 *Let $A \in \mathcal{S}^n$ and $\mathcal{P} \subseteq \mathcal{E}(A) \setminus \mathrm{ESS}(A)$. Fix $k \geq 1$ and let V be a linear subspace of $\mathcal{S}^n$ of dimension $k+1$, and for $a > 0$ define*

$$\mathcal{V}_{a,\mathsf{A},\mathcal{P}} := \{B \in V : \{I(\mathbf{p}) : \mathbf{p} \in \mathcal{P}\} \subseteq qpattern(A + B) \text{ and } 0 < \|B\| \leq a\} .$$

Suppose that $s := \inf_{a>0} \sigma^k \left(\left\{\frac{B}{\|B\|} : B \in \mathcal{V}_{a,\mathsf{A},\mathcal{P}}\right\}\right) > 0$, where σ^k denotes spherical measure. Then there is $\mathsf{B} \in V$ such that

$$h_{I,K}(B) := \left[(B_{K\times I} \mid \mathbf{o}) - (A_{K\times I} \mid \mathbf{e}) \begin{pmatrix} A_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} B_{I\times I} & \mathbf{o} \\ \mathbf{o}^\top & 0 \end{pmatrix}\right] \begin{pmatrix} A_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix} < \mathbf{o}, \tag{9}$$

holds for all $(I, K) \in \mathcal{I}_{\mathcal{P}} := \left\{\left(I(\mathbf{p}), J_A(\mathbf{p}) \setminus I(\mathbf{p})\right) : \mathbf{p} \in \mathcal{P}\right\}$.

Proof We have to show that the polyhedral cone $C := \{\mathsf{B} \in V : h_{I,K}(\mathsf{B}) \leq \mathbf{o}, \forall (I, K) \in \mathcal{I}_{\mathcal{P}}\}$ has nonempty interior. By assumption, for every $\ell \in \mathbb{N}$ there is $\mathsf{B}_\ell^{(1)} \in \mathcal{V}_{\frac{1}{\ell},\mathsf{A},\mathcal{P}}$. Then the sequence $\left(\frac{\mathsf{B}_\ell^{(1)}}{\|\mathsf{B}_\ell^{(1)}\|}\right)_{\ell\geq 1}$ of points on a k-sphere S^k of radius 1 has an accumulation point $\mathsf{B}^{(1)} \in V$. Now, by calculations similar to those that led to the L.H.S. of inequality (8), we have $h_{I,K}(\mathsf{B}_\ell^{(1)}) + \mathcal{O}\left(\|\mathsf{B}_\ell^{(1)}\|^2\right) < \mathbf{o}$ as $\ell \to \infty$. This implies that $h_{I,K}(\mathsf{B}^{(1)}) \leq \mathbf{o}$ is satisfied for all $(I, K) \in \mathcal{I}_{\mathcal{P}}$, therefore $\mathsf{B}^{(1)} \in C$.
Next define $N_1 := \{\mathsf{B} \in V \cap S^k : \det \Gamma(\mathsf{B}^{(1)}, \mathsf{B}) \leq \delta_1\}$, where $\Gamma(\cdot)$ denotes the Gram matrix of a set of vectors (or vectorized matrices), and $\delta_1 > 0$ is chosen such that $\sigma^k(N_1) \leq \frac{s}{2}$, and $C_1 := \mathrm{cone}(N_1)$. Further let $\mathsf{B}_\ell^{(2)} \in \mathcal{V}_{\frac{1}{\ell},\mathsf{A},\mathcal{P}} \setminus C_1 \neq \emptyset$ as $\sigma^k\left(\left\{\frac{\mathsf{B}}{\|\mathsf{B}\|} : \mathsf{B} \in \mathcal{V}_{\frac{1}{\ell},\mathsf{A},\mathcal{P}} \setminus C_1\right\}\right) > 0$. Again, the sequence $\left(\frac{\mathsf{B}_\ell^{(2)}}{\|\mathsf{B}_\ell^{(2)}\|}\right)_{\ell\geq 1}$ has an accumulation point $\mathsf{B}^{(2)} \in V$, that satisfies $\mathsf{B}^{(2)} \in C$. Moreover, $\mathsf{B}^{(1)}$ and $\mathsf{B}^{(2)}$ are linearly independent. Assume now that for $r \leq k$ we have found linearly independent elements $\mathsf{B}^{(1)}, \ldots, \mathsf{B}^{(r)} \in C$. Then define $N_r := \{\mathsf{B} \in V \cap S^k : \det \Gamma(\mathsf{B}^{(1)}, \ldots, \mathsf{B}^{(r)}, \mathsf{B}) \leq \delta_r\}$, where $\delta_r > 0$ is chosen such that $\sigma^k(N_r) \leq \frac{s}{2}$, and $C_r := \mathrm{cone}(N_r)$. Further let $\mathsf{B}_\ell^{(r+1)} \in \mathcal{V}_{\frac{1}{\ell},\mathsf{A},\mathcal{P}} \setminus C_r$. Such a sequence does indeed exist because of $\inf_{a>0} \sigma^k\left(\left\{\frac{\mathsf{B}}{\|\mathsf{B}\|} : \mathsf{B} \in \mathcal{V}_{a,\mathsf{A},\mathcal{P}} \setminus C_r\right\}\right) \geq \frac{s}{2} > 0$, and the sequence $\left(\frac{\mathsf{B}_\ell^{(r+1)}}{\|\mathsf{B}_\ell^{(r+1)}\|}\right)_{\ell\geq 1}$ has a limit point $\mathsf{B}^{(r+1)} \in C$. Summarizing, we have found $\mathsf{B}^{(1)}, \ldots, \mathsf{B}^{(k+1)} \in C$, spanning a subcone of C of full dimension $k+1$. Therefore C has nonempty interior C°, and any $\mathsf{B} \in C^\circ$ satisfies (9) for all $(I, K) \in \mathcal{I}_{\mathcal{P}}$. □

Does Lemma 3 also hold for $k = 0$, in which case we always have $s \in \{0, 1\}$? That is, if there is $\mathsf{B} \in \mathcal{S}^n$ and a sequence (ε_m) of positive reals such that $\varepsilon_m \to 0$ and $\{I(\mathbf{p}) : \mathbf{p} \in \mathcal{P}\} \subseteq \mathrm{qpattern}(\mathsf{A} + \varepsilon_m \mathsf{B})$ holds for all $m \geq 1$, does that imply that (9) is satisfied for all $(I, K) \in \mathcal{I}_{\mathcal{P}}$? The answer is no, as the following example shows.

Example 4 Let $\mathsf{A} := \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ and $\mathsf{B} := \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. Then $\mathcal{E}(\mathsf{A}) = \{\frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2), \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_3)\}$ and $\mathrm{ESS}(\mathsf{A}) = \emptyset$. We let $\mathcal{P} := \{\mathbf{p}\}$, where $\mathbf{p} = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$, $I = I(\mathbf{p}) = \{1, 2\}$, $J_{\mathsf{A}}(\mathbf{p}) = \{1, 2, 3\}$, $K = \{3\}$. It is easily checked that $h_{I,K}(\mathsf{B}) = \mathbf{o}$, i.e., (9) is violated. Next we compute $\begin{pmatrix} (\mathbf{p}_\varepsilon)_I \\ -v_\varepsilon \end{pmatrix} = \begin{pmatrix} -1+\varepsilon & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix} = \frac{1}{4-\varepsilon} \begin{bmatrix} 2 \\ 2-\varepsilon \\ -\varepsilon \end{bmatrix}$, with $\mathbf{p}_\varepsilon \in \Delta^3$ for $0 < \varepsilon \leq 2$ and $I(\mathbf{p}_\varepsilon) = \{1, 2\}$ for $0 < \varepsilon < 2$. Moreover, $(\mathsf{A} + \varepsilon\mathsf{B})_{K\times I}(\mathbf{p}_\varepsilon)_I = v_\varepsilon - \frac{\varepsilon^2}{4-\varepsilon} < v_\varepsilon$ for $0 < \varepsilon < 4$, meaning $J_{\mathsf{A}+\varepsilon\mathsf{B}}(\mathbf{p}_\varepsilon) = \{1, 2\}$, and $\mathcal{B}\big((\mathsf{A}+\varepsilon\mathsf{B})_{I\times I}\big) \succ \mathsf{O}$ for $0 < \varepsilon < 4$. Therefore $\mathbf{p}_\varepsilon \in \mathrm{ESS}(\mathsf{A} + \varepsilon\mathsf{B})$ is quasistrict, and thus $I \in \mathrm{qpattern}(\mathsf{A} + \varepsilon\mathsf{B})$ for all $\varepsilon > 0$ small enough.

Remark 2 If, in the setting of Lemma 3, we have $s_a := \sigma^k\left(\left\{\frac{\mathsf{B}}{\|\mathsf{B}\|} : \mathsf{B} \in \mathcal{V}_{a,\mathsf{A},\mathcal{P}}\right\}\right) > 0$ for all $a > 0$, but $\lim_{a\searrow 0} s_a = 0$ (this can happen, as the next example shows), then for each $\mathsf{B} \in \bigcap_{a>0} \mathrm{cl}\big(\mathcal{V}_{a,\mathsf{A},\mathcal{P}}\big)$ we have $h_{I,K}(\mathsf{B}) \leq \mathbf{o}$, but no such B will satisfy $h_{I,K}(\mathsf{B}) < \mathbf{o}$.

Example 5 Let $k = 1$, $\mathsf{A} := \begin{pmatrix} -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -1 & -2 \\ 2 & -2 & -2 & -4 \end{pmatrix}$, $\mathsf{B} := \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1/2 \\ -1 & 1 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \end{pmatrix}$, $\mathsf{C} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, and $V := \mathrm{span}\,(\mathsf{B}, \mathsf{C})$. We let $\mathbf{p} = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) \in \mathcal{E}(\mathsf{A}) \setminus \mathrm{ESS}(\mathsf{A})$, $I := I(\mathbf{p}) = \{1, 2\}$ and $K = \{3, 4\}$. We now look for $\mathbf{p}_{\varepsilon,\delta} \in \mathrm{ESS}(\mathsf{A}+\varepsilon\mathsf{B}+\delta\mathsf{C})$, obtaining $\mathbf{p}_{\varepsilon,\delta} = \frac{1}{4-\varepsilon}[2, 2-\varepsilon, 0, 0]^\top$, with $I(\mathbf{p}_{\varepsilon,\delta}) = \{1, 2\}$ for $0 < \varepsilon < 2$, and $v_{\varepsilon,\delta} = \frac{\varepsilon}{4-\varepsilon}$. Moreover, for $0 < \varepsilon < 2$ fixed, $(\mathsf{A}+\varepsilon\mathsf{B}+\delta\mathsf{C})_{K\times I}(\mathbf{p}_{\varepsilon,\delta})_I - v_{\varepsilon,\delta}\mathbf{e} = \frac{1}{4-\varepsilon}\begin{bmatrix} 2\delta-\varepsilon^2 \\ \delta(\varepsilon-2)-\varepsilon^2/2 \end{bmatrix} < \mathbf{o}$ if and only if $-\frac{1}{2}\frac{\varepsilon^2}{2-\varepsilon} < \delta < \frac{\varepsilon^2}{2}$. From this we conclude $s_a := \sigma^1\left(\left\{\frac{\mathsf{B}}{\|\mathsf{B}\|} : \mathsf{B} \in \mathcal{V}_{a,\mathsf{A},\mathcal{P}}\right\}\right) > 0$ for $a > 0$, as well as $s_a = \mathcal{O}(a)$, as $a \searrow 0$.

For $\mathsf{A} \in \mathcal{S}^n$ and $\mathcal{P} \subseteq \mathcal{E}(\mathsf{A})$ we now define the following convex cone

$$\begin{aligned} \mathcal{C}_{\mathsf{A}}(\mathcal{P}) := \{\mathsf{A}' \in \mathcal{S}^n : \mathsf{A}'\mathbf{x} \leq (\mathbf{x}^\top\mathsf{A}'\mathbf{x})\,\mathbf{e},\ J_{\mathsf{A}}(\mathbf{x}) \subseteq J_{\mathsf{A}'}(\mathbf{x}),\ \text{and} \\ |I(\mathbf{x})| > 1 \Rightarrow \mathcal{B}(\mathsf{A}'_{I(\mathbf{x})\times I(\mathbf{x})}) \succ \mathsf{O}\,,\ \text{for all } \mathbf{x} \in \mathcal{P}\}, \end{aligned}$$

which clearly satisfies $\mathcal{C}_{\mathsf{A}}(\mathcal{P}) = \bigcap_{\mathbf{p}\in\mathcal{P}} \mathcal{C}_{\mathsf{A}}(\{\mathbf{p}\})$. Observe also that $\mathsf{A} + \lambda\mathbf{e}\mathbf{e}^\top \in \mathcal{C}_{\mathsf{A}}(\mathcal{P})$ for all $\lambda \in \mathbb{R}$ due to (2).

Further members of $\mathcal{C}_{\mathsf{A}}(\mathcal{P})$, in case that A is negative-semidefinite, will be identified in the next result.

Proposition 6 *Let $A = -FF^\top$, where $F \in \mathbb{R}^{n\times k}$ for some k and assume $\Delta^n \cap \ker F^\top \neq \emptyset$. Then*

(a) for every $\mathbf{p} \in \mathcal{E}(A)$ we have $|I(\mathbf{p})| \leq \mathrm{rank}\ F + 1$, and for every positive-definite $\mathsf{L} \in \mathcal{S}^k$ we have $A' := -FLF^\top \in \mathcal{C}_A(\{\mathbf{p}\})$ with rank A = rank A'.
(b) For $m \geq 1$ fixed define

$$\mathcal{P}_m := \{\mathbf{p} \in \mathcal{E}(A) : |I(\mathbf{p})| = m\}\,.$$

If for some m the set $\mathcal{P}_m \neq \emptyset$, then there is a matrix $A'' = -HH^\top \in \mathcal{S}^n$ of rank $m - 1$, such that colspace$(H) \subseteq$ colspace(F), satisfying $\mathbf{p}^\top A''\mathbf{p} = 0$ for all $\mathbf{p} \in \mathcal{E}(A)$, and $\mathcal{P}_m \subseteq \mathcal{E}(A'')$.

Proof *(a)* follows by Proposition 1(d); note that $\mathsf{A}' = -\mathsf{F}'(\mathsf{F}')^\top$ with $\mathsf{F}' = \mathsf{F}\sqrt{\mathsf{L}}$ where $\sqrt{\mathsf{L}}$ denotes the symmetric square root factorization of L. *(b)* Assume that $\mathcal{P}_m$ is nonempty. Then from (a) we know that rank $\mathsf{F} \geq m-1$, and clearly $k \geq \operatorname{rank} \mathsf{F}$. If rank $\mathsf{F} = m-1$ we can simply choose $\mathsf{H} = \mathsf{F}$. So we assume rank $\mathsf{F} \geq m$, and consider $\mathsf{H} = \mathsf{FB}$, where $\mathsf{B} \in \mathbb{R}^{k \times (m-1)}$ satisfies rank $\mathsf{FB} = m-1$. (Those B constitute an open dense subset of $\mathbb{R}^{k\times(m-1)}$.) Then rank $\mathsf{A}'' = m-1$ and colspace$(\mathsf{H}) \subseteq$ colspace(F). Furthermore, $\operatorname{Ext}(\Delta^n \cap \ker \mathsf{H}^\top) \neq \emptyset$, and thus, by Proposition 1(d), we have $\mathcal{E}(\mathsf{A}'') = \operatorname{Ext}(\Delta^n \cap \ker \mathsf{H}^\top)$. To show $\mathcal{P}_m \subseteq \mathcal{E}(\mathsf{A}'')$ it is sufficient to have that $\mathbf{p} \in \mathcal{P}_m$ and $\mathbf{q} \in \Delta_{I(\mathbf{p})} \setminus \{\mathbf{p}\}$ together imply $\mathbf{q} \notin \ker \mathsf{H}^\top$. This would follow if the map from $\Delta_{I(\mathbf{p})}$ to $\mathbb{R}^{m-1}$ taking $\mathbf{x}$ to $\mathsf{B}^\top\mathsf{F}^\top\mathbf{x}$ is one-to-one, which again is satisfied by all B in an open dense subset of $\mathbb{R}^{k\times(m-1)}$. As $\mathcal{P}_m$ is a finite set, we find B as needed in a finite intersection of open dense subsets of $\mathbb{R}^{k\times(m-1)}$. □

So when searching for negative-semidefinite matrices A of order n with many points of support size m in their sets $\mathcal{E}(\mathsf{A})$, we may restrict our search to matrices of the minimal possible rank $m-1$. However, not every element of $\mathcal{C}_{\mathsf{A}}(\{\mathbf{p}\})$ need have a representation as in Proposition 6(a):

Example 7 Let $\mathsf{A} = -\mathsf{F}\mathsf{F}^\top$ and $\mathsf{A}' = -\mathsf{G}\mathsf{G}^\top$ where $\mathsf{F}^\top = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$ and $\mathsf{G}^\top = \left(\begin{smallmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 \end{smallmatrix}\right)$. Then $\mathcal{P} := \mathcal{E}(\mathsf{A}) = \{\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2), \frac{1}{2}(\mathbf{e}_3 + \mathbf{e}_4), \frac{1}{2}(\mathbf{e}_5 + \mathbf{e}_6)\} \subseteq \mathcal{E}(\mathsf{A}') = \mathcal{E}(\mathsf{A}) \cup \{\frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_4 + \mathbf{e}_6), \frac{1}{3}(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_5)\}$, furthermore $J_{\mathsf{A}}(\mathbf{p}) = J_{\mathsf{A}'}(\mathbf{p}) = [1:n]$ for all $\mathbf{p} \in \mathcal{P}$. Therefore $\mathsf{A}' \in \mathcal{C}_{\mathsf{A}}(\mathcal{P})$, but rank $\mathsf{A} \neq$ rank A'. Regarding Proposition 6(b), we have $n = 6, k = 3, m = 2, \mathcal{P}_2 = \mathcal{P}$ and we may choose $\mathsf{H}^\top = (1, -1, 1, -1, 1, -1)$, leading to $\mathcal{P}_2 \subseteq \mathcal{E}(\mathsf{A}'') = \{\frac{1}{2}(\mathbf{e}_{2i-1} + \mathbf{e}_{2j}) : i, j \in [1:3]\}$.

Suppose now that (9) is satisfied for certain $\mathsf{A}, \mathsf{B}, I, K$. In which ways can we change either A or B while keeping (9) intact? We start considering variations B' of B.

Theorem 8 *Let $A \in \mathcal{S}^n$ and $\mathbf{p} \in \mathcal{E}(A)$ such that $|I(\mathbf{p})| < |J_A(\mathbf{p})|$. Suppose that (9) holds for some $B \in \mathcal{S}^n$, where $I = I(\mathbf{p})$ and $K = J_A(\mathbf{p}) \setminus I$. Then for any $\bar{B} \in \operatorname{span} \mathcal{C}_A(\{\mathbf{p}\})$ and any $\lambda \in \mathbb{R}$, the matrix $B' := B + \lambda\bar{B}$ will also satisfy (9).*

Proof From the observations

$$\begin{pmatrix} \mathsf{A}_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{p}_I \\ -v \end{pmatrix}, \quad \text{where } v = \mathbf{p}_I^\top(\mathsf{A}_{I\times I})\mathbf{p}_I,$$
$$\begin{pmatrix} \bar{\mathsf{B}}_{I\times I} & \mathbf{o} \\ \mathbf{o}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_I \\ -v \end{pmatrix} = w \begin{pmatrix} \mathbf{e} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \bar{\mathsf{B}}_{K\times I} & \mathbf{o} \end{pmatrix} \begin{pmatrix} \mathbf{p}_I \\ -v \end{pmatrix} = w\mathbf{e}, \text{ where } w = \mathbf{p}_I^\top(\bar{\mathsf{B}}_{I\times I})\mathbf{p}_I,$$
$$\begin{pmatrix} \mathsf{A}_{K\times I} & \mathbf{e} \end{pmatrix} \begin{pmatrix} \mathsf{A}_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{e} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathsf{A}_{K\times I} & \mathbf{e} \end{pmatrix} \begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix} = \mathbf{e},$$

we conclude that $h_{I,K}$, defined in (9), satisfies

$$h_{I,K}(\bar{\mathsf{B}}) = \left[(\bar{\mathsf{B}}_{K\times I} \mid \mathbf{o}) - (\mathsf{A}_{K\times I} \mid \mathbf{e}) \begin{pmatrix} \mathsf{A}_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} \bar{\mathsf{B}}_{I\times I} & \mathbf{o} \\ \mathbf{o}^\top & 0 \end{pmatrix} \right] \begin{pmatrix} \mathsf{A}_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix} = \mathbf{o}.$$

Since $h_{I,K}$ is linear, we conclude $h_{I,K}(\mathsf{B}') = h_{I,K}(\mathsf{B})$, and this completes the proof. □

Now we turn to variations A' of A. Any matrix $\mathsf{A}' \in \mathcal{C}_{\mathsf{A}}(\mathcal{P})$ is a candidate for some perturbation $\mathsf{A}' + \varepsilon\mathsf{B}$ to yield $\{I(\mathbf{p}) : \mathbf{p} \in \mathcal{P}\} \subseteq \operatorname{qpattern}(\mathsf{A}' + \varepsilon\mathsf{B})$; however, in general, not every A', if any, will achieve this goal. Anyway, the search for a suitable A' can be restricted to the set $\mathcal{C}'_{\mathsf{A}}(\mathcal{P}) := \{\mathsf{A}' \in \mathcal{C}_{\mathsf{A}}(\mathcal{P}) : \mathsf{A}' \perp \mathbf{e}\mathbf{e}^\top,\ \|\mathsf{A}'\| = 1\}$.

Fixing now $\mathbf{p} \in \mathcal{E}(\mathsf{A})$ with $|J_{\mathsf{A}}(\mathbf{p})| > |I(\mathbf{p})|$, and fixing B satisfying (9) with $I = I(\mathbf{p})$, $K = J_{\mathsf{A}}(\mathbf{p}) \setminus I$, we ask whether (9) will stay satisfied if we replace A with some $\mathsf{A}' \in \mathcal{C}_{\mathsf{A}}(\{\mathbf{p}\})$. The answer will clearly be yes, if $J_{\mathsf{A}'}(\mathbf{p}) = J_{\mathsf{A}}(\mathbf{p}) =: J$ and $\mathsf{A}'_{J\times J} = \gamma \mathsf{A}_{J\times J} + \lambda \mathbf{e}\mathbf{e}^\top$ for some $\gamma > 0, \lambda \in \mathbb{R}$, but there are other cases as well, as the following theorem tells.

Theorem 9 *Let $A \in \mathcal{S}^n$ and $\mathbf{p} \in \mathcal{E}(A)$ such that $|I| < |J|$, where $I = I(\mathbf{p})$, $J = J_A(\mathbf{p})$, $K = J \setminus I$, and also assume $\mathbf{p}^\top A\mathbf{p} = 0$. Denote the L.H.S. of (9) by $g(A, B)$ [there it is denoted by $h_{I,K}(B)$ but we want to fix $B \in \mathcal{S}^n$ now]. Then for any $A' \in \mathcal{C}_A(\{\mathbf{p}\})$ satisfying $\mathbf{p}^\top A'\mathbf{p} = 0$ and colspace$(A'_{J\times I})$ = colspace$(A_{J\times I})$ we have $g(A, B) = g(A', B)$.*

Proof Our assumptions $\mathbf{p} \in \mathcal{E}(\mathsf{A})$ and $\mathbf{p}^\top \mathsf{A}\mathbf{p} = 0$ imply $\mathsf{A}_{J\times I}\mathbf{p}_I = \mathbf{o}$. Then also $\mathsf{A}_{I\times I}\mathbf{p}_I = \mathbf{o}$, and from $\mathcal{B}(\mathsf{A}_{I\times I}) \succ \mathsf{O}$ we deduce rank$(\mathsf{A}_{I\times I}) = |I| - 1$ and $\mathsf{A}_{I\times I} \preceq \mathsf{O}$. Moreover, every principal submatrix of order $|I| - 1$ of $\mathsf{A}_{I\times I}$ is negative definite.[2] Next we show that there is a full rank matrix $\mathsf{F} \in \mathbb{R}^{|J|\times(|I|-1)}$ such that $(-\mathsf{F}\mathsf{F}^\top)_{[1:|J|]\times[1:|I|]} = \mathsf{A}_{J\times I}$. In fact, denoting the upper left principal submatrix of order $|I| - 1$ of $\mathsf{A}_{I\times I}$ (corresponding to the index set $I' \subseteq I$) by $\tilde{\mathsf{A}}$, we have $-\tilde{\mathsf{A}} = \mathsf{R}\mathsf{R}^\top \succ \mathsf{O}$ for some invertible $\mathsf{R} \in \mathcal{S}^{|I|-1}$, and may choose $\mathsf{F} := \mathsf{A}_{J\times I'}(\mathsf{R}^\top)^{-1}$. Indeed, it is easy to see $(-\mathsf{F}\mathsf{F}^\top)_{[1:|J|]\times[1:|I|-1]} = \mathsf{A}_{J\times I'}$, and from $\big[(\mathsf{A} + \mathsf{F}\mathsf{F}^\top)\mathbf{p}\big]_{[1:|J|]} = \mathbf{o}$ we deduce exact match of the $|I|$th columns of both matrices. Clearly we have colspace$(\mathsf{A}_{J\times I})$ = colspace(F). Now the same conclusions can be made about A', resulting in a full rank matrix $\mathsf{F}' \in \mathbb{R}^{|J|\times(|I|-1)}$ spanning the same column space as F, so there has to be a positive-definite matrix $\mathsf{Q} \in \mathcal{S}^{|I|-1}$ such that $\mathsf{A}'_{J\times I} = (-\mathsf{F}\mathsf{Q}\mathsf{F}^\top)_{[1:|J|]\times[1:|I|]}$. We may and do assume that $\mathsf{Q} = \mathrm{Diag}(\lambda_1, \dots, \lambda_{|I|-1}) \succ \mathsf{O}$. It remains to show that the linear maps represented by $\mathsf{M} := (\mathsf{A}_{K\times I} \mid \mathbf{e})\begin{pmatrix} \mathsf{A}_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1}$ and $\mathsf{M}' := (\mathsf{A}'_{K\times I} \mid \mathbf{e})\begin{pmatrix} \mathsf{A}'_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1}$ coincide on a set of vectors spanning $\mathbb{R}^{|I|+1}$.

We choose the set $\{\binom{\mathbf{o}}{1}, \binom{\mathbf{e}}{0}, \binom{\mathbf{f}_1}{0}, \dots, \binom{\mathbf{f}_{|I|-1}}{0}\}$, where $\mathsf{F}_{[1:|I|]\times[1:|I|-1]} = (\mathbf{f}_1, \dots, \mathbf{f}_{|I|-1})$. Linear independence of those vectors is established by noting that $\binom{\mathbf{f}_i}{0}_{i\in[1:|I|-1]}$ are linearly independent because F has full column rank, and by observing $\mathsf{F}^\top\mathbf{p}_I = \mathbf{o}$ (implying $\mathbf{f}_i^\top\mathbf{p}_I = 0$ for $1 \le i \le |I| - 1$) and $\mathbf{e}^\top\mathbf{p}_I = 1$. First of all note

$$\mathsf{M}\binom{\mathbf{o}}{1} = (\mathsf{A}_{K\times I} \mid \mathbf{e})\binom{\mathbf{p}_I}{0} = \mathbf{o} = (\mathsf{A}'_{K\times I} \mid \mathbf{e})\binom{\mathbf{p}_I}{0} = \mathsf{M}'\binom{\mathbf{o}}{1}$$

because of $J_{\mathsf{A}} \subseteq J_{\mathsf{A}'}$, and

$$\mathsf{M}\begin{pmatrix}\mathbf{e}\\0\end{pmatrix} = (\mathsf{A}_{K\times I} \mid \mathbf{e})\begin{pmatrix}\mathbf{o}\\1\end{pmatrix} = \mathbf{e} = (\mathsf{A}'_{K\times I} \mid \mathbf{e})\begin{pmatrix}\mathbf{o}\\1\end{pmatrix} = \mathsf{M}'\begin{pmatrix}\mathbf{e}\\0\end{pmatrix}.$$

Next fix $i \in [1 : |I| - 1]$ and choose $\mathbf{v}_i \in \mathbb{R}^{|I|}$ such that (employing the Kronecker delta) $\mathbf{f}_j^\top\mathbf{v}_i = \delta_{ij}$ for $j \in [1 : |I| - 1]$ and $\mathbf{e}^\top\mathbf{v}_i = 0$. It can be easily checked that

$$\begin{pmatrix} \mathsf{A}_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}\begin{pmatrix}\mathbf{v}_i\\0\end{pmatrix} = \begin{pmatrix} (-\mathsf{F}\mathsf{F}^\top)_{[1:|I|]\times[1:|I|]} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}\begin{pmatrix}\mathbf{v}_i\\0\end{pmatrix} = \binom{-\mathbf{f}_i}{0} \text{ and}$$

$$\begin{pmatrix} \mathsf{A}'_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}\begin{pmatrix}\mathbf{v}_i\\0\end{pmatrix} = \begin{pmatrix} (-\mathsf{F}\mathsf{Q}\mathsf{F}^\top)_{[1:|I|]\times[1:|I|]} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}\begin{pmatrix}\mathbf{v}_i\\0\end{pmatrix} = \binom{-\lambda_i\mathbf{f}_i}{0}.$$

[2] Just note that $\mathcal{B}(\mathsf{A}_{I\times I}) \succ \mathsf{O}$ is equivalent to $\mathbf{x}^\top\mathsf{A}_{I\times I}\mathbf{x} < 0$ for all $\mathbf{x} \in \mathbf{e}^\perp \setminus \{\mathbf{o}\}$, and then, because of $\mathsf{A}_{I\times I}\mathbf{p}_I = \mathbf{o}$, we have $(\lambda\mathbf{p}_I + \mathbf{x})^\top\mathsf{A}_{I\times I}(\lambda\mathbf{p}_I + \mathbf{x}) \le 0$, for $\lambda \in \mathbb{R}$ and $\mathbf{x} \perp \mathbf{e}$, which means $\mathsf{A}_{I\times I} \preceq \mathsf{O}$. If for some $I' \subset I$ with $|I'| = |I| - 1$ the matrix $\mathsf{A}_{I'\times I'}$ had not full rank, there would be $\mathbf{q} \in \Delta^{|I|}$ satisfying $I(\mathbf{q}) = I'$, such that $\mathsf{A}_{I\times I}\mathbf{q} = \mathbf{o}$. Thus the dimension of the kernel of $\mathsf{A}_{I\times I}$ would be at least 2, and the rank of $\mathsf{A}_{I\times I}$ would have to be less than $|I| - 1$, which is a contradiction.

From this we obtain

$$\mathsf{M}\binom{\mathbf{f}_i}{0} = (\mathsf{A}_{K\times I} \mid \mathbf{e})\binom{-\mathbf{v}_i}{0} = \mathbf{g}_i \quad \text{and} \quad \mathsf{M}'\binom{\mathbf{f}_i}{0} = \lambda_i^{-1}(\mathsf{A}'_{K\times I} \mid \mathbf{e})\binom{-\mathbf{v}_i}{0} = \mathbf{g}_i\,,$$

where $\mathsf{F}_{[|I|+1:|J|]\times[1:|I|-1]} = (\mathbf{g}_1, \ldots, \mathbf{g}_{|I|-1})$, which finishes the proof. □

Corollary 10 *Let $A = -FF^\top$, where $F \in \mathbb{R}^{n\times k}$, and assume $\Delta^n \cap \ker F^\top \neq \emptyset$. Then for every fixed $\mathbf{p} \in \mathcal{E}(A) = Ext(\Delta^n \cap \ker F^\top)$ and every fixed diagonal matrix $\Lambda \succ O$ we have $A' := -F\Lambda F^\top \in \mathcal{C}_A(\{\mathbf{p}\})$, and $J_{A'}(\mathbf{p}) = J_A(\mathbf{p}) = [1 : n]$. Moreover, $colspace(A_{[1:n]\times I(\mathbf{p})}) = colspace(A'_{[1:n]\times I(\mathbf{p})})$, therefore, by Theorem 9, for fixed $B \in \mathcal{S}^n$, we have $g(A, B) = g(A', B)$.*

That is, for every $\mathcal{P} \subseteq \mathcal{E}(A)$ and for every $\Lambda \succ O$, if $\{I(\mathbf{p}) : \mathbf{p} \in \mathcal{P}\} \subseteq qpattern(A + \varepsilon B)$ for some $B \in \mathcal{S}^n$ and for $\varepsilon > 0$ small enough, then also $\{I(\mathbf{p}) : \mathbf{p} \in \mathcal{P}\} \subseteq qpattern(A' + \varepsilon B)$ for that same $B \in \mathcal{S}^n$ and for $\varepsilon > 0$ small enough.

2.3 Cyclic Symmetry

We call a subset $\mathcal{P} \subseteq \mathbb{R}^n$ closed under cyclic permutations, if $\mathbf{p} \in \mathcal{P} \Rightarrow \mathsf{P}\mathbf{p} \in \mathcal{P}$, where $\mathsf{P} = (p_{ij})$ satisfies $p_{ij} = 1$ for $j \equiv i + 1 (\bmod n)$, and otherwise $p_{ij} = 0$. We call $\mathsf{A} \in \mathcal{S}^n$ *cyclically symmetric* (or *symmetric circulant*), if it satisfies $\mathsf{A} = \mathsf{P}^\top\mathsf{A}\mathsf{P}$, and employ the notation $\mathsf{C}(\mathbf{a})$ for a cyclically symmetric matrix whose first column is $\mathbf{a}$. Note that for an $n \times n$ cyclically symmetric matrix A the set $\mathcal{E}(\mathsf{A})$ is closed under cyclic permutations. In case that A is cyclically symmetric, there are well-behaved subsets of $\mathcal{E}(\mathsf{A})$, where the "transfer" can be achieved by cyclically symmetric perturbations, whenever it can be achieved by certain symmetric perturbations, as the following theorem tells.

Theorem 11 *Let $A \in \mathcal{S}^n$ be cyclically symmetric, let $\mathcal{P} \subseteq \mathcal{E}(A)$ be closed under cyclic permutations, and let $\mathcal{P}' := \{\mathbf{p} \in \mathcal{P} : |I(\mathbf{p})| < |J_A(\mathbf{p})|\}$, which is then also closed under cyclic permutations.*

Suppose that there is $B \in \mathcal{S}^n$ satisfying (9) for all $(I, K) \in \mathcal{I}_{\mathcal{P}'}$. Then the matrix $C := \sum_{i=1}^n (P^i)^\top B P^i$ is cyclically symmetric and satisfies

$$h_{I,K}(C) = \left[(C_{K\times I} \mid \mathbf{o}) - (A_{K\times I} \mid \mathbf{e}) \begin{pmatrix} A_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} C_{I\times I} & \mathbf{o} \\ \mathbf{o}^\top & 0 \end{pmatrix} \right] \begin{pmatrix} A_{I\times I} & \mathbf{e} \\ \mathbf{e}^\top & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix} < \mathbf{o}\,, \tag{10}$$

also for all $(I, K) \in \mathcal{I}_{\mathcal{P}'}$.

Thus, by Lemma 2, for $\varepsilon > 0$ small enough, we have $\{I(\mathbf{p}) : \mathbf{p} \in \mathcal{P}\} \subseteq qpattern(A + \varepsilon C)$.

Proof From $\mathsf{C} = \mathsf{P}^\top\mathsf{C}\mathsf{P}$ it is clear that C is cyclically symmetric. Next observe that $(\mathsf{P}^\top\mathsf{B}\mathsf{P})_{K\times I} = \mathsf{B}_{K'\times I'}$, where $I = I(\mathbf{p})$, $I' = I(\mathsf{P}\mathbf{p})$, $K = J_\mathsf{A}(\mathbf{p})\setminus I$, and $K' = J_\mathsf{A}(\mathsf{P}\mathbf{p})\setminus I'$, as well as $(\mathsf{P}^\top\mathsf{B}\mathsf{P})_{I\times I} = \mathsf{B}_{I'\times I'}$, for some $\{\mathbf{p}, \mathsf{P}\mathbf{p}\} \subseteq \mathcal{P}$. Since A is cyclically symmetric, we have $\mathsf{A}_{K'\times I'} = \mathsf{A}_{K\times I}$ and $\mathsf{A}_{I'\times I'} = \mathsf{A}_{I\times I}$. Thus, if (9) is satisfied, it stays satisfied if we replace B by $(\mathsf{P}^i)^\top\mathsf{B}\mathsf{P}^i$. Summing over i then finishes the proof of (10). Finally, $I(\mathbf{p}) \in$ qpattern$(\mathsf{A}+\varepsilon\mathsf{C})$, for $\varepsilon > 0$ small enough, is true for $\mathbf{p} \in \mathcal{P}'$ by Lemma 2, and true for $\mathbf{p} \in \mathcal{P}\setminus\mathcal{P}'$ by a continuity argument (quasistrictness of NESs is preserved, as is negative-definiteness). □

Remark 3 Note that cyclic symmetry of C has the following implication. If for some $\mathbf{p} \in \Delta^n$ the inequalities (10) are satisfied for $(I, K) \in \mathcal{I}_{\{\mathbf{p}\}}$, those inequalities will be satisfied for all $(I, K) \in \mathcal{I}_{\mathcal{Z}_\mathbf{p}}$, where $\mathcal{Z}_\mathbf{p} := \{\mathbf{p}, \mathsf{P}\mathbf{p}, \ldots, \mathsf{P}^{n-1}\mathbf{p}, \mathsf{R}\mathbf{p}, \mathsf{P}\mathsf{R}\mathbf{p}, \ldots, \mathsf{P}^{n-1}\mathsf{R}\mathbf{p}\}$, with $\mathsf{R}\mathbf{p} := [p_n, p_{n-1}, \ldots, p_1]^\top$ denoting the vector with the coordinates of $\mathbf{p}$ in reverse order.

Remark 4 Note that Proposition 6 is not valid if we require A and A'' to be cyclically symmetric, as is demonstrated by the following example: For the cyclically symmetric matrix $\mathsf{A} := \mathsf{C}([-1, 0, 1, 0, -1, 0, 1, 0]^\top)$ of rank 2 we have $\mathcal{E}(\mathsf{A}) = \{\frac{1}{2}(\mathbf{e}_i + \mathbf{e}_{i+2}) : i \in [1:8]\}$, with $i+2 \in [1:8]$ computed modulo 8, so all supports have size 2. Now there is no cyclically symmetric matrix A' of rank 1 satisfying $\mathcal{E}(\mathsf{A}) \subseteq \mathcal{E}(\mathsf{A}')$, but we have $\mathcal{E}(\mathsf{A}) \subseteq \mathcal{E}(\mathsf{A}'')$ for the rank 1 matrix $\mathsf{A}'' := -\mathsf{H}\mathsf{H}^\top$, where $\mathsf{H}^\top := (1, 1, -1, -1, 1, 1, -1, -1)$. Also observe that $\mathsf{A}_\varepsilon := \mathsf{C}([-1, 0, 1+\varepsilon, 0, -1-\varepsilon, 0, 1+\varepsilon, 0]^\top)$ satisfies $\mathrm{ESS}(\mathsf{A}_\varepsilon) = \mathcal{E}(\mathsf{A})$ for arbitrarily small $\varepsilon > 0$, so in this example, perturbation of A yields a cyclically symmetric matrix as desired, whereas we can even get $\mathcal{E}(\mathsf{A}) \subseteq \mathrm{ESS}(\mathsf{A}''_\varepsilon)$ with $|\mathrm{ESS}(\mathsf{A}''_\varepsilon)| = 16$, where $\mathsf{A}''_\varepsilon := \mathsf{A}'' + \varepsilon \mathsf{I}_8$, when not insisting on cyclic symmetry.

The vector space $\mathcal{C}^n := \{\mathsf{A} \in \mathcal{S}^n : \mathsf{A} \text{ is cyclically symmetric}\}$, on which we now concentrate, has dimension $n'+1$, where $n' := \lceil \frac{n-1}{2} \rceil$, a basis being $\{\mathsf{P}^i + \mathsf{P}^{-i} : i \in [0:n']\}$. As $\mathcal{C}^n$ is a subspace of the space of all circulant matrices (see [14]) of order n, all members of $\mathcal{C}^n$ are simultaneously diagonalizable, and because of their symmetry, a common orthogonal basis of real eigenvectors can be used for that purpose. Those are the nonzero vectors in the set $\{\mathbf{c}_i, \mathbf{s}_i : i \in [0:n']\}$, where

$$\mathbf{c}_i := \left[\cos\frac{2i\pi}{n}, \cos\frac{4i\pi}{n}, \ldots, \cos\frac{2ni\pi}{n}\right]^\top \quad \text{and} \quad \mathbf{s}_i := \left[\sin\frac{2i\pi}{n}, \sin\frac{4i\pi}{n}, \ldots, \sin\frac{2ni\pi}{n}\right]^\top.$$

This leads to a basis for $\mathcal{C}^n$, consisting of low rank matrices $\mathsf{V}_i := \mathbf{c}_i\mathbf{c}_i^\top + \mathbf{s}_i\mathbf{s}_i^\top$, with those ranks adding up to n, that will be more convenient. The set $\{\mathsf{V}_i : i \in [0:n']\}$ indeed constitutes such a basis. Just note that $\mathsf{V}_0 = \mathbf{e}\mathbf{e}^\top$ has rank 1, $\mathsf{V}_{n'} = \mathbf{c}_{n'}\mathbf{c}_{n'}^\top$ in case of even n has rank 1 as well, and all remaining matrices V_i are seen to have rank 2 by orthogonality of $\{\mathbf{c}_i, \mathbf{s}_i\}$ for each $i \in [1:n']$ (and also for $i = n'$ if n is odd). Moreover, employing trigonometric identities, we obtain $(\mathsf{V}_i)_{k,\ell} = \cos\frac{2(k-\ell)i\pi}{n}$, which depends on k, ℓ only via the residue class of $|k-\ell| \mod n$, showing that $\mathsf{V}_i \in \mathcal{C}^n$ for all $i \in [0:n']$. Linear independence of the set $\{\mathsf{V}_i : i \in [0:n']\}$ follows from linear independence of corresponding column spaces, which follows from the observation that $\{\mathbf{c}_i, \mathbf{s}_i : i \in [0:n']\} \setminus \{\mathbf{o}\}$ constitutes a basis of $\mathbb{R}^n$. Note that with respect to the Frobenius inner product, we have $\mathsf{V}_i \bullet \mathsf{V}_j = 0$ if $i \neq j$.

2.4 Construction

Now consider negative-semidefinite matrices $\mathsf{A} \in \mathcal{C}^n$ satisfying $\mathsf{A} \bullet \mathsf{V}_0 = 0$. That is, there is a representation $\mathsf{A} = -\mathsf{F}\mathsf{F}^\top = -\sum_{i\in L} \lambda_i \mathsf{V}_i$ for some $L \subseteq [1:n']$, $\lambda_i > 0$ for $i \in L$, $\mathsf{F} \in \mathbb{R}^{n\times k}$, and the columns of F are nonzero multiples of vectors in the set $\{\mathbf{c}_i, \mathbf{s}_i \neq \mathbf{o} : i \in L\}$. (Note that, if n is odd then $k := \operatorname{rank}(\mathsf{A}) = 2|L|$ is even.) Also $\frac{1}{n}\mathbf{e} \in \Delta^n \cap \ker \mathsf{F}^\top$, therefore Corollary 10 applies. So when perturbing negative-semidefinite matrices $\mathsf{A} \in \mathcal{C}^n \setminus \{\mathsf{O}\}$ satisfying $\mathsf{A} \bullet \mathsf{V}_0 = 0$, we may restrict our attention to the finite set of *candidate matrices* $\mathcal{A}_n := \{\mathsf{A}_L : \emptyset \neq L \subseteq [1:n']\}$, where $\mathsf{A}_L := -\sum_{i\in L} \mathsf{V}_i$. Moreover, by Theorem 8, we may restrict the matrix B, used to perturb A_L via $\mathsf{A}_L + \varepsilon\mathsf{B}$, to span $\{\mathsf{V}_j : j \in [1:n'] \setminus L\}$.

As we can restrict ourselves to matrices with zero diagonal, yet another basis will even better serve our needs: $\{\mathsf{W}_i : i \in [0:n']\}$, where $\mathsf{W}_0 = \mathsf{V}_0$ and $\mathsf{W}_i = \frac{1}{2}(\mathbf{e}\mathbf{e}^\top - \mathsf{V}_i)$ for $i \in [1:n']$, with $(\mathsf{W}_i)_{k,\ell} = \sin^2\frac{(k-\ell)i\pi}{n}$. The last two sentences of the preceding paragraph remain true with $\mathsf{A}_L := \sum_{i\in L} \mathsf{W}_i$ and $\mathsf{B} \in \operatorname{span}\{\mathsf{W}_j : j \in [1:n'] \setminus L\}$.

We are only interested in the *size* of qpattern(A), so we would not distinguish between $\mathsf{A} \in \mathcal{C}^n$ and $\mathsf{A}' := \bar{\mathsf{P}}\mathsf{A}\bar{\mathsf{P}}^\top$, where $\bar{\mathsf{P}}$ is a permutation matrix. If $\mathsf{A}' = \bar{\mathsf{P}}\mathsf{A}\bar{\mathsf{P}}^\top \in \mathcal{C}^n$, but $\mathsf{A}' \neq \mathsf{A}$, this allows for a further reduction of the set of candidate matrices. In particular, if for some permutation τ of the set $[1:n']$, we have $\bar{\mathsf{P}}\mathsf{W}_i\bar{\mathsf{P}}^\top = \mathsf{W}_{\tau(i)}$ for $i \in [1:n']$, then for sets $L \subseteq [1:n']$ and $\tau L := \{\tau(i) : i \in L\}$ we have

$$\mathbf{p} \in \mathrm{ESS}(\mathsf{A}_L + \varepsilon\mathsf{B}) \Leftrightarrow \bar{\mathsf{P}}\mathbf{p} \in \mathrm{ESS}(\mathsf{A}_{\tau L} + \varepsilon\mathsf{B}'),$$

where $\mathsf{B}' = \bar{\mathsf{P}}\mathsf{B}\bar{\mathsf{P}}^\top$. That is, we can further restrict our set of candidate matrices to a subset $\bar{\mathcal{A}}_n \subseteq \mathcal{A}_n$ by ensuring that for every $L \subseteq [1:n']$ exactly one matrix from the set $\{\mathsf{A}_L, \mathsf{A}_{\tau L}, \mathsf{A}_{\tau^2 L}, \ldots\}$ is contained in $\bar{\mathcal{A}}_n$.

Example 12 For $n = 6$ we obtain

$$\mathsf{W}_0 = \begin{pmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{pmatrix}, \mathsf{W}_1 = \frac{1}{4}\begin{pmatrix} 0&1&3&4&3&1\\ 1&0&1&3&4&3\\ 3&1&0&1&3&4\\ 4&3&1&0&1&3\\ 3&4&3&1&0&1\\ 1&3&4&3&1&0 \end{pmatrix}, \mathsf{W}_2 = \frac{1}{4}\begin{pmatrix} 0&3&3&0&3&3\\ 3&0&3&3&0&3\\ 3&3&0&3&3&0\\ 0&3&3&0&3&3\\ 3&0&3&3&0&3\\ 3&3&0&3&3&0 \end{pmatrix}, \mathsf{W}_3 = \begin{pmatrix} 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{pmatrix}.$$

The only permutation τ that can be introduced via a permutation matrix $\bar{\mathsf{P}}$ as above is the identity, so the set of candidate matrices will be $\bar{\mathcal{A}}_6 = \mathcal{A}_6 = \{\mathsf{A}_L : \emptyset \neq L \subseteq [1:3]\}$.

For $n = 7$, we obtain

$$\mathsf{W}_0 = \mathbf{e}\mathbf{e}^\top,\ \mathsf{W}_1 = \mathsf{C}([0, \sin^2\tfrac{\pi}{7}, \sin^2\tfrac{2\pi}{7}, \sin^2\tfrac{3\pi}{7}, \sin^2\tfrac{3\pi}{7}, \sin^2\tfrac{2\pi}{7}, \sin^2\tfrac{\pi}{7}]^\top),$$
$$\mathsf{W}_2 = \bar{\mathsf{P}}\mathsf{W}_1\bar{\mathsf{P}}^\top,\ \mathsf{W}_3 = \bar{\mathsf{P}}\mathsf{W}_2\bar{\mathsf{P}}^\top,$$

where $\bar{\mathsf{P}} = (\bar{p}_{ij})$ satisfies $\bar{p}_{ij} = 1$ for $j \equiv 2i (\bmod n)$, and otherwise $\bar{p}_{ij} = 0$. Then also $\mathsf{W}_1 = \bar{\mathsf{P}}\mathsf{W}_3\bar{\mathsf{P}}^\top$ holds. So $\bar{\mathsf{P}}$ induces a non-trivial permutation τ on $[1:3]$, and the set of candidate matrices reduces to $\bar{\mathcal{A}}_7 := \{\mathsf{A}_{\{1\}}, \mathsf{A}_{\{1,2\}}, \mathsf{A}_{\{1,2,3\}}\}$.

Note that for the candidate matrix $\mathsf{A} := \mathsf{A}_{[1:n']} \in \bar{\mathcal{A}}_n$ we have $\mathcal{E}(\mathsf{A}) = \mathrm{ESS}(\mathsf{A}) = \{\frac{1}{n}\mathbf{e}\}$, and the latter set will not change, if A is slightly perturbed. Therefore we exclude that candidate matrix from further considerations. For the remaining candidate matrices A_L we have $L' := [1:n'] \setminus L \neq \emptyset$, and for $\boldsymbol{\alpha} \in \mathbb{R}^{L'}$ we define $\mathsf{B}_{\boldsymbol{\alpha}} := \sum_{i \in L'} \alpha_i \mathsf{W}_i$. For every $\mathbf{p} \in \mathcal{E}(\mathsf{A}_L)$, we denote by $\mathcal{L}_{L,\mathbf{p}}$ the solution set of inequality (9), where $I = I(\mathbf{p})$ and $K = J_{\mathsf{A}_L}(\mathbf{p}) \setminus I$, and clearly that inequality is linear in $\boldsymbol{\alpha}$. Next, we denote by $\mathcal{R}_L$ a maximal subset of $\mathcal{E}(\mathsf{A}_L)$ consisting only of $\mathbf{p}$ of maximal support size, i.e., $|I(\mathbf{p})| = \mathrm{rank}\,(\mathsf{A}_L) + 1$, and satisfying $\mathcal{Z}_{\mathbf{p}} \cap \mathcal{R}_L = \{\mathbf{p}\}$ for every $\mathbf{p} \in \mathcal{R}_L$, where $\mathcal{Z}_{\mathbf{p}}$ is introduced in Remark 3. Our aim is then to find a subset $\bar{\mathcal{R}}_L \subseteq \mathcal{R}_L$ of largest $\mathrm{size}(\bar{\mathcal{R}}_L) := \sum_{\mathbf{p} \in \bar{\mathcal{R}}_L} |\mathcal{Z}_{\mathbf{p}}|$, such that

$$\bigcap_{\mathbf{p} \in \bar{\mathcal{R}}_L} \mathcal{L}_{L,\mathbf{p}} \neq \emptyset.$$

Any $\boldsymbol{\alpha}$ in that nonempty intersection will then lead to a matrix $\mathsf{B}_{\boldsymbol{\alpha}}$, such that for $\varepsilon > 0$ small enough

$$|\mathrm{ESS}(\mathsf{A}_L + \varepsilon\mathsf{B}_{\boldsymbol{\alpha}})| \geq \mathrm{size}(\bar{\mathcal{R}}_L).$$

A natural further reduction of the set of candidate matrices is via the definition $\bar{\bar{\mathcal{A}}}_n := \{\mathsf{A}_L \in \bar{\mathcal{A}}_n : \mathcal{R}_L \neq \emptyset\}$.

Example 13 (Continuation of Example 12) For $n = 6$ we have $\mathcal{E}(\mathsf{A}_{\{1\}}) = \mathcal{Z}_{\mathbf{p}_1} \cup \mathcal{Z}_{\mathbf{p}_2}$, $\mathcal{E}(\mathsf{A}_{\{2\}}) = \mathcal{Z}_{\mathbf{p}_1} \cup \mathcal{Z}_{\mathbf{p}_3}$, $\mathcal{E}(\mathsf{A}_{\{3\}}) = \mathcal{Z}_{\mathbf{p}_2} \cup \mathcal{Z}_{\mathbf{p}_4}$, $\mathcal{E}(\mathsf{A}_{\{1,2\}}) = \mathcal{Z}_{\mathbf{p}_1}$, $\mathcal{E}(\mathsf{A}_{\{1,3\}}) = \mathcal{Z}_{\mathbf{p}_2}$, $\mathcal{E}(\mathsf{A}_{\{2,3\}}) = \mathcal{Z}_{\mathbf{p}_5}$, where

$$\mathbf{p}_1 := \tfrac{1}{3}(\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_5), \mathbf{p}_2 := \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_4),\ \mathbf{p}_3 := \tfrac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3),$$
$$\mathbf{p}_4 := \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{p}_5 := \tfrac{1}{6}(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 + \mathbf{e}_4),$$

resulting in the following sets $\mathcal{R}_L$ for $L \subseteq [1:3]$, $0 < |L| < 3$:

$$\mathcal{R}_{\{1\}} = \{\mathbf{p}_1\}\,,\ \mathcal{R}_{\{2\}} = \{\mathbf{p}_1, \mathbf{p}_3\},\ \mathcal{R}_{\{3\}} = \{\mathbf{p}_2, \mathbf{p}_4\}\,,\ \mathcal{R}_{\{1,2\}} = \emptyset\,,\ \mathcal{R}_{\{1,3\}} = \emptyset\,,\ \mathcal{R}_{\{2,3\}} = \{\mathbf{p}_5\}\,,$$

with maximizing sets $\bar{\mathcal{R}}_L = \mathcal{R}_L$ for all L and respective sizes

$$\mathrm{size}(\mathcal{R}_{\{1\}}) = 2, \quad \mathrm{size}(\mathcal{R}_{\{2\}}) = 8, \quad \mathrm{size}(\mathcal{R}_{\{3\}}) = 9, \quad \mathrm{size}(\mathcal{R}_{\{2,3\}}) = 6.$$

Corresponding matrices with 9 ESSs of support size 2 (resp. 8 ESSs of support size 3, resp. 6 ESSs of support size 4) are given in Table 2. Also note that we have just derived $\bar{\bar{\mathcal{A}}}_6 = \{\mathsf{A}_{\{1\}}, \mathsf{A}_{\{2\}}, \mathsf{A}_{\{3\}}, \mathsf{A}_{\{2,3\}}\}$.
Let us consider $L = \{2\}$ in more detail: We have $\mathsf{A}_{\{2\}} = \mathsf{W}_2$, $\mathsf{B}_{\boldsymbol{\alpha}} = \alpha_1 \mathsf{W}_1 + \alpha_2 \mathsf{W}_3$, and for $I = I(\mathbf{p}_3) = \{1, 2, 3\}$, $K = \{4, 5, 6\}$, inequality (9) reads

$$\frac{1}{4}\left[\begin{pmatrix} 4\alpha_1{+}4\alpha_2 & 3\alpha_1 & \alpha_1{+}4\alpha_2 & 0 \\ 3\alpha_1 & 4\alpha_1{+}4\alpha_2 & 3\alpha_1 & 0 \\ \alpha_1{+}4\alpha_2 & 3\alpha_1 & 4\alpha_1{+}4\alpha_2 & 0 \end{pmatrix} - \begin{pmatrix} 0&3&3&4 \\ 3&0&3&4 \\ 3&3&0&4 \end{pmatrix}\begin{pmatrix} 0&3&3&4 \\ 3&0&3&4 \\ 3&3&0&4 \\ 4&4&4&0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \alpha_1{+}4\alpha_2 & 3\alpha_1 & 0 \\ \alpha_1{+}4\alpha_2 & 0 & \alpha_1{+}4\alpha_2 & 0 \\ 3\alpha_1 & \alpha_1{+}4\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix} < \mathbf{o}\,,$$

simplifying to $\frac{1}{12}\begin{bmatrix} 4\alpha_1 + 4\alpha_2 \\ 8\alpha_1 - 4\alpha_2 \\ 4\alpha_1 + 4\alpha_2 \end{bmatrix} < \mathbf{o}$. Moreover, for $I = I(\mathbf{p}_1) = \{1, 3, 5\}$, $K = \{2, 4, 6\}$, inequality (9) reads

$$\frac{1}{4}\left[\begin{pmatrix} \alpha_1{+}4\alpha_2 & \alpha_1{+}4\alpha_2 & 4\alpha_1{+}4\alpha_2 & 0 \\ 4\alpha_1{+}4\alpha_2 & \alpha_1{+}4\alpha_2 & \alpha_1{+}4\alpha_2 & 0 \\ \alpha_1{+}4\alpha_2 & 4\alpha_1{+}4\alpha_2 & \alpha_1{+}4\alpha_2 & 0 \end{pmatrix} - \begin{pmatrix} 3&3&0&4 \\ 0&3&3&4 \\ 3&0&3&4 \end{pmatrix}\begin{pmatrix} 0&3&3&4 \\ 3&0&3&4 \\ 3&3&0&4 \\ 4&4&4&0 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 3\alpha_1 & 3\alpha_1 & 0 \\ 3\alpha_1 & 0 & 3\alpha_1 & 0 \\ 3\alpha_1 & 3\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ \alpha_2 \\ \alpha_2 \end{bmatrix} < \mathbf{o}\,,$$

This results in $\mathcal{L}_{\{2\},\mathbf{p}_3} = \{\boldsymbol{\alpha} \in \mathbb{R}^2 : \alpha_1 + \alpha_2 < 0, 2\alpha_1 - \alpha_2 < 0\}$ and $\mathcal{L}_{\{2\},\mathbf{p}_1} = \{\boldsymbol{\alpha} \in \mathbb{R}^2 : \alpha_2 < 0\}$. These two sets have nonempty intersection, containing $\boldsymbol{\alpha} = -[1, 1]^\top$, which gives rise to $\mathsf{B}_{\boldsymbol{\alpha}}$. Numerical experiments with varying ε yield that $\varepsilon := \frac{1}{4}$ is small enough for our purposes: With $\mathsf{M} := \mathsf{A}_{\{2\}} + \varepsilon \mathsf{B}_{\boldsymbol{\alpha}} = \mathsf{C}([0, \frac{7}{16}, \frac{9}{16}, -\frac{1}{2}, \frac{9}{16}, \frac{7}{16}]^\top)$ we have $\mathrm{ESS}(\mathsf{M}) = \mathcal{Z}_{\bar{\mathbf{p}}_3} \cup \mathcal{Z}_{\mathbf{p}_1}$, with $\bar{\mathbf{p}}_3 = \frac{1}{19}[7, 5, 7, 0, 0, 0]^\top$, and in particular $|\mathrm{ESS}(\mathsf{M})| = 8$. Likewise, by choosing $\boldsymbol{\alpha} \in \mathcal{L}_{\{2\},\mathbf{p}_3} \setminus \mathrm{cl}(\mathcal{L}_{\{2\},\mathbf{p}_1})$, resp. $\boldsymbol{\alpha} \in \mathcal{L}_{\{2\},\mathbf{p}_1} \setminus \mathrm{cl}(\mathcal{L}_{\{2\},\mathbf{p}_3})$, we can construct matrices having 6 resp. 2 ESSs.

Figure 1 shows part of the search space for $n = 6$. Note that we may get rid of two of the four parameters, that matrices from $\mathcal{C}^6$ depend on, by assuming a zero diagonal and a fixed sum of entries. In the figure we have fixed the sum of entries of the first row to 2. The interior of the white triangle corresponds to matrices with one single ESS, namely $\frac{1}{6}\mathbf{e}$. The boundary of the white triangle consists of negative-semidefinite matrices (up to an additive multiple of $\mathbf{e}\mathbf{e}^\top$), the three vertices corresponding to (positive multiples of) matrices W_1, W_2 and W_3. The matrix M from above then belongs to the dark gray region. There is a circle attached to each shaded region containing information regarding the pattern of matrices in the interior of that region, e.g., 8_3 indicates that there are 8 ESSs, each of support size 3. Note that the figure clearly misses matrices where entries sum to 0 or a negative value, and indeed the pattern 6_1 attained at the matrix $\mathsf{C}([0, -1, -1, -1, -1, -1]^\top)$ does not show up in the figure; see, however, [5, Figure 1] for the whole search space.

In terms of our perturbation approach, matrix $\mathsf{A}_{\{2,3\}}$ is our entry to region 6_4, matrix $\mathsf{A}_{\{3\}}$ to regions 3_2, 6_2 and 9_2, matrix $\mathsf{A}_{\{2\}}$ to regions 2_3, 6_3 and 8_3, and matrix $\mathsf{A}_{\{1\}}$ is our entry to the interior of $\mathrm{cl}(2_3 \cup 2_3 3_2)$. Note that, in the latter case, $\mathbf{p}_2 \in \mathcal{E}(\mathsf{A}_{\{1\}}) \setminus \mathcal{Z}_{\mathbf{p}_1}$ has insufficient support size and thus does not enter the inequalities restricting $\boldsymbol{\alpha}$. Matrices found near $\mathsf{A}_{\{1\}}$ may lie on either side of, or straight on, the boundary 2_3-$2_3 3_2$.

We conclude this example by some further remarks concerning the regions in Fig. 1, and their boundaries. For matrices A from one of the regions, we always have $\mathrm{ESS}(\mathsf{A}) = \mathcal{E}(\mathsf{A})$.

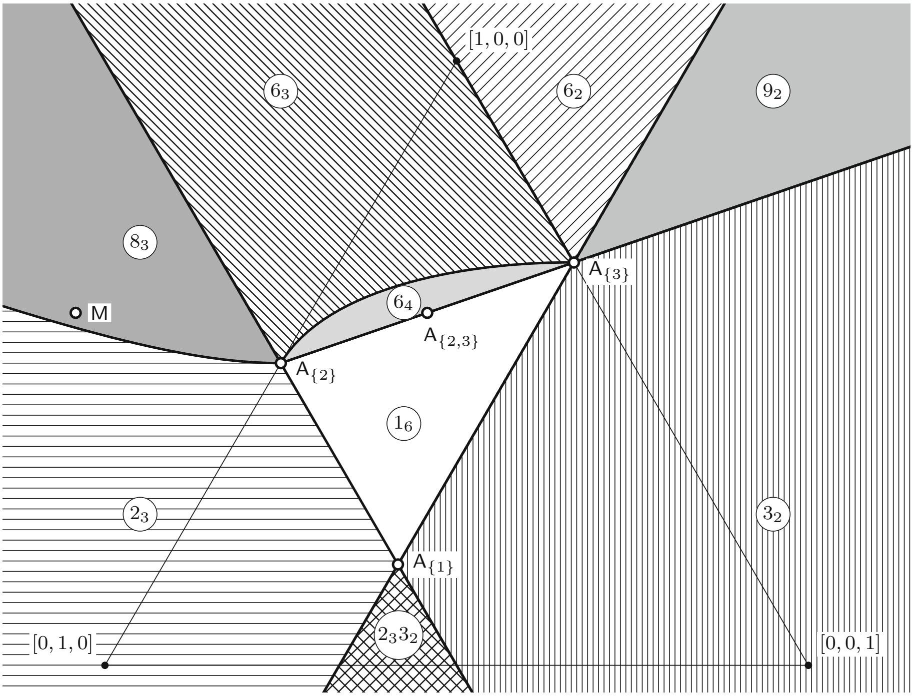

Fig. 1 Part of the search space for $n = 6$ on the hyperplane characterized by $a + b + c = 1$, where points $[a, b, c]$ depicted in barycentric coordinates correspond to positive multiples of matrices $\mathsf{C}([0, a, b, 2c, b, a]^\top)$

Regarding boundaries, we have $\mathrm{ESS}(\mathsf{A}) = \emptyset$, $|\mathrm{NSS}(\mathsf{A})| = \infty$ and $\frac{1}{6}\mathbf{e} \in \mathrm{NSS}(\mathsf{A})$ for every A on the boundary of the white triangle. There are 8 remaining boundary pieces in the complement of the closure of the white triangle, which are listed in Table 1. We observe two types of behavior. Either for a matrix A on a boundary, $|\mathrm{ESS}(\mathsf{A})|$ is the minimum of the numbers of ESSs found in the two regions separated by that boundary, while $|\mathcal{E}(\mathsf{A})|$ is the maximum of those, and $\mathrm{NSS}(\mathsf{A}) = \mathrm{ESS}(\mathsf{A})$. Or $\mathrm{ESS}(\mathsf{A}) = \emptyset$, while

$$\mathrm{NSS}(\mathsf{A}) = \bigcup_{\substack{\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{E}(\mathsf{A}) \\ I(\mathbf{p}_1) \subseteq J_{\mathsf{A}}(\mathbf{p}_2)}} \mathrm{conv}\,(\{\mathbf{p}_1, \mathbf{p}_2\})$$

is infinite. The relative interiors of the sides of the white triangle, that we added to the table, are also of the latter type, with the simplification, resulting from $J_{\mathsf{A}}(\mathbf{p}) = [1:6]$ in the second to last column, that $\mathrm{NSS}(\mathsf{A}) = \mathrm{conv}\,\mathcal{E}(\mathsf{A})$. There may be further types of $n' - 2$-dimensional boundaries for higher orders n.

In general, we would desire a more detailed picture, such as given in Fig. 2, where $\mathrm{pattern}(\mathsf{A})$ enters the definition of a "region" directly and not just via the sizes of the elements of $\mathrm{pattern}(\mathsf{A})$. This would, however, not change Fig. 1. The question arises, if those newly defined "regions", which in general have to be thought of as being confined to a hyperplane not containing the origin, are connected or even convex (the latter is not always the case, as can be seen in Fig. 1).

Table 1 Boundary pieces of regions in Fig. 1

Boundary piece	Equation satisfied	$\lvert\mathrm{ESS}(\mathsf{A})\rvert$	$\lvert\mathcal{E}(\mathsf{A})\rvert$	$\lvert J_{\mathsf{A}}(\mathbf{p})\rvert$ for $\mathbf{p} \in \mathcal{E}(\mathsf{A}) \setminus \mathrm{ESS}(\mathsf{A})$	Type
6_3-8_3	$a - b + c = 0$	6	8	6	1
3_2-$2_3 3_2$	$a - b + c = 0$	3	5	6	1
2_3-$2_3 3_2$	$a + b - 2c = 0$	2	5	6	1
2_3-8_3	$a^2 - ab - 2ac + bc = 0$	2	8	4	1
6_2-9_2	$a + b - 2c = 0$	6	9	6	1
3_2-9_2	$b + 2c - a = 0$	3	9	4	1
6_2-6_3	$b = 0$	0	6	4	2
6_3-6_4	$a^2 + b^2 - 2ab - 2ac = 0$	0	6	5	2
1_6-6_4	$b + 2c - a = 0$	0	6	6	2
1_6-2_3	$a - b + c = 0$	0	2	6	2
1_6-3_2	$a + b - 2c = 0$	0	3	6	2

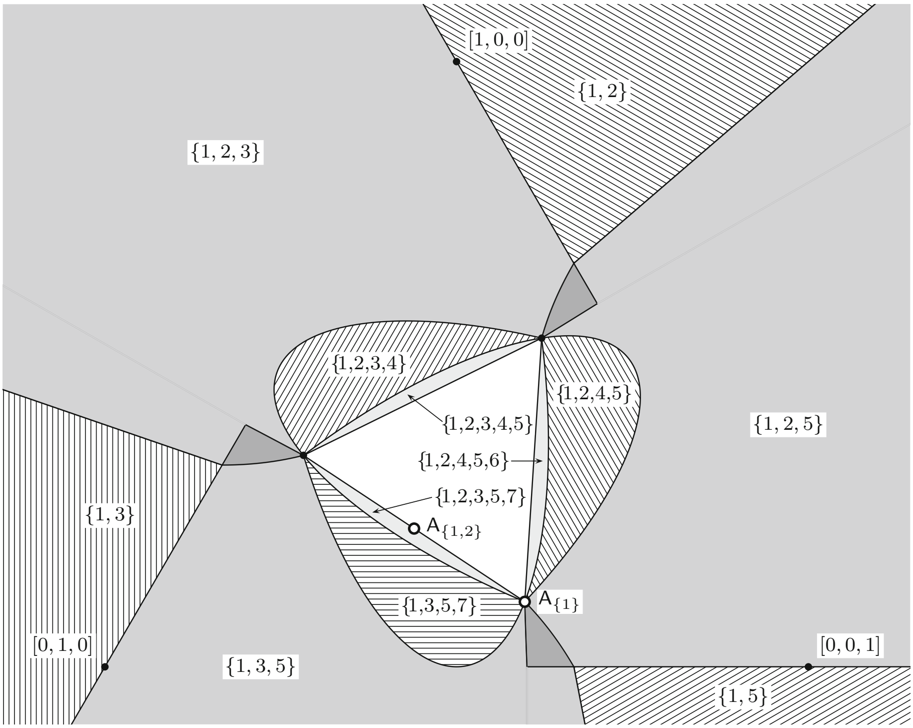

Fig. 2 Part of the search space for $n = 7$ on the hyperplane characterized by $a+b+c = 1$, where points $[a, b, c]$ depicted in barycentric coordinates correspond to positive multiples of matrices $\mathsf{C}([0, a, b, c, c, b, a]^\top)$

2.5 Results

We used the construction outlined in the previous subsection to find cyclically symmetric matrices of orders $n \in [4:23]$ with many ESSs of support size $k \in [2:n-2]$, with k either 2 or odd, when n is odd. We wanted those matrices to have integer entries to facilitate the computation of the set of ESSs. While our main goal was to maximize the number of ESSs, we also made efforts to keep the integer entries small. Our results can be found in Tables 2, 3 and 4, containing a box for each order $n \in [4:23]$, the order always being indicated by the bold number in the lower left of that box, with a row for each support size. The row $[8_5 2_4 \mid 1\ 5\ 4\ 2]$ in the box corresponding to order 8 tells us that the matrix $\mathsf{C}([0, 1, 5, 4, 2, 4, 5, 1]^\top)$ has 8 ESSs of support size 5, but also 2 "spurious" ESSs of support size 4. It was not our aim to get a "pure" result 8_5 by introducing further inequalities on $\boldsymbol{\alpha}$. Indeed, cases of "spurious" ESSs of slightly smaller support sizes than aimed at appear also in other places of the tables, mainly for even orders. In some cases that information was suppressed due to lack of space, but we give it here; the corresponding vector $\mathbf{a}$ can be found in row $[x|\mathbf{a}^\top]$ in Tables 2, 3

Table 2 Cyclically symmetric matrices of orders from 4 up to 19 with many ESSs of prescribed support size

$\mathbf{4}_2$	1	−1

5_2	1	−1
$\mathbf{5}_3$	1	3

9_2	1	−1	1
8_3	1	1	−1
$\mathbf{6}_4$	5	2	2

7_2	1	−1	−1
14_3	1	5	8
$\mathbf{7}_5$	2	4	3

16_2	1	−1	1	−1
$8_3 4_2$	0	1	2	2
20_4	10	4	17	9
$8_5 2_4$	1	5	4	2
$\mathbf{8}_6$	4	5	7	4

9_2	1	−1	−1	−1
30_3	1	5	10	13
27_5	19	26	17	39
$\mathbf{9}_7$	4	7	5	6

25_2	1	−1	1	−1	1
40_3	1	3	3	1	−1
50_4	21	14	30	5	15
32_5	1	1	1	1	−1
35_6	13	11	23	11	18
$10_7 2_5$	3	4	5	6	3
$\mathbf{10}_8$	9	6	11	9	8

22_2	1	−1	1	−1	−1
55_3	1	5	13	20	23
66_5	1	4	6	6	5
33_7	8	12	12	17	10
$\mathbf{11}_9$	5	8	6	7	6

36_2	1	−1	1	−1	1	−1
64_3	1	1	−1	1	1	−1
105_4	293	64	179	196	64	262
72_5	23	29	14	45	50	29
96_6	12	9	8	3	12	5
24_7	2	7	7	5	6	7
42_8	38	55	29	41	62	55
12_9	7	13	9	10	11	9
$\mathbf{12}_{10}$	7	7	8	6	10	6

26_2	1	−1	1	−1	−1	−1
91_3	1	5	13	21	28	31
169_5	793	2594	4132	4481	3768	2964
130_7	5	9	5	5	9	9
52_9	20	40	31	29	36	31
$\mathbf{13}_{11}$	7	10	7	9	8	8

49_2	1	−1	1	−1	1	−1	1
112_3	1	10	17	17	10	1	−1
196_4	170	102	301	161	244	30	137
224_5	2	4	3	3	4	2	−1
252_6	201	192	241	138	295	98	102
156_7	133	281	274	371	554	527	435
154_8	35	16	48	36	42	34	31
42_9	12	14	15	17	19	20	7
56_{10}	48	47	57	58	67	66	29
14_{11}	4	5	5	4	5	6	4
$\mathbf{14}_{12}$	4	3	5	3	4	4	4

45_2	1	−1	−1	1	−1	1	−1
140_3	1	9	23	39	54	66	71
303_5	127	390	571	640	753	994	1218
360_7	9	9	5	9	3	5	9
200_9	18	30	17	18	24	23	30
60_{11}	32	40	37	52	33	37	45
$\mathbf{15}_{13}$	9	13	10	12	11	12	11

64_2	1	−1	1	−1	1	−1	1	−1
112_3	1	2	12	24	37	48	55	55
336_4	1605	349	2265	1205	3197	2065	3850	2422
352_5	349	1235	2228	2879	2965	2596	2125	1916
512_6	4	5	7	4	7	5	4	−1
368_7	6319	4631	3244	10566	5940	9613	12828	7043
452_8	382	384	805	514	665	384	721	514
224_9	1894	3431	1985	1857	3500	3554	2655	2389
208_{10}	211	283	272	374	358	442	418	222
48_{11}	30	49	45	49	36	49	52	33
80_{12}	102	160	151	160	122	160	172	107
16_{13}	6	9	6	8	7	7	8	7
$\mathbf{16}_{14}$	27	41	30	37	34	33	36	31

51_2	1	−1	1	−1	1	−1	−1	−1
204_3	2	11	24	40	57	72	83	89
612_5	21	86	165	228	254	245	217	195
663_7	3837	4889	4501	9531	8131	6336	8789	5224
527_9	353	311	732	589	729	535	612	573
272_{11}	319	577	536	596	479	496	667	464
85_{13}	16	21	24	16	24	20	20	21
$\mathbf{17}_{15}$	6	8	6	8	7	7	7	7

81_2	1	−1	1	−1	1	−1	1	−1	1
240_3	1	5	10	13	13	10	5	1	−1
522_4	3204	3579	8722	8429	10628	6513	5787	1003	2184
864_5	19	26	17	39	39	17	26	19	−1
1080_6	1739	2040	5036	5337	7188	5789	6181	4191	5037
1152_7	4	7	5	6	6	5	7	4	−1
1062_8	2750	4660	4441	5865	4596	1622	3391	4019	2817
668_9	293	822	1006	822	702	822	920	822	730
729_{10}	23	19	23	26	23	17	23	39	23
342_{11}	340	353	373	397	423	448	468	481	148
258_{12}	581	294	539	448	702	431	676	568	431
72_{13}	192	237	267	259	234	228	257	302	151
81_{14}	5	4	5	7	5	5	5	6	5
18_{15}	7	9	8	8	9	8	7	10	7
$\mathbf{18}_{16}$	16	16	19	14	20	15	18	17	16

57_2	1	−1	1	−1	1	−1	−1	−1	−1
285_3	1	10	28	50	75	99	119	134	140
988_5	7579	27693	53247	75642	88144	88423	79256	66979	58593
1596_7	718	2376	3874	4379	3920	3232	3038	3433	3901
1197_9	5136	14818	19259	16630	13540	14726	17318	16829	14434
684_{11}	4366	6730	6024	8148	7126	7225	6829	4707	8813
304_{13}	43	75	66	69	58	70	75	52	74
95_{15}	58	83	73	72	73	82	61	86	70
$\mathbf{19}_{17}$	6	8	7	8	7	8	7	8	7

The bold number in the lower left of each box indicates the order of matrices in that box, and the row $[14_3 \mid 1\ 5\ 8]$ in the box corresponding to order 7 tells us that the matrix $\mathsf{C}([0, 1, 5, 8, 8, 5, 1]^\top)$ has 14 ESSs of support size 3

Table 3 Cyclically symmetric matrices of orders 20 and 21 with many ESSs of prescribed support size

100_2	1	−1	1	−1	1	−1	1	−1	1	−1
320_3	1	3	3	1	−1	1	3	3	1	−1
800_4	21	14	30	5	15	5	30	14	21	−1
$1184_5 25_4$	39	151	298	435	523	544	505	435	372	347
2240_6	13	11	23	11	18	11	23	11	13	−1
$1720_7 60_6 4_5$	670934	1565862	1774786	1277258	944237	1860904	2884941	3093863	2467220	1888590
2560_8	9	6	11	9	8	9	11	6	9	−1
$1580_9 70_8 4_5$	20802	59866	75781	58483	37194	40441	62568	77590	75786	71515
1520_{10}	9816	7149	12134	8738	8460	9400	12134	6486	9816	655
$840_{11} 40_{10}$	157024	188864	153287	201508	210994	316553	204951	172076	240613	175750
1225_{12}	14	13	14	11	14	23	14	11	14	18
$420_{13} 20_{12} 2_{10}$	29716	30986	41555	57575	31062	46923	32632	41639	67516	50743
360_{14}	8	6	7	4	7	7	7	8	8	6
$60_{15} 20_{14} 5_{12}$	427	720	525	649	695	553	631	615	597	675
100_{16}	8	9	8	6	8	11	8	9	8	8
$20_{17} 2_{10}$	14	19	14	18	15	17	16	15	17	15
$\mathbf{20}_{18}$	12	13	12	12	13	11	14	11	14	10
63_2	1	−1	1	−1	1	−1	−1	−1	−1	−1
385_3	1	11	34	64	97	131	163	191	212	219
1701_5	2	4	3	3	4	2	−1	2	4	3
2376_7	360975	592543	1150574	1539916	1725899	2137635	2261970	2907588	3197338	3582088
4410_9	15	15	7	15	15	7	7	15	7	15
$2079_{11} 14_9 3_7$	82591	217944	244806	188657	179951	229457	243152	205939	191442	213086
1029_{13}	544204	653438	465125	937091	818322	653739	729842	635706	769056	731486
406_{15}	4771	7913	7205	8412	7913	8031	5892	4771	8694	8412
126_{17}	3389	5684	4643	4418	4896	5497	5985	5949	4194	4924
$\mathbf{21}_{19}$	12	15	12	15	13	14	13	14	13	14

Table 4 Cyclically symmetric matrices of orders 22 and 23 with many ESSs of prescribed support size

121_2	1	−1	1	−1	1	−1	1	−1	1	−1	1
440_3	1	5	13	20	23	23	20	13	5	1	−1
1122_4	25541	24537	66953	69589	101311	82414	88522	48013	43471	6633	18857
2112_5	1	4	6	6	5	5	6	6	4	1	−1
3025_6	12	1	12	5	12	13	12	20	12	23	12
4224_7	8	12	12	17	10	10	17	12	12	8	−1
4444_8	2434323	1057798	2948856	695523	2698449	721840	2672139	972246	3034410	457710	1976371
5632_9	5	8	6	7	6	6	7	6	8	5	−1
4422_{10}	684876	1281651	2190757	1933610	1866035	1391942	1815986	1779453	2064073	1577524	1724349
2950_{11}	2226626	6164519	7513582	6164520	5308733	6164520	6830692	6164521	5580439	6164518	6724943
2772_{12}	11	8	11	8	11	8	11	8	11	8	2
1540_{13}	1819315	1858140	1919344	1997977	2087664	2181129	2270816	2349450	2410653	2449478	656656
1265_{14}	1372902	1150952	2144657	2037304	1546156	1866844	1440909	2072746	1798400	1740258	1660594
550_{15}	6368316	7582714	7226700	6214175	6671549	8258502	8715875	7703352	7347337	8561734	3981238
440_{16}	22801	21808	25273	17871	23675	23753	29557	22155	25619	24626	15933
88_{17}	162263	210812	219039	197387	195359	213733	211705	190053	198280	246830	144368
121_{18}	6	5	6	8	6	6	6	7	6	6	6
22_{19}	14	18	16	17	18	16	17	18	15	19	14
$\mathbf{22}_{20}$	20	20	21	18	23	17	22	19	20	21	19
92_2	1	−1	1	−1	1	−1	1	−1	−1	−1	−1
506_3	1	17	42	74	112	152	191	227	256	277	287
2760_5	13777	51750	104635	159847	204956	231032	234933	219898	194267	168766	153140
5612_7	1139411	3967973	7061004	8983982	9094095	7838202	6385670	5848673	6590399	8051221	9181803
5842_9	5433685	17400081	26917514	28433148	24074247	20336942	21173719	24682768	26276531	24230460	21371440
5267_{11}	379497	1084450	1390556	1190266	982320	1080275	1248913	1197078	1047012	1070000	1212305
2806_{13}	784170	1960130	2377665	2279891	2049365	2245727	2552376	2096509	1707404	2117294	2525749
1334_{15}	51352	99269	66787	54099	80291	70084	54462	72058	75098	66310	77901
460_{17}	159	242	234	199	224	246	180	255	203	222	222
115_{19}	134	143	173	140	151	150	161	148	135	181	137
$\mathbf{23}_{21}$	23	27	23	27	23	27	23	26	24	26	25

and 4, where x is the first part of the regular-type faced entries in the five row vectors of the following four lines:

$$(\mathbf{12}: 72_5 4_3, 24_7 26_3 4, 12_9 2_6), \qquad (\mathbf{14}: 42_9 14_8 2_7),$$
$$(\mathbf{16}: 112_3 8_4, 352_5 20_4, 368_7 32_6, 224_9 2_8, 48_{11} 16_{10} 2_8, 16_{13} 2_8),$$
$$(\mathbf{18}: 668_9 548 3_6, 342_{11} 18_{10} 8_9, 72_{13} 18_{12} 2_9, 18_{15} 2_9),$$
$$(\mathbf{22}: 2950_{11} 198_{10}, 1540_{13} 66_{12} 2_{11}, 550_{15} 44_{14} 2_{11}, 88_{17} 22_{16}, 22_{19} 2_{11}).$$

For instance, for $24_7 26_3 4$ with order $n = \mathbf{12}$ we have $\mathbf{a}^\top = [2, 7, 7, 5, 6, 7]$ from Table 2. All calculations were done with Maple$^{\text{TM}}$ and the matrices listed in the tables were double-checked with Reinhard Ullrich's program `ref` (rational equilibrium finder). For fixed n, the first task is to find the set $\bar{\mathcal{A}}_n$ of candidate matrices. Containing at most one matrix indexed by any of the subsets L of $[1:n']$, its size is bounded by $2^{n'}$, but a further reduction in size to roughly $\frac{2}{\varphi(n)} 2^{n'}$, where φ denotes the Euler totient function, is possible, by keeping just one member of each orbit $\{\mathsf{A}_L, \mathsf{A}_{\tau L}, \mathsf{A}_{\tau^2 L}, \ldots\}$, where τ is a permutation on $[1:n']$ generating a cyclic group of order $\frac{\phi(n)}{2}$. That resulted in, e.g., $|\bar{\mathcal{A}_{22}}| = 394$, which is much smaller then 2^{11}.

Next, we fixed $\mathsf{A} = \mathsf{A}_L \in \bar{\mathcal{A}}_n$ of rank $k-1$ and computed $\mathcal{R}_L$. Potential supports of members of $\mathcal{R}_L$ are all subsets of $[1:n]$ of size k. Among a certain subset and all its shifts and reflected shifts only one set has to be taken into account, which reduces the number of potential supports to roughly $\frac{1}{2n}\binom{n}{k}$. For any such support I we then checked if there is $\mathbf{p} \in \mathcal{E}(\mathsf{A})$ with $I(\mathbf{p}) = I$, i.e., we numerically solved $\mathsf{A}_{I\times I}\mathbf{p}_I = \mathbf{o}, \mathbf{e}^\top \mathbf{p}_I = 1$. If the system appeared to lack a unique solution, we discarded that support. We also did that, if the solution was outside the simplex Δ^k or apparently on its boundary. Only if the solution $\mathbf{p}$ was clearly in the interior of Δ^k, and $\mathcal{L}_{L,\mathbf{p}} \neq \emptyset$ was fulfilled, did we include I in $\mathcal{R}_L$. E.g., for $n = 22$ and $k = 10$ we obtain 14938 (shift- and reflection-reduced) potential supports, of which not more than 280 belong to any set $\mathcal{R}_L$: There are 42 candidate matrices of rank 9 in $\bar{\mathcal{A}_{22}}$. Corresponding sets $\mathcal{R}_L$ satisfy $280 \geq |\mathcal{R}_L| \geq 1$ and $11011 \geq \operatorname{size}(\mathcal{R}_L) \geq 22$ for L in a suitable 42-element set. For each such L we have to maximize $\operatorname{size}(\bar{\mathcal{R}}_L)$ under the constraints $\bar{\mathcal{R}}_L \subseteq \mathcal{R}_L, \bigcap_{\mathbf{p}\in\bar{\mathcal{R}}_L} \mathcal{L}_{L,\mathbf{p}} \neq \emptyset$. We would work our way down starting with the most promising L corresponding to $\operatorname{size}(\mathcal{R}_L) = 11011$, thus finding the largest value 4422 of $\operatorname{size}(\bar{\mathcal{R}}_L)$ for $L = \{1, 2, 3, 4, 11\}$, with $\operatorname{size}(\mathcal{R}_L) = 6138$ being the 28th-largest among the corresponding 42 values.

We replaced the "< 0"-inequalities originating in (9), that are present in the constraints of the maximization problem, by "$\leq -\delta$"-inequalities for some fixed $\delta > 0$. The problem can be regarded as a Maximum Feasible Subsystem Problem, see [16]. For small n, we solved the problem by exhaustive search, for larger n we used a heuristic approach to generate good lower bounds for the optimal value (basically, we repeatedly started with a list built from a random permutation of the points in $\mathcal{R}_L$ and, starting at the beginning of the list, greedily included points $\mathbf{p}$ while maintaining nonempty intersection of the corresponding sets $\mathcal{L}_{L,\mathbf{p}}$). For n up to 19 we also computed upper bounds for the optimal values, derived from formulations as Min IIS (irreducible inconsistent subsystem) Cover problems [16], with those upper bounds matching the lower bounds. So we can be sure to have solved the maximization problems for $n \in [4:19]$ to optimality, but we do not know whether some of our lower bounds for $n \in [20:23]$ leading to entries in Tables 3 and 4 are actually the optimal objective values for the corresponding problems.

Having solved (or having found a good lower bound $\underline{m}$ for) the maximization problem means that we now know a matrix $\mathsf{B}_{\boldsymbol{\alpha}}$ such that $\mathsf{A} + \varepsilon\mathsf{B}_{\boldsymbol{\alpha}}$ will have many quasistrict ESSs.

As we want integer entries for our matrices, we would rather consider matrices $r_1\mathsf{A} + r_2\mathsf{B}_{\boldsymbol{\alpha}}$ with r_1 large and r_2 of moderate size, and we would round the entries of that matrix to the nearest integers. Clearly r_2 may not be too small, otherwise $\mathsf{B}_{\boldsymbol{\alpha}}$'s influence will get wiped out by the rounding. Also $\frac{r_1}{r_2} = \frac{1}{\varepsilon}$ may not be too small, as we need ε small. So suitable r_1, r_2 will lead to a matrix with $\underline{m}$ quasistrict ESSs of support size k, having typically very large integer entries. Computing a series of ever smaller multiples of that matrix and rounding again—which is what we did—can reduce the entries of the matrix to some extent. Trying to have integer entries close to the minimal values possible would require considering all $\boldsymbol{\alpha}$ in the nonempty intersection of the sets $\mathcal{L}_{L,\mathbf{p}}$ and all matrices A' as defined in Corollary 10, for all L for which $\text{size}(\bar{\mathcal{R}}_L) = \underline{m}$. We did not pursue that approach.

Summarizing, for orders $n \in [4 : 19]$ and prescribed support sizes $k \in [2 : n-2]$, with k odd when n is odd, we have constructed matrices in $\mathcal{C}^n$ with a maximum number of quasistrict ESSs of support size k. Here maximality is meant among matrices in $\mathcal{C}^n$ occurring, in positive proportion, arbitrarily close to some negative-semidefinite matrix of rank $k-1$. We do, however, not claim that the matrices we found are close to negative-semidefinite ones in the sense that there is a continuous path in $\mathcal{C}^n$ from one to the other along which the pattern does not change, as we do not know if there may be disconnected regions within $\mathcal{C}^n$ that share the same pattern, so that our rounding procedure could have led to a jump from one region to another. This holds in particular in some cases where $n = n_1 n_2$ is not a prime number. If the best result we found for some fixed k was not better than what we got by constructing a matrix from suitable matrices of orders n_1 and n_2 using construction techniques from [5], we would list the latter in our table, when it had entries of smaller size. Note that, in view of Lemma 3 and Example 4, our approach need not have worked for $k = n-2$, but it did: It is known from [6, Thm. 5], that $\max_{\mathsf{A}\in\mathcal{C}^n} |\{\mathbf{p} \in \text{ESS}(\mathsf{A}) : |I(\mathbf{p})| = n-2\}| = n$, and our construction method produced matrices with integer entries achieving this maximal value for all $n \in [4:23]$.

Remark 5 Special attention to cyclically symmetric matrices has been payed by [6], from which we can deduce that

$$\max_{\mathsf{A}\in\mathcal{C}^n} |\{\mathbf{p} \in \text{ESS}(\mathsf{A}) : |I(\mathbf{p})| = k\}| = \max_{\mathsf{A}\in\mathcal{S}^n} |\{\mathbf{p} \in \text{ESS}(\mathsf{A}) : |I(\mathbf{p})| = k\}|\,, \text{if } k \in \{1, n-2, n\},$$

and also if n is even and $k = 2$. The L.H.S. is, however, strictly smaller than the R.H.S. if $n \geq 3$ and $k = n-1$, and also if $n \geq 3$ is odd and $k = 2$. (See also the next section for the case $k = 2$.) Furthermore, the best lower bound for the constant γ introduced in [6, Thm. 2], that is currently known, is $15120^{\frac{1}{24}} \approx 1.4933$, and originates from a matrix $\mathsf{A} \in \mathcal{C}^{24}$ satisfying $|\text{ESS}(\mathsf{A})| = 15120$, see [5]. The main reason we concentrated on cyclically symmetric matrices is however that they are particularly well suited for our perturbation method. There are only finitely many column spaces associated with matrices in $\mathcal{C}^n$, which, by invoking Corollary 10, led to finite sets $\bar{\mathcal{A}}_n$ of candidate matrices, something we cannot hope for if we replace $\mathcal{C}^n$ by (other interesting subsets of) $\mathcal{S}^n$.

3 Even Support Sizes of ESSs When the Order is Odd

The construction method described in Sect. 2 is not suited for finding cyclically symmetric matrices of odd order that have many ESSs of even support size, simply because $\bar{\mathcal{A}}_n$ does not contain matrices of odd rank when n is odd. Remark 4 above addresses a related issue. Figure 2 nevertheless shows that for order $n = 7$ matrices with 7 ESSs of support size 4 can be found by perturbing the matrix $\mathsf{A}_{\{1\}} \in \bar{\mathcal{A}}_7$. Furthermore, it is seen in Fig. 2 that for some matrices of order 7, there are 7 ESSs of support size 2, but those matrices do not result from small perturbations of matrices in the set $\bar{\mathcal{A}}_7$. In Fig. 2 regions of matrices sharing the same pattern are visualized by their shading, a particular support attached to some region always indicating just one member of a whole class of 7 cyclic shifts of that support. The (middle gray) 7_3 regions actually overlap, and their intersections, the dark gray triangle shaped regions, correspond to matrices with 14 ESSs of support size 3. The white triangle corresponds to matrices with 1 ESS of support size 7. The 7_4 regions are bounded by segments of the three ellipses $(b-c)^2 = ab$, $(c-a)^2 = bc$, and $(a-b)^2 = ca$. Two sides of the dark gray triangle shaped regions are actually curved, and are segments of the three hyperbolae $ac + 3bc = ab + 2c^2$, $ab + 3ac = bc + 2a^2$, and $bc + 3ab = ac + 2b^2$, the third sides are the straight lines $a = 0$, $b = 0$, $c = 0$. The 7_2 regions are bounded by the lines $a = 0$, $2b = c$, resp. $b = 0$, $2c = a$, resp. $c = 0$, $2a = b$. The union of the three lines forming the boundary of the 1_7 region satisfies the equation

$$a^3 + b^3 + c^3 + 3(a^2b + b^2c + c^2a) - 4(ab^2 + bc^2 + ca^2) - abc = 0.$$

Figure 2 suggests that for given odd n, matrices in $\mathcal{C}^n$ with many ESSs of even support size k could be found by perturbing a negative-semidefinite cyclically symmetric matrix A of rank $k-2$ in such a way that points $\mathbf{p} \in \operatorname{conv}\{\mathbf{p}_1, \mathbf{p}_2\}$, with $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{E}(\mathsf{A})$, $|I(\mathbf{p}_1)| = |I(\mathbf{p}_2)| = |I(\mathbf{p}_1) \cap I(\mathbf{p}_2)| + 1 = k-1$, get perturbed into quasistrict ESSs of the perturbed matrix. (This should also be considered in the case of even n and k of any parity, as suggested by Fig. 1, where we can enter the 6_3 region by perturbing $\mathsf{A}_{\{3\}}$.) Since there is no straightforward way to adapt our construction method to that setting, we leave this approach to future research. We do however have results for $k = 2$, which are also given in Tables 2, 3 and 4.

3.1 Support Size 2

Fix $\mathsf{A} \in \mathcal{S}^n$ and consider the graph G with vertex set $V(G) = [1:n]$ and an edge $\{i, j\} \in E(G)$ iff $\{i, j\} = I(\mathbf{p})$ for some $\mathbf{p} \in \mathrm{ESS}(\mathsf{A})$. Then we know from [19, Thm. 2 and Corollaries] that G is triangle-free. For a given triangle-free graph G let now $\mathsf{A}(G) = (a_{ij}) \in \mathcal{S}^n$ be defined by $a_{ii} = 0$ for $i \in [1:n]$, $a_{ij} = 1 \Leftrightarrow \{i, j\} \in E(G)$, $1 \le i < j \le n$, and $a_{ij} = -1$ otherwise. Then we know from [9, Thm. 1], that $\operatorname{pattern}(\mathsf{A}(G)) = E(G)$. So, as noted in [9, Thm. 5 and Corollaries], the maximum number of ESSs of support size 2 a matrix $\mathsf{A} \in \mathcal{S}^n$ can have is $\left\lceil \frac{n^2-1}{4} \right\rceil$, corresponding to a complete bipartite graph with maximally balanced partition. For even n, the matrix $\mathsf{A}(G)$, obtained from the complete bipartite graph G with $[1:n]$ partitioned into odd and even numbers, turns out to belong to $\mathcal{C}^n$, therefore $\frac{n^2}{4} = \max_{\mathsf{A} \in \mathcal{C}^n} |\{\mathbf{p} \in \mathrm{ESS}(\mathsf{A}) : |I(\mathbf{p})| = 2\}|$ is attained.

Not so for n odd. Note that $\mathsf{A}(G) \in \mathcal{C}^n$ means that G is a circulant graph on $V(G) = [1:n]$, being determined by a subset $S \subseteq [1:n-1]$ such that $\{i, j\} \in E(G) \Leftrightarrow i - j \equiv s \mod n$ for some $s \in S$. In order that G be triangle-free, S has to be a symmetric sum-free subset of $[1:n-1]$, i. e., $s \in S \Rightarrow n - s \in S$, and $s_1 + s_2 \equiv s \mod n$ has no solution with $s_1, s_2, s \in S$. For small odd n, the maximal sizes m_n of such symmetric sum-free subsets of $[1:n-1]$ are easily found,

$$\begin{aligned}&\left(\tfrac{m_{2k+1}}{2} : k \in [2:32]\right)\\ &\quad = (1, 1, 1, 2, 2, 3, 3, 3, 3, 4, 5, 4, 5, 5, 6, 7, 6, 6, 7, 7, 9, 8, 8, 9, 9, 11, 9, 10, 10, 10, 13),\end{aligned}$$

and it is observed that

$$\frac{n-3}{6} \leq \frac{m_n}{2} \leq \frac{n}{5}$$

holds for that range. Note that we are interested in $\frac{m_n}{2}$, because $|E(G)| = \frac{|S|}{2}n$. The inequality $\frac{n-3}{3} \leq m_n$ holds for all odd n, as is seen by observing that

$$\{2k-1 : k \in L_n\} \cup \{n+1-2k : k \in L_n\}, \text{ where } L_n = [1 : \lceil \tfrac{n-3}{6} \rceil],$$

is a symmetric sum-free subset of $[1 : n-1]$ of at least $\frac{n-3}{3}$ elements. The inequality $\frac{m_n}{2} \leq \frac{n}{5}$, on the other hand, is not surprising in the light of [10, Prop. 1.3, Thm. 1.5], from which we can deduce that the maximal size possible for a sum-free subset of $[1 : n-1]$ (without the requirement of being symmetric) is upper bounded by $\frac{n}{5}$. Interestingly, this upper bound is attained infinitely often also by symmetric sum-free sets, i. e., the equality $\frac{m_n}{2} = \frac{n}{5}$ holds for all $n \in \{10k-5 : k \geq 1\}$. This is seen by observing that $\{1, 4, 6, 9, 11, 14, \ldots, 10k-6\}$ (with differences of consecutive elements alternating between 3 and 2) is a symmetric sum-free subset of $[1 : 10k-6]$ of size $4k-2$.

So we conclude, that $M_n := \max_{\mathsf{A} \in \mathcal{C}^n} |\{\mathbf{p} \in \mathrm{ESS}(\mathsf{A}) : |I(\mathbf{p})| = 2\}|$ satisfies

$$M_n \geq \tfrac{n(n-3)}{6} \text{ for any odd } n,\ M_n = \tfrac{n^2}{5} \text{ for infinitely many odd } n, \text{ and } M_n > \tfrac{n^2}{5} \text{ for no odd } n,$$

in particular, the maximum $\frac{n^2-1}{4}$ for matrices constructed from bipartite graphs is never attained by M_n for odd $n \geq 3$.

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