Pursuit Strategy of Motion Camouflage in Dynamic Games

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Abstract

This work was inspired by an example of a pursuit strategy whereby a pursuer approaches an evader while appearing stationary to the latter. This effect is achieved due to the fact that the pursuer P stays on the line connecting some fixed reference point R and current position of the evader E. According to recent researches of biologists (Mizutani et al. in *Nature* 423(6940):604, 2003), such a strategy called motion camouflage is adopted by some insects, e.g., dragonflies. As it is shown in Anderson and McOwan (*Proc R Soc Lond B Biol Sci* 270(Suppl 1): S18–S20, 2003), humans can be also tricked in the same way. Interesting results on comparison of capture times for the motion camouflage strategy and some classic pursuit strategies are obtained in Glendinning (*Proc R Soc Lond B Biol Sci* 271(1538): 477–481, 2004) using computer simulation. This paper further expands results of Matichin and Chikriy (*J Autom Inf Sci* 37(3):1–5, 2005), providing explicit formulas for the pursuer control input implementing motion camouflage pursuit strategy with the reference point that does not coincide with pursuer's initial position.

Keywords Differential game · Pursuit strategy · Motion camouflage

1 Introduction

Although obtaining optimal (minimax) solution in differential games is an important and complicated problem, there exist a number of methods in differential games of pursuit aimed at guaranteed result rather than the optimal one. These methods, including the method of resolving functions [4], the extreme shift control principle [6], Pontryagin's first direct method, and others, often provide rigorous mathematical substantiation to pursuit strategies known from engineering practice. These methods are essentially based on the technique of set-valued maps and their selections [2, 10].

In this paper, a method for analysis of linear differential games of pursuit is described, providing rigorous mathematical substantiation for the motion camouflage strategy. This method is based on some ideas of the method of resolving functions, which provides rigorous substantiation for the parallel pursuit rule [4], heuristically known from engineering practice.

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It should be noted that under the fixed reference point at infinity the motion camouflage strategy reduces to the parallel pursuit rule, characterizing by the fact that lines of sight are parallel. The approach presented in the paper makes it possible to derive explicit formulas for the pursuer's control both when the fixed reference point coincides with pursuer's initial position and when it does not.

The paper is organized as follows. Section 2 presents overview of common pursuit/guidance strategies including the motion camouflage. Section 3 proposes a new method for solving differential games of pursuit, which employs the concept of integral inclusions. The main result is presented by Theorem 1, which provides sufficient conditions for capture in linear differential game of pursuit. Another major result of the paper is given in Sect. 4, where it is shown that the proposed method allows implementing the motion camouflage strategy when applied to the differential game of pursuit with simple motion. The use of the proposed method resulted in explicit formulas for pursuer's control implementing the motion camouflage for arbitrary reference point.

2 Pursuit Strategies in Dynamic Games

A number of pursuit strategies are known from the practice of guidance, navigation, and control, such as simple or pure pursuit, parallel pursuit, and proportional navigation. These strategies are used in missile guidance systems design and well described [7, 11].

Pure pursuit strategy was known long ago. It consists in choosing of such pursuer's velocity that always points directly toward the evader, i.e., the pursuer's velocity vector at each time instant is parallel to the line-of-sight (LOS) joining the current positions of the pursuer and evader (Fig. 1). The problem of determining the path of a pursuer, which applies the pure pursuit, was studied by Leonardo da Vinci and Pierre Bouguer in 1732 and by Du Boisaymé in 1811. Theoretical substantiation of the pure pursuit for the case of "simple motion" can be obtained using the extreme shift control principle [6].

Proportional navigation (PN) is perhaps the most widely used guidance law in sophisticated missiles. In the proportional navigation pursuit strategy, the rate of rotation of the pursuer velocity vector is proportional to the rate of rotation of the LOS. That is

Fig. 1 Pure pursuit

Fig. 2 Proportional navigation

Fig. 3 Parallel pursuit

$$\varphi' = k\theta',$$

where φ' is the rate of rotation of the pursuer velocity, θ' is the rate of rotation of the LOS, and k is the navigation constant. The geometry behind this pursuit strategy is shown in Fig. 2.

One obtains different forms of PN by choosing different values of the navigation constant k . It should be noted that for $k = 1$ PN yields pure pursuit, and for $k \rightarrow \infty$, PN reduces to the parallel pursuit. The latter is the pursuit strategy such that the LOS remains parallel to itself as time goes by (see Fig. 3).

The parallel pursuit strategy can be rigorously derived from the method of resolving functions known [4] in differential game theory.

According to recent researches of biologists [9], some predatory insects, e.g., dragonflies, apply a special pursuit strategy to stealthily approach the prey. This strategy called “motion camouflage” implies that the pursuer’s instantaneous position remains on the line joining the evader’s instantaneous position with some fixed reference point R as shown in Fig. 4. In other words, the LOS at each time instant passes through the point R . As a result, the

Fig. 4 Motion camouflage

pursuer approaches the evader while appearing stationary to the latter except for the inevitable perceived change in size of the pursuer as it approaches.

Let x^0 and y^0 be the initial positions of the pursuer and evader, respectively. The players' respective instantaneous positions at time t are given by vectors $x(t)$ and $y(t)$. Let us also denote by R a fixed reference point. Then, the fact that at each time t the pursuer keeps to the line joining R with the instantaneous position of the evader can be expressed in the form of the following equation

$$\frac{x(t) - y(t)}{\|x(t) - y(t)\|} = \frac{R - y(t)}{\|R - y(t)\|}. \quad (1)$$

Hereafter, $\|\cdot\|$ denotes the Euclidean norm.

Obviously, as the reference point R tends to infinity, the expression

$$\frac{R - y(t)}{\|R - y(t)\|}$$

approaches some constant value. As a result, Eq. (1) takes on the form

$$\frac{x(t) - y(t)}{\|x(t) - y(t)\|} = \text{const},$$

which means that the LOS remains parallel to itself for each t . Therefore, the parallel pursuit strategy is a particular case of the motion camouflage with the reference point at infinity.

3 Problem Statement and Main Theorem

Let us consider a linear dynamic game, the evolution of which is modeled by the following differential equation:

$$\dot{z} = Az + u - v, \quad z \in \mathbb{R}^n, \quad (2)$$

where z is a vector from n -dimensional Euclidean space $\mathbb{R}^n$, A is a constant square matrix of order n , and u and v are controls of pursuer and evader, respectively.

The pursuer P aims to transfer a trajectory of system (2) from the initial state

$$z^0 = z(0) \quad (3)$$

to the origin in a finite time by choosing its control u as a (Lebesgue) measurable function taking values in the compact set U , $U \subset \mathbb{R}^n$.

The evader E also chooses its control v as a measurable function of time taking values in the compact set V , $V \subset \mathbb{R}^n$. However, the evader's goal is the opposite, i.e., to prevent the pursuer from driving trajectory of (2) to the origin.

In what follows, we will denote by $\Omega_X[a, b]$ the set of all functions measurable on $[a, b]$ and taking values in the set X .

We assume that the pursuer at each time instant t forms its control as a quasi-strategy, i.e., on the basis of the information on initial state (3) and history of the evader's control $v_t(\cdot) = \{v(s) : s \in [0, t]\}$ up to the instant t :

$$u(t) = u(z^0, v_t(\cdot)), \quad u(t) \in U. \quad (4)$$

It should be noted that for fixed admissible players' controls $u(\cdot)$, $v(\cdot)$, the solution to the Cauchy problems (2), (3) is given by the expression

$$z(t) = e^{At} z^0 + \int_0^t e^{A(t-\tau)} (u(\tau) - v(\tau)) d\tau, \quad (5)$$

where e^{At} denotes the sum of the convergent series: $e^{At} = I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!} + \dots$

By analogy with the differential inclusions [3], let us call integral inclusion the following expression:

$$\beta(t, v(\cdot), z) \left[\int_0^t e^{-A\tau} v(\tau) d\tau - z \right] + \int_0^t \beta(\tau, v(\cdot), z) d\tau e^{-At} v(t) \in e^{-At} U, \quad (6)$$

where $\beta(t, v(\cdot), z)$ is some scalar function. By fixing some element $\bar{u} \in U$ in (6), we obtain an integral equation of Volterra type. Solutions to all such equations form the solution set of the inclusion (6).

Let us consider a set-valued map $B(t, v(\cdot), z)$, selections of which are nonnegative solutions of the integral inclusion (6):

$$B(t, v(\cdot), z) = \left\{ \beta(t, v(\cdot), z) \geq 0 : \beta(t, v(\cdot), z) \left[\int_0^t e^{-A\tau} v(\tau) d\tau - z \right] + \int_0^t \beta(\tau, v(\cdot), z) d\tau e^{-At} v(t) \in e^{-At} U \right\}.$$

Condition 1 The set $B(t, v(\cdot), z)$ is nonempty for any $t, v(\cdot), z, t \geq 0, v(\cdot) \in \Omega_V[0, t], z \in \mathbb{R}^n$.

Condition 1 is fulfilled in particular if the set U contains zero.
Let us introduce the function

$$\tilde{\beta}(t, v(\cdot), z) = \sup B(t, v(\cdot), z). \quad (7)$$

It should be noted that in virtue of Condition 1 the function $\tilde{\beta}(t, v(\cdot), z)$ is defined for any $t, t \geq 0$.

Condition 2 The function $\tilde{\beta}(t, v(\cdot), z)$ is measurable in t for any fixed $v(\cdot), z, v(\cdot) \in \Omega_V[0, t], z \in \mathbb{R}^n$.

Since the function $\tilde{\beta}(t, v(\cdot), z)$ is bounded by definition, Condition 2 implies its local integrability, i.e., $\tilde{\beta}(\cdot, v(\cdot), z) \in L^1_{loc}[0, t]$ for all $t \in \mathbb{R}_+, v(\cdot) \in \Omega_V[0, t], z \in \mathbb{R}^n$.

Now consider the function

$$T(z) = \min \left\{ t \geq 0 : \inf_{v(\cdot) \in \Omega_V[0, t]} \int_0^t \tilde{\beta}(\tau, v(\cdot), z) d\tau \geq 1 \right\}. \quad (8)$$

If the inequality in braces fails for all $t \geq 0$, we set $T(z) = +\infty$.

Theorem 1 Let Conditions 1 and 2 be fulfilled for system (2) and $T = T(z^0) < +\infty$ for some z^0 . Then, for any evader's control $v(\cdot), v(\cdot) \in \Omega_V[0, T]$, trajectory of system (2) can be transferred from the initial state z^0 to the origin by the time $T = T(z^0)$ using the pursuer's control of the form (4).

Proof Let us fix some evader's control $v(\cdot), v(\cdot) \in \Omega_V[0, T]$. Since $T < +\infty$, there exists a time instant $t^* = t^*(v(\cdot)), t^* \leq T$, such that

$$\int_0^{t^*} \tilde{\beta}(\tau, v(\cdot), z^0) d\tau = 1. \quad (9)$$

At each time instant $t, t \in [0, t^*]$, we select the pursuer's control $u(t)$ satisfying the equality

$$\begin{aligned} e^{-At} u(t) &= \tilde{\beta}(t, v(\cdot), z^0) \left[\int_0^t e^{-A\tau} v(\tau) d\tau - z^0 \right] \\ &\quad + \int_0^t \tilde{\beta}(\tau, v(\cdot), z^0) d\tau e^{-At} v(t). \end{aligned} \quad (10)$$

Equation (10) has solution by virtue of function's $\tilde{\beta}(t, v(\cdot), z^0)$ definition. It can be readily seen that the pursuer's control generated in such a way is a measurable function of time taking values in the compact set U .

Let us integrate the both sides of the equality (10) in t over the interval $[0, t^*]$, using integration by parts:

$$\begin{aligned}
 \int_0^{t^*} e^{-At} u(t) dt &= \int_0^{t^*} dt \int_0^t \tilde{\beta}(t, v(\cdot), z^0) e^{-A\tau} v(\tau) d\tau \\
 &\quad + \int_0^{t^*} dt \int_0^t \tilde{\beta}(\tau, v(\cdot), z^0) e^{-At} v(t) d\tau - \int_0^{t^*} \tilde{\beta}(t, v(\cdot), z^0) z^0 dt \\
 &= \int_0^{t^*} \tilde{\beta}(t, v(\cdot), z^0) dt \int_0^t e^{-At} v(t) dt - \int_0^{t^*} dt \int_0^t \tilde{\beta}(\tau, v(\cdot), z^0) e^{-At} v(t) d\tau \\
 &\quad + \int_0^{t^*} dt \int_0^t \tilde{\beta}(\tau, v(\cdot), z^0) e^{-At} v(t) d\tau - \int_0^{t^*} \tilde{\beta}(t, v(\cdot), z^0) z^0 dt \\
 &= \int_0^{t^*} \tilde{\beta}(t, v(\cdot), z^0) dt \int_0^t e^{-At} v(t) dt - \int_0^{t^*} \tilde{\beta}(t, v(\cdot), z^0) z^0 dt,
 \end{aligned}$$

whence, in view of (9), we have

$$\int_0^{t^*} e^{-At} u(t) dt = \int_0^{t^*} e^{-At} v(t) dt - z^0.$$

Let us multiply both sides of the above equality by the matrix e^{At^*} . As a result, we obtain

$$e^{At^*} z^0 + \int_0^{t^*} e^{A(t^*-t)} (u(t) - v(t)) dt = 0.$$

On the other hand, according to the Cauchy formula (5) for systems (2) and (3), the following expression is true:

$$z(t^*) = e^{At^*} z^0 + \int_0^{t^*} e^{A(t^*-t)} (u(t) - v(t)) dt,$$

whence it follows $z(t^*) = 0$, which completes the proof. $\square$

4 Mathematical Substantiation of the Motion Camouflage

Let us apply the above method to the differential pursuit–evasion game with “simple motions” [8]. Let two points P and E move in the space $\mathbb{R}^n$, $n \geq 2$, the coordinates of which are given by vectors x and y , $x, y \in \mathbb{R}^n$, respectively. The motion of the point P is governed by the pursuer, which can arbitrarily change the direction of the velocity vector u , bounded in norm by a constant a , $a > 1$. The evader E can arbitrarily change direction of the velocity vector

v with the norm not exceeding unity. Thus, the motion of the players is described by the following ODEs:

$$\begin{aligned}\dot{x} &= u, & x(0) &= x^0, \\ \dot{y} &= v, & y(0) &= y^0,\end{aligned}$$

where $\|u\| \leq a$, $a > 1$, $\|v\| \leq 1$.

The goal of the pursuer is to achieve an exact capture, i.e., fulfillment of the equality $x(T) = y(T)$, at a finite instant of time T . The goal of the evader is the opposite—to escape, i.e., to avoid an exact capture for all T , $T > 0$. The pursuer is choosing her control as a measurable function of time taking values in the closed ball of radius a centered at the origin, and the evader from the unit ball, i.e., $u \in U = aS$, $v \in V = S$, where $S = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

This problem can be reduced to the form (2) by the substitution $z = x - y$. As a result, we obtain the equation

$$\dot{z} = u - v \quad (11)$$

with the initial condition $z^0 = x^0 - y^0$. Now the capture means that $z(T) = 0$.

In this example, A is a null matrix; therefore, $e^{At} = I$. By definition, function $\tilde{\beta}(t, v(\cdot), z)$ satisfies the inclusion

$$\tilde{\beta}(s, v(\cdot), z) \int_0^s v(\tau) d\tau + \int_0^s \tilde{\beta}(\tau, v(\cdot), z) d\tau \quad v(s) - \tilde{\beta}(s, v(\cdot), z)z \in aS.$$

Let us integrate both sides of the above inclusion in s over the interval $[0, t]$. Using rules of integration of the set-valued maps [2], one can obtain that $\int_0^t aS ds = aSt$. Then, using integration by parts we have

$$\int_0^t \tilde{\beta}(s, v(\cdot), z) ds \left(\int_0^t v(s) ds - z \right) \in atS.$$

As before, by definition of function $\tilde{\beta}(t, v(\cdot), z)$ we can write down the following equality:

$$\int_0^t \tilde{\beta}(s, v(\cdot), z) ds \left\| \int_0^t v(s) ds - z \right\| = at,$$

whence

$$\int_0^t \tilde{\beta}(s, v(\cdot), z) ds = \frac{at}{\left\| \int_0^t v(s) ds - z \right\|}. \quad (12)$$

Expression (12) makes it possible to evaluate the capture time. One can readily see that infimum in (12) is attained if $v(s) = -\frac{z}{\|z\|}$ almost everywhere, i.e.,

$$\inf_{v(\cdot) \in \Omega_V[0, t]} \int_0^t \tilde{\beta}(s, v(\cdot), z) ds = \frac{at}{\left\| \int_0^t -\frac{z}{\|z\|} ds - z \right\|} = \frac{at}{t + \|z\|}.$$

Setting the latter expression equal to 1, according to (8), we obtain $T(z) = \frac{\|z\|}{a-1}$. Obviously, this estimate coincides with that obtained using classic strategies of pursuit such as pure and parallel pursuit discussed in Sect. 2 [4].

Now let us find formula for pursuer's control. From (10), we deduce:

$$u(s) = \tilde{\beta}(s, v(\cdot), z^0) \int_0^s v(\tau) d\tau + \int_0^s \tilde{\beta}(\tau, v(\cdot), z^0) d\tau v(s) - \tilde{\beta}(s, v(\cdot), z^0) z^0. \quad (13)$$

Let us integrate both sides of (13) in s :

$$\int_0^t u(s) ds = \int_0^t \tilde{\beta}(s, v(\cdot), z^0) ds \int_0^t v(s) ds - \int_0^t \tilde{\beta}(s, v(\cdot), z^0) ds z^0. \quad (14)$$

Let us add and subtract x^0 from the left-hand side of this equality. Denote $\mu(t) = \int_0^t \tilde{\beta}(s, v(\cdot), z^0) ds$; then, taking into account that $z^0 = x^0 - y^0$, we have:

$$x^0 + \int_0^t u(s) ds = \mu(t) \left(y^0 + \int_0^t v(s) ds \right) - (1 - \mu(t)) x^0.$$

The Cauchy formula (5) yields the following expressions:

$$x(t) = x^0 + \int_0^t u(s) ds, \quad (15)$$

$$y(t) = y^0 + \int_0^t v(s) ds. \quad (16)$$

As a result, we have:

$$x(t) = \mu(t)y(t) + (1 - \mu(t))x^0. \quad (17)$$

Since $\mu(t) \in [0, 1]$ is a nonnegative scalar function, one can see from expression (17) that $x(t)$ belongs to the line segment joining x^0 with $y(t)$. Hence, the pursuit strategy, represented by the pursuer's control (13), is the motion camouflage strategy [1–4].

In virtue of (12) and (16), we have

$$\mu(t) = \int_0^t \tilde{\beta}(s, v(\cdot), z^0) ds = \frac{at}{\left\| \int_0^t v(s) ds - x^0 + y^0 \right\|} = \frac{at}{\|y(t) - x^0\|}. \quad (18)$$

Hence, taking into account (14) we can write

$$\int_0^t u(s) ds = \frac{at}{\|y(t) - x^0\|} \left[\int_0^t v(s) ds - x^0 + y^0 \right] = at \frac{y(t) - x^0}{\|y(t) - x^0\|}. \quad (19)$$

Fig. 5 Motion camouflage with $R = x^0$

To obtain explicit formula for the pursuer's control, we differentiate the both sides of (19) and derive

$$u(t) = a \frac{y(t) - x^0}{\|y(t) - x^0\|} + at \left[\frac{v(t)}{\|y(t) - x^0\|} - \frac{y(t) - x^0}{\|y(t) - x^0\|} \left\langle \frac{y(t) - x^0}{\|y(t) - x^0\|}; \frac{v(t)}{\|y(t) - x^0\|} \right\rangle \right], \quad (20)$$

where $\langle \cdot; \cdot \rangle$ stands for the scalar product.

Thus, the instantaneous pursuer's control that implements the motion camouflage strategy can be generated on the basis of the information on instantaneous evader's control and position.

It should be noted that input given by (20) implements the version of motion camouflage, in which the reference point R coincides with the pursuer's initial state x^0 , as shown in Fig. 5. Figure 5 shows pursuer's trajectory (in green) when the evader moves along the x -axis with the constant speed and the pursuer's control input is given by (20). The blue lines join players' instantaneous positions.

Since Eq. (11) is time invariant, the control input (20) can be adjusted to implement the motion camouflage strategy for the case when $R \neq x^0$, provided that x^0 belongs to the line segment joining R with y^0 . Let us apply substitution $t' = t + t_0$, where

$$t_0 = \frac{\|R - x^0\|}{a}.$$

That is, we assume that the game started at the instant $t' = 0$, when initial position of the pursuer coincided with R . At the time instant $t' = t_0$, the pursuer arrives at the point x^0 under assumption that the evader does not move up to that moment.

Fig. 6 Motion camouflage with $R \neq x^0$

Thus, the control input of the form

$$u(t) = a \frac{y(t) - R}{\|y(t) - R\|} + a(t + t_0) \left[\frac{v(t)}{\|y(t) - R\|} - \frac{y(t) - R}{\|y(t) - R\|} \left\langle \frac{y(t) - R}{\|y(t) - R\|}, \frac{v(t)}{\|y(t) - R\|} \right\rangle \right] \quad (21)$$

implements a general motion camouflage strategy with $R \neq x^0$.

Figure 6 shows pursuer's trajectory (in green) when the evader moves along the x -axis with the constant speed and the pursuer's control input is given by (21). The reference point is at $(0, 2)$, while x^0 is at $(0, 1)$. The blue lines join players' instantaneous positions.

5 Conclusion

The paper deals with the motion camouflage in differential games of pursuit, whereby the pursuer's position remains on the line joining current evader's position with a fixed reference point. A method utilizing the concept of integral inclusion for solving differential games of pursuit is proposed. The presented method is used to derive sufficient conditions for game termination in a finite time. It is shown that the proposed method implements the motion camouflage pursuit strategy in an example with "simple motion" dynamics of the players. One of the main results of the paper consists in obtaining explicit formulas (20), (21) for pursuer's control input implementing motion camouflage for the both cases: When the reference point coincides with the pursuer initial position and when it does not. Future research will be devoted to extending these results to more complicated dynamics.

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