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ERRATUM

Corrections to *Matrix polynomial generalizations of the sample variance-covariance matrix when* $pn^{-1} \to y \in (0, \infty)$, **Indian Journal of Pure and Applied Mathematics**, **48**(4) (2017), 575-607 by Monika Bhattacharjee and Arup Bose.

Statement of Theorem 4.2 *is not correct as stated. The corrected statement (and its proof) is given below in Theorem C4.2. Lemma* 4.3 *was used to prove Theorem 4.2. Now this needs to be replaced by Lemma C4.3 given below. Specifically, texts between* (4.11) *and* (4.22) *and the proofs given in Sections* 5.4 *and* 5.5 *should be considered deleted. This does not affect any of the subsequent corollaries and consequences of Theorem 4.2. They continue to remain valid with the new Theorem C4.2.*

Define

$$\begin{aligned}
\bar{R}_{j,j_1,j_2,j_3}(\underline{f},\bar{\Pi}) &= (1+y)\bar{\varphi}(\bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3}\bar{\delta}^{j-1})\underline{f}_{j_2},\ R_{j,j_1,j_2,j_3}(f,\Pi) = \frac{1+y}{y}\bar{\varphi}(\bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3}\bar{\delta}^{j-1})f_{j_2}, \\
\bar{A}_{j_1,j_2,j_3}(z,\underline{f},\bar{\Pi}) &= \sum_{i=1}^{\infty} z^{-i}\bar{R}_{i,j_1,j_2,j_3}(\underline{f},\bar{\Pi}),\ A_{j_1,j_2,j_3}(z,f,\Pi) = \sum_{i=1}^{\infty} z^{-i}R_{i,j_1,j_2,j_3}(f,\Pi).
\end{aligned}$$

The following lemma guarantees existence of the above sums. Recall the states φ_{odd} and φ_{even} defined on the spaces $\mathcal{A}_{\text{odd}}$ and $\mathcal{A}_{\text{even}}$ respectively. Clearly $\{d_j, e_j\}$ are in $\mathcal{A}_{\text{odd}}$ and $\{f_j\}$ are in $\mathcal{A}_{\text{even}}$.

Lemma C4.3 — Suppose for all sufficiently large $|z|$, $z \in \mathbb{C}^+$, and for some $C > 0$, $|\varphi_{\text{odd}}((\Pi\Pi^*)^r)| \leq C^r$ for all $r \geq 1$. Then $\bar{A}_{j_1,j_2,j_3}(z,\underline{f},\bar{\Pi})$ and $A_{j_1,j_2,j_3}(z,f,\Pi)$ exist in the sense of (4.10).

PROOF : There exists a $C > 0$ such that for any $\{a_{2i-1}\} \in \{b_{2i-1}, b^*_{2i-1}\}$, $\{a_{2i}\} \in \{b_{2i}, b^*_{2i}\}$ and $h \geq 1$,

$$|\bar{\varphi}(\bar{a}_1\bar{a}_3\cdots\bar{a}_{2h-1})| \leq C^h,\ \ |\bar{\varphi}(\underline{a}_2\underline{a}_4\cdots\underline{a}_{2h})| \leq C^h,\ \ |\bar{\varphi}(\bar{\delta}^h)| \leq C^h. \tag{1}$$

Proof of (1) is along the same lines as the proof of Carleman's condition (C) in Theorem 4.1. Hence we omit it. Now by (1), for some $C_1 > 0$, we have

$$\begin{aligned}
|\varphi_{\text{even}}((A_{j_1,j_2,j_3}(z,f,\Pi))^r)| &\leq C_1|\varphi_{\text{even}}(f_{j_2}^r)| \sum_{i_1,i_2,\ldots,i_r=1}^{\infty} \Big(\prod_{k=1}^{r} |z|^{-i_k} |\bar\varphi(\bar\Pi \bar d_{j_1} \bar e_{j_3} \bar\delta^{i_k-1})|\Big) \\
&\leq C_1|\varphi_{\text{even}}(f_{j_2}^r)|(\bar\varphi(\bar\Pi\bar\Pi^*))^{r/2}(\bar\varphi((\bar e_{j_3}^* \bar d_{j_1}^* \bar d_{j_1} \bar e_{j_3})^2))^{r/4} \\
&\qquad \times \sum_{i_1,i_2,\ldots,i_r=1}^{\infty} \Big(\prod_{k=1}^{r} |z|^{-i_k} (\bar\varphi(\bar\delta^{4i_k-4}))^{1/4}\Big) \\
&\leq C_1 C^{3r} \sum_{i_1,i_2,\ldots,i_r=1}^{\infty} \Big(\prod_{k=1}^{r} |z|^{-i_k} C^{i_k}\Big) \\
&= C_1 C^{4r}(|z|-C)^{-r}, \quad \text{for sufficiently large } |z|.
\end{aligned}$$

This completes the proof of the Lemma. □

***Theorem* C4.2** — *Assume (A1)-(A3) hold and $p, n \to \infty$, $p/n \to y > 0$.*

(a) *The following recursive relation holds:*

$$\begin{aligned}
\bar A_{j_1,j_2,j_3}(z,\underline{f},\bar\Pi) &= \frac{1+y}{z}\big[\bar\varphi(\bar\Pi \bar d_{j_1} \bar e_{j_3})\underline{f}_{j_2} \\
&+ \sum_{t=1}^{\infty} \sum_{1} \bar\varphi\big(\bar A_{l_0,l_{t-1},l_{t-1}}(z,\underline{f},\bar\Pi \bar d_{j_1} \bar e_{j_3}) \prod_{u=0}^{t-2} \bar A_{l_{u+1},l_u,l_u}(z,\underline{f},1)\big)\underline{f}_{j_2}\big].
\end{aligned} \tag{2}$$

As a consequence, for $z \in \mathbb{C}^+$, $|z|$ large, $m_{\bar\mu}(z)$ is given by

$$m_{\bar\mu}(z) = -\frac{1}{z}\big[1 + \sum_{t=1}^{\infty} \sum_{1} \bar\varphi\big(\prod_{u=1}^{t} \bar A_{l_u,l_{u-1},l_{u-1}}(z,\underline{f},1)\big)\big]. \tag{3}$$

(b) The following recursive relation holds:

$$\begin{aligned}
A_{j_1,j_2,j_3}(z,f,\Pi) &= \frac{1}{z}\big[\varphi_{\text{odd}}(\Pi d_{j_1} e_{j_3}) f_{j_2} \\
&+ \sum_{t=1}^{\infty} \sum_{1} y^{t-1} \varphi_{\text{even}}\big(A_{l_0,l_{t-1},l_{t-1}}(z,f,\Pi d_{j_1} e_{j_3}) \prod_{u=0}^{t-2} A_{l_{u+1},l_u,l_u}(z,f,1)\big) f_{j_2}\big].
\end{aligned} \tag{4}$$

As a consequence, for $z \in \mathbb{C}^+$, $|z|$ large, $m_{\mu}(z)$ is given by

$$m_{\mu}(z) = -\frac{1}{z}\big[1 + \sum_{t=1}^{\infty} \sum_{1} y^{t-1} \varphi_{\text{even}}\big(\prod_{u=1}^{t} A_{l_u,l_{u-1},l_{u-1}}(z,f,1)\big)\big]. \tag{5}$$

PROOF : To prove the theorem we shall prove a lemma that provides expression for $\bar\varphi(\bar\Pi\bar\delta^r)$.

Define

$$\sum_1 = \sum_{l_0,l_1,\ldots,l_{t-1}=1}^{q}, \quad \sum_2 = \sum_{1=k_0<k_1<k_2<\ldots<k_{t-1}\le r},$$
$$\sum_3 = \sum_{l_1,l_2,\ldots,l_r=1}^{q}, \quad \bar{d}_{l_t} = \bar{\Pi}\bar{d}_{l_0}, \ \ k_t = r+1.$$

Lemma 1 — Suppose (A1)-(A3) hold and $p, n = n(p) \to \infty$, $p/n \to y > 0$. Then

$$\bar{\varphi}(\bar{\Pi}\bar{\delta}^r) = \sum_{t=1}^{r}\sum_1\sum_2 (1+y)^t \bar{\varphi}\Big(\prod_{u=0}^{t-1}\bar{\varphi}(\bar{e}_{l_u}\bar{\delta}^{k_{u+1}-k_u-1}\bar{d}_{l_{u+1}})\underline{f}_{l_u}\Big). \tag{6}$$

PROOF : By Lemma C4.3, we have

$$\begin{aligned}
\bar{\varphi}(\bar{\Pi}\bar{\delta}^r) &= (1+y)^r\bar{\varphi}(\bar{\Pi}(\sum_{i=1}^{q}\bar{d}_i s\underline{f}_i s\bar{e}_i)^r) \\
&= (1+y)^r\sum_3\bar{\varphi}(\bar{\Pi}\prod_{u=1}^{r}\bar{d}_{l_u}s\underline{f}_{l_u}s\bar{e}_{l_u})) \\
&= (1+y)^r\sum_3\sum_{\sigma\in NC_2(2r)}\bar{\varphi}_{K(\sigma)}[\underline{f}_{l_1},\bar{e}_{l_1}\bar{d}_{l_2},\underline{f}_{l_2},\bar{e}_{l_2}\bar{d}_{l_3},\ldots,\bar{e}_{l_r}\bar{\Pi}\bar{d}_{l_1}] \\
&= \sum_{\sigma\in NC_2(2r)}\tau_\sigma
\end{aligned} \tag{7}$$

where $K(\sigma)$ is the Kreweras complement of σ (see Definition 9.21 in [1]) and

$$\tau_\sigma = (1+y)^r\sum_3\bar{\varphi}_{K(\sigma)}[\underline{f}_{l_1},\bar{e}_{l_1}\bar{d}_{l_2},\underline{f}_{l_2},\bar{e}_{l_2}\bar{d}_{l_3},\ldots,\bar{e}_{l_r}\bar{\Pi}\bar{d}_{l_1}]. \tag{8}$$

Now to compute (7), we consider the decomposition of $NC_2(2r) = \cup_{t=1}^{r}\mathcal{P}_t^{2r}$, where

$$\mathcal{P}_1^{2r} = \{\sigma \in NC_2(2r) : \{1,2\} \in \sigma\}$$

and for all $2 \le t \le r$,

$$\begin{aligned}
\mathcal{P}_t^{2r} = \ &\{\sigma \in NC_2(2r) : \sigma = \{2k_0-1, 2k_{t-1}\}, \{2k_0, 2k_1-1\}, \{2k_1, 2k_2-1\}, \\
&\ldots, \{2k_{t-2}, 2k_{t-1}-1\}, \ 1 = k_0 < k_1 < \cdots < k_{t-1} \le r\}.
\end{aligned}$$

Hence, (7) is equivalent to

$$\bar{\varphi}(\bar{\Pi}\bar{\delta}^r) = \sum_{t=1}^{r}\mathcal{T}_t, \tag{9}$$

where for all $1 \leq t \leq r$,

$$\begin{aligned}\mathcal{T}_t &= \sum_{\sigma \in \mathcal{P}_t^{2r}} \tau_\sigma = (1+y)^r \sum_{\sigma \in \mathcal{P}_t^{2r}} \sum_3 \bar{\varphi}_{K(\sigma)}[\underline{f}_{l_1}, \bar{e}_{l_1}\bar{d}_{l_2}, \underline{f}_{l_2}, \bar{e}_{l_2}\bar{d}_{l_3}, \ldots, \bar{e}_{l_r}\bar{\Pi}\bar{d}_{l_1}] \\ &= (1+y)^t \sum_1 \sum_2 \bar{\varphi}(\prod_{u=0}^{t-1} \underline{f}_{l_u}) \prod_{u=0}^{t-1} \bar{\varphi}(\bar{e}_{l_u}\bar{\delta}^{k_{u+1}-k_u-1}\bar{d}_{l_{u+1}}) \\ &= (1+y)^t \sum_1 \sum_2 \bar{\varphi}(\prod_{u=0}^{t-1} \bar{\varphi}(\bar{e}_{l_u}\bar{\delta}^{k_{u+1}-k_u-1}\bar{d}_{l_{u+1}})\underline{f}_{l_u}). \end{aligned} \tag{10}$$

Note $\sigma \in \mathcal{P}_t^{2r}$ implies that one block of $K(\sigma)$ is $\{2k_0-1, 2k_1-1, \ldots, 2k_{t-1}-1\}$ and its other blocks are subset of $\{2k_u, 2k_u+1, \ldots, 2k_{u+1}-2,\} : \ 0 \leq u \leq t-2$. Thus by using Lemma M5.1(b), above third equality holds. Moreover, recall (4.7) and note that each $\bar{\delta}$ involves one $(1+y)$. In the third equality we have $\sum_{u=0}^{t-1}(K_{u+1}-k_u-1) = r-t$ many $\bar{\delta}$ and therefore they absorb $(1+y)^{r-t}$. Hence the lemma follows by (9) and (10). □

PROOF OF (2) :

$$\bar{A}_{j_1,j_2,j_3}(z, \underline{f}, \bar{\Pi}) = \sum_{r=1}^{\infty} z^{-r}(1+y)\bar{\varphi}(\bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3}\bar{\delta}^{r-1})\underline{f}_{j_2} = T_1 + T_2, \text{ (say)} \tag{11}$$

where

$$T_1 = \frac{1+y}{z}\bar{\varphi}(\bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3})\underline{f}_{j_2}, \text{ and } T_2 = \sum_{r=1}^{\infty} z^{-(r+1)}(1+y)\bar{\varphi}(\bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3}\bar{\delta}^r)\underline{f}_{j_2}. \tag{12}$$

Let $\sum_4 = \sum_{i_1,i_2,\ldots,i_t=1}^{\infty}$. Now by Lemma 1, $\bar{d}_{l_t} = \bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3}\bar{d}_{l_0}$ and

$$\begin{aligned} T_2 &= \frac{1+y}{z} \sum_{r=1}^{\infty} \sum_{t=1}^{r} \sum_1 \sum_2 z^{-r}(1+y)^t \bar{\varphi}(\prod_{u=0}^{t-1} \bar{\varphi}(\bar{e}_{l_u}\bar{\delta}^{k_{u+1}-k_u-1}\bar{d}_{l_{u+1}})\underline{f}_{l_u})\underline{f}_{j_2} \\ &= \frac{1+y}{z} \sum_{t=1}^{\infty} \sum_{r=t}^{\infty} \sum_1 \sum_2 z^{-r}(1+y)^t \bar{\varphi}(\prod_{u=0}^{t-1} \bar{\varphi}(\bar{e}_{l_u}\bar{\delta}^{k_{u+1}-k_u-1}\bar{d}_{l_{u+1}})\underline{f}_{l_u})\underline{f}_{j_2} \\ &= \frac{1+y}{z} \sum_{t=1}^{\infty} \sum_1 \sum_4 z^{-(i_1+i_2+\cdot+i_t)}(1+y)^t \bar{\varphi}(\prod_{u=0}^{t-1} \bar{\varphi}(\bar{e}_{l_u}\bar{\delta}^{i_{u+1}-1}\bar{d}_{l_{u+1}})\underline{f}_{l_u})\underline{f}_{j_2} \\ &= \frac{1+y}{z} \sum_{t=1}^{\infty} \sum_1 \bar{\varphi}(\prod_{u=0}^{t-1} \sum_{i_{u+1}=1}^{\infty} z^{-i_{u+1}}(1+y)\bar{\varphi}(\bar{e}_{l_u}\bar{\delta}^{i_{u+1}-1}\bar{d}_{l_{u+1}})\underline{f}_{l_u})\underline{f}_{j_2} \end{aligned}$$

$$= \frac{1+y}{z}\sum_{t=1}^{\infty}\sum_{1}\bar{\varphi}\Big(\sum_{i_t=1}^{\infty} z^{-i_t}\bar{R}_{i_t,l_0,l_{t-1},l_{t-1}}(\underline{f},\bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3}) \prod_{u=0}^{t-2}\sum_{i_{u+1}=1}^{\infty} z^{-i_{u+1}}\bar{R}_{i_{u+1},l_{u+1},l_u,l_u}(\underline{f},1)\Big)$$

$$= \frac{1+y}{z}\sum_{t=1}^{\infty}\sum_{1}\bar{\varphi}\big(\bar{A}_{l_0,l_{t-1},l_{t-1}}(z,\underline{f},\bar{\Pi}\bar{d}_{j_1}\bar{e}_{j_3})\prod_{u=0}^{t-2}\bar{A}_{l_{u+1},l_u,l_u}(z,\underline{f},1)\big). \tag{13}$$

Hence (2) is established by (11)-(13).

PROOF OF (3) : Note that

$$m_{\bar{\mu}}(z) = \bar{\varphi}((\delta - z)^{-1}) = -\sum_{i=1}^{\infty} z^{-i}\bar{\varphi}(\bar{\delta}^{i-1}) = -(1+y)^{-1}\bar{A}_{0,0,0}(z,\underline{f},1).$$

Thus (3) follows by (2) and observing that $\bar{d}_0 = 1$, $\bar{e}_0 = 1$ and $\underline{f}_0 = 1$.

This completes the proof of the theorem $C4.2(a)$.

PROOF OF (**b**) : By (1), note that for any polynomials $\Pi_1, \Pi_2, \ldots, \Pi_k$, of the same form as Π, we have

$$\begin{aligned}
&\bar{\varphi}(\prod_{k=1}^{r}\bar{R}_{j_k,j_{1k},j_{2k},j_{3k}}(\underline{f},\bar{\Pi}_k)) \\
&= (1+y)^r\bar{\varphi}(\underline{f}_{j_{21}}\underline{f}_{j_{22}}\cdots\underline{f}_{j_{2r}})\prod_{k=1}^{r}\bar{\varphi}(\bar{\Pi}_k\bar{d}_{j_{1k}}\bar{e}_{j_{3k}}\bar{\delta}^{j_k-1}) \\
&= \frac{y^r}{1+y}\varphi_{\text{even}}(f_{j_{21}}f_{j_{22}}\cdots f_{j_{2r}})\prod_{k=1}^{r}\big(\frac{1+y}{y}\bar{\varphi}(\bar{\Pi}_k\bar{d}_{j_{1k}}\bar{e}_{j_{3k}}\bar{\delta}^{j_k-1})\big) \\
&= \frac{y^r}{1+y}\varphi_{\text{even}}(\prod_{k=1}^{r}R_{j_k,j_{1k},j_{2k},j_{3k}}(f,\Pi_k)).
\end{aligned}$$

Thus

$$\bar{\varphi}(\prod_{k=1}^{r}\bar{A}_{j_{1k},j_{2k}j_{3k}}(z,\underline{f},\bar{\Pi}_k)) = \frac{y^r}{1+y}\varphi_{\text{even}}(\prod_{k=1}^{r}A_{j_{1k},j_{2k}j_{3k}}(z,f,\Pi_k)). \tag{14}$$

Therefore, by (2) and (14), we have

$$\begin{aligned}
\bar{A}_{j_1,j_2,j_3}(z,\underline{f},\bar{\Pi}) &= \frac{1}{z}\big[y\varphi_{\text{odd}}(\Pi d_{j_1}e_{j_3})\underline{f}_{j_2} \\
&+ \sum_{t=1}^{\infty}\sum_{1} y^t\varphi_{\text{even}}\big(A_{l_0,l_{t-1},l_{t-1}}(z,f,\Pi d_{j_1}e_{j_3})\prod_{u=0}^{t-2}A_{l_{u+1},l_u,l_u}(z,f,1)\big)\underline{f}_{j_2}\big].
\end{aligned} \tag{15}$$

Hence, (4) follows from the above equation and (1).

Now by (4.9), (3) and (14), we have

$$\frac{y}{1+y}m_\mu(z) - \frac{1}{1+y}\frac{1}{z} = -\frac{1}{z}\big[1 + \sum_{t=1}^{\infty}\sum_{1}\frac{y^t}{1+y}\varphi_{\text{even}}\big(\prod_{u=1}^{t}A_{l_u,l_{u-1},l_{u-1}}(z,f,1)\big)\big].$$

Simplifying the above equation,

$$m_\mu(z) = -\frac{1}{z}\big[1 + \sum_{t=1}^{\infty}\sum_{1}y^{t-1}\varphi_{\text{even}}\big(\prod_{u=1}^{t}A_{l_u,l_{u-1},l_{u-1}}(z,f,1)\big)\big].$$

This establishes (5) and hence completes the proof of Theorem $C4.2$.

Reference

1. A. Nica and R. Speicher, *Lectures on the combinatorics of free probability*, Cambridge University Press, Cambridge, UK, 2006.