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ORIGINAL ARTICLE

Influence networks and public goods

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Abstract We consider a model of local public goods in a random network context. The influence network determines (exogenously) who observes whom every period and comprises a wide array of options depending on the degree distribution and the in/out-degree correlations. We show that there exists a unique equilibrium level of public good provision and compare it with the efficient level. We derive further insights for this problem by performing a comparative statics analysis.

Keywords Influence networks · Public goods · Out-degree · In-degree · Best-shot game

JEL Classification D85 · H41

1 Introduction

The study of networks has been of significant importance in diverse academic fields such as sociology, physics and computer science (see, e.g., Wasserman and Faust 1994; Newman 2003, and the long list of references cited therein). The last two decades have witnessed how numerous phenomena of economic relevance have also been studied using the paradigm of networks. Instances are network formation (e.g., Jackson and Wolinsky 1996; Bala and Goyal 2000), diffusion of behaviors (Morris 2000; López-Pintado 2008a), labor markets (Calvó-Armengol 2004; Calvó-Armengol and

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Jackson 2007), peer effects in education (Calvó-Armengol et al. 2009), crime activities (Ballester et al. 2006; Goyal and Virgier 2014), financial markets (Elliot et al. 2014) and microfinance credits (Benerjee et al. 2013).[1]

We consider a model of local public goods in a random network context. There are many socioeconomic situations which can be described as a local public good, that is, a good that is non-rival and non-excludible among neighbors in a relevant network. Some examples are innovations among collaborating firms, complementarity of skills within social contacts, the provision of an open source *product* (e.g., software) or information in the Internet (e.g., websites or blogs). We shall be concerned with *directed* networks, i.e., networks in which the benefits from interacting with an agent providing the public good is only one way. We define the *out-degree* (observability) of an agents as the number of agents she observes, whereas her *in-degree* (visibility) indicates how many agents observe this agent. The correlation between agent's out-degree and in-degree might depend on the application. For instance, in friendship networks, this correlation tends to be high because friendship is mostly bidirectional, whereas for other types of networks such as the WWW this correlation could potentially be lower.

As standard in this literature, we assume that the network is exogenously given in order to isolate the decision to contribute to the local public good from the network formation issue. We depart, however, from the fixed network approach and consider random networks instead (see e.g., Pastor-Satorrás and Vespignani 2001; Jackson and Rogers 2007; López-Pintado 2008b, 2012, 2013; Galeotti and Goyal 2009, among others). That is, the network is characterized by its large-scale properties which are fixed, although the connections evolve randomly over time. This paper focuses on two of such properties: the (out-)degree distribution and the in/out-degree correlation.

Agents have to decide whether to invest, or not, in the public good. If they enjoy the public good -either because they have invested in it or because they free ride on some other agent in the population that has done so- they obtain some benefits. This determines a game known as the best-shot game in which an agent has incentives to invest in the public good only if no other agent observed by her has already done so. The static version of this model on a fixed (directed) underlying network structure has some limitations. On the one hand, for most networks structures there will be a large set of equilibrium outcomes and therefore we would incur in multiplicity problems. On the other hand, for some simple networks an equilibrium, in pure strategies, will not exist. We therefore propose and focus on an alternative approach based on a dynamics influence process which leads to a unique equilibrium prediction. In particular, there exists a unique *globally stable outcome* (fraction of individuals investing in the public good) of this dynamics characterized by the (out-)degree distribution and the in/out-degree correlation. Thus, some comparative statics results can be provided.

There are two outcomes of interest: (1) the fraction of public good providers in equilibrium, and (2) the fraction of links that reach a public good provider in equilibrium. The comparative statics results lead to the following conclusions. On the one hand, for any given out-degree distribution, an increase in the in/out-degree correlation increases measure (1) and decreases measure (2). On the other hand, for any

[1] See Goyal (2007), Vega-Redondo (2007) and Jackson (2008) for comprehensive treatments of the general topic of social and economic networks.

in/out-degree correlation, if the network is sufficiently dense, a First Order Stochastic Dominance shift (Mean Preserving Spread) of the out-degree distribution decreases (increases) measure (2).

We partially compare the efficient and equilibrium outcomes. We find that in the case of an homogeneous population in equilibrium there is underprovision of the public good. We also show that if we allow for some heterogeneity regarding out-degrees the efficient and equilibrium outcomes provide opposite results; the probability of contributing to the public good increases with respect to out-degree in equilibrium, whereas it decreases in the efficient state.

Public goods in a network context was first analyzed in the seminar paper by Bramoullé and Kranton (2007) who characterized the (multiple) Nash equilibria of a public good game in a static network framework. Boticinelli and Pin (2012) addressed the issue of equilibrium selection by introducing a dynamic model. Galeotti et al. (2010) also shrink considerably the potentially large set of equilibria that arise under complete information by assuming incomplete knowledge by part of the consumers with respect to the network. Bramoullé et al. (2014) focus on games of strategic substitutes on networks with linear best-reply functions which has recently been extended to non-linear settings by Allouch (2015). We contribute to this vast literature by analyzing local public goods in a random network context.

The rest of the paper is organized as follows. In Sect. 2 we describe the model. In Sect. 3 we characterize the equilibrium. The comparative statics of the equilibrium outcome are discussed in Sect. 4. In Sect. 5 we provide some results on efficiency. Finally, in Sect. 6 we conclude.

2 The model

Before presenting the full specification of the model, let us start with the static benchmark case which will be useful to motivate our dynamic approach.

2.1 The static model

Let us consider a population with $N = \{1, \ldots, n\}$ agents, where n is sufficiently large. Agents observe each other due to what we describe as an *influence network*, which is exogenously given. Let $N_i \subseteq N$ be the set of agents observed by i, where d_i, the out-degree, is simply the cardinality of N_i. The network is directed. That is, if an agent i observes (or is influenced by) j this does not imply that j observes i. For instance, an agent can observe the webpage of another agent but not vice versa. Each agent $i \in N$ chooses an action indicating whether or not to provide a (costly) local public good. More precisely, an agent chooses $a_i \in \{0, 1\}$, where $a_i = 1$ (0) is interpreted as the decision of (not) providing a public good. Let $a = (a_1, \ldots, a_n) \in \{0, 1\}^n$ denote an action profile. An agent providing the public good ($a_i = 1$) pays a cost of $C > 0$, whereas an agent enjoying the public good (from at least one other agent, or herself) obtains a benefit of $A > 0$, where $A > C$. Let $U_i(a_i, a_{-i})$ be the utility function of agent i given the action profile $a = (a_i, a_{-i})$, where a_{-i} stands for the action chosen by all agents except for i. Then:

$$U_i(a_i, a_{-i}) = A - a_i C \quad \text{if} \sum_{j \in N_i \cup \{i\}} a_j \geq 1$$
$$U_i(a_i, a_{-i}) = 0 \quad \text{otherwise}$$

There are at least two key assumptions implicit in this specification. On the one hand, there is no congestion, and thus the benefit of observing the public good is independent on how many other agents also benefit from it (i.e., the good is non-rival). On the second hand, there is no extra benefit derived from observing the public good more than once. [2]

We can now analyze the Nash equilibria of the induced game, where all agents decide simultaneously whether or not to provide the public good. An agent would decide to provide the public good if and only if nobody observed by her is doing so. Formally, one can characterize the equilibrium as follows: a strategy profile $(a_1^*, a_2^*, \ldots, a_n^*)$ is a Nash equilibrium if and only if all agents providing the public good, i.e., $\{i \in N, \text{ s.t. } a_i^* = 1\}$, form a *maximal independent set*. A maximal independent set is a set of agents satisfying that no agent observes any other agent in the set, and all agents out of the set observe at least one agent within the set. [3]

Notice that, this model has little predictive power since for many complex networks there will be a large set of possible equilibrium outcomes. [4] In addition to this problem, it is easy to construct simple networks where there exists no maximal independent set, and thus the issue of non-existence of (pure strategy) Nash equilibria arises. [5] This motivates the study of a dynamic approach for which there always exists a unique prediction.

2.2 The dynamic model

We consider a dynamic model to describe the evolution of agents' choices through time. We differ from the static approach in several directions. On the one hand, agents update their decision to provide the public good over time. On the other hand, the influence network is going to change (non-strategically) every period due to the randomness assumed in the linking process. Agents are characterized by their out-degree d_i, representing for instance, exogenous time constraints, where $P(d)$ denotes the out-degree distribution in the population. At each time step the network is randomly

[2] The equilibrium predictions of this model are robust to other types of utility functions, as long as the main features remain (i.e., agents are only willing to provide the public good if nobody observed by them has done so). The welfare analysis described in Sect. 5 will obviously depend on the specific utility function considered.

[3] This concept was introduced as a formal way of describing the equilibrium for local public good games by Bramoullé and Kranton (2007). Later, López-Pintado (2013) extended the concept to directed networks with no in/out-degree correlations.

[4] Consider, as an example, the undirected star network. Notice that, the case where only the center contributes to the public good is a Nash equilibrium, as well as the case where all agents except for the center contribute to the public good.

[5] An example where there is no (pure strategy) Nash equilibrium is the following. Consider a network composed by three agents $\{i, j, k\}$ and three directed links. In particular, agent i observes j, j observes k and k observes i.

generated given two primitives of the process that remain fixed: (a) the degree distribution $P(d)$ and (b) a parameter $\alpha \in \mathbb{R}$ which introduces linking biases with respect to the out-degree. In particular, the probability that an agent links with (or observes) an agent with out-degree d is equal to

$$\frac{1}{\langle d^{\alpha}\rangle_P} d^{\alpha} P(d),$$

where $\langle d^{\alpha}\rangle_P = \sum_d d^{\alpha} P(d)$. The *expected* in-degree of an agent is defined as the *expected* number of agents that observe this agent. It is straightforward to show the expected in-degree (d_{in}) of an agent with out-degree $d_{out} = d$ is equal to $\langle d_{in} \mid d_{out} = d\rangle = \frac{d^{\alpha}}{\langle d^{\alpha}\rangle_P} \langle d\rangle_P$.[6] With some abuse of terminology, throughout the paper, the expected in-degree will simply be referred to as the in-degree. Two special cases could be singled out regarding α. If $\alpha = 0$ agents observe others uniformly at random and as a consequence the out-degree and in-degree are uncorrelated, in fact, all agents have the same in-degree or visibility (see López-Pintado 2013 for the analysis of such extreme case). If, instead, $\alpha = 1$ then an agent with twice the out-degree of another agent is selected twice as often. This reflects the idea that agents that have a higher out-degree are also more visible for others and therefore will have a higher in-degree. Hereafter the in/out-degree correlation can be roughly identified with parameter α when $\alpha \in [0, 1]$ as an increase in such parameter corresponds with an increase in the similarity between the out-degree and in-degree of agents.

Notice that, if $\alpha > 1$ agents with a high out-degree have an even higher in-degree, whereas $\alpha < 0$ implies that agents with a high out-degree will typically have a very low in-degree. For most of the paper we will concentrate on the case $\alpha \in [0, 1]$, although we will point out which results can be extended to other values of α.

Consider a continuous-time dynamics to describe the evolution of the provision of the local public good in the population. At each time t an agent, say i, revises his action a_i at a rate $\lambda \geq 0$. This agent decides whether or not to contribute to the public good given the behavior of those agents observed by agent i in the influence network realized at time t, i.e., applying a myopic best response.

Let $\rho_d(t)$ denote the proportion of agents with out-degree d that are choosing action 1 at time t. A state is determined by the profile $\{\rho_d(t)\}_{d\geq 1}$, where we assume that all agents have at least out-degree 1. There are two measures that will be of particular importance for our analysis.

First, the overall fraction of agents choosing 1 (non-conditional on degree). This measure is denoted by $\rho(t)$ and can be computed as follows:

$$\rho(t) = \sum_{d\geq 1} P(d)\rho_d(t).$$

6 Notice that $\langle d_{in} \mid d_{out} = d\rangle = \frac{\frac{1}{\langle d^{\alpha}\rangle_P} d^{\alpha} P(d) \sum_k n P(k) k}{n P(d)}$, where the numerator is the total number of links that would reach the set of agents with out-degree d and the denominator is the number of agents with out-degree d in the population.

Second, the probability that a link reaches an agent choosing 1. This probability is represented by $\theta(t)$ and can be computed as follows:

$$\theta(t) = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} d^\alpha P(d) \rho_d(t). \tag{1}$$

The computation of $\theta(t)$ is derived from the fact that $\frac{d^\alpha P(d)}{\langle d^\alpha \rangle}$ is the probability of observing an agent with out-degree d and, conditional on having out-degree d, the probability of providing the public good is $\rho_d(t)$. Notice that, in the extreme case where $\alpha = 0$ (i.e., in/out-degree correlation is zero) then $\theta(t) = \rho(t)$ but, in general, these two measures will differ.

For each degree d, the deterministic approximation of the evolution of $\rho_d(t)$ is given by the following differential equation:

$$\frac{d\rho_d(t)}{dt} = (1 - \rho_d(t))\lambda(1 - \theta(t))^d - \rho_d(t)\lambda(1 - (1 - \theta(t))^d). \tag{2}$$

Notice that the positive term in the equation accounts for transitions from action 0 to 1, whereas the negative term accounts for the reverse transitions (i.e., from action 1 to 0). The first term can be interpreted as follows: $1 - \rho_d(t)$ is the probability that an agent with out-degree d is choosing 0 at time t. This agent revises at a rate λ and switches to 1 if nobody observed by her is choosing 1 , something which occurs with probability $(1 - \theta(t))^d$. The second term can be interpreted in a similar way: $\rho_d(t)$ is the probability that an agent with out-degree d is choosing 1 at time t. This agent revises at a rate λ and switches to 0 if at least one agent observed by her is choosing 1, something which occurs with probability $1 - (1 - \theta(t))^d$.

After simplifications of Eq. (2) we find that, for each d,

$$\frac{d\rho_d(t)}{dt} = \lambda(-\rho_d(t) + (1 - \theta(t))^d). \tag{3}$$

Given this system of differential equations we can now address the issue of equilibrium.

3 Results: the equilibrium

In the stationary state (or equilibrium) of this dynamics, $\frac{d\rho_d(t)}{dt} = 0$, for all d, which implies that:

$$\rho_d(\theta) = (1 - \theta)^d. \tag{4}$$

Therefore, given Eq. (1), θ^* must be a solution of the following (fixed-point) equation:

$$\theta = H_{P,\alpha}(\theta), \tag{5}$$

where we define

$$H_{P,\alpha}(\theta) \equiv \sum_{d \geq 1} \frac{d^\alpha P(d)}{\langle d^\alpha \rangle_P} (1-\theta)^d.$$

Once we know θ^* in equilibrium we can also determine ρ_d^* for each d and, consequently, the overall fraction of public good contributors ρ^*.

Notice that, in the dynamics described above, agents can switch actions (from 0 to 1 and vice-versa) in equilibrium, given that the influence network is randomly generated every period. Therefore, the concept of stationary state, only refers to stationary values for θ, $\{\rho_d\}_d$ and ρ. The next result addresses the issue of existence and uniqueness of the equilibrium.

Proposition 1 *Given an influence network characterized by P and $\alpha \in \mathbb{R}$, there exists a unique equilibrium of the dynamic model. Furthermore, this equilibrium is globally stable.*

Proof Note that $\frac{dH_{P,\alpha}(\theta)}{d\theta} = -\frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} d^\alpha P(d) d (1-\theta)^{d-1} \leq 0$ for all $\theta \in [0, 1]$. Therefore, $H_{P,\alpha}(\theta)$ is a (continuous and) decreasing function of θ. Furthermore, $H_{P,\alpha}(0) = 1$ and $H_{P,\alpha}(1) = 0$. Thus, there exists a unique solution $\theta^* \in (0, 1)$ of Eq. (5) and therefore a unique value for θ (and ρ) in equilibrium. To conclude, let us show that θ^* is globally stable, i.e., starting from any initial fraction of agents choosing 1 ($\rho = \rho_0$), the dynamics converges to a state where $\theta = \theta^*$. To do so notice that

$$\frac{d\theta(t)}{dt} = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} d^\alpha P(d) \frac{d\rho_d(t)}{dt},$$

and, substituting $\frac{d\rho_d(t)}{dt}$ for its value determined by (3), we obtain that

$$\frac{d\theta(t)}{dt} = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} d^\alpha P(d) \lambda(-\rho_d + (1-\rho(t))^d);$$

or, equivalently,

$$\frac{d\theta(t)}{dt} = \lambda(H_{P,\alpha}(\theta(t)) - \theta(t)),$$

from where the desired conclusion follows. □

In the next section we develop comparative statics results with respect to the in/out-degree correlation α and the out-degree distribution P (the two primitives of the influence network).

4 Comparative statics

Let $\rho^*(P, \alpha)$ and $\theta^*(P, \alpha)$ denote the equilibrium values for ρ and θ, respectively, given P and α.

Consider first the comparative statics with respect to α.

Proposition 2 *Let P be a given out-degree distribution. Also, let $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$ be two levels of in/out-degree correlation. If $\alpha_1 \leq \alpha_2$ then $\theta^*(P, \alpha_1) \geq \theta^*(P, \alpha_2)$ and $\rho^*(P, \alpha_1) \leq \rho^*(P, \alpha_2)$.*

Notice that an increase in α leads to opposite effects on θ^* and ρ^*, something which, at first, is quite counter-intuitive. The intuition for such a result is the following. It is always the case that, in equilibrium, the fraction of agents contributing to the public good decreases with respect to the out-degree (i.e., ρ_d^* is decreasing with respect to d) since the probability of observing an agent providing the public good is lower for agents with smaller out-degrees (see Eq. 4). Thus, if α increases, agents with low out-degree are observed by relatively fewer agents which implies that the probability of observing through one of the links the public good (i.e., θ^*) would also decrease. As a consequence, to compensate for such a decrease in θ^*, the fraction of agents contributing to the public good (i.e., ρ^*) increases as α increases.

The formal proof is the following.

Proof Recall that the fixed point equation characterizing $\theta^*(P, \alpha)$ is

$$\theta = \sum_{d \geq 1} \frac{d^\alpha P(d)}{\langle d^\alpha \rangle_P} (1 - \theta)^d.$$

Let $Q_{\alpha,P}(d) = \frac{d^\alpha P(d)}{\langle d^\alpha \rangle_P}$. We can interpret $Q_{\alpha,P}(d)$ as the out-degree distribution of observed agents. We show next that $Q_{\alpha_2,P}(d)$ First Order Stochastic Dominates $Q_{\alpha_1,P}(d)$ if and only if $\alpha_1 \leq \alpha_2$. Intuitively, this should hold as a higher in/out-degree correlation implies that agents with high out-degree are observed more often which also implies that the out-degree distribution of observed agents must take larger values. Formally, we must find that the cumulative distribution function of $Q_{\alpha_2,P}(d)$ is always below the cumulative distribution function of $Q_{\alpha_1,P}(d)$. That is, for all $D > 1$

$$\sum_{d \geq 1}^{D} \frac{d^{\alpha_2} P(d)}{\langle d^{\alpha_2} \rangle_P} \leq \sum_{d \geq 1}^{D} \frac{d^{\alpha_1} P(d)}{\langle d^{\alpha_1} \rangle_P},$$

or, analogously, that

$$\sum_{d \geq 1}^{D} d^{\alpha_2} P(d) \left\langle d^{\alpha_1} \right\rangle_P \leq \sum_{d \geq 1}^{D} d^{\alpha_1} P(d) \left\langle d^{\alpha_2} \right\rangle_P. \tag{6}$$

Condition (6) can be written as follows:

$$\sum_{d_1 \geq 1}^{D} \sum_{d_2 \geq 1} d_1^{\alpha_2} d_2^{\alpha_1} P(d_1) P(d_2) \leq \sum_{d_1 \geq 1}^{D} \sum_{d_2 \geq 1} d_1^{\alpha_1} d_2^{\alpha_2} P(d_1) P(d_2).$$

Note that the two expressions coincide, as long as d_2 is bounded below D. That is, we know that

$$\sum_{d_1 \geq 1}^{D} \sum_{d_2 \geq 1}^{D} d_1^{\alpha_2} d_2^{\alpha_1} P(d_1) P(d_2) = \sum_{d_1 \geq 1}^{D} \sum_{d_2 \geq 1}^{D} d_1^{\alpha_1} d_2^{\alpha_2} P(d_1) P(d_2).$$

For the part of the sum where d_2 exceeds D, however, this is no longer the case (since a permutation of the indices no longer appears in the sum). For such cases, as $d_1 < d_2$ and $\alpha_1 < \alpha_2$, then $d_1^{\alpha_2} d_2^{\alpha_1} < d_1^{\alpha_1} d_2^{\alpha_2}$. Thus,

$$\sum_{d_1 \geq 1}^{D} \sum_{d_2 \geq D} d_1^{\alpha_2} d_2^{\alpha_1} P(d_1) P(d_2) < \sum_{d_1 \geq 1}^{D} \sum_{d_2 \geq D} d_1^{\alpha_1} d_2^{\alpha_2} P(d_1) P(d_2)$$

which proves condition (6).

To complete the proof we use that $\rho_d = (1-\theta)^d$ is decreasing as a function of d (for all $\theta \in [0, 1]$), which implies that

$$\sum_{d \geq 1} \frac{d^{\alpha_2} P(d)}{\langle d^{\alpha_2} \rangle_P} (1-\theta)^d \leq \sum_{d \geq 1} \frac{d^{\alpha_1} P(d)}{\langle d^{\alpha_1} \rangle_P} (1-\theta)^d,$$

and thus that $\theta^*(P, \alpha_2) \leq \theta^*(P, \alpha_1)$. Then, $\rho_d^*(P, \alpha_1) \leq \rho_d^*(P, \alpha_2)$ for all d and therefore $\rho^*(P, \alpha_1) \leq \rho^*(P, \alpha_2)$. □

Notice that the result is true for all possible values of $\alpha \in \mathbb{R}$, not just $\alpha \in [0, 1]$.

In order to present the next result let us define first the meaning of a free rider in this context. An agent is a free rider if it observe the public good but it does not provide it herself. The next result indicates how the fraction of free riders depends on the parameter α.

Proposition 3 *The fraction of free riders in equilibrium decreases with* α*, where* $\alpha \in [0, 1]$.

Proof Notice that the fraction of free riders, denoted by y, is equal to

$$y = \sum_{d \geq 1} P(d)(1-\rho_d)(1-(1-\theta)^d)$$

since $(1-(1-\theta)^d)$ is the probability that an agent with out-degree d has of enjoying the public good. In equilibrium $\rho_d^* = (1-\theta^*)^d$ and therefore

$$y^* = \sum_{d \geq 1} P(d)(1 - (1 - \theta^*)^d)^2$$

which is increasing with respect to θ^*. Notice that, due to Proposition 2, θ^* decreases with α which proofs the result. □

Consider now the comparative statics with respect to P. To do so, we assume a fixed value of $\alpha \in [0, 1]$ and analyze how different out-degree distributions lead to different outcomes. We first study the effect of a First Order Stochastic Dominance shift of the out-degree distribution, and then analyze the effect of a Mean Preserving Spread.

Proposition 4 *Let $\alpha \in [0, 1]$ and let d_m denote the minimum degree in the network. If $\overline{P}(d)$ First Order Stochastic Dominates $P(d)$ and*

$$1 - e^{\frac{-\alpha}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle} \sum_{d \geq d_m} d^\alpha \overline{P}(d) e^{-\frac{d\alpha}{d_m}}, \tag{7}$$

then $\theta^*(\overline{P}, \alpha) \leq \theta^*(P, \alpha)$.

Notice that condition (7) is satisfied as long as d_m is sufficiently high. Roughly speaking, Proposition 3 implies that if the network becomes denser then, in equilibrium, the fraction of links reaching the public good decreases. This result also implies that $\rho_d^*(\overline{P}, \alpha) \geq \rho_d^*(P, \alpha)$ for all d. Comparative statics, however, with respect to ρ^* might depend on further properties of the out-degree distributions.[7]

The proof of the result is provided next.

Proof We aim to show that $H_{\overline{P},\alpha}(\theta) \leq H_{P,\alpha}(\theta)$ for all $\theta \in [\theta^*(\overline{P}, \alpha), 1]$ as this would imply that $\theta^*(\overline{P}, \alpha) \leq \theta^*(P, \alpha)$. We first rewrite $H_{P,\alpha}(\theta)$ as

$$H_{P,\alpha}(\theta) = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} P(d) g_\theta(d),$$

where

$$g_\theta(d) = d^\alpha (1 - \theta)^d.$$

Let us show next that if $\theta \in [1 - e^{\frac{-\alpha}{d_m}}, 1]$ then $g_\theta(d)$ is decreasing for all $d \geq d_m$. Notice that

$$g'_\theta(d) = d^{\alpha - 1}(1 - \theta)^d [\alpha + d \ln(1 - \theta)],$$

[7] Note that the result is straightforward if we consider the simpler framework where we compare two homogeneous populations, one with a low out-degree and another one with a high out-degree. In this case $\theta = \rho$ and thus we find that the fraction of public good contributors is higher for the low density network, as expected.

which is negative if and only if

$$d \geq \frac{-\alpha}{\ln(1-\theta)}.$$

Let θ_m be such that

$$d_m = \frac{-\alpha}{\ln(1-\theta_m)}.$$

That is, $\theta_m = 1 - e^{\frac{-\alpha}{d_m}}$. We can easily check that $\frac{-\alpha}{\ln(1-\theta)}$ is a decreasing function of θ which then implies that $g(d)$ is decreasing for all $d \geq d_m$, provided that $\theta \geq \theta_m$. Therefore, as $\overline{P}(d)$ first order stochastic dominates $P(d)$,

$$\sum_{d \geq d_m} \overline{P}(d) g(d) \leq \sum_{d \geq d_m} P(d) g(d),$$

for all $\theta \in [\theta_m, 1]$. In addition, as d^α is an increasing function of d we know that $\langle d^\alpha \rangle_{\overline{P}} \geq \langle d^\alpha \rangle_P$. Thus,

$$H_{\overline{P},\alpha}(\theta) \leq H_{P,\alpha}(\theta) \quad \text{for all } \theta \in [\theta_m, 1].$$

To complete the proof we must show that $\theta_m \leq \theta^*(\overline{P}, \alpha)$, which is the case if $\theta_m \leq H_{\overline{P},\alpha}(\theta_m)$, or, analogously, if the next condition holds:

$$1 - e^{\frac{-\alpha}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle_{\overline{P}}} \sum_{d \geq d_m} d^\alpha \overline{P}(d) e^{-\frac{d\alpha}{d_m}}.$$

□

It is straightforward to show that Proposition 3 holds for all values of $\alpha > 0$, but does not apply to the case $\alpha < 0$, as d^α would be an increasing function of d and thus $\langle d^\alpha \rangle_{\overline{P}} \leq \langle d^\alpha \rangle_P$.

Finally we compare two out-degree distributions where one is a Mean Preserving Spread of the other one. In particular these two distribution have the same average out-degree, but different variance. Which case would lead to a larger contribution in equilibrium? The next result partially addresses this question.

Proposition 5 *Let $\alpha \in [0, 1]$ and let d_m denote the minimum degree in the network. If $\overline{P}(d)$ is a Mean Preserving Spread of $P(d)$, and*

$$1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle} \sum_{d \geq d_m} d^\alpha P(d) e^{-d\frac{\alpha+\sqrt{\alpha}}{d_m}}, \tag{8}$$

then $\theta^*(P, \alpha) \leq \theta^*(\overline{P}, \alpha)$.

Note that condition (8) holds as long as d_m is sufficiently high. Proposition 4 indicates that the number of links reaching the public good (i.e., θ^*) increases with the heterogeneity of the network. This result also implies that $\rho_d^*(\overline{P}, \alpha) \leq \rho_d^*(P, \alpha)$ for all d. Again, comparative statics with respect to ρ might depend on further properties of the out-degree distributions.

Proof We aim to show that $H_{P,\alpha}(\theta) \leq H_{\overline{P},\alpha}(\theta)$ for all $\theta \in [\theta^*(P, \alpha), 1]$ as this would imply that $\theta^*(P, \alpha) \leq \theta^*(\overline{P}, \alpha)$. Consider again

$$g_\theta(d) = d^\alpha (1-\theta)^d$$

and let us show next that if $\theta \in [1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}}, 1]$ then $g(d)$ is convex for all $d \geq d_m$. We have that

$$g_\theta''(d) = (1-\theta)^d d^{\alpha-2} f_\theta(d),$$

where

$$f_\theta(d) = \left[(\ln(1-\theta))^2 d^2 + 2\alpha \ln(1-\theta) d + \alpha(\alpha - 1) \right].$$

Notice that $f_\theta(d)$ is a parabolic function. It is straightforward to see that $g_\theta''(d) \geq 0$ if and only if $f_\theta(d) \geq 0$. Moreover, the positive solution of $f_\theta(d) = 0$ is $\widehat{d} = \frac{\alpha+\sqrt{\alpha}}{-\ln(1-\theta)}$. Therefore, if $d_m = \widehat{d}$ then $g_\theta''(d)$ is positive for all $d \geq d_m$. Let θ_m be such that

$$d_m = \frac{\alpha + \sqrt{\alpha}}{-\ln(1-\theta_m)}.$$

That is, $\theta_m = 1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}}$. Since $\frac{\alpha+\sqrt{\alpha}}{-\ln(1-\theta)}$ is a decreasing function of θ then $g(d)$ is convex for all $d \geq d_m$, provided that $\theta \geq \theta_m$. Therefore,

$$\sum_{d \geq d_m} P(d) g(d) \leq \sum_{d \geq d_m} \overline{P}(d) g(d),$$

for all $\theta \in [\theta_m, 1]$. In addition, we know that $\langle d^\alpha \rangle_P \geq \langle d^\alpha \rangle_{\overline{P}}$, as d^α is a concave function of d. Thus,

$$H_{P,\alpha}(\theta) \leq H_{\overline{P},\alpha}(\theta) \quad \text{for all } \theta \in [\theta_m, 1].$$

To complete the proof we must show that $\theta_m \leq \theta^*(P, \alpha)$, which is the case if $\theta_m \leq H_{P,\alpha}(\theta_m)$ or, analogously, if the next condition holds:

$$1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle} \sum_{d \geq d_m} d^\alpha P(d) e^{-d\frac{\alpha+\sqrt{\alpha}}{d_m}}.$$

□

It is straightforward to show that Proposition 4 does not apply to the cases $\alpha < 0$ nor $1 < \alpha$, since d^α would be a convex function of d and thus $\langle d^\alpha \rangle_P \leq \langle d^\alpha \rangle_{\overline{P}}$.

5 Efficiency

In this section we focus on efficiency. We study the simplest possible version of utility aggregation which is utilitarianism. That is, we say that a state is efficient if it maximizes the sum of the utilities of all the agents in the population. A state in this context is characterized by the vector $\{\rho_d\}_{d\geq 1}$. In particular, let the expected welfare be defined (when normalized by the population size n) as:

$$W(\{\rho_d\}_{d\geq 1}) = A(1-x) - C\sum_d P(d)\rho_d$$

where

$$x = \sum P(d)(1-\rho_d)(1-\theta)^d$$

is the fraction of agents that do not enjoy the public good. Notice that, this is analogous to the computation of the expected utility of an agent chosen uniformly at random from the population given $\{\rho_d\}_{d\geq 1}$.

To simplify matters we focus first on the homogeneous case where all agents have the same out-degree d. By definition α plays no role in such an homogenous framework. In particular, in this case the fraction of links pointing to a public good coincides with the fraction of agents contributing to the public good, i.e., $\theta = \rho$. We obtain the following result.

Proposition 6 *If all agents have a degree equal to d then the fraction of agents that contribute to the public good in the efficient state is*

$$\rho^e = 1 - \left[\frac{1}{\frac{A}{C}(d+1)}\right]^{1/d}.$$

Moreover, in equilibrium there is underprovision of the public good, that is, $\rho^* < \rho^e$.

The proof comes next.

Proof We must solve the following maximization problem

$$\max_{0\leq\rho\leq 1} W(\rho) = A(1-(1-\rho)^{\overline{d}+1}) - C\rho.$$

The second order condition is satisfied and the first order condition ($W'(\rho) = 0$) provides the efficient outcome $\rho^e = 1 - \left[\frac{1}{\frac{A}{C}(d+1)}\right]^{1/d}$. Moreover, the equilibrium value of ρ should satisfy the fixed point equation $\rho^* = H(\rho^*) = (1-\rho^*)^d$. It is straightforward to see that $H(\rho^e) < \rho^e$ which implies, as H is decreasing, that $\rho^* < \rho^e$. □

The previous result illustrates that the efficient and equilibrium outcomes do not typically coincide. It also shows that the efficient value of ρ^e increases with respect to the revenue/cost ratio $\frac{A}{C}$ of the public good. Thus, as $\frac{A}{C}$ increases the tension between efficiency and equilibrium is augmented.[8] The computation of the efficient state in a more general setting can be quite cumbersome. We present next the case of a population with two types of agents; agents with high out-degree $\overline{d}$ and agents with low out-degree $\underline{d}$ to show that not only the efficient and equilibrium outcomes do not coincide, but also that they exhibit distinctive properties. For simplicity let us assume that an agent can have high or low out-degree with equal probability. That is,

$$P(\overline{d}) = 1/2 \quad \text{and} \quad P(\underline{d}) = 1/2.$$

We show the following.

Proposition 7 *Consider a population where half of the agents have a high degree $\overline{d}$, and the other half a low out-degree $\underline{d}$ and let $\alpha > 0$ then the efficient state is such that $\rho^e_{\underline{d}} < \rho^e_{\overline{d}}$.*

This result shows that the efficient and equilibrium states are different in a fundamental property. The probability that an agent invests in the public good decreases with the out-degree in the equilibrium state, whereas it increases in the efficient state (recall from Eq. 4 that $\rho^*_{\overline{d}} < \rho^*_{\underline{d}}$).

Proof In this case the expected welfare is equal to

$$W(\rho_{\underline{d}}, \rho_{\overline{d}}) = A\left(1 - \frac{1}{2}((1-\rho_{\underline{d}})(1-\theta)^{\underline{d}} + (1-\rho_{\overline{d}})(1-\theta)^{\overline{d}})\right) - C\frac{1}{2}(\rho_{\underline{d}} + \rho_{\overline{d}})$$

where

$$\theta = \frac{1}{\frac{1}{2}(\underline{d}^\alpha + \overline{d}^\alpha)} \frac{1}{2}(\underline{d}^\alpha \rho_{\underline{d}} + \overline{d}^\alpha \rho_{\overline{d}})$$

It is straightforward to show that $W(\rho_{\underline{d}}, \rho_{\overline{d}})$ is a continuous, differentiable and concave function of $\rho_{\underline{d}}$ and $\rho_{\overline{d}}$. The efficient state should satisfy the first order conditions and thus,

$$\frac{\partial W}{\partial \rho_{\underline{d}}} = \frac{\partial W}{\partial \rho_{\overline{d}}} = 0.$$

In particular,

$$\frac{\partial W}{\partial \rho_{\underline{d}}} = \frac{\partial W}{\partial \rho_{\overline{d}}}$$

8 In particular, if $d = 1$ then $\rho^* = 1/2$ and $\rho^e = 1 - \frac{C}{2A}$.

if and only if

$$(1-\theta)^{\underline{d}} - (1-\rho_{\underline{d}})B - (1-\rho_{\underline{d}})D = (1-\theta)^{\overline{d}} - (1-\rho_{\overline{d}})B - (1-\rho_{\overline{d}})D$$

or analogously,

$$(1-\theta)^{\underline{d}} - (1-\theta)^{\overline{d}} = (B+D)(\rho_{\overline{d}} - \rho_{\underline{d}})$$

where

$$B = \underline{d}(1-\theta)^{\underline{d}-1}\frac{\underline{d}^{\alpha}\frac{1}{2}}{\frac{1}{2}(\underline{d}^{\alpha}+\overline{d}^{\alpha})} \quad \text{and} \quad D = \overline{d}(1-\theta)^{\overline{d}-1}\frac{\overline{d}^{\alpha}\frac{1}{2}}{\frac{1}{2}(\underline{d}^{\alpha}+\overline{d}^{\alpha})}$$

Since B and D are positive values then it must be the case that $\rho_{\overline{d}} - \rho_{\underline{d}} > 0$. □

6 Discussion

In this paper we propose a stylized model of public good provision in a random network context. The influence network is characterized by its degree distribution and the correlation between agents' out-degree (observation level) and in-degree (visibility level). In particular, because nowadays personal interaction and influence is being substituted by online social networks it seems reasonable to assume that such correlations are not trivial. We find that, in equilibrium, an increase in the in/out-degree correlation increases the number of public good providers (i.e., ρ), but, on the contrary, it decreases the number of links that reach a public good provider (i.e., θ). Moreover, the number of free riders decreases with in/out-degree correlations.

We have also analyzed the effect that a variation of the out-degree distribution has on the equilibrium outcomes. Our results in this respect show that, if degrees are sufficiently large, an increase in the average level of information (i.e., an increase in the average out-degree) decreases the fraction of links reaching the public good, whereas an increase in the dispersion of information (i.e., an increase in the variance of P) increases such fraction.

Finally we show that there is misalignment between the efficient outcome and the equilibrium outcome which becomes more important as the revenue/cost ratio of the public good (A/C) increases (at least in the homogeneous case). We illustrate an additional tension between efficient and equilibrium states; the probability of contributing to the public good decreases with respect to out-degree in equilibrium, whereas it increases in the efficient state.

This paper contributes to the growing literature on public goods in networks by bridging the work developed in statistical physics (where random networks are commonly used) with the literature in economics, for which the problem of public good provision is a major topic of study. The assumption that the network is randomly generated every period is quite strong and thus, one possible direction for further study would be to enrich our model by allowing for clustering and community structures. This extension has already been addressed for contagion models with heterogenous agents and homophily providing fruitful results (e.g., Jackson and López-Pintado 2013).

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