SERIEs (2011) 2:401–421
DOI 10.1007/s13209-011-0037-8

ORIGINAL ARTICLE

# Generalized marginal rate of substitution in multiconstraint consumer's problems and their reciprocal expenditure problems

**Manuel Besada · Javier García · Miguel Mirás · Carmen Vázquez**

Received: 28 April 2010 / Accepted: 23 December 2010 / Published online: 2 March 2011


**Abstract** The aim of this paper is to explore several features concerning the generalized marginal rate of substitution (GMRS) when the consumers utility maximization problem with several constraints is formulated as a quasi-concave programming problem. We show that a point satisfying the first order sufficient conditions for the consumer's problem minimizes the associated quasi-convex reciprocal cost minimization problems. We define the GMRS between endowments and show how it can be computed using the reciprocal expenditure multipliers. Additionally, GMRS is proved to be a rate of change between different proportion bundles of initial endowments. Finally, conditions are provided to guarantee a decreasing GMRS along a curve of initial endowments while keeping the consumer's utility level constant.

**Keywords** Quasi-concave programming · Indirect utility function · Marginal rates of substitution · Multiple constraint optimization problems

**JEL Classification** C61 · D11

## 1 Introduction

In modern theory, a utility function is a convenient mathematical concept to convey exactly the same information about consumer's preferences as it does about the relation itself. As the ability to rank consumption bundles is all that we require of consumer's preference ordering, a utility function representing those preferences is a purely ordinal tool and should, therefore, simply reflect this same ranking on a numerical scale. So, within this framework, and because that property is not preserved by positive

M. Besada (✉) · J. García · M. Mirás · C. Vázquez
Departamento de Matemáticas, Universidade de Vigo, Vigo, Spain
e-mail: mbesada@uvigo.es

monotonic transformations of utility, the idea of diminishing marginal utility of goods is of no use, thus economists have had to come up with the concept of diminishing marginal rates of substitution between goods. The equivalence between diminishing marginal rates of substitution and quasi-concavity of the utility function, rigorously proved by Arrow and Enthoven (1961), transforms quasi-concavity into a fundamental property of all utility functions.

The problem of the consumer maximizing utility subject to several constraints has received some attention lately, both from the application and theory points of view. In the classical single constraint case, the familiar atomistic consumer is endowed with a fixed monetary income $R > 0$ and faces the budget constraint $px \leq R$, where $p \in \mathbb{R}^n_+$ is the vector of market prices of the goods. If one assumes non-satiation of the preferences and quasi-concavity of the utility function $U$ then the usual first order conditions are necessary and sufficient for a constraint maximum.

Moreover, if the Lagrange multiplier associated to the budget constraint at the optimum $x_0$ is strictly positive, then $x_0$ also minimizes the cost of attaining utility $U(x_0)$ and the marginal cost of utility is the reciprocal of the marginal utility of income. See Silberberg (1990); Jehle (1991), among many others, for a complete presentation of these facts. As pointed out by Arrow and Enthoven (1961), the quasi-concavity of $U$ and the linearity of the budget constraint make quasi-concave programming the natural mathematical framework for this analysis.

It is also quite common to add other constraints that have to be explicitly modeled in many interesting and important problems. The first of such models dates back to the studies of demand under conditions of rationing during wartime, particularly after World War II, or other emergencies by Samuelson (1974); Tobin and Houthakker (1951) and the survey article by Tobin (1952). In Sect. 8 we discuss one example of coupon rationing constraints to illustrate our results. To study the implications of the scarcity of time in consumer's behavior, many authors (see Becker 1965; Baumol 1973; Atkinson and Stern 1979; Steedman 2001, to name just a few) have worked with models in which two budget constraints are imposed: one incorporating monetary prices; the other incorporating time prices. Suppose that the consumption of one unit of good $k$ costs $p_k$ units of money and $t_k$ hours. Since activities cannot be taken simultaneously, the consumer's problem is then:

$$\begin{aligned} &\text{Maximize} \quad U(x) \\ &\text{subject to :} \begin{cases} px \leq R \\ tx \leq T \end{cases} \end{aligned}$$

where $T$ is the total endowment of available hours. Diamond and Yaari (1972) model portfolio choice under uncertainty as a problem of maximizing a two-period utility function over a period 1 level of consumption with certainty and a state contingent consumption in period 2. Alternatively, one may suppose that there are several states of nature. Within each state a consumer may allocate income among the currently available goods. But when allocation between states is concerned, if assets cannot be traded freely between states, i.e., there is no opportunity to give up one unit of a commodity in state 1 in exchange for one unit in state 2 hence the consumer will have as many separate budget constraints as there are states. Shefrin and Heineke (1979);

Cornes and Milne (1989) also analyze models with incomplete securities markets in a multiple constraint setting. The list of applications of models with multiple constraints would include the demand of foods subject to the standard budget constraint as well as to dietary requirements such as a minimum or maximum intake of calories, Gilley and Karels (1991); the impure public goods model, Cornes and Sandler (1984); the analysis of own substitution effects, inferior goods or Giffen behavior, (Lancaster 1968; Lipsey and Rosenbluth 1971; Johnson and Larson 1994). Chapter 7 of Cornes (1992) provides a number of examples and references for applications of the multiple constraint setting. In Weber (2005) there is an updated list.

The interest of these applications has triggered several papers dealing with some theoretical aspects of these problems. Initially, the difficulty in obtaining comparative static properties from the direct utility function favored the systematic application of dual techniques, Neary and Roberts (1980), or the search for other functional forms, Deaton and Muellbauer (1980). More recently, Partovi and Caputo (2006) present a new comparative statics formalism for any differentiable, constraint optimization problem. Weber (1998, 2001); Caputo (2001) and Besada and Mirás (2002) study the relationships between the utility maximization problem and the, so called, reciprocal expenditure or cost minimization problems by analyzing the equality constraint optimization problem formed by those constraints binding at the optimum. They examine whether an optimal solution of the primal problem is also a solution for all the reciprocal problems and study the reciprocal relations among the corresponding multipliers and their interpretation, as in the single constraint case.

If the first constraint, say the budget constraint, is binding at the optimum, the corresponding Lagrange multiplier is the marginal utility of income, i.e., the derivative of the indirect utility function with respect to the first endowment. Again, as utility is purely ordinal and the sign of the derivatives of the primal multipliers with respect to the endowments varies with the application of monotone increasing transformations to the direct utility function, the concept of diminishing marginal utility of income is meaningless. As in the case of the direct utility, it can be replaced by the concept of diminishing MRS of income and connected to quasi-concavity of the indirect utility function. In our quasi-concave programming framework we can prove that if all the constraint functions are convex, a common situation in the economic literature, then the indirect utility function is also quasi-concave. We then establish that the quasi-concavity of the indirect utility function implies, if marginal utility of income is positive, diminishing MRS of income. In fact, it also implies diminishing GMRS of income or any other initial endowment. We additionally provide two examples to illustrate that if either quasi-concavity of $U$ or convexity of $g_i$ are not guaranteed then the indirect utility function need not be quasi-concave. These results question the claim in Weber (2005) that the indirect utility function is “always” quasi-concave.

In any optimization problem the constraints restrict the feasible set. In general if the number of constraints increases, the “size” of the feasible set decreases. Nevertheless, a lower optimum value does not necessarily imply lower welfare, as could happen in a problem with dietary requirements to improve consumer’s health.

Therefore, our aim in this paper is to further explore the multiple constraint optimization results of Besada and Mirás (2002); Caputo (2001); Weber (2001, 1998) and Besada and Vázquez (1999) in the context of quasi-concave programming problems.

Our main contribution is to adapt the GMRS, defined by Besada and Vázquez (1999), to our framework and to interpret it as a rate of change between different proportion bundles of initial endowments. We also show how the GMRS can be computed using the reciprocal expenditure multipliers and provide conditions to guarantee a decreasing GMRS along a curve of initial endowments while keeping the consumer's utility level constant.

The paper is organized as follows. Section 2 introduces the general notations and provides an overview on quasi-concave programming problems. The utility maximization primal problem and the associated expenditure minimization reciprocal problems are presented in Sect. 3 where, in addition, we establish the relations among the optimal solutions and the Lagrange multipliers of the primal and the reciprocal problems. The quasi-concavity of the indirect utility function is studied in Sect. 4. The next section addresses the definition of the concept of GMRS between endowment proportions of the primal problem, whereas the interpretation of the reciprocal multipliers as MRS occupies Sect. 6. In Sect. 7, we show that quasi-concavity of the indirect utility function implies monotonicity of the GMRS. Finally, in Sect. 8 the paper concludes with an illustrative example to summarize the results.

## 2 Quasi-concave programming background

We start by introducing some notations and by listing the definitions and results on quasi-concave functions and quasi-concave programming problems that will be used in our work. Let us make the following notational convention: if $x, y \in \mathbb{R}^p$, $x \geq y$ means $x_i \geq y_i$ for all $i = 1, \dots, p$, where $x_i$ and $y_i$, respectively, represent the $i^{\text{th}}$ element of $x$ and $y$. We will denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$.

### 2.1 Quasi-concave functions

Let $f\colon A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued function defined on a convex set $A$ in $\mathbb{R}^n$. Function $f$ is said to be quasi-concave on $A$ if for all $x, x_0 \in A$ and $0 \leq t \leq 1$,

$$f(x) \geq f(x_0) \text{ implies } f(tx + (1-t)x_0) \geq f(x_0). \tag{1}$$

A function is quasi-concave if for each real number $\alpha$ the set $A^\alpha = \{x \in A : f(x) \geq \alpha\}$ is convex. The level set $A^\alpha$ is sometimes referred to as the upper-level set. Function $f$ is said to be quasi-convex on $A$ if $-f$ is quasi-concave on $A$. It is clear from the definition that all concave functions are quasi-concave, but the converse is not true. Thus quasi-concavity is a generalization of the notion of concavity.

### 2.2 Local and global maxima

Function $f$ is said to be explicitly quasi-concave on $A$ if it is quasi-concave[1] and if for all $x, x_0 \in A$, $x \neq x_0$, and $0 < t \leq 1$,

[1] If $f$ is upper-semicontinuous, then property (2) implies the quasi-concavity of $f$.

$$f(x) > f(x_0) \text{ implies } f(tx + (1-t)x_0) > f(x_0). \tag{2}$$

Function $f$ is said to be strictly quasi-concave on $A$ if for all $x, x_0 \in A, x \neq x_0$, and $0 < t < 1$,

$$f(x) \geq f(x_0) \text{ implies } f(tx + (1-t)x_0) > f(x_0). \tag{3}$$

Every explicitly quasi-concave function is quasi-concave and every strictly quasi-concave function is explicitly quasi-concave. The converse of these statements does not necessarily hold.

In general, a local maximum of a quasi-concave function is not necessarily a global maximum. Nevertheless, if $f$ is explicitly quasi-concave, then every local maximum of $f$ in $A$ is also a global maximum. If $f$ is strictly quasi-concave, then every local maximum of $f$ in $A$ is also a unique global maximum.

### 2.3 Quasi-concave programming

Consider a continuously differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ which is quasi-concave and a vector-function $g: \mathbb{R}^n \to \mathbb{R}^m$, $g = (g_1, \ldots, g_m)$ such that $g_i$ is quasi-convex and continuously differentiable for all $i = 1, \ldots, m$. The problem of choosing $x \in \mathbb{R}^n$ so as to

$$\begin{aligned} &\text{Maximize} \quad f(x) \\ &\text{subject to}: \begin{cases} g(x) \leq 0 \\ x \geq 0 \end{cases} \end{aligned}$$

is known as a quasi-concave programming problem and was thoroughly discussed by Arrow and Enthoven (1961). Denote by $S$ the constraint set of this problem and given $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ define the lagrangian $L(x, \lambda) = f(x) - \lambda_1 g_1(x) - \cdots - \lambda_m g_m(x)$. Denote by $\nabla L(x, \lambda)$ the gradient vector of $L$ with respect to $x$ evaluated at $(x, \lambda)$.

**Definition 1** A point $x^* \in S$ satisfies the first order conditions (FOC) if there exists $\lambda^* \in \mathbb{R}^m$ such that

1. $\lambda^* \geq 0$
2. $\nabla L(x^*, \lambda^*) \leq 0$
3. $\nabla L(x^*, \lambda^*) \cdot x^* = 0$
4. $\lambda^* \cdot g(x^*) = 0$

The FOC alone does not necessarily imply that $x^*$ maximizes $f$ subject to $x \in S$. The FOC are sufficient conditions for the existence of a maximum if the Kuhn-Tucker constraint qualification condition is satisfied. Different constraint qualifications have been proposed in the literature since the seminal work of Arrow and Enthoven (1961) which are basically improvements, refinements and generalizations of this work. Yet, these conditions are sometimes used carelessly in the economic literature (Giorgi 1995). We will state the qualification condition obtained in Takayama (1994).

**Proposition 1** *If $f$ is quasi-concave, $S$ is convex and condition*

$$\nabla f(x^*) \neq 0, \quad \textit{for } x^* \in S$$

*holds, then the (FOC) at $x^*$ imply that $x^*$ maximizes $f$ subject to $x \in S$.*

If the functions $f$ and $g_i$ are quasi-convex, for all $i$, the problem of choosing $x \in \mathbb{R}^n$ so as to minimize $f(x)$ subject to $g(x) \leq 0$ and $x \geq 0$ is known as a quasi-convex programming problem. In this problem, a point $x^* \in S$ satisfies the first order conditions (FOC) if there exists $\lambda^* \in \mathbb{R}^m$ such that $\lambda^* \leq 0$, $\nabla L(x^*, \lambda^*) \geq 0$, $\nabla L(x^*, \lambda^*) \cdot x^* = 0$ and $\lambda^* \cdot g(x^*) = 0$. In this case, Proposition 1 says that (FOC) at $x^*$ imply that $x^*$ minimizes $f$ subject to $x \in S$.

## 3 The consumer's problem and expenditure reciprocal problems

Our starting point is the familiar atomistic consumer whose preferences over the consumption set $\mathbb{R}^n_+$ can be represented by a $\mathcal{C}^1$-utility function $U: \mathbb{R}^n \to \mathbb{R}$. Assume also that the indifference curves define convex sets, or equivalently diminishing marginal rates of substitution between goods, that is, the utility function $U$ is quasi-concave on $\mathbb{R}^n_+$.

Let $g = (g_1, \ldots, g_m)$ be a vector-valued constraint function such that $g_i$ is a continuously differentiable, quasi-convex function for all $i = 1, \ldots, m$ and $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ a vector of initial endowments. The general multiple constraint utility maximization problem can be stated as

$$\begin{aligned} &\text{Maximize} \quad U(x) \\ &\text{subject to}: \begin{cases} g(x) \leq c \\ x \geq 0 \end{cases} \end{aligned} \tag{$P(c)$}$$

We will refer to problem $P(c)$ as the primal problem and the associated lagrangian function will be denoted by

$$L(x, \lambda) = U(x) - \lambda_1(g_1(x) - c_1) - \cdots - \lambda_m(g_m(x) - c_m).$$

Denote by $S_c$ the constraint set of the problem $P(c)$ and the optimal solution to problem $P(c)$ will be denoted by $x(c)$, with Lagrange multipliers given by $\lambda(c)$. For each of these solutions there is a corresponding optimal value $V(c) = U(x(c))$, the highest level of utility that can be achieved with endowment $c \in \mathbb{R}^m$. $V$ is called the indirect utility function.

In the single budget constraint case, $g(x) = px = c > 0$, one can define a companion measure to the indirect utility function to complete the picture of the consumer's problem: the expenditure function. Basically, one asks what minimum level of monetary expenditure the consumer must make to achieve a given level of utility $u$. Mathematically, the expenditure function can be expressed as $e(u) = \min\{px : U(x) \geq u, x \geq 0\}$. Naturally, there is a close and well known relationship between the indirect utility function and the expenditure function (Jehle 1991). Our aim is to generalize this concept to the multiple constraint case and study which relations with the indirect utility function still stand.

Let us introduce the notation $b_{-k}$ to represent the vector obtained from $b = (b_1, \ldots, b_m)$ deleting the $k$-th coordinate. Associated with the primal problem $P(c)$, there are $m$ different expenditure minimization problems

$$\begin{aligned} &\text{Minimize} \quad g_k(x) \\ &\text{subject to}: \begin{cases} U(x) \geq u \\ g_{-k}(x) \leq c_{-k} \\ x \geq 0 \end{cases} \end{aligned} \qquad (R(u, c_{-k}))$$

where $k \in \{1, \ldots, m\}$. Adopting the terminology proposed by Caputo (2001), we refer to this problem as the $k$-th reciprocal expenditure problem. Note that any solution for the problem $R(u, c_{-k})$ will depend on the parameters $u$ and $c_{-k}$. Let us write the Lagrangian function corresponding to $R(u, c_{-k})$ as

$$L^k(x, \lambda^k) = g_k(x) - \mu^k(-U(x) + u) - \sum_{j \neq k} \lambda_j^k (g_j(x) - c_j),$$

where both the objective function $g_k(x)$ and constraints functions, $-U(x)$ and $g_j(x)$ $j \neq k$, are quasi-convex functions, so that the problem $R(u, c_{-k})$ is a quasi-convex programming problem, and where $\lambda^k = (\mu^k, \lambda_j^k) \in \mathbb{R}^m$, $j \neq k$. The optimal solution of the $k$-th reciprocal problem $R(u, c_{-k})$ will be denoted by $x^k(u, c_{-k})$. We define the $k$-th expenditure function $e^k$ as the minimum value function corresponding to problem $R(u, c_{-k})$, that is, $e^k(u, c_{-k}) = g_k(x^k(u, c_{-k}))$.

Next, we will pursue the relationship between utility maximization and expenditure minimization by choosing the utility level in the reciprocal problems equal to $u = V(c)$, the optimal level of utility the consumer achieves at $c$. Without loss of generality, we will state our results for the first reciprocal problem. Again, to simplify notation, we will write $R(c)$ instead of $R(V(c), c_{-1})$ and $S_c^1 = \{x \in \mathbb{R}^n : G(x) \leq B\}$, where $G(x) = (-U(x), g_2(x), \ldots, g_m(x))$ and $B = (-V(c), c_2, \ldots, c_m)$, to denote the constraint set of $R(c)$. Analogously, the multipliers for the reciprocal problem $R(c)$ will be denoted by $\lambda^1(c) = (\mu^1(c), \lambda_2^1(c), \ldots, \lambda_m^1(c))$. Note that $\lambda_1^1$ does not exist.

Observe that all the reciprocal problems are quasi-convex programming problems. From a practical perspective, these reciprocal problems would be of some interest if the solution to the utility maximization problem were also a solution to all the reciprocal expenditure minimization problems. Caputo (2001) and Besada and Mirás (2002) provided results in this line that we extend to the general framework of problems $P(c)$ and $R(u, c_{-k})$.

Since the primal problem is a quasi-concave programming problem, the constraint qualification condition is:

$$x \text{ satisfies the (FOC) for the problem } P(c) \text{ and } \nabla U(x) \neq 0. \qquad (4)$$

The 1-reciprocal problem is a quasi-convex programming problem, the constraint qualification condition is:

$$x \text{ satisfies the (FOC) for the problem } R(c) \text{ and } \nabla g_1(x) \neq 0. \tag{5}$$

We can now formulate and prove the main result of this section.

**Theorem 1** *If $x(c) \in S_c$ satisfies (4) with $\lambda_1(c) > 0$ then $x(c)$ also satisfies the (FOC) for the 1-reciprocal problem $R(u, c_{-1})$ where the utility level in the reciprocal problem is taken to be $u = V(c)$, the optimal level of utility the consumer achieves at $c$.*

*Proof* Since $x(c) \in S_c$ satisfies the (FOC) for problem $P(c)$, there exists $\lambda(c) \in \mathbb{R}^m$ such that

1. $\lambda(c) \geq 0$
2. $\nabla L(x(c), \lambda(c)) \leq 0$
3. $\nabla L(x(c), \lambda(c)) \cdot x(c) = 0$
4. $\lambda(c) \cdot (g(x(c)) - c) = 0$

First, because condition (4) holds, according to Proposition 1, $x(c)$ maximizes $U$ subject to $x \in S_c$. Consequently, $V(c) = U(x(c))$ and $x(c) \in S_c^1$.

1'. By hypothesis, $\lambda_1(c) > 0$, so we can divide $\nabla L(x(c), \lambda(c)) \leq 0$ by $\lambda_1(c)$ to obtain

$$\frac{1}{\lambda_1(c)} \nabla U(x(c)) - \nabla g_1(x(c)) - \sum_{j=2}^{m} \frac{\lambda_j(c)}{\lambda_1(c)} \nabla g_j(x(c)) \leq 0$$

or, equivalently,

$$\nabla g_1(x(c)) + \frac{1}{\lambda_1(c)} \nabla(-U(x(c))) + \sum_{j=2}^{m} \frac{\lambda_j(c)}{\lambda_1(c)} \nabla g_j(x(c)) \geq 0. \tag{6}$$

If we define

$$\mu^1(c) = -\frac{1}{\lambda_1(c)} \tag{7}$$

$$\lambda_j^1(c) = -\frac{\lambda_j(c)}{\lambda_1(c)}, \ \ j = 2, \ldots, m \tag{8}$$

it is clear that $\lambda^1(c) \leq 0$.

2'. The expression (6) reads $\nabla L^1(x(c), \lambda^1(c)) \geq 0$.

3'. Using the same arguments as we did before, it is easy to see that

$$\frac{\partial L}{\partial x_i}(x(c), \lambda(c)) = 0 \text{ if and only if } \frac{\partial L^1}{\partial x_i}(x(c), \lambda^1(c)) = 0.$$

Thus, $\nabla L^1(x(c), \lambda^1(c)) \cdot x(c) = 0$.

4'. Obviously, $\lambda_j^1(c) = 0$ if and only if $\lambda_j(c) = 0$. Moreover, $\mu^1(c) < 0$ and $U(x(c)) = V(c)$. Therefore, $\lambda^1(c) \cdot (G(x(c)) - B) = 0$.

So, we have just proved that $x(c)$ satisfies the (FOC) for the quasi-convex programming problem $R(c)$. □

If both conditions (4) and (5) hold,[2] a point $x(c)$ satisfying the (FOC) will be a maximum of $U$ subject to $x \in S_c$ and also a minimum of $g_1$ subject to $x \in S_c^1$. If, in addition, $U$ is strictly quasi-concave, $x(c)$ is the unique global maximum of $U$ subject to $x \in S_c$. Moreover, since $\lambda_1(c) > 0$, necessarily $g_1(x(c)) = c_1$, which in turn implies that

$$e^1(c) = e^1(V(c), c_{-1}) = c_1. \tag{9}$$

In words, the expenditure in resource 1 necessary to achieve the maximum utility $V(c)$ must be an amount equal to the initial endowment $c_1$.

Let $U$ and $g_i, i = 1, \ldots, m$ be twice continuously differentiable and $(x^*, c^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $x^*$ is a solution for the problem $P(c^*)$. Fixed $k \in \{1, \ldots, m\}$, the expenditure function $e^k$ viewed only as a function of $u$ is a real function of one real variable defined in a neighborhood of $u^* = V(c^*)$, $e^k(\_, c_{-k})$. Moreover, by the envelope theorem,

$$\frac{\partial e^k}{\partial u}(V(c), c_{-k}) = \frac{\partial L^k}{\partial u}(x(c), c_{-k}) = -\mu^k > 0,$$

so that $e^k$ is strictly increasing in $u$. Then it has an inverse $(e^k)^{-1}$. Now, from (9) we have $e^k(V(c), c_{-k}) = c_k$. If we apply the inverse function $(e^k)^{-1}$ to both sides of this identity we conclude the following result.

**Proposition 2** *Fixed $k \in \{1, \ldots, m\}$ and take $c \in B(c^*, r)$. Then $V(c) = (e^k)^{-1}(c_k)$.*

This result emphasizes the close connection between utility maximization and expenditure minimization even in the multiple constraint case. Intuitively, the indirect utility function is simply the inverse of the expenditure function.

Equality (7), along with equality (8) first stated by Weber (1998) in a simplified setting, extends to the multiple constraint case the well-known result concerning the reciprocal relationship between the Lagrange multipliers of the classical single-constraint utility maximization and expenditure minimization problems: the marginal utility of income is the reciprocal of the marginal cost of utility. The exact meaning of (8) will be the object of the next sections.

[2] If one assumes non-satiation of the preferences, that is, $\frac{\partial U}{\partial x_j} > 0$ for some $j$, then (4) will be satisfied. If, as usual, $g_1(x) = px$ is the budget constraint, then $\nabla g_1(x) = p > 0$ will be the price vector of the $n$ goods and (5) will also be satisfied.

## 4 Quasi-concavity of the indirect utility function

We assume that $x(c)$ is the optimal solution of problem $P(c)$ for all $c$ in a certain convex region $D \subset \mathbb{R}^m$ and that $x(c)$ and $\lambda(c)$, which satisfied the (FOC), are continuously differentiable for all $c \in D$. Then, $V(c) = U(x(c))$, the indirect utility function is continuously differentiable on $D$ as well.

It is known, (Sydsaeter 1981; Silberberg 1990) for example, that if $U$ is concave and $g_1, \ldots, g_m$ are all convex, then $V$ is concave on $D$. But concavity is not preserved by positive monotonic transformations of the utility function so, in our setting this result is of little help. Weber (2005) claims that $V$ is always quasi-concave, even if $U$ is not quasi-concave and imposing no restriction to the constraint function $g$. The following example shows that in general and even if the constraints are neat linear functions, if $U$ is not quasi-concave the indirect utility function $V$ itself does not necessarily have to be quasi-concave.

*Example 1* Let $U(x, y, z) = xy + z$ be the utility function, $c^* = (1, 1)$ the vector of initial endowments and consider the linear constraints $g_1(x, y, z) = x + y + z$, $g_2(x, y, z) = \frac{1}{2}x + y + 4z$. Note that $U$ is not quasi-concave on $\mathbb{R}^3_+$.

The bundle $(x^*, y^*, z^*) = \left(\frac{1}{2}, \frac{5}{12}, \frac{1}{12}\right)$ with multipliers $(\lambda_1^*, \lambda_2^*) = \left(\frac{1}{3}, \frac{1}{6}\right)$ is the unique global maximum of problem $P(c^*)$, so, both constraints are binding at the optimum. After some computations we obtain that

$$\begin{aligned} x(c_1, c_2) &= \frac{1}{14}(8c_1 - 2c_2 + 1) \\ y(c_1, c_2) &= \frac{1}{12}(8c_1 - 2c_2 - 1) \\ z(c_1, c_2) &= \frac{1}{84}(-20c_1 + 26c_2 + 1) \end{aligned}$$

are the components of the demand function near $c^*$ and

$$V(c_1, c_2) = \frac{8}{21}c_1^2 + \frac{1}{42}c_2^2 - \frac{4}{21}c_1c_2 - \frac{5}{21}c_1 + \frac{13}{42}c_2 + \frac{1}{168}$$

is the indirect utility function in an open neighborhood of $(1, 1)$. Function $V$ is not quasi-concave near $(1, 1)$. In fact, $V$ is convex in $\mathbb{R}^2$.

The above example shows that, in spite of Weber (2005) claim, we can not expect $V$ to be quasi-concave in general. Nevertheless, in many practical applications of nonlinear programming theory to economics, either the constraints are all linear or they are all convex. In such a neat situation we can prove the quasi-concavity of the indirect utility function.

**Theorem 2** *If $U$ is quasi-concave and all the constraint functions $g_1, \ldots, g_m$ are either linear or convex, then the indirect utility function $V$ is quasi-concave on $D$.*

*Proof* Let $c, c_0 \in D$ such that $V(c) \geq V(c_0)$. We have to prove that $V(tc+(1-t)c_0) \geq V(c_0)$ for all $t \in [0, 1]$. Remember that $x(c)$ and $x(c_0)$ are the optimal solutions for problems $P(c)$ and $P(c_0)$ respectively, so $g(x(c)) \leq c$ and $g(x(c_0)) \leq c_0$. Denote by $c_t = tc + (1-t)c_0$ and $x_t = tx(c) + (1-t)x(c_0)$. We know that $U(x(c)) \geq U(x(c_0))$. Since $U$ is quasi-concave, we have that

$$U(x_t) \geq U(x(c_0)). \tag{10}$$

If all $g_1, \ldots, g_m$ are convex, we can write, $g(x_t) \leq tg(x(c)) + (1-t)g(x(c_0)) \leq tc + (1-t)c_0 = c_t$, in particular, if all $g_1, \ldots, g_m$ are linear $g(x_t) = c_t$. Thus, $x_t \in S_{c_t}$, that is, $x_t$ is in the constraint set for problem $P(c_t)$ and, consequently, $V(c_t) \geq U(x_t)$. Combining the last inequality with (10) we have

$$V(c_t) \geq U(x_t) \geq U(x(c_0)) = V(c_0).$$

and this concludes the proof. □

We would like to point out that, though in our formulation we have imposed differentiability on $u$ and $g$, Theorem 2 is quite general in the sense that no regularity requirements are needed to prove the quasi-concavity of $V$ beyond the existence of $V$ itself. To end this section, we provide an example that shows that convexity of the constraint functions cannot be dropped in Theorem 2. Here, the utility function $U$ and restriction $g_1$ are both linear but $g_2$ is a quasi-convex function that is not convex.

*Example 2* Let $U(x, y, z) = x + 2y + 3z$, $g_1(x, y, z) = x + y + z$, $g_2(x, y, z) = \frac{1}{xy+1}$ and $c = (c_1, c_2)$. Utility function $U$ and restriction $g_1$ are both linear. Restriction $g_2$ is not convex on $\mathbb{R}^3_+$ because $H_{g_2}(x, y, z)$ is indefinite for all $(x, y, z) \in \mathbb{R}^3_+$ such that $xy \leq \frac{1}{3}$. But since $g_2$ is the reciprocal of a positive quasi-concave function, $g_2$ is quasi-convex on $\mathbb{R}^3_+$.

One can check that the point with coordinates

$$x(c_1, c_2) = \frac{1}{2}\sqrt{2\frac{1-c_2}{c_2}}$$

$$y(c_1, c_2) = \sqrt{2\frac{1-c_2}{c_2}}$$

$$z(c_1, c_2) = c_1 - \frac{3}{2}\sqrt{2\frac{1-c_2}{c_2}}$$

satisfies the (FOC) with multipliers $\lambda_1 = 3$, $\lambda_2 = \sqrt{\frac{2}{c_2^3(1-c_2)}}$ in an open neighborhood of $\left(10, \frac{3}{4}\right)$. Since the constraint qualification condition holds, according to Proposition 1, this bundle is the maximum of problem $P(c)$. Therefore

$$V(c_1, c_2) = 3c_1 - 2\sqrt{2\frac{1-c_2}{c_2}}$$

is the indirect utility function in an open neighborhood of $\left(10, \frac{3}{4}\right)$. Function $V$ is not quasi-concave near $\left(10, \frac{3}{4}\right)$. To see this, just observe that, unless empty, the intersection of the upper level set $\{(c_1, c_2) : V(c_1, c_2) \geq \alpha\}$, $\alpha \in \mathbb{R}$, with an open neighborhood of $\left(10, \frac{3}{4}\right)$ cannot be convex because $x_0 = \frac{3}{4}$ is an inflection point for $c_1 = h(c_2) = \frac{2}{3}\sqrt{2\frac{1-c_2}{c_2}}$.

The previous example shows that Theorem 2 is the best you can get, in the sense that if just one of the constraints is not convex, then $V$ is not necessarily quasi-concave.

## 5 Marginal rates of substitution between initial endowments

There are, of course, certain conditions that must be satisfied for the indirect utility function $V$ to be continuously differentiable. Let us impose some assumptions on problem $P(c)$.

(A1) $U \in \mathcal{C}^2(\mathbb{R}^n)$ and $G\colon \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $G(x, c) = g(x) - c$, belongs to $\mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$.

(A2) There exists $(x^*, c^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $x^*$ satisfies the (FOC) for problem $P(c^*)$.

(A3) Let $I$ be the set of all $i = 1, \ldots, m$ for which $g_i(x^*) = c_i^*$. Then the jacobian matrix $\left(\partial g_i(x^*)/\partial x_j\right)_{i \in I, j=1,\ldots,n}$ has maximal rank.[3]

(A4) The Lagrange multipliers associated to the binding constraints at $x^*$ are strictly positive, that is, $\lambda_i^* > 0$ for all $i \in I$.

(A5) The solution $x^*$ satisfies the second-order sufficient conditions for a strict local maximum.[4] Equivalently,

$$\sum_{i,j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j}(x^*, \lambda^*) p_i p_j < 0 \ \text{ for all } \ p = (p_1, \ldots, p_n) \ \text{ such that}$$

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x^*) p_i = 0.$$ [5]

In particular, the bordered Hessian matrix has nonzero determinant.

Then, by (A1) and (A5) and implicit function theorem, $x(c)$, $\lambda(c)$, and hence $V(c)$, become $\mathcal{C}^1$-functions in a neighborhood of $c^*$ that can be chosen sufficiently small so that the same $g_i$ constraints are active "all the time". Thus, without losing generality, the consumer's non-linear programming problem with inequality constraints could be reduced to a problem with equality constraints, those binding at the optimum. For simplicity, assume that the active constraints are $g_1, \ldots, g_p$, $p < n$. Keeping all these

[3] The constraint $g_i(x) \leq c_i$ is called active or binding at $x^*$ provided $g_i(x^*) = c_i^*$.

[4] If $U$ is assumed to be explicitly quasi-concave or strictly quasi-concave, then $x^*$ will be a global maximum or the unique global maximum respectively.

[5] These conditions may be stated in terms of the signs of the principal minors of the bordered Hessian matrix, Takayama (1994).

considerations in mind, consider the problem of maximizing function $U$ subject to the $p$-vector constraints $g(x) = c$, where $c = (c_1, \ldots, c_p) \in \mathbb{R}^p$ is the vector of parameters:

$$\begin{array}{l}\text{Maximize } U(x)\\ \text{subject to: } g(x) = c\end{array} \qquad (EP(c))$$

Then, for all $c \in B(c^*, r)$, $x(c)$ is a strict local maximum of the equality constraints problem $EP(c)$ with strictly positive Lagrange multipliers given by $\lambda(c)$.

**Proposition 3** *Let $U$ and $G$ satisfy (A1) and (A4) and be the bordered Hessian with nonzero determinant at $(x^*, \lambda^*)$, the optimal solution for the problem $EP(c)$. Then $V$ is strictly increasing in each variable $c_j$; in fact, $\frac{\partial V}{\partial c_j}(c) = \lambda_j(c)$ for all $j = 1, \ldots, p$.*

*Proof* The envelope theorem (Sydsaeter 1981) states that, for all $c \in B(c^*, r)$,

$$\frac{\partial V}{\partial c_j}(c) = \frac{\partial L}{\partial c_j}(x(c), \lambda(c)) = \lambda_j(c), \ \ j = 1, \ldots, p \qquad (11)$$

Now, $V$ is strictly increasing in $c_j$ because of (A4). □

The Lagrange multiplier $\lambda_j(c)$ is the marginal utility of endowment $c_j$, it measures the rate at which utility changes as resource $c_j$ is increased.

If all we required of the preference ordering is that rankings among bundles be meaningful, then any utility function representing that ordering is only capable of carrying ordinal information. No significance can be attached to the actual numbers assigned by a given utility function to particular bundles, only to the relative sizes of those numbers. In this respect, let us examine the effect of a positive monotonic transformation of the utility function to the indirect utility function $V$ and the Lagrange multipliers $\lambda(c)$. Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that $h'(x) > 0$ for all $x \in \mathbb{R}$ and denote $W = h \circ U$ the new utility function representing the same preference ordering as $U$. First of all, one can easily argue that $x(c)$ is an optimal solution to problem $EP(c)$ when taking $W$ instead of $U$ as the objective function. Thus, the relation between the indirect utility functions $V$ and $V_W$ for $U$ and $W$, respectively, is given by $V_W = h \circ V$. By the chain rule,

$$\frac{\partial V_W}{\partial c_j}(c) = h'(V(c)) \frac{\partial V}{\partial c_j}(c), \ \ j = 1, \ldots, p \qquad (12)$$

The above equation illustrates a well-known fact: the Lagrange multipliers are not preserved by positive monotonic transformations of the utility function but the signs of the Lagrange multipliers are invariant. Combining (12) and (11) produces

$$\frac{\frac{\partial V_W}{\partial c_j}(c)}{\frac{\partial V_W}{\partial c_k}(c)} = \frac{h'(V(c))\frac{\partial V}{\partial c_j}(c)}{h'(V(c))\frac{\partial V}{\partial c_k}(c)} = \frac{\frac{\partial V}{\partial c_j}(c)}{\frac{\partial V}{\partial c_k}(c)} = \frac{\lambda_j(c)}{\lambda_k(c)}.$$

So, any given ratio $\rho_j^k(c) = \frac{\lambda_j(c)}{\lambda_k(c)}$, $j \neq k$, is invariant to positive monotonic transformations of the utility function even though the Lagrange multipliers $\lambda_j(c)$ themselves are not.

**Theorem 3** *Let $U$ and $G$ satisfy (A1) and (A4) and be the bordered Hessian with nonzero determinant at $(x^*, \lambda^*)$, the optimal solution for the problem $EP(c)$. Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that $h'(x) > 0$ for all $x \in \mathbb{R}$ and denote $W = h \circ U$. Then, for all $c \in B(c^*, r)$, $\rho_j^k(c)$, $j \neq k$, is invariant for $U$ and $W$.*

At this point let us make a highly informal reasoning. Recall that $\rho_j^k = \frac{\lambda_j}{\lambda_k}$, $\lambda_j = \frac{\partial V}{\partial c_j}$ and $\lambda_k = \frac{\partial V}{\partial c_k}$. Combining these expressions and making a bold simplification leads us to $\rho_j^k = \frac{\frac{\partial V}{\partial c_j}}{\frac{\partial V}{\partial c_k}} = \frac{\partial c_k}{\partial c_j}$. Hence, it seems that the multiplier ratios can be seen as marginal rates of substitution among the initial endowments. Indeed that is the case, already pointed out by Weber (2001), but some formal mathematical work must be carried out to make a precise statement. We do the job adapting to our setting the definition of a generalized marginal rate of substitution among commodities given by Besada and Vázquez (1999). The general idea is simple to grasp. Imagine the consumer loses part of her initial endowments, $c^*$, by certain percentages, say $\omega^2 = (\omega_1^2, \ldots, \omega_p^2)$, and therefore decreases her optimal level of utility. Now, the consumer tries to compensate that loss, and maintain her original level of utility $u^* = V(c^*)$, by increasing the initial resources by other proportions, say $\omega^1 = (\omega_1^1, \ldots, \omega_p^1)$. How many "units" of $\omega^1$ must she buy to do so? That number would be the generalized marginal rate of substitution between endowment proportions $\omega^1$ and $\omega^2$ at $c^*$, $GMRS_{\omega^1,\omega^2}(c^*)$. Saying that the proportions in $\omega^1$ and $\omega^2$ are different is the same as saying that the vectors are linearly independent.

Consider two unitary linearly independent parameter vectors $\omega^1, \omega^2 \in \mathbb{R}_+^p$. We claim that if $|\alpha| < \frac{r}{2}$ and $|\beta| < \frac{r}{2}$ then $c^* + \alpha\omega^1 + \beta\omega^2 \in B(c^*, r)$. Indeed,

$$\|c^* - (c^* + \alpha\omega^1 + \beta\omega^2)\| \leq |\alpha|\|\omega^1\| + |\beta|\|\omega^2\| < r,$$

Then, we can define function $F\colon (-\frac{r}{2}, \frac{r}{2}) \times (-\frac{r}{2}, \frac{r}{2}) \longrightarrow \mathbb{R}$, as $F(\alpha, \beta) = V(c^* + \alpha\omega^1 + \beta\omega^2)$. By the chain rule,

$$\begin{aligned} DF(\alpha, \beta) &= DV(c^* + \alpha\omega^1 + \beta\omega^2)(\omega^1,\ \omega^2) \\ &= D_{\omega^1} V(c^* + \alpha\omega^1 + \beta\omega^2), D_{\omega^2} V(c^* + \alpha\omega^1 + \beta\omega^2)\,, \end{aligned}$$

where $D_{\omega_i} V$ is the directional derivative of $V$ in the direction of unit vector $\omega_i$, $i = 1, 2$.

Now, we study the structure of the set of endowments $c$ near $c^*$ that assure a level of utility equal to $u^*$. Consider the function $H\colon \left(-\frac{r}{2}, \frac{r}{2}\right) \times \left(-\frac{r}{2}, \frac{r}{2}\right) \longrightarrow \mathbb{R}$, $H(\alpha, \beta) = F(\alpha, \beta) - V(c^*)$. Trivially, $H$ is a $\mathcal{C}^1$ function, $H(0, 0) = 0$ and

$$\frac{\partial H}{\partial \beta}(0, 0) = D_{\omega^2} V(c^*) = \sum_{j=1}^{p} \lambda_j^* \omega_j^2 > 0.$$

By virtue of the implicit function theorem, there exists a unique $\mathcal{C}^1$-function $\beta = \varphi(\alpha)$ such that $c^* + \alpha\omega^1 + \varphi(\alpha)\omega^2$ belongs to the $u^*$-level set of $V$, i.e. the indirect utility function remains constant at $u^*$. Besides, for $\alpha$ small,

$$\varphi'(\alpha) = -\frac{D_1 F(\alpha, \varphi(\alpha))}{D_2 F(\alpha, \varphi(\alpha))} = -\frac{D_{\omega^1} V(c^* + \alpha\omega^1 + \varphi(\alpha)\omega^2)}{D_{\omega^2} V(c^* + \alpha\omega^1 + \varphi(\alpha)\omega^2)}. \tag{13}$$

**Definition 2** We define the generalized marginal rate of substitution between endowment proportions $\omega^1$ and $\omega^2$ at the point $c^*$ as the real number

$$GMRS_{\omega^1,\omega^2}(c^*) = -\varphi'(0) = \frac{D_{\omega^1} V(c^*)}{D_{\omega^2} V(c^*)}.$$

Let $C = \{e_1, \ldots, e_p\}$ be the canonical base of $\mathbb{R}^p$. If we take $\omega^1 = e_j$ and $\omega^2 = e_k$, $j \neq k$, the GMRS will be named the marginal rate of substitution of $c_j$ for $c_k$, and denoted as $MRS_{j,k}(c^*) = GMRS_{e_j,e_k}(c^*)$.

As said before, $GMRS_{\omega^1,\omega^2}(c^*)$ represents the change in the endowments in the proportions given by $\omega^1$ that would be necessary to compensate for a loss in initial resources proportional to $\omega^2$, keeping the utility level constant and equal to $u^* = V(c^*)$. As for $MRS_{j,k}(c^*)$ we have that $MRS_{j,k}(c^*) = \frac{\frac{\partial V}{\partial c_j}(c^*)}{\frac{\partial V}{\partial c_k}(c^*)} = \frac{\lambda_j^*}{\lambda_k^*} = \rho_j^k(c^*)$. Hence, in fact, the ratio $\rho_j^k$ is the rate at which we can exchange endowments $c_j^*$ for $c_k^*$ while holding $V$ constant at $u^* = V(c^*)$. In general, if $\omega^i = (\omega_1^i, \ldots, \omega_m^i)$, $i = 1, 2$, then

$$GMRS_{\omega^1,\omega^2}(c^*) = \frac{\lambda_1^*\omega_1^1 + \cdots + \lambda_m^*\omega_m^1}{\lambda_1^*\omega_1^2 + \cdots + \lambda_m^*\omega_m^2}.$$

Dividing this expression by $\lambda_1^*$, yields[6]

$$GMRS_{\omega^1,\omega^2}(c^*) = \frac{\omega_1^1 + MRS_{2,1}(c^*)\omega_2^1 + \cdots + MRS_{m,1}(c^*)\omega_m^1}{\omega_1^2 + MRS_{2,1}(c^*)\omega_2^2 + \cdots + MRS_{m,1}(c^*)\omega_m^2}.$$

Let us summarize our findings in the following result.

**Theorem 4** *Under assumptions* (*A*1)–(*A*5)*, given any linearly independent parameter vectors* $\omega^1, \omega^2 \in \mathbb{R}_+^p$ *such that* $||\omega^1|| = ||\omega^2|| = 1$*, there exists a unique* $\mathcal{C}^1$*-function* $\beta = \varphi(\alpha)$ *defined in a neighborhood of* $(0, 0)$ *such that the indirect utility function* $V$ *remains constant at* $u^* = V(c^*)$ *along the curve* $c^* + \alpha\omega^1 + \varphi(\alpha)\omega^2$*. Besides, for all* $c = c^* + \alpha\omega^1 + \varphi(\alpha)\omega^2$*,*

[6] Obviously, one can derive analogous expressions dividing by $\lambda_j^*$, $j = 1, \ldots, m$.

$$GMRS_{\omega^1,\omega^2}(c) = \frac{\omega_1^1 + \rho_2^1(c)\omega_2^1 + \cdots + \rho_m^1(c)\omega_m^1}{\omega_1^2 + \rho_2^1(c)\omega_2^2 + \cdots + \rho_m^1(c)\omega_m^2}, \tag{14}$$

$$MRS_{j,k}(c) = \rho_j^k(c). \tag{15}$$

We end this section by differentiating expression (13), if $F$ is twice continuously differentiable function, to obtain,

$$\varphi''(\alpha) = \frac{-1}{(D_2F)^3}\Big((D_2F)^2 D_{11}F - 2D_1FD_2FD_{12}F + (D_1F)^2 D_{22}F\Big). \tag{16}$$

where all the derivatives must be evaluated at $(\alpha, \varphi(\alpha))$.

## 6 Expenditure minimization multipliers as MRS between initial endowments

Let $u^* = V(c^*)$ be the optimal level of utility the consumer achieves at $c^*$. Applying the main theorem of Besada and Mirás (2002) we know that $x^* = x(c^*)$ is a minimum of all the equality constraint reciprocal expenditure problems $ER(u^*, c^*_{-k})$, $k \in \{1, \ldots, p\}$, given by

$$\begin{array}{ll} \text{Minimize} & g_k(x) \\ \text{subject to :} & \begin{cases} u - U(x) = 0 \\ g_{-k}(x) = c_{-k} \end{cases} \end{array} \tag{$ER(u, c_{-k})$}$$

that satisfies the second-order sufficiency conditions for a strict local minimum. The implicit function theorem guarantees the existence of an open neighborhood $B = B((u^*, c^*_{-k}), s) \subset \mathbb{R}^m$ and unique $\mathcal{C}^1$-functions $x^k$ and $\lambda^k$, such that, for all $(u, c_{-k}) \in B$, $x^k(u, c_{-k})$ is a strict local minimum of the $k$-th equality constraint reciprocal problem $ER(u, c_{-k})$ with strictly negative Lagrange multipliers given by $\lambda^k(u, c_{-k})$. The $k$-th expenditure function $e^k$ is the minimum value function corresponding to problem $ER(u, c_{-k})$, that is, $e^k(u, c_{-k}) = g_k(x^k(u, c_{-k}))$. Next, we detail some straightforward properties of the expenditure functions $e^k$.

**Proposition 4** *Let $U$ and $G$ satisfy (A1) and (A4) and be the bordered Hessian with nonzero determinant at $(x^*, \lambda^*)$, the optimal solution for the problem $EP(c)$. For each $k \in \{1, \ldots, m\}$, the expenditure function $e^k$ is strictly increasing in $u$ and strictly decreasing in each coordinate $c_j$, $j \neq k$. In fact, for all $(u, c_{-k}) \in B$, $\frac{\partial e^k}{\partial u}(u, c_{-k}) = -\mu^k$
$(u, c_{-k}) > 0$ and $\frac{\partial e^k}{\partial c_j}(u, c_{-k}) = \lambda_j^k(u, c_{-k}) < 0$ whenever $j \neq k$.*

Then, higher levels of utility required higher levels of expenditure in the $k$ resource to achieve, while if one increases the endowment $c_j$ the cost incurred in the $k$ resource will decrease. We are now ready to provide an interpretation for the reciprocal multipliers $\lambda_j^k$, $j \neq k$, that follows directly from equality (8) and the fundamental reciprocal relation (15).

**Proposition 5** *Under assumptions* $(A1)$–$(A5)$*, we have that, for all* $c = c^* + \alpha e_j + \varphi(\alpha) e_k$,

$$\lambda_j^k(c) = -MRS_{j,k}(c) \tag{17}$$

Consequently, the reciprocal multipliers $\lambda_j^k$, $j \neq k$, are, up to the sign, the MRS between initial endowments $c_j$ and $c_k$ in the indirect utility function, i.e. they measure the units of endowment $c_j$ that would be necessary to compensate for a loss of one unit of initial endowment $c_k$ while keeping the utility level constant and equal to $u^* = V(c^*)$.

## 7 Quasi-concavity and diminishing MRS between endowments

It is a known fact (Silberberg 1990) that the sign of the derivatives of the Lagrange multipliers $\lambda_j(c)$, the marginal utility of endowment $c_j$ in the primal problem, with respect to the other resources varies with the application of monotone increasing transformations to the direct utility function $U$. As a consequence, the concept of diminishing marginal utility of income (or any other endowment) is meaningless. But according to Theorem 3 and (17) the multipliers of the reciprocal problems $\lambda_j^k$ are invariant under monotonic transformations and, in turn, can be interpreted as marginal rates of substitution. Thus, we can explore the possibility of having a diminishing marginal rate of substitution between the initial endowments in the indirect utility function.

On the other hand, in speaking of a quasi-concave function one assumes a specific convex domain of definition. When talking about utility functions, this domain is taken to be $\mathbb{R}_+^n$. A quasi-concave function $f$ is one that has a diminishing marginal rate of substitution, if $\nabla f > 0$. In the previous section, we have established the quasi-concavity of the indirect utility function $V$ on a certain convex domain $D$ where $V$ is well defined. In applications we will rarely have $D = \mathbb{R}_+^m$ so we must be careful not to imply at once that $V$ exhibits diminishing marginal rates of substitution.

**Theorem 5** *If* $V$ *is quasi-concave on* $B(c^*, r)$ *then* $F$ *is quasi-concave on* $I_r = \left(-\frac{r}{2}, \frac{r}{2}\right) \times \left(-\frac{r}{2}, \frac{r}{2}\right)$*, for all* $\omega^1, \omega^2 \in \mathbb{R}_+^p$ *such that* $||\omega_1|| = ||\omega_2|| = 1$.

*Proof* Recall that, given two linearly independent $\omega^1, \omega^2 \in \mathbb{R}_+^p$ such that $||\omega_1|| = ||\omega_2|| = 1$ we define $F(\alpha, \beta) = V(c^* + \alpha\omega_1 + \beta\omega_2)$. Now, take $(\alpha, \beta) \in I_r$, $(\alpha', \beta') \in I_r$ and $0 \leq t \leq 1$.

$$\begin{aligned} F(t(\alpha, \beta) + (1-t)(\alpha', \beta')) &= F(t\alpha + (1-t)\alpha', t\beta + (1-t)\beta') \\ &= V(c^* + (t\alpha + (1-t)\alpha')\omega_1 + (t\beta + (1-t)\beta')\omega_2) \\ &= V(t(c^* + \alpha\omega_1 + \beta\omega_2) + (1-t)(c^* + \alpha'\omega_1 + \beta'\omega_2)) \\ &\geq \min\{V(c^* + \alpha\omega_1 + \beta\omega_2), V(c^* + \alpha'\omega_1 + \beta'\omega_2)\} \\ &= \min\{F(\alpha, \beta), F(\alpha', \beta')\} \end{aligned}$$

where the inequality holds because $V$ is quasi-concave on $B(c^*, r)$. The expression obtained characterizes the quasi-concavity of $F$ on $I_r$. □

We state the following result readily adapted from Theorem 4 in Arrow and Enthoven (1961).

**Theorem 6** *If $F \in \mathcal{C}^2(I_r)$ and $F$ is quasi-concave on $I_r$ then*

$$\det \begin{pmatrix} 0 & D_1F(\alpha,\beta) & D_2F(\alpha,\beta) \\ D_1F(\alpha,\beta) & D_{11}F(\alpha,\beta) & D_{12}F(\alpha,\beta) \\ D_2F(\alpha,\beta) & D_{21}F(\alpha,\beta) & D_{22}F(\alpha,\beta) \end{pmatrix} \geq 0$$

*for all $(\alpha,\beta) \in I_r$.*

Observe that expression (16) can be written in terms of the determinant in Theorem 6. Indeed,

$$\varphi''(\alpha) = \frac{1}{(D_2F)^3} \det \begin{pmatrix} 0 & D_1F & D_2F \\ D_1F & D_{11}F & D_{12}F \\ D_2F & D_{21}F & D_{22}F \end{pmatrix}$$

The following result is straightforward.

**Corollary 1** *Assume $V$ is $\mathcal{C}^2$ and $\nabla V(c) > 0$. Under assumptions of Theorem 4, if $V$ is quasi-concave on $B(c^*, r)$ then given any parameter vectors $\omega^1, \omega^2 \in \mathbb{R}^p_+$ such that $||\omega_1|| = ||\omega_2|| = 1$ the generalized marginal rate of substitution $GMRS_{\omega_1,\omega_2}(c(\alpha))$ decreases*[7] *along the curve $c(\alpha) = c^* + \alpha\omega_1 + \varphi(\alpha)\omega_2$.*

In fact, what we know is that $\varphi''(\alpha) = -\frac{d}{d\alpha}(GMRS_{\omega_1,\omega_2}(c(\alpha))) \geq 0$. Remember that the marginal rate of substitution is $MRS_{j,k}(c) = \rho^k_j(c) = -\lambda^k_j(c)$ so, alternatively one can try to compute the derivatives of the reciprocal multipliers $\lambda^k_j$ and the ratios of primal multipliers $\rho^k_j$ and analyze their sign to imply monotonicity of MRS. Certainly, one can obtain expressions for the derivatives $\frac{\partial \lambda^k_j}{\partial c_i}$ by applying the implicit function theorem to the first order necessary conditions of problem $ER(c)$. Analogously, the derivatives of $\rho^k_j$ can be computed by applying the implicit function theorem to the first order necessary conditions of the primal problem and then using the quotient rule of differentiation. In any case, the expressions are so general that no information on the sign of these derivatives, and subsequently on the property of diminishing marginal rates of substitution, can be inferred from the simple hypotheses of our setting.

## 8 Summarizing example and conclusions

We will summarize our results by means of a very simple example with just two constraints. An atomistic consumer has complete, reflexive, transitive, continuous, monotonic, convex and twice continuously differentiable preferences over the consumption set $\mathbb{R}^n_+$. Then, they can be represented by a $\mathcal{C}^2$ quasi-concave utility function $U\colon \mathbb{R}^n_+ \to \mathbb{R}$. The consumer is endowed with a fixed monetary income $R > 0$ and

[7] If $\nabla V(c) < 0$ then the GMRS would be increasing.

faces the budget constraint $px \leq R$, where $p \in \mathbb{R}^n_+$ is the vector of market prices of the $n$ goods.

Now, assume that the consumer lives in a time of severe scarcity and that the authorities restrict consumption by issuing some rationing coupons. The consumer then faces another constraint, the rationing constraint, $g(x) \leq c$, where $c > 0$ is the coupon endowment and $g$ is a convex function. The simplest form is given by the linear constraint $g(x) = qx$ where $q_i \geq 0$ is the price of good $i$ in ration coupons. The consumer's problem is

$$\text{Maximize } U(x)$$

$$\text{subject to: } \begin{cases} px \leq R \\ g(x) \leq c \\ x \geq 0 \end{cases}$$

There are 2 associated reciprocal problems. One is that of minimizing rent expenditure whilst guaranteeing a utility level $u_0$ and complying with the rationing constraint

$$\text{Minimize } \; px$$

$$\text{subject to: } \begin{cases} U(x) \geq u_0 \\ g(x) \leq C \\ x \geq 0 \end{cases}$$

The other is the problem of minimizing rationing expenditure while guaranteeing a utility level $u_0$ and complying with the budget constraint

$$\text{Minimize } \; g(x)$$

$$\text{subject to: } \begin{cases} px \leq R \\ U(x) \geq u_0 \\ x \geq 0 \end{cases}$$

In this example, monotonicity is a hypothesis of desirability on the preference relation, in the sense that all goods are not harmful. This condition is very common and guarantees that $\nabla U(x) > 0$. In any case, it will suffice that $\frac{\partial U}{\partial x_i}(x) > 0$, for some relevant variable $i$. Then (4) holds and, trivially, (5) is also satisfied. Then, any consumption bundle $x(R, c)$ satisfying the $(FOC)$ solves the primal and the 1-reciprocal problem (if the budget constraint is binding). Moreover, since both constraints in the primal problem are convex and $U$ is quasi-concave, the indirect utility function $V$ is quasi-concave in its domain.

Under some technical conditions that assure existence and differentiability of the indirect utility function $V$ in the neighbourhood of a known solution $x^*$, and always taking the wanted level of utility in the reciprocal problems as being equal to the optimal utility in the primal problem, we can affirm that

1. The marginal utility of income (multiplier $\lambda_1$) is the reciprocal of the marginal cost of utility in the 1-reciprocal problem (multiplier $\mu_1$).

2. Multiplier $\lambda_2^1$ in the 1-reciprocal problem is the MRS between income $R$ and the rationing coupon endowment $c$. Multiplier $\lambda_2^1$ coincides with the primal problem multipliers ratio $\frac{\lambda_2}{\lambda_1}$ which is invariant by positive monotonic transformations of utility.

Multiplier $\lambda_2^1$ represents the units of monetary income that the consumer is willing to give up to compensate for a loss of one unit of the coupon endowment while keeping utility constant.

Finally, since preferences are monotonic, $\nabla U > 0$, and the indirect utility function $V$ is quasi-concave, the MRS between income $R$ and the rationing coupon endowment $c$ decreases, that is, as income rises the rate of exchange between income and the other endowment to hold utility constant decreases. This property does not depend on the utility representation of the consumer's preferences chosen.

To conclude, we have developed a framework where the concept of diminishing marginal utility of income (or any other initial endowment) can be derived from basic assumptions on the consumer's preferences. The fundamental property is quasi-concavity of the direct utility function that, under adequate conditions, implies quasi-concavity of the indirect utility function. We have also explored the relationship between the consumer's utility maximization problem and the associated cost minimization reciprocal problems. Given the extensive use of these kinds of models in the literature, our results have a potentially wide range of application.

**Acknowledgements** We thank an anonymous referee for his/her comments on a previous version of this work. We have benefited from the financial support of the Spanish Ministry of Education through DGICYT grants ECO2008-03004 and SEJ2006-15401-C04-01/ECON.



## References

Arrow KJ, Enthoven AC (1961) Quasi-concave programming. Econometrica 29(4):779–800
Atkinson A, Stern N (1979) A note on the allocation of time. Econ Lett 3:119–123
Baumol W (1973) Income and substitution effects in the Linder Theorem. Q J Econ 87:629–633
Becker G (1965) A theory of the allocation of time. Econ J 75:493–517
Besada M, Mirás M (2002) A note on Lagrange multipliers in the multiple constraint case. Econ Lett 75(1):141–145
Besada M, Vázquez C (1999) The generalized marginal rate of substitution. J Math Econ 31:553–560
Boadway RW, Bruce N (1989) Welfare economics. Basil Blackwell, Oxford
Caputo MR (2001) Further results on Lagrange multipliers with several binding constraints. Econ Lett 70:335–340
Cornes R (1992) Duality and modern economics. Cambridge University Press, New York
Cornes R, Milne F (1989) A simple analysis of mutually disadvantage trading opportunities. Econ Stud Q 40:122–134
Cornes R, Sandler T (1984) The comparative statics properties of the impure public good model. J Publ Econ 94:580–598
Deaton A, Muellbauer J (1980) Economics and consumer behavior. Cambrige University Press, Cambridge
Diamond PA, Yaari M (1972) Implications of the theory of rationing for consumer choice under uncertainty. Am Econ Rev 62(3):333–343
Gilley O, Karels G (1991) In search of Giffen behavior. Econ Inq 29:182–189

Giorgi G (1995) On first order sufficient conditions for constraint optima. In: Maruyama T, Takahashi W Nonlinear and convex analysis in economic theory. Springer, Berlin, pp 53–66
Jehle GA (1991) Advanced microeconomic theory. Prentice Hall, New Jersey
Johnson R, Larson D (1994) Focusing the search for Giffen behavior. Econ Enq 32:168–174
Lancaster K (1968) Mathematical economics. MacMillan, New York
Lipsey R, Rosenbluth G (1971) A contribution to the new theory of demand: a rehabilitation of the Giffen good. Can J Econ 4:131–163
Neary J, Roberts K (1980) The theory of household behavior under rationing. Eur Econ Rev 13:25–42
Partovi HM, Caputo MR (2006) A complete theory of comparative statics for differentiable optimization problems. Metroeconomica 57(1):31–67
Samuelson P (1974) Foundations of economic analysis. Harvard University Press, Cambridge
Shefrin H, Heineke J (1979) On duality theory in intertemporal choice with uncertainty. Econ Lett 3:19–23
Silberberg E (1990) The structure of economics: a mathematical analysis. McGraw-Hill, New York
Steedman I (2001) Consumption takes time: implications for economic theory. Routledge, London
Sydsaeter K (1981) Topics in mathematical analysis for economists. Academic Press, Inc, London
Takayama A (1994) Analytical methods in economics. Harvester Wheatsheaf, New York
Tobin J (1952) A survey on the theory of rationing. Econometrica 20:512–553
Tobin J, Houthakker H (1951) The effects of rationing on demand elasticities. Rev Econ Stud 18:140–453
Weber CE (1998) A note on Lagrange multipliers with several binding constraints. Econ Lett 59:71–75
Weber CE (2001) Lagrange multipliers as marginal rates of substitution in multi-constraints optimization problems. Metroeconomica 52(1):79–94
Weber CE (2005) Diminishing marginal value of income without apology. Oxf Econ Pap 57:586–609