Riser and well casing analysis during drift-off: a coupled solution in the time domain

Fabiano Guimarães¹

Received: 9 September 2021 / Accepted: 14 December 2021 / Published online: 7 January 2022
© The Author(s) 2022

Abstract

One of the most serious incidents that can occur in offshore drilling and exploration is damage to the well structure and subsea components which can result in uncontrolled hydrocarbon release to the environment and present a safety hazard to rig personnel. Over decades, there have been substantial developments to the mathematical models and algorithms used to analyze the stresses on the related structure and to define the operational and integrity windows in which operations can proceed safely and where the mechanical integrity of the well is preserved. The purpose of this work is to present a time-domain solution to the system of equations that model the dynamic behavior of the riser and casing strings, when connected for well drilling/completion during the event of drift-off of the rig. The model combines a solution using finite differences for the riser dynamics and a recursive method to analyze the behavior of the casing in the soil. It allows for the coupling between the equations related to the riser and casing and for the coupling with the equations that describe the dynamics of the rig when station keeping capabilities are lost. The use of the forward–backward finite-differences coupled with the recursive method does not require linearization of the forces acting on the structure making it an ideal methodology for riser analysis while improving convergence. The findings of this study can help improve understanding of the impact of the watch circle limits to riser/well integrity, whether these limits are set based on a quasi-static drive-off/drift-off or fully dynamic. The gain in accuracy in using the fully coupled equations of drift-off dynamics, where there is interaction between the rig and the top of the riser during drive-off/drift-off, is evaluated, and the effects of varying the riser top tension and the compressive loads on the casing string are also analyzed. In particular, it is shown that the results of the fully coupled system of equations representing the dynamics of the riser and casing during drift-off/drive-off are less conservative than the quasi-static approach. Another important finding is that the gain in accuracy in coupling the top of the riser and the rig during drift-off/drive-off is not substantial, which indicates that solving separately the rig dynamics equations and the riser-casing equations is an approach that provides reasonable results with less computational effort. The model can also be used to evaluate wellhead and casing fatigue during the life of the intervention. Finally, the model limitations are discussed.

Keywords Riser · Riser analysis · Casing · Finite difference method · Drift-off · Drive-off

Introduction

Riser Analysis has been a subject of great interest to the Oil and Gas Industry. Drilling riser, subsea components and well casing are subject to stresses that depend on several factors, most notably the environmental forces acting on the riser (current and wave) and the rig offset referenced to the point at sea level right above the wellhead. Figure 1 represents the basic configuration under analysis. During an event of drive-off/drift-off, when the rig is no longer able to keep its fixed

position and starts to move away from the original position due to the environmental forces (current, wind and wave), the increasing horizontal offset between the rig and the wellhead causes additional stresses on the riser and well components (wellhead, surface conductor, casing, welds, etc.). The question that arises is related to the maximum allowable offset (referred to as POD – point of disconnect) in a scenario of drift-off/drive-off before the structural limits of the components of the system are reached and when the Emergency Disconnect Sequence (EDS) needs to be activated to prevent damage to the well, which could have catastrophic consequences. The EDS closes the rams on the BOP stack so that the well is isolated from the sea and then it disconnects the LMRP from the BOP stack, and as a result, the riser and the

✉ Fabiano Guimarães
fab_guimaraes@yahoo.com.br

¹ Independent Researcher, Niteroi, Brazil

Fig. 1 Marine Riser deflection during drift-off/drive-off

LMRP would remain attached to the rig as it moves away from the wellhead. By defining the “red alert” as the maximum point at which the EDS needs to be initiated so that it is completed before the rig reaches the POD, mechanical integrity is maintained.

Patel and Seyed (Patel & Seyed, 1995) presented a comprehensive historical review of riser analysis techniques dating back to the 1970’s and discussed the particularities and challenges of each method. In 1973, Burke (Burke, 1973) built the first models for the two basic philosophies used in riser analysis: static and dynamic analysis. The static analysis is performed considering the rig at a specific offset measured from the wellhead and subject to current forces only. Dynamic cases are similar to the static cases with the addition of dynamic wave forces (for a review on various dynamic riser analysis see (Keum, 2018)). Both analyses aim at finding the operability envelopes at which well operation could be safely maintained and the critical envelopes for maintaining equipment integrity. Originally riser analysis only considered the equipment located between and including the wellhead and the upper flex joint, conductor and casing stresses were not evaluated.

One of the first methods employing finite differences for riser analysis was reported by Botke in 1975 (Botke,

1975). In 1976, Gardner and Kotch (Gardner & Kotch, 1976) developed a riser analysis model using the finite element method. In 1977, the API (American Petroleum Institute) released the first edition of bulletin 2 J (which is no longer maintained by API) on comparison of marine drilling riser analysis. In 1985, Langer (Langer, 1985) published his work on suspended catenary pipes, also using finite differences. Except on the extremities, bending moment distribution was accurately predicted.

By the 1990’s, the number of different algorithms used for riser analysis was so large that in 1992, API replaced bulletin 2 J with bulletin 16 J, as an effort to standardize the practice (bulletin 16 J is no longer maintained). It is worth mentioning that the water depth was limited to 6,000 ft in this study. Nowadays, several areas in the world commonly operate at 10,000 ft water depths. The bulletin also reported that the results of the static analysis were similar among the companies which participated in the study, while the results of the dynamic models varied substantially.

The first edition of API 16Q (API, 1993) for the design, selection, operation, and maintenance of marine drilling riser systems was issued in 1993 and provided guidelines for static and frequency and time-domain dynamic riser analysis. The publication acknowledged that at that time, most riser programs used finite element modeling. VIV (Vortex Induced Vibration) was briefly covered in the document. Vortex Induced Vibration is riser movement transversal to the direction of the main current, induced by the shedding of vortices and can substantially increase stresses on the riser and well structure making them more susceptible to failure and reducing lifetime (for a review on VIV see (Keum-S and Umer 2018)).

The first edition of API RP 2RD (API, 1998) was released in 1998 with recommendations on the design of risers and provided guidelines on the frequency and time domain riser analysis, providing details for the coupling between the rig and the riser on a fixed offset using the RAO (Response Amplitude Operators), and providing details on VIV. The focus was naturally on finite element modeling as the most used technique for riser analysis.

In the 2000s, some riser analysis models started incorporating the analysis of the conductor and surface casing since they were also subject to high stresses due to the deformation of the casing in the soil, especially as the rig moves away from the zero-offset position. As a result, riser analysis evolved to incorporate these structures and the stress concentration factors of the casing configuration. Wellhead and casing fatigue can also be analyzed by these models. In addition, sensors can be installed on the riser and BOP stack to provide a real-time fatigue assessment.

In 2004, Michel Dib et al. (2004) reported results of a FEM model for riser-casing analysis during drift off.

In 2008, Chatjigeorgiou (2008) developed a finite differences formulation for the dynamics of 2D catenary risers.

In 2012, WANG et al. (2012) developed a finite difference approximation for hanging risers to analyze the landing of the BOP or LMRP on the wellhead. As most of the models at the time, these analyses did not consider the loads transferred from the riser to the wellhead and casing-soil.

The second revision of the API 16Q (API, 2017) issued in 2017 for the design, selection, operation, and maintenance of marine drilling riser systems highlighted the importance of the riser analysis to well integrity due to the loads transferred from the riser system to the BOP stack, and to the wellhead and casing. It also provided guidelines for the drive-off and drift-off analysis.

Kanhua Su et al. (2018) used finite differences method to analyze the dynamics of the riser coupled to the casing of a Deepwater Surface BOP Drilling System. The force interaction between the rig and the top of the riser as the rig moves away from the zero-offset position was not modeled.

Very often the wellhead and the casing are the components more susceptible to failure during an event of drift-off. Figure 2 shows that as the rig moves away from what can be referred to as the zero-offset position, the bending moments and mechanical stresses increase on the wellhead and casing.

This work presents a time-domain solution to the system of equations that model the dynamic behavior of the riser and casing strings, when connected for well drilling/

completion during an event of drift-off of the rig. The model combines a solution using finite differences for the riser dynamics and a recursive method to analyze the behavior of the casing in the soil. This method provides a complete description of the dynamics of the casing and its interaction with the soil, the wellhead, BOP stack, riser string and coupling to the rig, whether the rig is stationary or in drive-off/drift-off scenario. Additionally, the use of the finite-differences coupled with the recursive method does not require approximations/linearization, making it an ideal methodology for the analysis since the forces acting on the riser and casing are nonlinear in nature. The use of the forward–backward formulation improves convergence. The model can also be used for riser analysis when the rig is operating on a fixed offset (no drive-off/drift-off) and for wellhead and fatigue assessment.

This is the structure of this paper. Using a case study, some particularities of the solution will be presented and important topics highlighted:

- Riser-rig coupling
- Drive-off/Drift-off analysis
- Casing-soil analysis
- Riser-casing coupling
- Riser dynamics, Wellhead and Casing Fatigue Analysis

Fig. 2 Forces acting on the marine riser and casing

- The systems of equations and solution methodology are presented in detail in the appendices

Case study

The case study was performed for well conditions commonly encountered at the Santos Basin in Brazil.

Parameters:

Water Depth: 1,927 m.

Rotational Stiffness – Lower Flex Joint – 111,210 lbf.ft/°

Rotational Stiffness – Upper Flex Joint – 18,000 lbf.ft/°

Table 1 presents the parameters for each environmental condition:

Figure 3 presents the current profiles for each environmental condition:

Jonswap Spectrum:

$\gamma = 3.3$

$\sigma_a = 0.07$.

$\sigma_b = 0.09$.

Well Construction:

Conductor casing – Diameter 36 in / thickness 1.5 in.

Surface casing – Diameter 22 in / thickness 1.125 in.

$L_c = 100$ – length of the surface casing (m).

Maximum wellhead bending moment – 5250 ft.kips.

Maximum casing stress – 331 MPa.

Soil:

$\epsilon_c = 0.02$ – Strain at 50% of maximum stress.

$J_{clay} = 0.5$ – dimensionless empirical constant.

$\gamma = 5150$ – effective weight (N/m³).

$a = 1000$ Pa; $b = 1350$ Pa in the equation $c = a + bx$; x (m).

Rig Displacement = 46,500mT (operating draft).

EDS time = 73 s (from the moment the EDS is activated to the physical release of the riser connector).

Weight of the BOP stack in water = 540,000 lbf.

Mud Weight = 12 ppg.

Riser Top Tension = 1600 kips—1650 kips—1700 kips.

Compressive force at the top of the casing = 290,000 lbf (this is the highest compression value as a direct result of the applied riser top tension value of 1600 kips).

Table 1 Current profiles 99% nexc and 1-year environment

	99% NEXC	1 year
wind (m/s)	12.5	20.4
Hs (m)	4.7	6.9
Tp (s)	11.4	12
current (m/s)	0.94	1.24

Fig. 3 99% nexc and 1-year current profiles

Riser-rig coupling

Figure 4 shows that as the rig drifts away from the zero-offset position, the horizontal force exerted by the riser on the rig is dependent on the angle at the upper flex joint. At the same time, the dynamics of the riser is dependent of the drift-off velocity. Ideally, the equations that describe the riser and rig dynamics should be solved simultaneously.

One model of interest is to define the direction of the riser top tension as the direction defined by a straight line passing

Fig. 4 Details of riser tension models 1 and 2

through the lower and upper flex joints. As we can see from the graph, this approximation is not accurate for small drift-off distances, but we are going to compare it against the more accurate model for larger drift-off distances.

Please refer to appendix 2. It presents details and limitations of the tension models named Model 1 (straight line approximation) and Model 2 (real angle at the UFJ). Figure 5 and Fig. 6 show the results obtained when solving the coupled riser-rig drift-off equations for the 99% and 1-year environmental conditions.

We see that the results obtained with model 1 are close to the more accurate model 2, for distances of interest on the drift-off analysis. This allows for the possibility of solving the riser and casing equations independently of the drift-off equations while still arriving at reasonably accurate results. It is also interesting to notice that while for 99% NEXC conditions, model 2 results in higher drift-off velocities, for the 1-year conditions, model 1 results in higher drift-off velocities. The reason for this difference is the riser angle at the flex joint—for the 99% NEXC conditions, the real riser angle used in model 2 results in lower horizontal forces than model 1 acting on the rig against the drift-off direction, while for the 1-year conditions, model 2 results in higher horizontal forces.

Drive-off/drift-off analysis

In the solution, wind, current and wave are considered colinear and varied at 1° steps from -30° to 30° relative to bow.

For the analysis, the incident direction chosen for Wind, Current and Wave is the one that results in maximum drift-off for an arbitrary distance (8% of water depth). For comparison, Fig. 7 shows the drift-off with and without the riser

Fig. 5 Drift-off velocity and distance vs time for the 99% nexc

Fig. 6 - Drift-off velocity and distance vs time for the 1-year environment

force acting on the rig as modeled in Appendix 2. They were obtained using the 99% NEXC environmental conditions.

Casing-soil analysis

In this example, the weight of the BOP is supported by the soil using a linear relationship to casing depth, but other assumptions can be used.

Let us assume that it was possible to change the axial force acting on the casing while keeping the bending moment and the horizontal force acting at the top of the

Fig. 7 - Drift-off distance and velocity vs time with and without riser force

Fig. 8 Casing deflection for different compressive loads

Fig. 10 Casing deflection for different tension loads

Fig. 9 Casing von mises stress for different compressive loads

Fig. 11 Casing von mises stress for different tension loads

casing constant. While not possible in practice since a change in one parameter would affect all variables in different ways, this would show how casing displacements and stresses would vary as a function of the compressive and tension loads, under these hypothetical conditions.

Figures 8, 9, 10 and 11 are obtained using the same soil properties used in the Case Study. Here, the bending moment and the horizontal force acting at the top of the casing are considered constant and equal to $M_0 = 3.6 \times 10^6 \text{ Nm}$ and $V_0 = 1.7 \times 10^5 \text{ N}$. The axial force on the

casing is varied in units of $F = 290,000 \text{ lbf}$ with negative values indicating casing under compression and positive values indicating tension. To exemplify, a curve labelled as $0.2 \cdot F$ was obtained with a tension force of $58,000 \text{ lbf}$ on the wellhead, while a curve labelled as $-0.2 \cdot F$ was obtained with a compressive force of $58,000 \text{ lbf}$ on the wellhead.

The curves are obtained for each of the six recursive steps used for convergence of the equations used in the analysis of the casing. Convergence is obtained after just a few

Fig. 12 Riser movement due to rig oscillation and environmental forces

iterations. The casing is subject to minimum overall stress when the axial force is zero. Since the casing strength is often the weakest link of the well-riser system, it is recommended to solve the equations riser-casing-drift-off equations with different values of top tension as an attempt to find the top tension value that results in the largest POD and EDS activation offset.

Riser-casing coupling

The system of equations used to model the coupling between the riser and the casing are described in detail in appendix 6.

Fig. 14 Casing stresses due to bending due to rig oscillation and environmental forces

Riser dynamics, wellhead and casing fatigue analysis

The solution to the riser analysis equation corresponding to $f = f(x, t)$ not only provides more accurate results but is fundamental to fatigue evaluation. The system of equations presented in the appendices can be used to analyze wellhead and casing fatigue.

For exemplification, a RAO (response amplitude operator) of 0.4 related to the wave period of 11.4 s will be used. Considering the rig oscillating around the zero-offset position, we obtain the following graphics for the displacements along the riser string and the casing stresses due to bending,

Fig. 13 Riser movement due to rig oscillation and wave

Fig. 15 Casing stresses due to bending due to rig oscillation and wave

during 1 full oscillation of the rig. Figures 12 and 14 are obtained using the 99% NEXC environmental forces, while Fig. 13 and Fig. 15 are obtained similarly to the previous case without current.

The stresses over the riser and casing are obtained from which fatigue analysis can be performed.

With the stress values obtained, the number of cycles to failure can be calculated.

Premise validation

Since the main limitation of the model is that the environmental forces are assumed to be collinear and that the drive-off/drift-off path can be approximated as a straight line, at least for relatively short drift-off/drive-off distances, it is important to validate this assumption. As an example, for the 99% NEXC case (TT = 1600 kips), when the drift-off distance on the x-axis reaches 200 m, the distance travelled on the y-axis is less than 2.5 m showing that the assumption

is satisfactory, which is commonly the case when considering collinear wave, current and wind.

Simulation and results

These were the scenarios evaluated and the results obtained, using both solution philosophies

Model validation

To allow for a direct comparison with a report from one of the companies that performs analysis of riser and casing during drift-off, additional calculations were performed with the inclusion of a Production Adapting Base (height = 2.22 m), using Philosophy 2. The tension used in these calculations was 1,600 kips. The results are shown on the next table. The limiting component was the conductor casing in all scenarios.

Table 2 Simulation and results

WHBM ^a	Casing Strength ^a	POD ^a	EDS point ^a	EDS Time (s) ^b
Philosophy 1—1600 kips—1 year—EDS with Model 1				
6.1%	6.1%	6.1%	0.9%	37
Philosophy 1—1600 kips—99% NEXC—EDS with Model 1				
6.5%	6.5%	6.5%	2.8%	104
Philosophy 1—1650 kips—99% NEXC—EDS with Model 1				
6.3%	6.5%	6.3%	2.6%	100
Philosophy 1—1700 kips—99% NEXC – EDS with Model 1				
6.0%	6.6%	6.0%	2.4%	96
Philosophy 2—1600 kips 1 year – Model 1				
9.4%	9.1%	9.1%	2.6%	67
Philosophy 2—1600 kips 99% NEXC – Model 1				
7.6%	7.3%	7.3%	3.5%	119
Philosophy 2—1650 kips 99% NEXC – Model 1				
7.3%	7.3%	7.3%	3.5%	120
Philosophy 2—1700 kips 99% NEXC – Model 1				
7.1%	7.3%	7.1%	3.3%	115
Philosophy 2—1600 kips 1 year – Model 2				
9.3%	8.9%	8.9%	2.6%	67
Philosophy 2—1600 kips 99% NEXC – Model 2				
7.6%	7.3%	7.3%	3.5%	119
Philosophy 2—1650 kips 99% NEXC – Model 2				
7.3%	7.3%	7.3%	3.5%	119
Philosophy 2—1700 kips 99% NEXC – Model 2				
7.1%	7.3%	7.1%	3.3%	114

^a% Water Depth

^bafter commencement of drift-off

Table 3 Comparison with reference model

	POD ^a	EDS point ^a	EDS Time (s) ^b	POD Time (s) ^b
1 year—with Model 1	9.1%	2.6%	67	140
1 year—with Model 2	8.9%	2.6%	67	140
1 year—Reference Analysis	8.9%	2.1%	55	128
99% NEXC—with Model 1	7.3%	3.5%	119	192
99% NEXC—with Model 2	7.3%	3.5%	119	192
99% NEXC – Reference Analysis	7.5%	3.1%	101	174

^a% Water Depth^bafter commencement of drift-off

Discussion

As it can be seen on Table 2, the results obtained with philosophy 1 (quasi-static) are more conservative than those obtained with philosophy 2 (fully coupled dynamics). This is mostly due to the decrease in the “apparent current velocity” relative to the riser as the drift-off progresses, using philosophy 2. Using philosophy 2, we also notice that more severe environmental conditions increase the velocity at which the rig moves away from the zero-offset position during drift-off and result in a larger POD, but a lower EDS activation point, since now the EDS needs to be activated at an earlier stage in order to be completed before the rig reaches the POD.

Care must be taken when using these assumptions, and when in doubt, a more conservative approach should be used. It is also seen that the values of the POD and the EDS activation point are dependent on the model of riser force used on the drift-off equations, but model 1 provides reasonable results with less computational effort. Appendix 2 describes the limitations regarding the riser force used in the model.

The equations presented in Appendixes 2 and 4 are coupled using a Runge–Kutta method and an appropriate delta t for stability, this has allowed for the riser force to be more properly modeled, especially on the first 20 m away from the zero-offset.

For the case study under evaluation, a relatively small increase in the riser top tension reduces the POD and the EDS activation offset, all the other parameters remaining the same. The maximum stress on the casing is affected by the compressive axial force on the casing and by the bending moment and horizontal force acting at the top of the casing. In this example, a slight increase in the riser top tension slightly reduces the maximum tension on the casing but at the same time it increases the wellhead bending moment and the overall effect is a reduction of the POD and EDS activation offsets.

The model can benefit from the development of a solution in finite differences for the equation describing the behavior of the casing in the soil, which would then be coupled to the finite differences equations for the riser and rig dynamics.

The results on Table 3 show that the model correlates relatively well with the reference model. The differences are probably due to the fact that usually not all parameters and coefficients are presented on the reference reports, especially the ones used to model the dynamics of the unit under drift off.

Summary and conclusions

The time-domain solution using finite differences presented in this paper is a satisfactory solution to the system of equations that model the dynamic behavior of a riser string connected to a casing strings, during the event of a drift-off of the rig. It represents a robust alternative to the finite element analysis which is more commonly used in this type of analysis. The conclusions can be summarized as follows:

- Results obtained with philosophy 1 are more conservative than those obtained with philosophy 2
- When using philosophy 2, we notice that more severe environmental conditions result in a larger POD, but a lower EDS activation point
- When using philosophy 2, model 1 provides reasonable results with less computational effort
- Riser top tension has a noticeable effect on the POD and on the EDS activation offset, all the other parameters remaining the same. The sensitivity on the top tension needs to be evaluated on case-by-case basis
- Wellhead and casing fatigue analysis can also be performed with the mathematical model presented

Appendices

1. Details of the algorithms used in both calculation strategies

Nomenclature

x vertical coordinate of the riser element (m)

y	riser element horizontal displacement of (m)
E	modulus of elasticity of riser main tube (Pa)
I	Imoment of area riser transversal section (m ⁴)
T_e	Effective Tension (N)
m	linear density of mass riser element (Kg/m)
C_a	added mass coefficient
ρ	density of sea water (Kg/m ³)
A	area of a transversal section of the riser (m ²)
C_D	drag coefficient
D	diameter of an element of the riser element (m)
u_c	current speed (m/s)
u_w	wave induced current (m/s)
T	wave period (s)
d	water depth (m)
x_c	vertical coordinate of each casing element (m)
y_c	casing element horizontal displacement (m)
E_c	modulus of elasticity of the casing (Pa)
I_c	moment of area casing transversal section (m ⁴)
$Q(x_c)$	casing compressive force (N)
$F_{soil}(x_c, y_c)$	soil lateral reaction force (N/m)
K_I	rotational rigidity of the LFJ (Nm/rad)
K_S	rotational rigidity of the UFJ (Nm/rad)
δ	function of the RAO of the unit (rad)

This work presents two basic strategies:

Philosophy 1—The rig is placed on increasing horizontal offsets from the wellhead and the system of equations for the dynamics of the riser and casing is solved for each offset (API, 1998; Spanos and Chen, 1980; Shamsheer, 1990).

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(T_e \frac{\partial y}{\partial x} \right) + (m + \rho C_a A) \frac{\partial^2 y}{\partial t^2} = F(x, t) \quad (1)$$

$$F(x, t) = \frac{1}{2} \rho C_D D \left(u - \frac{\partial y}{\partial t} \right) \left| u - \frac{\partial y}{\partial t} \right| + \rho C_m A \frac{\partial u}{\partial t} \quad (2)$$

$$E_c I_c \frac{\partial^4 y_c}{\partial x_c^4} + Q(x_c) \frac{\partial^2 y_c}{\partial x_c^2} = F_{soil}(x_c, y_c) \quad (3)$$

With the boundary conditions:

$$y(0, t) = y_c(0) - h \frac{dy_c}{dx}(0) \quad (4)$$

$$EI \frac{\partial^2 y(0, t)}{\partial x^2} = K_I \frac{\partial y(0, t)}{\partial x} \quad (5)$$

$$y(l, t) = \text{offset} \quad (6)$$

$$EI \frac{\partial^2 y(l, t)}{\partial x^2} = -K_S \left(\frac{\partial y(l, t)}{\partial x} - \delta(t) \right) \quad (7)$$

where

$u = u_c + u_w$ is the velocity of sea current and the horizontal component due to wave, as per the Airy Linear Wave Theory (m/s)

$$u_w = \frac{\pi H}{T} \frac{\cosh k(y + d)}{\sinh kd} \cos [k(x - ct)] \quad (8)$$

$$k = \frac{2\pi}{L}; L = \frac{gT^2}{2\pi} \tanh \left(\frac{2\pi d}{L} \right); kc = \frac{2\pi}{T} \quad (9)$$

The general solution will be given by

$$y(x, t) = y_1(x) + y_2(x, t) \quad (10)$$

With

$$F(x, t) = F_x(x) + F_{xt}(x, t) \quad (11)$$

The solution to y_1 is presented in appendix 3, while the solution to y_2 is presented in appendix 4.

Using the equations presented in appendix 2, the point at which the lowest mechanical limit of the system casing/riser is reached is obtained.

Philosophy 2—The dynamics of the riser and the casing is evaluated during an event of drift-off, with (Kanhua et al. 2018; Xiuquan et al. 2016; Maolin et al. 2020):

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(T_e \frac{\partial y}{\partial x} \right) + (m + \rho C_a A) \frac{\partial^2 y}{\partial t^2} = F(x, t) \quad (12)$$

$$F(x, t) = \frac{1}{2} \rho C_D D \left(u - \frac{\partial y}{\partial t} \right) \left| u - \frac{\partial y}{\partial t} \right| + \rho C_m A \frac{\partial u}{\partial t} \quad (13)$$

$$E_c I_c \frac{\partial^4 y_c}{\partial x_c^4} + Q(x_c) \frac{\partial^2 y_c}{\partial x_c^2} = F_{soil}(x_c, y_c) \quad (14)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & M + m_{xx} & m_{xy} & m_{xz} \\ 0 & 0 & 0 & m_{yx} & M + m_{yy} & m_{yz} \\ 0 & 0 & 0 & m_{zx} & m_{zy} & I_z + m_{zz} \end{bmatrix} \begin{bmatrix} \dot{X}_c \\ \dot{Y}_c \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix} \quad (15)$$

$$= \begin{bmatrix} u \cos \psi - v \sin \psi \\ u \sin \psi + v \cos \psi \\ r \\ F_{curr}^x + F_{wind}^x + F_{drift}^x + F_{riser}^x + d_x r + Mvr \\ F_{curr}^y + F_{wind}^y + F_{drift}^y + F_{riser}^y + d_y r - Mur \\ M_z^{curr} + M_z^{wind} + M_z^{drift} + M_z^{riser} + d_z r \end{bmatrix}$$

With the boundary conditions:

$$y(0, t) = y_c(0) - h \frac{dy_c}{dx}(0) \quad (16)$$

$$EI \frac{\partial^2 y(0, t)}{\partial x^2} = K_I \frac{\partial y(0, t)}{\partial x} \quad (17)$$

$$y(l, t) = \text{offset} \quad (18)$$

$$\frac{\partial y(l, t)}{\partial t} = v_{rig} = \sqrt{u^2 + v^2} \quad (19)$$

$$EI \frac{\partial^2 y(l, t)}{\partial x^2} = -K_S \left(\frac{\partial y(l, t)}{\partial x} - \delta(t) \right) \quad (20)$$

Note: if desired, the dynamics associated with the response amplitude operator can be superimposed to the solution to (15).

2. Solution to the system of equations representing the drift-off

Nomenclature

X_c	x coordinate center of mass (m)
Y_c	y coordinate center of mass (m)
M	total mass of the rig (Kg)
I_z	moment of inertia in relation to z-axis (Kg.m ²)
u	component of rig velocity on Ox (m/s)
v	component of rig velocity on Oy (m/s)
ψ	rig heading (rad)
m	additional mass
d	dissipation coefficients
ρ_a	density of air (Kg/m ³)
A_s	submersed area (in x and y) (m ²)
A_v	area above water line (in x and y) (m ²)
L_x	rig length (m)
u_c	current speed (in x and y) (m/s)
u_a	wind speed (in x and y) (m/s)
C_{Dh}	current force drag coefficient
C_{Da}	wind force drag coefficient
C_{Mh}	current moment drag coefficient
C_{Ma}	wind moment drag coefficient
d_{rtcm}	distance rotary table center or mass (m)
d_{rtcmx}	d_{rtcm} projected on Ox (m)
d_{rtcmx}	d_{rtcm} projected on Oy (m)
X_{wh}	x coordinate of wellhead (m)
Y_{wh}	y coordinate of wellhead (m)
T_{px}	riser force projected on OX (N)
T_{py}	riser force projected on OY (N)
$S(w)$	Wave spectrum
$C_{driftx,y,z}$	Wave drift coefficients

Fig. 16 - Angle of attack of current and wind and relative velocity

Given the fixed system of coordinates OXY and the system moving with the Rig Oxy, with origin coincident with the center of mass, as per Fig. 16:

The system of equations can be written in the form:

$$\begin{bmatrix} \dot{X}_c \\ \dot{Y}_c \\ \dot{\psi} \\ M\dot{u} \\ M\dot{v} \\ I_z\dot{r} \end{bmatrix} = \begin{bmatrix} u \cos \psi - v \sin \psi \\ u \sin \psi + v \cos \psi \\ r \\ F_x + Mvr \\ F_y - Mur \\ M_z \end{bmatrix} \quad (21)$$

Or:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & M + m_{xx} & m_{xy} & m_{xz} \\ 0 & 0 & 0 & m_{yx} & M + m_{yy} & m_{yz} \\ 0 & 0 & 0 & m_{zx} & m_{zy} & I_z + m_{zz} \end{bmatrix} \begin{bmatrix} \dot{X}_c \\ \dot{Y}_c \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} u \cos \psi - v \sin \psi \\ u \sin \psi + v \cos \psi \\ r \\ F_x^{curr} + F_x^{wind} + F_x^{drift} + F_x^{riser} + d_x r + Mvr \\ F_y^{curr} + F_y^{wind} + F_y^{drift} + F_y^{riser} + d_y r - Mur \\ M_z^{curr} + M_z^{wind} + M_z^{drift} + M_z^{riser} + d_z r \end{bmatrix} \quad (22)$$

Forces and moments due to current and wind.

The forces and moments acting on the rig can be written as

$$F_x^{curr} = \frac{1}{2} \rho_w C_{Dhx} (\beta_{curr}, \psi, v_{rel}) A_{Sx} [(u_{cx} - u)^2 + (u_{cy} - v)^2] \quad (23)$$

$$F_y^{curr} = \frac{1}{2} \rho_w C_{Dhy} (\beta_{curr}, \psi, vrel) A_{Sy} \left[(u_{cx} - u)^2 + (u_{cy} - v)^2 \right] \quad (24)$$

$$M_z^{curr} = \frac{1}{2} \rho_w C_{Mh} (\beta_{curr}, \psi, vrel) A_{Sy} L_x \left[(u_{cx} - u)^2 + (u_{cy} - v)^2 \right] \quad (25)$$

$$F_x^{wind} = \frac{1}{2} \rho_a C_{Dax} (\beta_{wind}, \psi, vrel) A_{vx} \left[(u_{ax} - u)^2 + (u_{ay} - v)^2 \right] \quad (26)$$

$$F_y^{wind} = \frac{1}{2} \rho_a C_{Day} (\beta_{wind}, \psi, vrel) A_{vy} \left[(u_{ax} - u)^2 + (u_{ay} - v)^2 \right] \quad (27)$$

$$M_z^{wind} = \frac{1}{2} \rho_a C_{Ma} (\beta_{wind}, \psi, vrel) A_{vy} L_x \left[(u_{ax} - u)^2 + (u_{ay} - v)^2 \right] \quad (28)$$

Wave drift forces and moments (Faltinsen, 1998)

$$F_x^{drift} = 2 \int_0^\infty S(w) C_x^{drift}(w, \beta_{wave}) dw \quad (29)$$

$$F_y^{drift} = 2 \int_0^\infty S(w) C_y^{drift}(w, \beta_{wave}) dw \quad (30)$$

$$M_z^{drift} = 2 \int_0^\infty S(w) C_z^{drift}(w, \beta_{wave}) dw \quad (31)$$

Riser forces and moment—model 1 (Bhalla and Cao, 2005).

For small displacements between the rig and the wellhead, relative to the water depth, we can write the riser forces and moment acting on the rig as

$$F_x^{riser} = -k_r (X_c - X_{wh}) \cos \psi - k_r (Y_c - Y_{wh}) \sin \psi \quad (32)$$

$$F_y^{riser} = k_r (X_c - X_{wh}) \sin \psi - k_r (Y_c - Y_{wh}) \cos \psi \quad (33)$$

$$M_z^{riser} = F_y^{riser} drtcm \quad (34)$$

$$k_r = \frac{toptension}{waterdepth} \quad (35)$$

Riser forces and moment—model 2.

When solving the riser and drift-off equations simultaneously, the instantaneous angle of the riser at the upper flex joint can be used to compute the instantaneous force on the rig caused by the riser. Riser forces and moment can be written as

$$F_x^{riser} = T_{pX} \cos \psi + T_{pY} \sin \psi \quad (36)$$

$$F_y^{riser} = -T_{pX} \sin \psi + T_{pY} \cos \psi \quad (37)$$

$$M_z^{riser} = -F_x^{riser} drtcm_y + F_y^{riser} drtcm_x \quad (38)$$

With

$$T_{pX} = -T \sin \left(\text{atan} \left(\frac{dy}{dx} \right) \right) \frac{(X_c - X_{wh})}{\sqrt{(X_c - X_{wh})^2 + (Y_c - Y_{wh})^2}} \quad (39)$$

$$T_{pY} = -T \sin \left(\text{atan} \left(\frac{dy}{dx} \right) \right) \frac{(Y_c - Y_{wh})}{\sqrt{(X_c - X_{wh})^2 + (Y_c - Y_{wh})^2}} \quad (40)$$

It is important to notice that as riser force modeled in this way is only valid for relatively short drift-off distances. Modeled in this way the riser, wellhead and casing could stop the drift-off at some point far away from the wellhead but one of these components would fail before that point is reached.

Another limitation is in the assumptions used to write the equations. The model assumes that as the rig moves away from the zero-offset position, the riser profile can be approximated by a straight line. But as it can be seen in Fig. 4, which is a zoom in in Fig. 9, we can see that in the first 20 m of drift-off, the riser profile is not well approximated by a straight line, moreover, during these first 20 m, approximately, the angle at the upper flex joint would make the riser force acting on the rig accelerate the drift-off and not act against it, as described by the equations.

Coordinates of the rotary table

To find the point of critical failure of the System Riser String/Wellhead/Casing, it is necessary to find the equations that relate the movement of the Rotary table to the movement of the center of mass of the rig. Once the drift-off equations are solved and X_c , Y_c e ψ are found, the coordinates of the Rotary table are found as

$$X_{mr} = X_c + drtcm \cos \psi \quad (41)$$

$$Y_{mr} = Y_c + drtcm \sin \psi \quad (42)$$

3. Solution to the riser analysis equation with a non time-dependent force

The equation

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y_1}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(T_e \frac{\partial y_1}{\partial x} \right) = F_x(x) \quad (43)$$

can be solved as a system of difference equations as

$$\begin{aligned} \frac{y_{1i+2}}{\Delta x^4} - \left(\frac{4}{\Delta x^4} + \frac{p1}{\Delta x^2} + \frac{p2}{2\Delta x} \right) y_{1i+1} + \left(\frac{6}{\Delta x^4} + \frac{2p1}{\Delta x^2} \right) y_{1i} \\ - \left(\frac{4}{\Delta x^4} + \frac{p1}{\Delta x^2} - \frac{p2}{2\Delta x} \right) y_{1i-1} + \frac{y_{1i-2}}{\Delta x^4} = r \end{aligned} \quad (44)$$

where

$$p1 = \frac{T_e}{EI}, p2 = \frac{1}{EI} \frac{\Delta T_e}{\Delta x}, r = \frac{F_x(x)}{EI} \quad (45)$$

For a riser string divided in $n-1$ segments of length Δx , there will be n nodes where the equations in finite differences can be applied. Two extra imaginary nodes will be needed before node 1 and 2 extra imaginary nodes will be needed after node n .

Proceeding in the manner, the indexes 1 and 2 will be used for the imaginary points before the lower flex joint, the indexes 3 to $n+2$ will be used for the nodes along the riser string and the indexes $n+3$ and $n+4$ will be used for the imaginary points above the upper flex joint.

The boundary conditions are written as.

Lower Flex Joint

$$y_{13} = 0 \quad (46)$$

$$\left(\frac{1}{\Delta x^2} - \frac{K_I}{2EI\Delta x} \right) y_{14} - 2 \frac{y_{13}}{\Delta x^2} + \left(\frac{1}{\Delta x^2} + \frac{K_I}{2EI\Delta x} \right) y_{12} = 0 \quad (47)$$

and.

Upper Flex Joint

$$y_{1n+2} = \text{offset} \quad (48)$$

And

$$\begin{aligned} \frac{y(i+3)}{\Delta x^4} - \left(\frac{4}{\Delta x^4} + \frac{p1(i+1)}{\Delta x^2} + \frac{p2(i+1)}{2\Delta x} \right) y(i+2) \\ + \left(\frac{6}{\Delta x^4} + \frac{2p1(i+1)}{\Delta x^2} \right) y(i+1) - \left(\frac{4}{\Delta x^4} + \frac{p1(i+1)}{\Delta x^2} - \frac{p2(i+1)}{2\Delta x} \right) y_{a1} + \frac{y_{a2}}{\Delta x^4} = r_{i+1} \end{aligned} \quad (54)$$

$$\begin{aligned} \frac{y(i+2)}{\Delta x^4} - \left(\frac{4}{\Delta x^4} + \frac{p1_a}{\Delta x^2} + \frac{p2_a}{2\Delta x} \right) y(i+1) \\ + \left(\frac{6}{\Delta x^4} + \frac{2p1_a}{\Delta x^2} \right) y_{a1} - \left(\frac{4}{\Delta x^4} + \frac{p1_a}{\Delta x^2} - \frac{p2_a}{2\Delta x} \right) y_{a2} + \frac{y_{a3}}{\Delta x^4} = r_i \end{aligned} \quad (55)$$

$$\begin{aligned} \frac{y(i+2)}{\Delta x^4} - \left(\frac{4}{\Delta x^4} + \frac{p1_a}{\Delta x^2} + \frac{p2_a}{2\Delta x} \right) y(i+1) \\ + \left(\frac{6}{\Delta x^4} + \frac{2p1_a}{\Delta x^2} \right) y_{a1} - \left(\frac{4}{\Delta x^4} + \frac{p1_a}{\Delta x^2} - \frac{p2_a}{2\Delta x} \right) y_{a2} + \frac{y_{a3}}{\Delta x^4} = r_i \end{aligned} \quad (56)$$

$$\left(\frac{1}{\Delta x^2} + \frac{K_S}{2EI\Delta x} \right) y_{1n+3} - 2 \frac{y_{1n+2}}{\Delta x^2} + \left(\frac{1}{\Delta x^2} - \frac{K_S}{2EI\Delta x} \right) y_{1n+1} = \frac{K_S \delta}{EI} \quad (49)$$

The solution of this system of $n+4$ equations and $n+4$ unknowns yields $y_1(x)$.

Discontinuities

In points where

$$EI(i+1) \neq EI(i) \quad (50)$$

we can define in the vicinity of the point “p” of discontinuity, the quantities

$$y_p^a(i-2), y_p^a(i-1), y_p^a(i) \quad (51)$$

Before point “p,” and

$$y_p^d(i+1), y_p^d(i+2) \quad (52)$$

After the point “p.”

According to Fig. 17

Defining

$$\begin{aligned} p1_a &= \frac{T_e(i)}{EI(i+1)} & p1_d &= \frac{T_e(i)}{EI(i)} \\ p2_a &= \frac{1}{EI(i+1)} \frac{\Delta T_e}{\Delta x} & p2_d &= \frac{1}{EI(i)} \frac{\Delta T_e}{\Delta x} \\ r_a &= \frac{F_x(i)}{EI(i+1)} & r_d &= \frac{F_x(i)}{EI(i)} \end{aligned} \quad (53)$$

we can write the equations

Fig. 17 Virtual displacements at a discontinuity

$$\begin{aligned}
& \frac{y_{d2}}{\Delta x^4} - \left(\frac{4}{\Delta x^4} + \frac{p_{1d}}{\Delta x^2} + \frac{p_{2d}}{2\Delta x} \right) y_{d1} \\
& + \left(\frac{6}{\Delta x^4} + \frac{2p_{1d}}{\Delta x^2} \right) y(i) \\
& - \left(\frac{4}{\Delta x^4} + \frac{p_{1d}}{\Delta x^2} - \frac{p_{2d}}{2\Delta x} \right) y(i-1) \\
& + \frac{y(i-2)}{\Delta x^4} = r_i
\end{aligned} \quad (57)$$

And

$$EI_a \frac{\partial^3 y_{p-}^{j+1}}{\partial x^3} = EI_d \frac{\partial^3 y_{p+}^{j+1}}{\partial x^3} \quad (58)$$

$$EI_a \frac{\partial^2 y_{p-}^{j+1}}{\partial x^2} = EI_d \frac{\partial^2 y_{p+}^{j+1}}{\partial x^2} \quad (59)$$

$$\frac{\partial y_{p-}^{j+1}}{\partial x} = \frac{\partial y_{p+}^{j+1}}{\partial x} \quad (60)$$

$$y_p^a(i) = y(i) \quad (61)$$

where

$$\frac{\partial^3 y_{p-}^{j+1}}{\partial x^3} = \frac{y_{d2} - 2y_{d1}(i+1) + 2y^{j+1}(i-1) - y^{j+1}(i-2)}{2\Delta x^3} \quad (62)$$

$$\frac{\partial^3 y_{p+}^{j+1}}{\partial x^3} = \frac{y^{j+1}(i+2) - 2y^{j+1}(i+1) + 2y_{a2} - y_{a3}}{2\Delta x^3} \quad (63)$$

$$\frac{\partial^2 y_{p-}^{j+1}}{\partial x^2} = \frac{y_{d1} - 2y^{j+1}(i) + y^{j+1}(i-1)}{\Delta x^2} \quad (64)$$

$$\frac{\partial^2 y_{p+}^{j+1}}{\partial x^2} = \frac{y^{j+1}(i+1) - 2y_{a1} + y_{a2}}{\Delta x^2} \quad (65)$$

$$\frac{\partial y_{p-}^{j+1}}{\partial x} = \frac{y_{d1} - y^{j+1}(i-1)}{2\Delta x} \quad (66)$$

$$\frac{\partial y_{p+}^{j+1}}{\partial x} = \frac{y^{j+1}(i+1) - y_{a2}}{2\Delta x} \quad (67)$$

Once the velocity profile and the acceleration profile along the riser are obtained, the current velocity along the riser can be adjusted to yield the current velocity relative to the riser. Considering the inertia forces, the equation is rewritten as

$$\begin{aligned}
& \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y_1}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(T_e \frac{\partial y_1}{\partial x} \right) \\
& = \frac{1}{2} \rho C_D D (u_{c,relative}) |u_{c,relative}| - (m + \rho C_a A) \frac{\partial^2 y}{\partial t^2}
\end{aligned} \quad (68)$$

This equation is then coupled to the casing equations.

4. Solution to the riser analysis equation with a time-dependent force

To solve the equation

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y_2}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(T_e \frac{\partial y_2}{\partial x} \right) + (m + \rho C_a A) \frac{\partial^2 y_2}{\partial t^2} = F_{xt}(x, t) \quad (69)$$

$$v = \frac{\partial y_2}{\partial t} \quad (70)$$

we define (Richtmyer, 1968).

and

$$w = \alpha \frac{\partial^2 y_2}{\partial x^2} + \beta \frac{\partial y_2}{\partial x} + \gamma y_2 \quad (71)$$

where

$$\begin{aligned}
a &= \frac{EI}{(m + \rho C_a A)}, b = \frac{T_e}{(m + \rho C_a A)}, \\
c &= \frac{\frac{dT_e}{dx}}{(m + \rho C_a A)}, d = \frac{F(x, t)}{(m + \rho C_a A)}
\end{aligned} \quad (72)$$

$$\alpha^2 = a, \beta^2 - \alpha\gamma = b, \gamma\beta = c \quad (73)$$

The equation can be rewritten as

$$\frac{\partial v}{\partial t} = -\alpha \frac{\partial^2 w}{\partial x^2} + \beta \frac{\partial w}{\partial x} + d \quad (74)$$

$$\frac{\partial w}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2} + \beta \frac{\partial v}{\partial x} + \gamma v \quad (75)$$

With

$$\beta^3 - b\beta - \sqrt{ac} = 0 \quad (76)$$

which is an equation with three real roots. The value of β to be used is the one that results in the minimum potential energy of the riser configuration.

Boundary conditions:

Lower Flex Joint:

$$y_2(0, t) = 0 \quad (77)$$

$$\frac{\partial w}{\partial t}(0, t) = \left(\alpha \frac{K_I}{EI} + \beta \right) \frac{\partial v}{\partial x}(0, t) \quad (78)$$

Upper Flex Joint – fixed offset:

$$y_2(l, t) = 0 \quad (79)$$

$$\frac{\partial w}{\partial t}(l, t) = \left(-\alpha \frac{K_S}{EI} + \beta \right) \frac{\partial v}{\partial x}(l, t) \quad (80)$$

Upper Flex Joint – Drift-off:

$$\frac{\partial w}{\partial t}(l, t) = (-\alpha K_S + \beta) \frac{\partial v}{\partial x}(l, t) + \gamma v(l, t) \quad (81)$$

$$y_2(l, t) = \text{offset} \quad (82)$$

where $v(l, t)$ is the drift-off velocity.

Using the finite difference method Forward–Backward as in Richtmyer, 1968, we write

$$w_i^{j+1} = w_i^j - \frac{\alpha t}{\Delta x^2} (w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}) + \frac{\beta \Delta t}{2\Delta x} (w_{i+1}^{j+1} - w_{i-1}^{j+1}) + \Delta t d(x, t) \quad (83)$$

$$w_i^{j+1} = w_i^j + \frac{\alpha \Delta t}{\Delta x^2} (v_{i+1}^j - 2v_i^j + v_{i-1}^j) + \frac{\beta \Delta t}{2\Delta x} (v_{i+1}^j - v_{i-1}^j) + \gamma \Delta t v_i^j \quad (84)$$

Given the initial conditions:

$$y_2(x, 0) = f(x), 0 < x < l \quad (85)$$

$$\frac{\partial y_2}{\partial t}(x, 0) = 0, 0 < x < l \quad (86)$$

By means of successive iterations, the equations in finite differences can be solved. The displacements are obtained from

$$y_2^{j+1} = y_2^j + \Delta t v_i^j \quad (87)$$

Discontinuities

In points where

$$EI(i+1) \neq EI(i) \text{ or } M(i+1) \neq M(i) \quad (88)$$

we can define virtual displacements in the vicinity of point “p” of discontinuity:

$$y_p^a(i-2), y_p^a(i-1), y_p^a(i) \quad (89)$$

Before the point of discontinuity, and

$$y_p^d(i+1), y_p^d(i+2) \quad (90)$$

After the point.

Please refer to Fig. 17 once again for the schematics showing the virtual displacements. By defining the quantities:

$$\begin{aligned} & \frac{w_{p-}}{\frac{\partial y_{p-}}{\partial x}} \quad \frac{w_{p+}}{\frac{\partial y_{p+}}{\partial x}} \\ & \frac{\frac{\partial^2 y_{p-}}{\partial x^2}}{\frac{\partial^4 y_{p-}}{\partial x^4}} \quad \frac{\frac{\partial^2 y_{p+}}{\partial x^2}}{\frac{\partial^4 y_{p+}}{\partial x^4}} \\ & \frac{\frac{\partial^4 y_{p-}}{\partial x^4}}{\frac{\partial^2 y_{p-}}{\partial x^2}} \quad \frac{\frac{\partial^4 y_{p+}}{\partial x^4}}{\frac{\partial^2 y_{p+}}{\partial x^2}} \end{aligned} \quad (91)$$

We can write the equations

$$\frac{v_p^{j+1} - v_p^j}{\Delta t} = \frac{F(x, t)}{M_a} - \frac{EI_a}{M_a} \frac{\partial^4 y_{p-}^{j+1}}{\partial x^4} + \frac{Te_a}{M_a} \frac{\partial^2 y_{p-}^{j+1}}{\partial x^2} + \frac{1}{M_a} \frac{dTe_a}{dx} \frac{\partial y_{p-}^{j+1}}{\partial x} \quad (92)$$

$$\frac{v_p^{j+1} - v_p^j}{\Delta t} = \frac{F(x, t)}{M_d} - \frac{EI_d}{M_d} \frac{\partial^4 y_{p+}^{j+1}}{\partial x^4} + \frac{Te_d}{M_d} \frac{\partial^2 y_{p+}^{j+1}}{\partial x^2} + \frac{1}{M_d} \frac{dTe_d}{dx} \frac{\partial y_{p+}^{j+1}}{\partial x} \quad (93)$$

$$EI_a \frac{\partial^3 y_{p-}^{j+1}}{\partial x^3} = EI_d \frac{\partial^3 y_{p+}^{j+1}}{\partial x^3} \quad (94)$$

$$EI_a \frac{\partial^2 y_{p-}^{j+1}}{\partial x^2} = EI_d \frac{\partial^2 y_{p+}^{j+1}}{\partial x^2} \quad (95)$$

where

$$\frac{\partial^4 y_{p-}^{j+1}}{\partial x^4} = \frac{y_{d2} - 4y_{d1} + 6y^{j+1}(i) - 4y^{j+1}(i-1) + y^{j+1}(i-2)}{\Delta x^4} \quad (96)$$

$$\frac{\partial^4 y_{p+}^{j+1}}{\partial x^4} = \frac{y^{j+1}(i+2) - 4y^{j+1}(i+1) + 6y^{j+1}(i) - 4y_{a2} + y_{a3}}{\Delta x^4} \quad (97)$$

$$\frac{\partial^3 y_{p-}^{j+1}}{\partial x^3} = \frac{y_{d2} - 2y_{d1} + 2y^{j+1}(i-1) - y^{j+1}(i-2)}{2\Delta x^3} \quad (98)$$

$$\frac{\partial^3 y_{p+}^{j+1}}{\partial x^3} = \frac{y^{j+1}(i+2) - 2y^{j+1}(i+1) + 2y_{a2} - y_{a3}}{2\Delta x^3} \quad (99)$$

$$\frac{\partial^2 y_{p-}^{j+1}}{\partial x^2} = \frac{y_{d1} - 2y^{j+1}(i) + y^{j+1}(i-1)}{\Delta x^2} \quad (100)$$

$$\frac{\partial^2 y_{p+}^{j+1}}{\partial x^2} = \frac{y^{j+1}(i+1) - 2y^{j+1}(i) + y_{a2}}{\Delta x^2} \quad (101)$$

$$\frac{\partial y_p^{-j+1}}{\partial x} = \frac{y_{d1} - y^{j+1}(i-1)}{2\Delta x} \quad (102)$$

$$\frac{\partial y_p^{+j+1}}{\partial x} = \frac{y^{j+1}(i+1) - y_{a2}}{2\Delta x} \quad (103)$$

With this system of 5 equations and 5 unknowns, we obtain the value v_p^{j+1} at the discontinuity. The displacement at the discontinuity is

$$y_p^{j+1} = y_p^j + \Delta t v_p^j \quad (104)$$

$$v_i^{j+1} = (1 - e\Delta t)v_i^j - \frac{\alpha\Delta t}{\Delta x^2}(w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}) + \frac{\beta\Delta t}{2\Delta x}(w_{i+1}^{j+1} - w_{i-1}^{j+1}) + \Delta t d(x, t) \quad (105)$$

Dissipative effects:

Considering structural friction effects in the riser the equations become

$$v_i^{(j+1)} = (1 - e\Delta t)v_i^j - \alpha\Delta t/\Delta x^2(w_{(i+1)}^{(j+1)} - 2w_i^{(j+1)} + w_{(i-1)}^{(j+1)}) + \beta\Delta t/2\Delta x(w_{(i+1)}^{(j+1)} - w_{(i-1)}^{(j+1)}) + \Delta t d(x, t) \quad (106)$$

$$w_i^{j+1} = w_i^j + \frac{\alpha\Delta t}{\Delta x^2}(v_{i+1}^j - 2v_i^j + v_{i-1}^j) + \frac{\beta\Delta t}{2\Delta x}(v_{i+1}^j - v_{i-1}^j) + \gamma\Delta t v_i^j \quad (107)$$

Stability of the solution

To obtain the Amplification Matrix, we write

$$v_i^j = v_0 e^{ipx} \quad (108)$$

$$v_i^{j+1} = \lambda_1 v_0 e^{ipx} \quad (109)$$

$$w_i^j = w_0 e^{iqx} \quad (110)$$

$$w_i^{j+1} = \lambda_2 w_0 e^{iqx} \quad (111)$$

If we define

$$k_1 = \frac{\alpha\Delta t}{\Delta x^2} \quad (112)$$

$$k_2 = \frac{\beta\Delta t}{2\Delta x} \quad (113)$$

And

$$A = (1 - e\Delta t) \quad (114)$$

$$B = 2k_1(1 - \cos q\Delta x) + 2k_2 i \sin q\Delta x \quad (115)$$

$$C = 1 \quad (116)$$

$$D = 2k_1(\cos p\Delta x - 1) + 2k_2 i \sin p\Delta x \quad (117)$$

The equations can be written as

$$\begin{bmatrix} v_i^{j+1} \\ w_i^{j+1} \end{bmatrix} = \begin{bmatrix} A + BD & BC \\ D & C \end{bmatrix} \begin{bmatrix} v_i^j \\ w_i^j \end{bmatrix} + \begin{bmatrix} \Delta t d(x, t) \\ 0 \end{bmatrix} \quad (118)$$

The amplification matrix is defined as

$$[G] = \begin{bmatrix} A + BD & BC \\ D & C \end{bmatrix} \quad (119)$$

The Von Neumann condition for the stability of this system of equations requires that (Richtmyer, 1968):

$$|\lambda_i| < 1 + O(t) \quad (120)$$

where.

λ_i are the eigenvalues of the amplification matrix.

We can adopt an even more restrictive condition

$$|\lambda_i| < 1 \quad (121)$$

5. Solution to the casing analysis equation

The effects of inertia and accelerations of the casing and cement are not going to be considered, and we will write the following equation for the casing subject to the weight of the BOP and to the reaction of the soil

Fig. 18 P-y curves and E(x,y)

$$E_c I_c \frac{\partial^4 y_c}{\partial x_c^4} + Q(x_c) \frac{\partial^2 y_c}{\partial x_c^2} = F_{soil}(x_c, y_c) \quad (122)$$

We will make use of the $P - y$ curves represented in Fig. 18.

The equation will be re-written as (API, 2004)

$$EI \frac{\partial^4 y}{\partial x^4} + Q(x) \frac{\partial^2 y}{\partial x^2} + E(x, y)y = 0 \quad (123)$$

Soil with different characteristics have different curves $E(x, y)$. Some basic models are available in the literature as in Shamsheer 1990; API (2002).

Solution to the Casing Equation on Elastic Foundation.

To solve the Casing Equation on Elastic Foundation, we will first divide the casing in small elements so that the equation can be solved for the element i :

$$EI \frac{\partial^4 y}{\partial x^4} + Q(x) \frac{\partial^2 y}{\partial x^2} + E(x_i, y)y = 0 \quad (124)$$

But since the value $E(x_i, y)$ is also dependent of y , which we intend to calculate, we will use a method based on iterations, where an initial state will be assumed for the curve y (e.g., a straight vertical line $y(x)=0$). With these assumptions, the initial values of $E(x_i, y)$ will correspond to the inclination of the P - y curves at $y=0$, for each depth x_i . In other words, for the first interaction:

$$E(x_i, y) = \frac{dP}{dy}(x_i, y=0) \quad (125)$$

So that the problems become redefined as solving sequentially the equation:

$$y_i(x) = C_{1i} e^{\beta_{1i} x} \cos \beta_{2i} x + C_{2i} e^{\beta_{1i} x} \sin \beta_{2i} x + C_{3i} e^{-\beta_{1i} x} \cos \beta_{2i} x + C_{4i} e^{-\beta_{1i} x} \sin \beta_{2i} x \quad (126)$$

where

$$\beta_{1i} = \sqrt[4]{(ct_1)^2 + (ct_2)^2} \cos \left(\frac{\pi}{2} - \frac{1}{2} \arcsin \left(\frac{ct_2}{\sqrt{(ct_1)^2 + (ct_2)^2}} \right) \right) \quad (127)$$

$$\beta_{2i} = \sqrt[4]{(ct_1)^2 + (ct_2)^2} \sin \left(\frac{\pi}{2} - \frac{1}{2} \arcsin \left(\frac{ct_2}{\sqrt{(ct_1)^2 + (ct_2)^2}} \right) \right) \quad (128)$$

With

$$ct_1 = \frac{q_i}{2EI} \quad (129)$$

$$ct_2 = \frac{\sqrt{4EI k_{ij} - q_i^2}}{2EI} \quad (130)$$

Considering $l = \Delta x$ the length of each casing element, for element i and the subsequent element $i + 1$, we can write the equations

$$\begin{aligned} y_{i+1}(0) &= y_i(l) \\ y'_{i+1}(0) &= y'_i(l) \\ y_{i+1}(l) &= y_i(0) \\ y'''_{i+1}(0) &= y'''_i(l) \end{aligned} \quad (131)$$

Performing the differentiation, we arrive at the equations:

$$\begin{bmatrix} y_i(x) \\ y'_i(x) \\ y''_i(x) \\ y'''_i(x) \end{bmatrix} = \begin{bmatrix} f_{11}(x) & f_{12}(x) & f_{13}(x) & f_{14}(x) \\ f_{21}(x) & f_{22}(x) & f_{23}(x) & f_{24}(x) \\ f_{31}(x) & f_{32}(x) & f_{33}(x) & f_{34}(x) \\ f_{41}(x) & f_{42}(x) & f_{43}(x) & f_{44}(x) \end{bmatrix} \begin{bmatrix} C_{1i} \\ C_{2i} \\ C_{3i} \\ C_{4i} \end{bmatrix} \quad (132)$$

where

$$f_{11}(\beta_i, x) = e^{\beta_{1i} x} \cos \beta_{2i} x \quad (133)$$

$$f_{21}(\beta_i, x) = \beta_{1i} e^{\beta_{1i} x} \cos \beta_{2i} x - \beta_{2i} e^{\beta_{1i} x} \sin \beta_{2i} x \quad (134)$$

$$\begin{aligned} f_{31}(\beta_i, x) &= \beta_{1i}^2 e^{\beta_{1i} x} \cos \beta_{2i} x - 2\beta_{1i} \beta_{2i} e^{\beta_{1i} x} \sin \beta_{2i} x \\ &\quad - \beta_{2i}^2 e^{\beta_{1i} x} \cos \beta_{2i} x \end{aligned} \quad (135)$$

$$\begin{aligned} f_{41}(\beta_i, x) &= \beta_{1i}^3 e^{\beta_{1i} x} \cos \beta_{2i} x - 3\beta_{1i}^2 \beta_{2i} e^{\beta_{1i} x} \sin \beta_{2i} x \\ &\quad - 3\beta_{1i} \beta_{2i}^2 e^{\beta_{1i} x} \cos \beta_{2i} x + \beta_{2i}^3 e^{\beta_{1i} x} \sin \beta_{2i} x \end{aligned} \quad (136)$$

$$f_{12}(\beta_i, x) = e^{\beta_{1i} x} \sin \beta_{2i} x \quad (137)$$

$$f_{22}(\beta_i, x) = \beta_{1i} e^{\beta_{1i} x} \sin \beta_{2i} x + \beta_{2i} e^{\beta_{1i} x} \cos \beta_{2i} x \quad (138)$$

$$\begin{aligned} f_{32}(\beta_i, x) &= \beta_{1i}^2 e^{\beta_{1i} x} \sin \beta_{2i} x + 2\beta_{1i} \beta_{2i} e^{\beta_{1i} x} \cos \beta_{2i} x \\ &\quad - \beta_{2i}^2 e^{\beta_{1i} x} \sin \beta_{2i} x \end{aligned} \quad (139)$$

$$\begin{aligned} f_{42}(\beta_i, x) &= \beta_{1i}^3 e^{\beta_{1i} x} \sin \beta_{2i} x + 3\beta_{1i}^2 \beta_{2i} e^{\beta_{1i} x} \cos \beta_{2i} x \\ &\quad - 3\beta_{1i} \beta_{2i}^2 e^{\beta_{1i} x} \sin \beta_{2i} x - \beta_{2i}^3 e^{\beta_{1i} x} \cos \beta_{2i} x \end{aligned} \quad (140)$$

$$f_{13}(\beta_i, x) = e^{-\beta_{1i} x} \cos \beta_{2i} x \quad (141)$$

$$f_{23}(\beta_i, x) = -\beta_{1i} e^{-\beta_{1i} x} \cos \beta_{2i} x - \beta_{2i} e^{-\beta_{1i} x} \sin \beta_{2i} x \quad (142)$$

$$f_{33}(\beta_i, x) = \beta_1^2 e^{-\beta_1 x} \cos \beta_2 x + 2\beta_1 \beta_2 e^{-\beta_1 x} \sin \beta_2 x - \beta_2^2 e^{-\beta_1 x} \cos \beta_2 x \quad (143)$$

$$f_{43}(\beta_i, x) = -\beta_1^3 e^{-\beta_1 x} \cos \beta_2 x - 3\beta_1^2 \beta_2 e^{-\beta_1 x} \sin \beta_2 x + 3\beta_1 \beta_2^2 e^{-\beta_1 x} \cos \beta_2 x + \beta_2^3 e^{-\beta_1 x} \sin \beta_2 x \quad (144)$$

$$f_{14}(\beta_i, x) = e^{-\beta_1 x} \sin \beta_2 x \quad (145)$$

$$\begin{bmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{bmatrix} = [MatrixA] \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \end{bmatrix} \quad (153)$$

$$f_{24}(\beta_i, x) = -\beta_1 e^{-\beta_1 x} \sin \beta_2 x + \beta_2 e^{-\beta_1 x} \cos \beta_2 x \quad (146)$$

$$f_{34}(\beta_i, x) = \beta_1^2 e^{-\beta_1 x} \sin \beta_2 x - 2\beta_1 \beta_2 e^{-\beta_1 x} \cos \beta_2 x - \beta_2^2 e^{-\beta_1 x} \sin \beta_2 x \quad (147)$$

$$f_{44}(\beta_i, x) = -\beta_1^3 e^{-\beta_1 x} \sin \beta_2 x + 3\beta_1^2 \beta_2 e^{-\beta_1 x} \cos \beta_2 x + 3\beta_1 \beta_2^2 e^{-\beta_1 x} \sin \beta_2 x - \beta_2^3 e^{-\beta_1 x} \cos \beta_2 x \quad (148)$$

The equations that link the elements i and $i + 1$ can then be written as

$$\begin{bmatrix} y_{i+1}(0) \\ y'_{i+1}(0) \\ y''_{i+1}(0) \\ y'''_{i+1}(0) \end{bmatrix} = \begin{bmatrix} y_i(l) \\ y'_i(l) \\ y''_i(l) \\ y'''_i(l) \end{bmatrix} \quad (149)$$

$$\begin{bmatrix} f_{11}(\beta_i, l) & f_{12}(\beta_i, l) & f_{13}(\beta_i, l) & f_{14}(\beta_i, l) \\ f_{21}(\beta_i, l) & f_{22}(\beta_i, l) & f_{23}(\beta_i, l) & f_{24}(\beta_i, l) \\ f_{31}(\beta_i, l) & f_{32}(\beta_i, l) & f_{33}(\beta_i, l) & f_{34}(\beta_i, l) \\ f_{41}(\beta_i, l) & f_{42}(\beta_i, l) & f_{43}(\beta_i, l) & f_{44}(\beta_i, l) \end{bmatrix} \quad (150)$$

Defining

$$[Matrix2]_i = \begin{bmatrix} f_{11}(\beta_i, 0) & f_{12}(\beta_i, 0) & f_{13}(\beta_i, 0) & f_{14}(\beta_i, 0) \\ f_{21}(\beta_i, 0) & f_{22}(\beta_i, 0) & f_{23}(\beta_i, 0) & f_{24}(\beta_i, 0) \\ f_{31}(\beta_i, 0) & f_{32}(\beta_i, 0) & f_{33}(\beta_i, 0) & f_{34}(\beta_i, 0) \\ f_{41}(\beta_i, 0) & f_{42}(\beta_i, 0) & f_{43}(\beta_i, 0) & f_{44}(\beta_i, 0) \end{bmatrix} \quad (151)$$

The coefficients for $i + 1$ can be obtained from

$$\begin{bmatrix} C_{1i+1} \\ C_{2i+1} \\ C_{3i+1} \\ C_{4i+1} \end{bmatrix} = [Matrix2]_{i+1}^{-1} \cdot [Matrix1]_i \begin{bmatrix} C_{1i} \\ C_{2i} \\ C_{3i} \\ C_{4i} \end{bmatrix} \quad (152)$$

The coefficients at the lower extremity of the casing can be expressed as

Boundary conditions:

The boundary conditions at the upper extremity of the casing are written in terms of the bending moment and the horizontal force acting at this point. If we denote M_0 e V_0 , this bending moment and horizontal force, respectively, can be written as

$$\frac{d^2 y_1}{dx^2}(0) = \frac{M_0}{EI} \quad (154)$$

$$\frac{d^3 y_1}{dx^3}(0) = \frac{V_0}{EI} \quad (155)$$

The boundary conditions for the lower extremity of the casing are of zero bending moment and zero horizontal force, which correspond to.

Defining the matrix

$$[MatrixP1] = \begin{bmatrix} MatrixP1_{11} & MatrixP1_{12} & MatrixP1_{13} & MatrixP1_{14} \\ MatrixP1_{21} & MatrixP1_{22} & MatrixP1_{23} & MatrixP1_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (156)$$

With

$$MatrixP1_{11} = \beta_1^2 - \beta_2^2 \quad (157)$$

$$MatrixP1_{12} = 2\beta_1 \beta_2 \quad (158)$$

$$MatrixP1_{13} = \beta_1^2 - \beta_2^2 \quad (159)$$

$$\text{Matrix}P1_{14} = -2\beta 1_i \beta 2_i \quad (160)$$

$$\text{Matrix}P1_{21} = \beta 1_i^3 - 3\beta 1_i \beta 2_i^2 \quad (161)$$

$$\text{Matrix}P1_{22} = 3\beta 1_i^2 \beta 2_i - \beta 2_i^3 \quad (162)$$

$$\text{Matrix}P1_{23} = 3\beta 1_i \beta 2_i^2 - \beta 1_i^3 \quad (163)$$

$$\text{Matrix}P1_{24} = 3\beta 1_i^2 \beta 2_i - \beta 2_i^3 \quad (164)$$

And

$$[\text{Matrix}P2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \text{Matrix}P2_{31} & \text{Matrix}P2_{32} & \text{Matrix}P2_{33} & \text{Matrix}P2_{34} \\ \text{Matrix}P2_{41} & \text{Matrix}P2_{42} & \text{Matrix}P2_{43} & \text{Matrix}P2_{44} \end{bmatrix} \quad (165)$$

where

$$\text{Matrix}P2_{31} = \beta 1_n^2 e^{\beta 1_n l} \cos \beta 2_n l - 2\beta 1_n \beta 2_n e^{\beta 1_n l} \sin \beta 2_n l - \beta 2_n^2 e^{\beta 1_n l} \cos \beta 2_n l \quad (166)$$

$$\text{Matrix}P2_{32} = \beta 1_n^2 e^{\beta 1_n l} \sin \beta 2_n l + 2\beta 1_n \beta 2_n e^{\beta 1_n l} \cos \beta 2_n l - \beta 2_n^2 e^{\beta 1_n l} \sin \beta 2_n l \quad (167)$$

$$\text{Matrix}P2_{33} = \beta 1_n^2 e^{-\beta 1_n l} \cos \beta 2_n l + 2\beta 1_n \beta 2_n e^{-\beta 1_n l} \sin \beta 2_n l - \beta 2_n^2 e^{-\beta 1_n l} \cos \beta 2_n l \quad (168)$$

$$\text{Matrix}P2_{34} = \beta 1_n^2 e^{-\beta 1_n l} \sin \beta 2_n l - 2\beta 1_n \beta 2_n e^{-\beta 1_n l} \cos \beta 2_n l - \beta 2_n^2 e^{-\beta 1_n l} \sin \beta 2_n l \quad (169)$$

$$\text{Matrix}P2_{41} = \beta 1_n^3 e^{\beta 1_n l} \cos \beta 2_n l - 3\beta 1_n^2 \beta 2_n e^{\beta 1_n l} \sin \beta 2_n l - 3\beta 1_n \beta 2_n^2 e^{\beta 1_n l} \cos \beta 2_n l + \beta 2_n^3 e^{\beta 1_n l} \sin \beta 2_n l \quad (170)$$

$$\text{Matrix}P2_{42} = \beta 1_n^3 e^{\beta 1_n l} \sin \beta 2_n l + 3\beta 1_n^2 \beta 2_n e^{\beta 1_n l} \cos \beta 2_n l - 3\beta 1_n \beta 2_n^2 e^{\beta 1_n l} \sin \beta 2_n l - \beta 2_n^3 e^{\beta 1_n l} \cos \beta 2_n l \quad (171)$$

$$\text{Matrix}P2_{43} = -\beta 1_n^3 e^{-\beta 1_n l} \cos \beta 2_n l - 3\beta 1_n^2 \beta 2_n e^{-\beta 1_n l} \sin \beta 2_n l + 3\beta 1_n \beta 2_n^2 e^{-\beta 1_n l} \cos \beta 2_n l + \beta 2_n^3 e^{-\beta 1_n l} \sin \beta 2_n l \quad (172)$$

$$\text{Matrix}P2_{44} = -\beta 1_n^3 e^{-\beta 1_n l} \sin \beta 2_n l + 3\beta 1_n^2 \beta 2_n e^{-\beta 1_n l} \cos \beta 2_n l + 3\beta 1_n \beta 2_n^2 e^{-\beta 1_n l} \sin \beta 2_n l - \beta 2_n^3 e^{-\beta 1_n l} \cos \beta 2_n l \quad (173)$$

And

$$[\text{Matrix}P2] \cdot \begin{bmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (175)$$

Performing a few operations, we arrive at

$$\begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \end{bmatrix} = \{[\text{Matrix}P1] + [\text{Matrix}P2] \cdot [\text{Matrix}A]\}^{-1} \begin{bmatrix} \frac{M_0}{EI} \\ \frac{V_0}{EI} \\ 0 \\ 0 \end{bmatrix} \quad (176)$$

Having obtained the coefficients

$$\begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \end{bmatrix}$$

The boundary conditions can be written in matrix form

$$[\text{Matrix}P1] \cdot \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \end{bmatrix} = \begin{bmatrix} \frac{M_0}{EI} \\ \frac{V_0}{EI} \\ 0 \\ 0 \end{bmatrix} \quad (174)$$

All the other coefficients are determined, as is the curve representing casing deflection. From the curves P-y for each depth interval x, the coefficients k_{ij} are obtained for the next iteration.

6. System of equations for the coupling Riser-Casing.

Nomenclature

$$C_{41} = \text{MatrixPinv}(4, 1) \frac{M_0}{EI} + \text{MatrixPinv}(4, 2) \frac{V_0}{EI} \quad (184)$$

And as a result

$$y_c(0) = f_1(M_0, V_0) = C_{11} + C_{31} = [\text{MatrixPinv}(1, 1) + \text{MatrixPinv}(3, 1)] \frac{M_0}{EI} + [\text{MatrixPinv}(1, 2) + \text{MatrixPinv}(3, 2)] \frac{V_0}{EI} \quad (185)$$

And

$$\begin{aligned} \frac{dy_r}{dx}(0) = f_2(M_0, V_0) &= \beta_{11}C_{11} + \beta_{21}C_{21} - \beta_{11}C_{31} + \beta_{21}C_{41} \\ &= [\beta_{11}\text{MatrixPinv}(1, 1) + \beta_{21}\text{MatrixPinv}(2, 1) - \beta_{11}\text{MatrixPinv}(3, 1) + \beta_{21}\text{MatrixPinv}(4, 1)] \frac{M_0}{EI} \\ &\quad + [\beta_{11}\text{MatrixPinv}(1, 2) + \beta_{21}\text{MatrixPinv}(2, 2) - \beta_{11}\text{MatrixPinv}(3, 2) + \beta_{21}\text{MatrixPinv}(4, 2)] \frac{V_0}{EI} \end{aligned} \quad (186)$$

y_{-1} virtual deflection of the riser before LFJ (m) (m)
 y_{-2} virtual deflection of the riser before LFJ (m) (m)
 h elevation of the flex joint relative to the seabed (m)
 T_e Effective Tension (N)

The complete set of equations for the coupling riser-casing is

$$y_c(0) = f_1(M_0, V_0) \quad (187)$$

To simultaneously solve the dynamic equations of the riser and the casing, we first need to express the deflection and inclination of the top of the casing in terms of the bending moment and the horizontal force acting at this point.

Using equation:

$$\frac{dy_c}{dx}(0) = f_2(M_0, V_0) \quad (188)$$

$$y_i(x) = C_{1i}e^{\beta_{1i}x} \cos \beta_{2i}x + C_{2i}e^{\beta_{1i}x} \sin \beta_{2i}x + C_{3i}e^{-\beta_{1i}x} \cos \beta_{2i}x + C_{4i}e^{-\beta_{1i}x} \sin \beta_{2i}x \quad (177)$$

we can write

$$y_c(0) = f_1(M_0, V_0) = C_{11} + C_{31} \quad (178)$$

$$EI \frac{y_1 - 2y_0 + y_{-1}}{\Delta x^2} = R_I \left(\frac{y_1 - y_{-1}}{2\Delta x} + \frac{dy_c}{dx}(0) \right) \quad (189)$$

$$\frac{dy_c}{dx}(0) = f_2(M_0, V_0) = \beta_{11}C_{11} + \beta_{21}C_{21} - \beta_{11}C_{31} + \beta_{21}C_{41} \quad (179)$$

$$\begin{aligned} M_0 = T_e(y_0)h \left(\frac{y_1 - y_{-1}}{2\Delta x} + \frac{dy_c}{dx}(0) \right) - EIh \frac{y_2 - 2y_1 + 2y_{-1} - y_{-2}}{2\Delta x^3} \\ + R_I \left(\frac{y_1 - y_{-1}}{2\Delta x} + \frac{dy_c}{dx}(0) \right) - W_{BOP} \frac{h}{2} \frac{dy_c}{dx}(0) \end{aligned} \quad (190)$$

Since these coefficients can be obtained from Eq. (176), by denoting

$$\text{MatrixPinv} = \{[\text{MatrixP1}] + [\text{MatrixP2}][\text{MatrixA}]\}^{-1} \quad (180)$$

$$y_0 = y_c(0) - h \frac{dy_c}{dx}(0) \quad (191)$$

We have

$$C_{11} = \text{MatrixPinv}(1, 1) \frac{M_0}{EI} + \text{MatrixPinv}(1, 2) \frac{V_0}{EI} \quad (181)$$

$$C_{21} = \text{MatrixPinv}(2, 1) \frac{M_0}{EI} + \text{MatrixPinv}(2, 2) \frac{V_0}{EI} \quad (182)$$

$$C_{31} = \text{MatrixPinv}(3, 1) \frac{M_0}{EI} + \text{MatrixPinv}(3, 2) \frac{V_0}{EI} \quad (183)$$

$$\begin{aligned} V_0 = T_e(y_0) \left(\frac{y_1 - y_{-1}}{2\Delta x} \right) - EI \frac{y_2 - 2y_1 + 2y_{-1} - y_{-2}}{2\Delta x^3} \\ - (T_e(y_0) - W_{BOP}) \frac{dy_c}{dx}(0) \end{aligned} \quad (192)$$

$M_0, V_0, y_{-2}, y_{-1}, y_c(0)$ and $dy_c(0)/dx$ are the unknowns in this system of equations.

This system of 6 equations and 6 unknowns of riser-casing coupling can be added to the matrix $n_{riser} + 4 \times n_{riser} + 4$. The solution will provide as results $y_1(x), y_{c1}(x), M_0$ and

V_0 The solution of $y_2(x,t)$ by means of finite differences will yield the values ΔM and ΔV acting on the top of the casing.

These additional values are added to M_0 and V_0 , and another interaction is performed using the casing equations, the solution will yield $y_c(x)$.

As $E(x,y)$ is dependent on the depth in the soil and on the casing deflection, a way to solve the system of equations is to use a series of iterations until the solution for casing deflections converges. The initial value for $E(x,y)$ used will be the derivative of the soil reaction curve for 0 deflection. In general, convergence is obtained with a small number of iterations, as it can be seen in Fig. 7

Funding The author received no financial support for the research, authorship, and/or publication of this article.

Data availability The author ensures that all data and materials as well as software application or custom code support his published claims.

Declarations

Conflict of interest The author states that there is no conflict of interest.

Consent for publication The publication of the manuscript is approved by the author.

Ethical approval This article reflects the authors' own research and analysis in a truthful and complete manner.

Human or animals rights This article does not contain any studies with human participants or animals.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

API 16Q Design (1993) Selection, operation, and maintenance of marine drilling riser systems. Washington DC, USA~ American Petroleum Institute, 19(9), p.3

- API 2RD (1998) Design of risers for floating production systems (FPSs) and Tension-Leg Platforms (TLPs)
- API 2A (2002) Recommended Practice for Planning, Designing and Constructing Fixed Offshore Platforms – Working Stress Design, Errata and supplement
- API 16Q Design (2017) Selection, operation, and maintenance of marine drilling riser systems
- Bhalla K, Cao Y (2005) Watch circle assessment of drilling risers during a drift-off and drive-off event of a dynamically positioned vessel, In: Dynamic Positioning Conference, Houston, TX, USA (pp. 15-16)
- Botke JC (1975) An analysis of the dynamics of marine risers. Delco Electron
- Burke BG (1973) An analysis of marine risers for deep water. In: Offshore Technology Conference. OnePetro
- Chatjigeorgiou IK (2008) A finite differences formulation for the linear and nonlinear dynamics of 2D catenary risers. Ocean Eng 35(2008):616–636
- Faltinsen OM (1998) Sea loads on ships and offshore structures
- Gardner TN, Kotch MA (1976) Dynamic analysis of risers and caissons by the finite element method. In Offshore Technology Conference. OnePetro
- Kanhua S, Stephen B, Jianming Y, Hongyuan Q (2018) Coupled dynamic analysis for the riser-conductor of deepwater surface BOP drilling system. Shock and Vib
- Keum-SH, Umer HS (2018) Vortex-induced vibrations and control of marine risers: a review, Ocean Eng
- Langer CG (1985) Relationships for deep water suspended pipe spans
- Michel D, Enda O, Julian S (2004) Fully coupled EDS/Drift-off analysis for a harsh environment, deepwater site. In: International conference on offshore mechanics and arctic engineering, vol 37432. pp 1133–1142
- Maolin L, Gaowei W, Zhiying G, Yipeng Z, Ruifeng L (2020) Mathematical modelling and dynamic analysis of an offshore drilling riser. Shock Vib
- Patel MH, Seyed FB (1995) Review of flexible riser modeling and analysis techniques. Eng Struct, 17(4): 293-304
- Shamsher P (1990) Pile Foundations in Engineering Practice
- Spanos P-TD, Chen TW (1980) Vibrations of marine riser systems
- Richtmyer R (1968) Difference methods for initial-value problems
- WANG Sw, XU X-s, YAO B-h, LIAN L (2012) A finite difference approximation for dynamic calculation of vertical free hanging slender risers in re-entry application. China Ocean Eng 26(4):637-652
- Xiuquan L, Guoming C, Yuanjiang C, Jingqi J, Jingjie F, Qiang S (2016) Drift-off warning limits for deepwater drilling platform/riser coupling system. Petrol Explor Develop, 43(4):701-707

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.