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Unmoved movers: a very simple and novel form of indeterminism

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Abstract
It is common knowledge that the Aristotelian idea of an unmoved mover (Primum Mobile) was abandoned definitively (from a mechanical standpoint, at least) with the advent of modern science and, in particular, Newton's precise formulation of mechanics. Here I show that the essential attribute of an unmoved mover (in a non-trivial sense, and in the context of infinite systems theory) is not incompatible with such mechanics; quite the contrary, it makes this possible. The unmoved mover model proposed does not involve supertasks, and (perhaps precisely for this reason) leads both to an outrageous form of indeterminism and a new, accountable form of interaction. The process presents a more precise characterization of the crucial going-to-the-limit operation (which will admittedly require further development in future research). It has long been acknowledged in the existing literature that, theoretically, in infinite Newtonian systems, masses can move from rest to motion through supertasks. Numerous minor variations on the original schemes have already been published. Against this backdrop, this paper introduces three significant additions: 1) It formulates for the first time a limit postulate for systematically addressing infinite systems; 2) It shows that an Aristotelian unmoved mover (with no supertask) is possible in systems of infinitely many particles that interact with each other solely by contact collision; 3) It shows how interaction at a distance can emerge in systems of infinitely many particles (at relative rest) that interact with each other solely by contact.

Keywords Unmoved mover · Infinite system · Newtonian possible world · Limit · Contact · Collision

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1 Introduction

Attempting to model classical problems in philosophy in a detailed and precise way by making use of physical and/or mathematical theories, whose formal structure is reasonably well understood, is an expeditious and intellectually stimulating task. This translation process usually entails some irreparable loss of content, but also perhaps some gain in new intuitions which (if so) could yield greater conceptual enrichment of the debates involved and, consequently, a broader vision regarding the nature of precisely such problems.

Perhaps the best known domain where the above strategy has been developed is that of the classical Zeno paradoxes (see, for example, Angel, 2001, from the important body of literature). Various kinds of supertasks have also been widely used as a modeling tool, even to deal with problems unrelated (or at least not directly) to infinity. One of the most recent representative proposals in this regard is Shackel (2018), where the author considers the age-old question of what happens when the immovable object meets the irresistible force. The supertask "in a Newtonian universe" that he proposes to represent it illustrates (supposedly) what he terms "the Nothing from infinity paradox" (infinite mass and energy disappear completely, leaving an empty space). Corral-Villate (2020) criticized (rightly in my view) the technical aspects of Shackel's analysis. The way in which the immovable object and the irresistible force are modeled as particle systems in the context of collision mechanics is also open to criticism (for example, it was found that Shackel's immovable object can even move by itself, as shown by Laraudogoitia, 1998, although I will not elaborate on this here). Nonetheless, the idea of modeling as such seems valuable and thought-provoking.

This paper argues that a Newtonian modeling of the Aristotelian idea of unmoved mover exists. Given the incommensurability of the Aristotelian and Galilean-Newtonian notions of motion, this claim must be qualified. I do so in section 2 (without intending it to be a scholarly analysis of the sources), where I clarify what should be understood by unmoved mover in the Newtonian sense, and the resulting loss of content. Section 3 is central as it rigorously defines the notion of unmoved mover and introduces the limit postulate in detail. Section 4 shows (in simple collision mechanics terms) how unmoved movers are possible in Newtonian worlds of colliding particles. The reasoning, which is technically very simple (as is the Newtonian model that eventually follows), is developed in four stages, in close parallelism with a well-known supertask. Even though the way the unmoved mover operates is not via a supertask, it does consist of an infinite particle system, hence the relevance of this parallelism. To conclude, section 5 reflects on the significance of the results obtained. In addition to the particular importance of going to the limit in the study of infinite systems, it also reveals the surprising way in which interaction at a distance emerges from contact interaction under the limit postulate.

Finally, I would like to stress that this paper can be read (completely aside from historical considerations) as the justification of a series of unexpected outcomes regarding the causes of motion that clash with our most elementary Newtonian intuitions.

2 A look at unmoved movers in the history of philosophy

The aim of this section is neither historical comprehensiveness nor strict adherence to texts by classical authors, which would be inappropriate for a paper such as this, whose focus is on conceptual analysis rather than the interpretation of the foundational works of philosophy.

The idea of the unmoved mover dates back to Aristotle, and is closely linked to his proof of the existence of a prime mover (Primum Mobile). Writing on the superlunary stage, he says:

> "... for it is impossible that there should be an infinite series of movements, each of which is itself moved by something else, since in an infinite series there is no first term - if then everything that is in motion is moved by something, and the first movement is moved but not by anything else, it must be moved by itself." (Aristotle, 1957, 256a).

Scholars have distinguished up to three different contexts in which Aristotle supplies as many proofs of the existence of the Primum Mobile, with subtle differences between them (Lang, 1978) in terms of the scope of their conclusion, i.e., in terms of the prime mover's attributes. A narrower and more useful perspective for our interests can be seen (according to Sauvé, 1987) when considering:

"…the three principal claims that Aristotle makes of the capacity in virtue of which something is an UMM [unmoved mover]:

1. that its exercise is not a movement
2. that it causes movement
3. that it stops the regress of causal explanation" (p. 175)

In light of these capacities, Thomas Aquinas' argument in the first of his Five Ways is crystal clear. It is based on what has come to be termed the Aristotelian mover-causality principle: everything which is in movement is moved by another (implying that the causal chain of movements is linear).

There has been considerable discussion surrounding the Aristotelian mover-causality principle. However, viewed from the perspective that Galileo initiated in physics (with his attention turned to primary qualities and measurability to the detriment of essences and causes), which Newton consolidated, it clearly seems to be false.

Newton introduced three different notions of force in the Principia: vis insita (innate force of matter), vis impressa (impressed force) and centripetal force. The first two are of interest here. He formulates the three laws of motion explicitly for impressed forces (in the third law this is implicit, though not explicit). What is the role of vis insita? Newton also calls it the force of inertia (but here it has a different meaning from that of today), and he certainly saw the concepts of mass and force as two primary notions (Dugas, 1988) rather than considering mass and acceleration as primary, as is usual today. Vis insita is not seen as a cause of acceleration, and Jammer (1999) interprets it as a mere concession to pre-Galilean mechanics. McGuire (1994) also agrees with this assessment by arguing that Newton does not completely break with the Aristotelian analysis of motion and its causes. In his view, on the surface of Newton's ideas, vis insita is identifiable with inertia, but deeper down embodies the idea that it is the inherent cause of a material body's motion. In any event, this idea of innate force of matter has been used to justify that there is self-motion in Newton's conception of mechanics ("It is clear that Newton conceives natural motion as a species of self-motion", 1994, p. 329). Nevertheless, this does not imply the possibility of unmoved movers. As stated above, the Aristotelian mover-causality principle is false in Newtonian mechanics (while it is true that some moving objects are moved by others, not all moving objects are moved by others; the principle is false as a universal statement.). There is no infinite regress in the causal explanation of motion to be found. So, of the three conditions characterizing an unmoved mover (according to Sauvé), only the first two (1. that its exercise is not a movement; 2. that it causes movement) must be maintained in a Newtonian context. Even so, there seems to have always been an implicit consensus that the existence of an unmoved mover is incompatible (at least under usual contact forces[1]) with the laws of classical mechanics. This idea will be challenged below. I will show that the falsity of the Aristotelian mover-causality principle in Newtonian mechanics does not prevent the possibility of unmoved movers (even though for Aristotle, and Aquinas in his first way, their existence was deduced using this principle).

3 Defining unmoved mover and introducing the limit postulate

The brief historical overview given in the previous section allows us to extract a notion of unmoved mover which is valid in a Newtonian context. However, in order to address the results hereafter (which are more technical and precise than those mentioned so far), it would be wise to hone our ideas at this point. No commitment to a definite conception (among the several existing) of what Newtonian mechanics is is required to do so. Nor does the most general possible concept of unmoved mover within it need to be considered. For the purposes of this paper, confining oneself (bearing in mind footnote 1 and similar, easily-imaginable situations) exclusively to those cases where gravitation is not considered is sufficient. Thus, the idea of unmoved mover outlined up to this point can be described in greater detail as follows:

> Unmoved mover: In the absence of gravitational interaction, an unmoved mover M is a material body at rest in an inertial reference frame R (where all of M's parts are also at rest) with the following property: it is possible for material bodies M* at rest in R (both the bodies themselves or any of their parts) to be set in motion by M's causal action without M or any of its parts ceasing to be at rest in R.

Although no specific or detailed conception of Newtonian mechanics is ascribed to in this article, as mentioned above, it is nonetheless important to realize that Newtonian

[1] Consider the uniform gravitational field (g) created by a material plane Π of infinite extension and constant mass density. Any particle P with mass m placed on it will move under the action of force mg exerted by Π. However (given its infinite mass), Π will not move, thus acting (trivially) as an unmoved mover.

mechanics (and not, for example, Hamiltonian mechanics) is specifically considered. So, when the definition of unmoved mover refers to causal action, this concept is more clearly circumscribed. Still, even within this narrower scope, there is no need for a general definition of what a causal action is. In order to understand the definition of unmoved mover and causal discourse that (partially) permeates my argument, one need only acknowledge that, in a Newtonian formulation of mechanics: a) only a force can be the causal agent of a material body's change in velocity (i.e. change in the state of motion) within an inertial reference frame; b) only a material body can exert a force within an inertial reference frame. From a) and b), it follows that material body M can be referred to, in a derived sense, as the causal agent of material body M*'s change in velocity within inertial reference frame R: M exerts force F, which is the causal agent of this change. Correspondingly, also in a derived sense, M's causal action is force F's causal action, exerted by M. Given these clarifications, the above definition of unmoved mover has been refined so as to better understand its role within the framework of subsequent arguments.

Moreover, as stated in the introduction, the unmoved mover proposed involves the intervention of material systems consisting of an infinite number of particles. Something should therefore be said at this point as to how to approach such systems. Clear indications to this effect can be found in the theory of supertasks; indications that incorporate a going-to-the-limit process based on finite configurations of particles. However, these indications prove to be insufficient for the kind of infinite systems that model unmoved movers. More specifically, an explicit explanation will be given below on how to perform going to the infinite limit in the number of particles based on a system with a finite number of particles. This indication shall be referred to as the limit postulate (LP). It is an essential postulate for what follows, yet not particularly committing. It basically states that in order to analyze the time evolution of an infinite system, the limits of numerical sequences, in the sense of elementary mathematical analysis, and limits of sequences of sets, in the sense of measure theory, must be computed. For its proximate application, already performed in subsection 4.1, it is presented as follows:

LP: In order to analyze the time evolution of a physical system S of infinite particles from the initial instant t_0 to a later instant $t > t_0$, first consider the evolution from t_0 to t of a subsystem S_n with only n particles. Then:

- (I) take limit $n \to \infty$ in the following, precise sense: a) according to the mathematical meaning of the limit of a real numerical function of real variable on all this class' relevant functions that depend on n (for example, the real functions of real variable $x^t_m(n)$ and $v^t_m(n)$, which respectively give the position and velocity of subsystem S_n's particle p_m at instant t, $1 \leq m \leq n$); b) according to the mathematical meaning of the limit of a sequence of sets (a sequence that is a set-valued function of real variable) on all the relevant increasing sequences of sets that depend on n (for example, the sequence of particle sets $S_1 = \{p_1\}$, $S_2 = \{p_1, p_2\}$, ..., $S_n = \{p_1, p_2, ..., p_n\}$, ... or sequences of particle parameter sets).
- (II) finally, if the limits mentioned exist, these therefore provide possible values for the relevant physical magnitudes corresponding to system $S = \lim_{n \to \infty} S_n$ of infinite particles (for example, for each particle $p_m \in \lim_{n \to \infty} S_n = S$, with $m \geq 1$, limits $x^t_m(S) = \lim_{n \to \infty} x^t_m(n)$ and $v^t_m(S) = \lim_{n \to \infty} v^t_m(n)$ respectively provide

possible values for the position and velocity, $x^t_m(S)$ and $v^t_m(S)$, of system S's particle p_m at instant t).

Note that in the limit postulate (where $n \to \infty$ is performed) there are functions of n whose values are real numbers (corresponding to sequences of numbers) and functions of n whose values are sets (corresponding to sequences of sets). This entails very careful consideration of the going-to-the-limit process, which (to the best of my knowledge) has been specified here for the first time. For its part, the condition that sequences of sets shall be increasing is eminently natural in view of the fact that the limit postulate's purpose is to study the evolution of sets with an infinite number of particles.

Note also that the LP seeks to describe the situation with $n \to \infty$ on the basis of what is known for finite n, i.e., relating it to what is known for finite n. Not by relating it in any fashion, rather by means of a rigorous and formally manageable procedure: going to the limit. On the one hand, the going to the limit used by the LP is neither ad hoc nor artificial. Quite the contrary, it uses notions that are familiar and used widely in conventional mathematics. I believe this to be the strongest argument in favor of its plausibility. On the other, it seems to me that employing some notion of limit is the only way to refer judiciously to infinite systems on the basis our experience with finite systems.

In the LP applications that follow, not only real functions such as $x^t_m(n)$ and $v^t_m(n)$ will be considered, namely functions giving positions and velocities. However, it will be clear from the context that the use of some other different real functions (e.g. $t_n \equiv t(n)$ functions with values in time, basically for the sake of convenience) could always be avoided in favor of standard $x^t_m(n)$ and $v^t_m(n)$-type functions (positions and velocities). Hence, only trivial transformations would mediate.

4 The road to an unmoved mover

4.1 First stage

The unmoved mover model presented here involves the use of an infinite particle system (using the term in its broad sense; a particle is in general a rigid body of finite size which, for the sake of explanatory simplicity, can be regarded as having zero volume). However, this is not a supertask; there is no infinite sequence of interactions or collisions. Even so, some of the well-established theory of supertasks will be useful to illustrate the going-to-the-limit processes applied. So as to avoid misunderstandings, it may be useful to briefly specify what is meant by supertask. In my understanding of the term (which I believe to be in line with most of the literature), a supertask requires the execution of an infinite sequence of tasks in a finite time, yet not in null time! (If the tasks are simultaneous, there is no sequence). Moreover, the unmoved mover acts instantaneously (in null time) and therefore does not execute a supertask in the usual sense. A broader notion of supertask (excessively broad, in my opinion) dispenses with the idea of sequence: an event (not process) could be a supertask in this sense. In contrast, the narrower concept advocated here affords a very characteristic and exclusive type of well-known conceptual difficulties, which justifies introducing it apart.

The most suitable (and, at the same time, simplest) example of a supertask for our purposes is provided by Laraudogoitia, 1996. Let us consider an infinite set of identical point particles p_n ($n > 1$), each of unit mass, at rest at coordinate points $x = 1/(n - 1)$. The system is approached from the right by another identical particle p_1 at unit velocity. The collision between p_1 and p_2 occurs at $t = 0$. This gives rise to a perturbation, which propagates through the system as an infinite sequence of successive collisions (a supertask), and ends at $t = 1$ with all particles at rest (p_n at point $x = 1/n$, $n \geq 1$). The outcome is a 1/2 magnitude loss of energy and unit magnitude loss of momentum. Although this outcome is justified on the basis of our initial intuitions regarding the type of infinite system considered, it will be interesting (for what follows) to deduce it as a consequence of going to the limit ($n \to \infty$) based on the analysis of a finite n-particle system. For this purpose, we shall take a finite set of identical n particles $\{p_1, p_2, p_3, ..., p_n\}$ of unit mass. $p_2, p_3, ..., p_n$ are initially at rest at their respective points 1/1, 1/2, ..., $1/(n - 1)$ while p_1 approaches them at unit velocity from the right and collides with p_2 at $t = 0$. A perturbation now also occurs, which propagates through the system as a finite sequence of successive collisions, and ends at $t = (1/1) - (1/(n - 1)) = (n - 2)/(n - 1)$ with particles $p_1, p_2, ..., p_{n-1}$ at rest at their respective points 1/1, 1/2, ..., $1/(n - 1)$. But that is not all: as of $t = (n - 2)/(n - 1)$, the "last particle", p_n, (moving leftwards at unit velocity) takes away all the energy and momentum initially held by p_1. Going to the limit $n \to \infty$ now has a precise sense according to the limit postulate (LP). It is performed: a) according to the mathematical meaning of the limit of a real numerical function of real variable on all this class' relevant functions that depend on n; b) according to the mathematical meaning of the limit of a sequence of sets on all the relevant increasing sequences of sets that depend on n. The only relevant numerical function is $t = (n - 2)/(n - 1)$, and clearly $\lim_{n \to \infty} (n - 2)/(n - 1) = 1$. The two relevant sequences of sets are: a) the sequence of sets of particles $P_2, P_3, P_4, ... P_i, ...$ where $P_i = \{p_1, p_2, p_3, ..., p_i\}$ and b) the sequence of initial locations $I_2, I_3, I_4, ... I_i, ...$ where $I_i = \{1/1, 1/2, ..., 1/(i - 1)\}$. Since both sequences are increasing ($P_2 \subseteq P_3 \subseteq P_4 \subseteq ... \subseteq P_i \subseteq ...$ and $I_2 \subseteq I_3 \subseteq I_4 \subseteq ... \subseteq I_i \subseteq ...$), one reasonable definition for a limit[2] is, in each case, the union (Edgar, 1992): $P = \cup_{k>1} P_k$ and $I = \cup_{k>1} I_k$. Clearly, $P = \{p_1, p_2, p_3, ..., p_i, ...\}$ and $I = \{1/1, 1/2, ..., 1/i, ...\}$. Therefore, the description of the resulting state of affairs after going to the limit is clear. Now based on an infinite set of identical particles

[2] This definition is not arbitrary but rather a particular case of the standard, general definition of the limit of a sequence of sets (see, e.g., Resnick, 1999, p.6). Given the arbitrary (infinite) sequence of sets $B_1, B_2, B_3, ... B_n, ...$ it is said that $\lim_{n \to \infty} B_n = B$ if and only if $\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} B_k = B$. As can be easily seen, $\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k$ is the set formed by the elements belonging to all the sets of the sequence, save perhaps a finite number of them. Analogously, $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} B_k$ is the set formed by all the elements belonging to infinite sets in the sequence. When the sequence is increasing ($B_1 \subseteq B_2 \subseteq B_3 \subseteq ... \subseteq B_i \subseteq ...$), it is evident that both sets are equal to $\cup_{k \geq 1} B_k$, which is therefore the limit of the sequence. The general definition of the limit of a sequence of sets is of interesting application to Ross's paradox. Suppose that we possess an infinitely large urn and an infinite collection of balls labeled ball number 1, number 2, number 3, and so on. Consider an experiment performed as follows: "At 1 minute to 12 P.M., balls numbered 1 through 10 are placed in the urn and ball number 1 is withdrawn; at 1/2 minute to 12 P.M., balls numbered 11 through 20 are placed in the urn and ball number 2 is withdrawn; at 1/4 minute to 12 P.M., balls numbered 21 through 30 are placed in the urn and ball number 3 is withdrawn; ... and so on. .... how many balls are in the urn at 12 P.M.?" (Ross, 2010, p. 46). The surprising answer that the urn is empty at 12 P.M. is an immediate consequence of the general definition of the limit of a sequence of sets. If U_n designates the set of balls in the urn at $t_n = 12 - (3/2)(1/2^n)$, then $\lim_{n \to \infty} U_n = \varnothing$, i.e. the urn is empty at 12 P.M. = $\lim_{n \to \infty} t_n$.

$\{p_1, p_2, p_3, ..., p_n, ...\}$ of unit mass. $p_2, p_3, ..., p_n, ...$ are initially at rest at their respective points 1/1, 1/2, ..., 1/(n - 1), ... while p_1 approaches them at unit velocity from the right and collides with p_2 at $t = 0$. A perturbation then occurs, which propagates through the system in the form of an infinite sequence of successive collisions, and ends at $t = 1$ with particles $p_1, p_2, ..., p_n, ...$ at rest at their respective points 1/1, 1/2, ..., 1/n, ... That is all. As there is no "last particle", p_∞, nothing can take away the energy and momentum initially held by p_1. So, as from $t = 1$, this energy and momentum have been lost. Thus, the outcome first obtained on the basis of a direct intuition of the situation is retrieved. Reconsidering such a situation in terms of a more elaborate limit calculation may seem futile at this point, but will prove useful shortly.

Naturally, whether the consequences of the limit calculation performed correspond to "the facts" is something that cannot be guaranteed a priori. Introducing such a calculation for prediction purposes actually implies the assumption of an additional postulate (referred to above as the "limit postulate", LP). This, added to the underlying Newtonian theoretical structure, enables precise conclusions to be drawn on infinite systems. Some authors (Atkinson, 2007; Laraudogoitia et al., 2002) have speculated on the possibility that neither energy nor momentum is lost in situations such as the beautiful supertask. I will not discuss such alternatives here, although it seems clear that LP leads to conclusions more in line with the stability of matter characterizing Newtonian possible worlds (according to Peijnenbug & Atkinson, 2010), which are of interest to me here. Moreover, one might expect that a suitable LP formulation for more complex infinite systems than those considered in this paper would require greater sophistication: in order to obtain not only reasonable but also well-defined outcomes (see note (2) to this effect). This would be the road to a general infinite systems theory. However, my goals here are more unassuming, and the LP version outlined above is sufficient for the infinite systems to be considered.

4.2 Second stage. Sudden stop

The LP will now be applied to the analysis of a simple infinite system where no supertask is executed. Figure 1 shows a rigid ball, B, of unit radius (and unit mass) moving (at unit velocity) towards Cartesian plane YZ (plane $x = 0$). B's geometric center is displaced on the X-axis (points (x, 0,0)) and there is a denumerable infinity of rigid rings a_n on the YZ plane (of zero thickness, for the sake of simplicity) at rest all centered at the origin of coordinates O(0,0,0) and with respective radiuses $r_n = 1/n$ (only the first four rings are shown in the Fig. 1). All the a_n are independent of one another, and of the same mass, equal to B's mass. Assuming that, in its rightward movement, the center of B reaches point (1,0,0) at instant $t = 0$. How will the infinite system evolve henceforth? The LP will again be applied to the finite situation where the number of rigid rings is finite. Note that, in principle, only the $a_1, a_2, a_3, ..., a_n$ are present. In these conditions of finitude the future is clearly prescribed. B will continue to make further progress rightwards with no perturbation for a certain additional distance, a, up to a point where it will collide with ring a_n. The value of a is easy to calculate (see Appendix I), and the result: $a = 1 - (1/n)\sqrt{(n^2-1)}$. After B's collision with a_n (obviously at instant $t_a = 1 - (1/n)\sqrt{(n^2-1)}$), B will be stopped, while a_n will take away all its kinetic energy and momentum moving rightwards thereafter at unit velocity

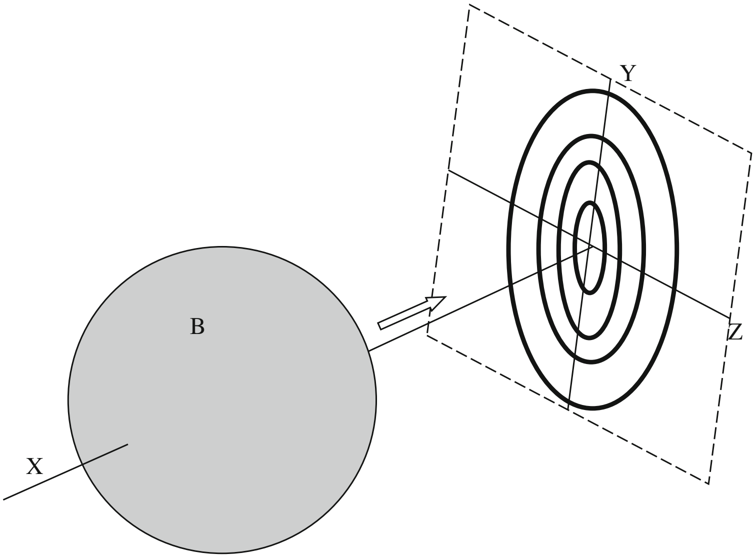

Fig. 1 A rigid ball on collision course with a denumerable infinity of rigid rings

(note that the masses of B and a_n are equal). The other rings present remain forever at rest. It will be seen below that the evolution from t = 0 in the infinite case is also clearly prescribed by applying the LP. The only relevant numerical function is $a = t_a = 1 - (1/n)\sqrt{(n^2-1)})$, and obviously $\lim_{n \to \infty} [1 - (1/n)\sqrt{(n^2-1)})] = 0$. And the only relevant sequence of sets is $A_1, A_2, A_3, \ldots A_i, \ldots$ where $A_i = \{a_1, a_2, a_3, \ldots, a_i\}$. Since it is increasing ($A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \subseteq A_i \subseteq \ldots$), as before $\lim_{n \to \infty} A_n = A = \cup_{k \geq 1} A_k = \{a_1, a_2, a_3, \ldots, a_i, \ldots\}$. The description of the resulting state of affairs after going to the limit is then clear. Starting now from B and an infinite set of rings $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ with radii $r_n = 1/n$, each of unit mass. $a_1, a_2, a_3, \ldots, a_n, \ldots$ are initially at rest on the YZ plane centered at point (0,0,0) while B approaches them at unit velocity from the left, its geometric center reaching point (1,0,0) at instant t = 0. Since $a = t_a = 0$, B will make no further progress rightwards, so it will collide at that instant and stop. That is all. As there is no "last ring", a_∞, B collides with no ring, nor is there any ring that can take away the energy and momentum that B initially held. So, as of t = 0, this energy and momentum are lost. All that has happened is that B has stopped instantaneously, with the subsequent loss of energy and momentum, but no ring has moved at any time throughout the whole process. The collision stopping B is an example of what is termed in the literature as a global collision (Laraudogoitia, 2005): B collides with all the rings (i.e. collides with the set of rings, with the physical system formed by the rings) without colliding with any one ring separately. To date, the circumstances in which energy and/or momentum could be lost involved infinite sequences of collisions in the context of supertasks (excluding inelastic collisions since, at all times, only frictionless rigid-body dynamics are considered). What is surprising is that no supertask is executed in our example. Just one collision (without the required presence of others) leads to such a loss. The existence of a collision between rigid bodies (no supertask and no friction) where no energy is conserved reveals, surprisingly, that the philosophically interesting properties of infinite systems reach far beyond what the theory of supertasks can tell us about them. This judgment will be more forcefully sustained when the "Aristotelian" unmoved mover model is later discussed.[3]

If the idea of global collision is considered problematic, there will be some reservations concerning the argument above, which supports it. One may then suspect that perhaps the configuration in Fig. 1 (upon which the argument above is built) is not in fact well defined; a fundamental indefiniteness that the usual notion of limit would be unable to address. In response, I reject that Fig. 1's configuration is poorly defined. Rather, the usual notion of limit (confined to calculating the limits of real functions of real variable) is insufficient in itself to study this configuration. In my view, potential critics would be considering the problem from this constrained notion of limit (which only takes clause (I) a) of the limit postulate LP into account); hence the impression of indefiniteness. When studied with the help of the limit postulate LP "in full", all indefiniteness vanishes and the approach taken in this section can immediately be followed.

Ideally, a precise definition of the term "collision" would dispel all doubts about the legitimacy or otherwise of the discourse on global collisions. For example, providing the necessary and sufficient conditions for a collision would settle such doubts. This is a difficult task that will not be discussed here, given the enormous complexity involved in the countless infinite-particle systems imaginable. Fortunately, neither is this necessary. A conjunction of conditions simply needs to be provided which, as a conjunction, is clearly sufficient for a collision. It is assumed that the conjunction of the following four conditions is clearly sufficient to conclude that rigid body D collides at instant t with material system C (being itself a rigid body or system consisting of rigid bodies). 1) Contact condition: at instant t, D is in contact with C (i.e. at zero distance from C). 2) Discontinuity condition: at instant t, D undergoes a discontinuous change in its velocity. 3) Condition of relative motion: at t - 0 (also usually denoted as t^{-}), there is relative motion between D and at least one part of C. 4) Condition of causal effectiveness: without C's presence, there would have been no discontinuous change in D's velocity at t.

If rigid ball B is taken as D and system A of the infinite concentric rings as C, it is clear that (given the analysis in this subsection) B collides with A at $t = 0$. And, as noted earlier, the collision's "global" character simply reflects the fact that B does not collide at $t = 0$ with any of the rings a_n forming part of A.

4.3 The road to an unmoved mover. Third stage

Even more philosophically significant than the beautiful supertask described in 4.1 is its time reversal. Here we start from identical particle system p_n at rest at points $x = 1/n$ ($n \geq 1$) and invert the process there observed. Suddenly (at an undetermined instant in time, say $t = 0$), the system is self-excited by an infinite sequence of binary collisions where different p_i successively gain and lose unit velocity until finally (for $t > 1$) all of them are at rest (p_n at point $x = 1/(n - 1)$, with $n > 1$) barring p_1, which moves away

[3] The sudden stop in this section could perhaps be avoided by questioning the use of the limit postulate (LP) in this section. However, there would then be a problem: explaining why such use is unsuitable here and not in the case of the beautiful supertask discussed in section 4.1. And, to my mind, considering that it is also unsuitable in the latter case entails rejecting the beautiful supertask as such.

from the others at unit velocity. Energy and momentum have been spontaneously created for $t > 0$, which are transported through the system until they eventually reach p_1. Since the direct process was an infinite sequence of reversible collisions, the possibility of this reverse process follows in an intuitively evident way. Even so, it will be instructive to see how it can be reached by applying the going-to-the-limit procedure ($n \to \infty$) based on the analysis of a finite n-particle system (using the limit postulate, LP). To this end, let us take a finite set of identical n particles $\{p_1, p_2, p_3, ..., p_n\}$ of unit mass. At instant $t_{initial} = 1/n$, all particles $p_1, p_2, p_3, ..., p_n$ are at their respective points 1/1, 1/2, ..., 1/n but, while the first n - 1 are at rest, p_n (the "last particle") approaches them, moving at unit velocity from the left. p_n collides with p_{n-1} at $t_{perturbation} = 1/n + [1/(n - 1) - 1/n] = 1/(n - 1)$, causing a perturbation that propagates through the system (in the form of a finite sequence of successive binary collisions) and ends at $t_{final} = (1/1) - (1/(n - 1)) = (n - 2)/(n - 1)$ with particles $p_2, p_3, p_4, ..., p_n$ at rest at their respective points 1/1, 1/2, ..., 1/(n - 1), and p_1 moving rightwards at unit velocity. p_1 takes away all the energy and momentum that the "last particle" p_n initially held. Going to the limit $n \to \infty$ is now also clear. The only relevant numerical functions are $t_{initial} = 1/n$ (with $\lim_{n \to \infty} t_{initial} = 0$), $t_{perturbation} = 1/(n - 1)$ (with $\lim_{n \to \infty} t_{perturbation} = 0$) and $t_{final} = (n - 2)/(n - 1)$ (with $\lim_{n \to \infty} t_{final} = 1$). The two relevant sequences of sets are: a) sequence of particle sets $P_2, P_3, P_4, ... P_i, ...$ where $P_i = \{p_1, p_2, p_3, ..., p_i\}$ and b) sequence of initial locations $I_2, I_3, I_4, ... I_i, ...$ where $I_i = \{1/1, 1/2, ..., 1/i\}$. Clearly $\lim_{n \to \infty} P_n = P = \cup_{k>1} P_k = \{p_1, p_2, p_3, ..., p_i, ...\}$ and $\lim_{n \to \infty} I_n = I = \cup_{k>1} I_k = \{1/1, 1/2, 1/3, ..., 1/i, ...\}$. Therefore, the description of the resulting state of affairs after going to the limit is unambiguous. Starting from an infinite set of identical particles $\{p_1, p_2, p_3, ..., p_n, ...\}$ of unit mass, all of which are at rest at their respective points 1/1, 1/2, ..., 1/n ... at instant $t_{initial} = 0$ (there is no "last particle" in motion, p_∞, approaching them). A perturbation then occurs at $t_{perturbation} = 0$ (albeit not caused by any specific collision, there is no "first collision" between p_∞ and $p_{\infty - 1}$ because there are no such particles) which propagates through the system in the form of an infinite sequence of successive collisions, and ends at $t_{final} = 1$ with particles $p_2, p_3, p_4, ..., p_n, ...$ at rest at their respective points 1/1, 1/2, ..., 1/(n - 1), ... and p_1 moving rightwards at unit velocity. p_1 takes away all the energy and momentum that initially emerged from the perturbation. Since this perturbation is internal to particle system P (it does not derive from any external action) and does not result from a specific collision, the term self-excitation is justifiable for describing it. The outcome first obtained on the basis of a direct intuition of reversibility in the beautiful supertask is thus retrieved.

Use of the limit postulate (LP) in this subsection will be the pattern to follow in order to reach the unmoved mover in the following subsection.

4.4 The road to an unmoved mover. Last stage

Here it will be seen how the key which leads to the unmoved mover is to apply the LP as used in the previous subsection (4.3) to the analysis of an infinite system where no supertask is executed (as in 4.2).

Figure 2 shows a rigid ball, B, with unit radius (and unit mass) at rest and in point contact with Cartesian plane YZ (plane $x = 0$). B's geometric center is located on the X axis (at point (1,0,0)), and on the YZ plane there is a denumerable infinity of rigid rings a_n (of zero thickness, for the sake of simplicity) at rest, all of which are centered at the

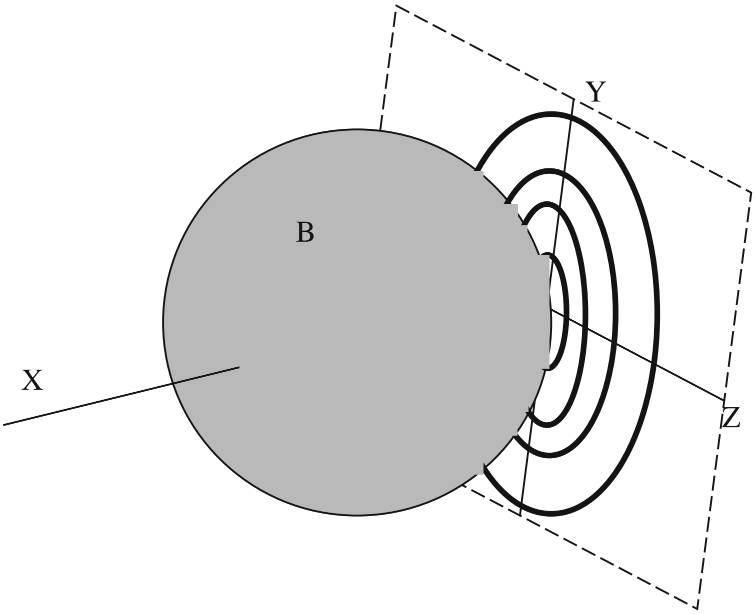

Fig. 2 A rigid ball at rest in contact with a denumerable infinity of rigid rings

origin of coordinates O(0,0,0) and with respective radiuses $r_n = 1/n$ (only the first four rings are shown in the Figure.). All the a_n are independent of one another and have the same mass, equal to B's mass. How will the infinite system evolve henceforth? One obvious possibility is the perpetuation of all its components' state of rest. Surprisingly, however, this is not the only possibility. Another possibility is that the infinite system of rings a_n acts as an unmoved mover.

In order to observe this, we shall start by considering a situation similar to the one just described (shown in Fig. 2) but with two modifications: a) the set of rings present is finite $\{a_1, a_2, a_3, ..., a_n\}$; b) at instant $t_{initial} = 0$ there is only one single ring, a_n (the "last ring") which, albeit momentarily centered at O(0,0,0), is moving leftwards at unit velocity, progressively approaching B. The calculation made in the Appendix I clearly shows that a_n will collide with B at $t_{final} = 1 - (1/n)\sqrt{(n^2–1)}$, with B moving leftwards from then on at unit velocity (and a_n, of course, is at rest from that instant, as are the other rings). B takes away all the energy and momentum that the "last ring" a_n initially held. Going to the limit $n \to \infty$ in this case is also equally clear. The only relevant numerical function is $t_{final} = 1 - (1/n)\sqrt{(n^2–1)}$ (with $\lim_{n \to \infty} t_{final} = 0 = t_{initial}$). And the only relevant sequence of sets is $A_1, A_2, A_3, ... A_i, ...$ where $A_i = \{a_1, a_2, a_3, ..., a_i\}$. As before in 4.2, $\lim_{n \to \infty} A_n = A = \cup_{k \geq 1} A_k = \{a_1, a_2, a_3, ..., a_i, ...\}$. The description of the resulting state of affairs after going to the limit is therefore unambiguous. Now based on from B and an infinite set of concentric rings $\{a_1, a_2, a_3, ..., a_n, ...\}$ of unit mass, all of which are at rest and centered at O(0,0,0) at instant $t_{initial} = 0$ (there is no "last ring" in motion, a_∞, approaching B). A perturbation then occurs at $t_{final} = t_{initial} =$ 0 (albeit not caused by any specific collision, there is no collision between a_∞ and B because there are no a_∞) which sets B in motion leftwards at unit velocity, whilst the rings remain at rest. From the outset, B takes away all the energy and momentum originated from the perturbation. Since this perturbation is internal to the system formed by B and the infinite rings (it does not derive from any external action on the system), and does not result from a specific collision, the term self excitation is now also justifiable for its description. There is, however, a qualitatively and philosophically relevant difference between this self excitation and that discussed in section 4.3. In 4.3 the cause of each movement is a previous movement: what set p_n in motion was the previous (causally responsible) movement of p_{n+1}, which was altered accordingly (particle system P as such does not move because its center of mass does not move). Now, however, what sets B in motion is not a previous movement that thus becomes altered. The system of infinite rings $A = \{a_1, a_2, a_3, ..., a_i, ...\}$ is causally responsible for B's movement, but neither A nor any of the a_i alters its permanent state of rest as a consequence. A is literally an unmoved mover. Note that it is system A of rings that interacts at a distance with B, but no one ring in particular. This is consistent with the implicit assumption throughout this discussion that any ring can only interact with B (or with any other ring) by contact collision.[4]

Even though the above analysis of the unmoved mover only considered the relevant numerical function $t_{final} = 1 - (1/n)\sqrt{(n^2-1)}$, there is nothing preventing the study of other additional numerical functions. To better illustrate the procedure, two others will be briefly considered: radius(a_n) ≡ radius(a_n in A_n) = 1/n and, at instant $t_{initial} = 0$, velocity (a_n in A_n) = 1 (velocity(a_n) is not well defined, and is therefore not a function). At limit $n \rightarrow \infty$ these functions reach the respective values $\lim_{n \to \infty}$ radius(a_n in A_n) = 0 and $\lim_{n \to \infty}$ velocity(a_n in A_n) = 1. Therefore, there still seems to be a "ring" at velocity 1 (and radius 0) at the limit, which would not make B's subsequent movement surprising and, incidentally, would dismantle the assumed unmoved mover model. Yet here the same mistake is being made as alluded to at the end of section 4.2: that is, considering only the limit of real functions of real variable, whilst overlooking the fact that the limit postulate LP also requires that the limit of sequences of sets be taken (which are also functions of real variable, although their range is not real numbers). When this is done, the problem disappears. As previously seen, the limit of the sequence of sets $A_1, A_2, A_3, ... A_i, ...$ is $\lim_{n \to \infty} A_n \equiv A_\infty \equiv A = \{a_1, a_2, a_3, ..., a_n, ...\}$, it transpires that in limit set A (responsible for B's movement), there is in fact no a_∞, that is, no ring with radius 0 or at velocity 1. Therefore, the description of the resulting state of affairs after going to the limit is clear: at the infinite limit there is no ring with radius 0 or at velocity 1.

It is interesting to note that the LP is not only a formal calculation resource but also serves as a guide to refer clearly and unambiguously to causality, as can be seen in this presentation of the unmoved mover. The LP connects the case of a finite number of rings (where the causal relation is clear) with the case of an infinite number. It is precisely the going-to-the-limit process that the LP entails which guides our intuition on causal efficacy when taking this "leap to infinity". It is precisely along this path that the Limit Postulate provides solid grounds to contend that the ball is just a passive receiver of the causal influence from the infinitely many rings.

[4] The unmoved mover's action in this section could perhaps be avoided by questioning the use of the limit postulate (LP) therein. However, there would then be a problem: explaining why such use is unsuitable here and not in the case of the spontaneous self-excitation supertask discussed in section 4.3. And, to my mind, considering that it is also unsuitable in the latter case entails rejecting spontaneous self excitation as such.

The set of rings' action on B is yet another example of global action (Laraudogoitia, 2005). It shows is that an unmoved mover acts globally (although it is a uniquely interesting and surprising type of global action). It can be seen that none of the rings does anything (performs no task). Indeed, if the ring, take a_{10}, were to act in some way on ball B (and, in a Newtonian context such as this, to act is to do so by means of a force) then the Newtonian law of action and reaction would require that also B act on a_{10}, yet the latter does not occur because a_{10} is at permanent rest and separated from the other bodies all the time! Hence, only the set of rings acts on ball B. Naturally, B also acts on the set of rings but, since this set has infinite total mass, the reaction force produces no acceleration and, since the rings were initially at rest, no movement.

4.4.1 Unmoved mover and inertia

By way of supplementary explanation, it is interesting to consider the role of Newton's first law (FNL, principle of inertia) in this analysis of the unmoved mover. Since infinite systems are being addressed, the general formulation of this principle may be controversial. Fortunately, for present purposes, nothing resembling a general formulation is needed. A special case will suffice, which, one assumes, will be free of controversy in a Newtonian formulation of mechanics:

(Special FNL): the state of rest or uniform motion of a particle P (not necessarily a point particle; it may be a rigid body or even a system consisting of a finite number of rigid bodies) within an inertial reference frame can only be altered by an external force acting on P.

Appealing to forces is not essential in all formulations of classical mechanics (for example, it is not in the Lagrangian formulation), but it is essential in all formulations of Newtonian mechanics, which is the theory considered in this article. Reference to external forces in (Special FNL) is also essential. If the qualifier "external" is dispensed with, (Special FNL) therefore becomes virtually devoid of any content because, in such a case, any non-point particle could, in principle, self-accelerate at any instant and at any acceleration. The self-acceleration "mechanism" is simple, and was first posited by Laraudogoitia (2002) (see also Lee, 2011 for a further application of the same idea). Without going into details that would detract from the main point, it is based on the fact that parts of any finite rigid body R can suddenly exert finite forces on other parts of R in such a way that the body as such undergoes arbitrary, and arbitrarily time-varying, acceleration. The internal forces developing in R do, of course, obey Newton's third law (principle of action and reaction), but their resultant (determined by the acceleration experienced by R) is non-zero, and thus a reflection of R's action on itself. (Special FNL) effectively blocks this type of pathology, due precisely to the explicit reference to external forces in its formulation. Note, however, that LP is useless to this effect: dealing, as we know, with possible forms of evolution of infinite-particle systems, it says nothing about the dynamics of a single (or finite number of) particles and, consequently, nothing about their self-acceleration. LP is of no use to circumvent this kind of problem. Note also that (Special FNL) does not prevent a material body from exerting a force on itself (in a trivial sense). System U + V formed by particles U and V colliding at instant t exerts a force on itself at t by means of the force that U exerts on V, and also by means of the force that V exerts on U. What (Special FNL) prevents is that the force exerted by a material body on itself alters its state of motion within an inertial reference frame.

Another good reason to believe that ball B, studied in this section, is a passive recipient of the causal influence of infinite-ring system A follows from this analysis of the principle of inertia (see Fig. 2). B goes from being at rest in an inertial system for $t < 0$ to moving at unit velocity for $t > 0$. According to (Special FNL), this can only be due to the action of an external force on B. Given that, in Newtonian mechanics, forces in an inertial system are exerted only by material bodies, it is clear that the instantaneous force acting on B is exerted by A (which is consistent with the LP-based analysis of the unmoved mover performed above).

5 Philosophical intimations with the unmoved mover

Much of the philosophical literature on infinite dynamical systems has been heavily influenced for some time by the use of supertasks. However, the unmoved mover model discussed above is not of this type. It does not resort to any infinite sequence of de facto, real operations (typically, in these contexts, collisions). Nor do infinite sets of merely potential operations need to be considered (even when none of them ever become actualized, as in Angel, 2001, and several other variants of Benardete's paradox (Pruss, 2018)). Now infinitude only appears as the mere cardinality of a set of rings that is not altered at any instant. None of the rings changes its state or interacts with anything. Set A, containing all the rings, simply enables a rigid ball to separate "cleanly" from A. I believe that, even for a scrupulous finitist, such economy of means for description must be striking, to say the least.

While the unmoved mover can be a cause of movement, it can also be a cause of stoppage (transition to rest), as seen in section 4.2. Ball B colliding with A in the manner considered here transitions directly to rest, without transferring its energy elsewhere. This is interesting because, if the rings in A were rigidly attached to one another, then the outcome of the collision would be completely different: the energy would be conserved in B's backward movement (at the same speed as it was originally traveling but heading in the opposite direction).[5] Contrary to what might be expected, freeing the rigid union constraint fails to open up further possibilities as to how the energy can be distributed among the new degrees of freedom present. Quite the opposite: the energy cannot be distributed in any way, but it can disappear. Note the extent to which there was no "experience" of this in the classical literature on the subject. The significant explanatory and predictive role of the Limit Postulate can be very clearly seen here. If the rings are rigidly bound together, the LP leads to B bouncing (it should not be forgotten that in this case there is still an infinite denumerable number of rings, albeit not free), but if the rings are subject to no constraints (such as, for example, being rigidly bound together), then the LP implies that the ball can simply be stopped. Thus, B's behavior depends on the characteristics of the target of rings (so as not to break continuity, see a detailed discussion of this in Appendix II.). Furthermore, this dependence does not undermine the LP; but rather the LP reveals it.

[5] This would simply be B's standard binary collision with another body of infinite mass (composed, however, of infinite rings rigidly bound together).

The spontaneous self-excitation process in section 4.3 could have been obtained (according to our direct intuitions concerning reversibility) solely from the beautiful supertask direct process (section 4.1) by simple time reversal: the time reversal of the collisions in the former becomes the collisions in the latter. However, it is debatable whether the unmoved mover's action in section 4.4 could also have been obtained from the sudden stop in section 4.2 by time reversal. This is because the time reversal of the collision that the sudden stop consists of (and which formally leads to the unmoved mover's action) does not seem to be a collision, at least at first sight. So, if it is not a collision, neither is it clear that it should be admitted as a possible process in classical collision mechanics (were it not admitted, classical collision mechanics would definitively cease to be a time reversal invariant theory). This potential source of criticism was avoided by deducing the possibility of self excitation (in section 4.3) in a second way: by referring to a sufficiently detailed form of the LP. Later, in section 4.4, the same postulate was applied in an essentially similar way to reach the unmoved mover. This point highlights the particular importance of going to the limit (and its precise definition) in the study of infinite systems.

5.1 How interaction at a distance emerges from contact

So, is the way the unmoved mover (as such) operates a collision? It does not seem so. Intuitively, the idea of collision is associated with "reciprocal thrust" relating to the impenetrable nature of matter. Collision avoids interpenetration. This is what occurs, for example, in the sudden stop in section 4.2. If no collision were to take place here, ball B would interpenetrate an infinity of a_n rings. However, the way that the unmoved mover operates by setting B in motion does not avoid any interpenetration. If B were not set in motion, simply nothing would happen. Everything would remain eternally in place. This new form of interaction is puzzling. It should be remembered that, as stated at the beginning of section 3, only a material body can exert a force within an inertial reference frame. Furthermore, also note that, in Newtonian mechanics, there are no three-body forces nor, in general, n-body forces, except for $n = 2$ (the $n = 1$ case of a body's force on itself is trivial and, as seen from the example in section 4.4.1, ultimately amounts to the $n = 2$ case). Consequently, an interaction in Newtonian mechanics is a reciprocal action between two material bodies performed by forces obeying the principle of action and reaction (Newton's third law). If, however, gravitational interaction is excluded (as it is here from the outset), the only scope would seem to be for interactions under conditions of contact between the material bodies involved (at least, if only configurations of material bodies with uniformly bounded velocities are permitted). Indeed, it is precisely under these conditions (namely under contact conditions) that the interaction's puzzling nature, which sets B in motion, is revealed in the first place. It does not appear to be a collision even though it is formally the time reversal of a collision. The same conclusion is reached if it is argued that the condition of relative motion presented at the end of subsection 4.2 ("at t - 0, also usually denoted as t^-, there is relative motion between D and at least one part of C") is a necessary condition for collision at t between D and C provided that joint system D + C is an isolated system (not subject to external perturbations). It is eminently plausible that what is termed here the condition of relative motion is a necessary condition for collision between D and C, when D + C is an isolated system. Besides being plausible, it is also relevant to the situation at hand, because D and C are now B (the ball) and A (the set of rings) respectively, and B + A is an isolated system in the model. Having acknowledged this, it then becomes clear that unmoved mover A does not set B in motion by collision. Of course, if the condition of relative motion is not considered a necessary condition for collision (not even in the specific circumstances of an isolated total system as mentioned), in principle, the possibility remains open that the model in section 4.4 is still a collision model. It would certainly be a qualitatively different collision to usual collisions (B's spontaneous and unpredictable motion would not be preceded by any previous relative motion between certain parts of total system B + A). It could be termed spontaneous collision. Whether collision is mentioned or not in this case, the interesting problems do not end here, as shall be seen below. Is the way the unmoved mover operates (as such) at least a contact interaction? Neither does it seem so. In a contact interaction (collision, friction), contact is essential, not circumstantial. If it disappears, so does this form of interaction (although other forms, such as gravitation, may be present). The problem is that when unmoved mover A sets B in motion, contact is irrelevant. They do not have to be in contact initially, as Fig. 2 and the analysis in Section 4.4 may misleadingly suggest. A can act on B from a distance as an unmoved mover by spontaneously setting it in motion. In other words, if the plane containing the rings is separated from ball B by distance D (as graphically shown in Fig. 3), then the unmoved mover can operate at that distance (regardless of D's value). To see why this is so, the same reasoning as in section 4.4 needs only to be followed with the following modification. When considering the finite set of rings $\{a_1, a_2, a_3, ..., a_n\}$ at instant $t_{initial} = 0$, a_n (the "last ring") will not be centered at O(0,0,0), but at (D,0,0) (although it is moving, as before, at unit velocity leftwards, progressively approaching B). Everything remains the same from here on: again a_n will collide with B at $t_{final} = 1 - (1/n)\sqrt{(n^2–1)}$, moving B leftwards thereafter at unit velocity ...... etc.. And the conclusion (having applied the LP) is also the same: a) We start from B and from an infinite set of concentric rings $\{a_1, a_2, a_3, ..., a_n, ...\}$ of unit mass, all of which are at rest and centered at O(0,0,0) at instant $t_{initial} = 0$ (there is no "last ring" a_∞ in motion, approaching B); b) A perturbation then occurs at $t_{final} = t_{initial} = 0$ which sets

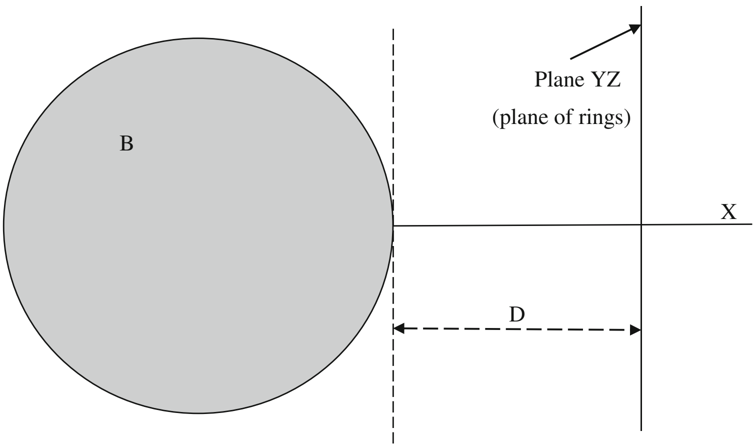

Fig. 3 How the unmoved mover can operate at a distance

B in motion leftwards at unit velocity, while the rings remain at rest. Despite the initial separation distance D, from the outset, B takes away all the energy and momentum arising from the perturbation. Consequently, the way the unmoved mover operates (as such) does not appear to be a contact interaction because it can take place at a distance. However, it cannot be characterized as a distance interaction in the usual sense either, because distance magnitude is irrelevant to the interaction (moreover, given that it can also take place under contact conditions). In any case, note that it is system A of rings that interacts at a distance with B, but no one ring in particular. This is consistent with the assumption implicit throughout this discussion that any ring can only interact with B (or with any other ring) by contact.

Finally, if B's movement, caused by the unmoved mover, is not a collision, we are then presented with a qualitatively new (and I would also say outrageous) case of indeterminism. The two main varieties of indeterminism known thus far in classical mechanics were linked either to the non-uniqueness of the solution to the differential equation of motion (e.g. Norton, 2008, with precedents in the early nineteenth century, as described in detail in Van Strien, 2014) or to the non-existence (at least in the set of ordinary functions) of the differential equation of motion itself due to the presence of rigid collisions (trivial indeterministic multiple collisions or indeterministic sequences of deterministic collisions, as in the case of supertasks).[6] If B's spontaneous movement caused by the unmoved mover is not a collision, it is therefore neither of these types. In any event, this novel form of indeterminism (whose rejection would seriously compromise time-reversal invariance in classical mechanics, as stated above) points to the need for a more thorough investigation of the significance, scope and limitations of mathematical going-to-the-limit processes.

The above conclusion is strengthened when it is considered that contact is also irrelevant in B's sudden stop by A seen in section 4.2. B and A do not necessarily have to end up in contact, as Fig. 1 and the analysis in section 4.2 may misleadingly suggest. A can act on B at any distance D, spontaneously stopping its movement without coming into contact with it. To see why this is so, the same reasoning as in section 4.2 needs only to be followed with the following modification. When considering the finite set of rings $\{a_1, a_2, a_3, \ldots, a_n\}$ at instant $t = 0$, a_n (the "last ring") will not be centered at O(0,0,0) but at (D,0,0), with $D \geq 0$ (and B is moving rightwards at unit velocity, progressively approaching a_n). After B's collision with a_n, the ball will stop with its geometric center at point $(D + (1/n)\sqrt{(n^2{-}1)}, 0, 0)$ (and a_n will take away all its

[6] Although the aim of this paper is to present the dynamics of the unmoved mover, and not to place it within the literature on indeterminism in classical infinite systems, it is interesting to mention that several notable and well-known models of indeterminism in infinite systems can be obtained simply as a direct application of the Limit Postulate LP. Moreover, their analysis closely parallels the discussion already considered on our road to the unmoved mover (third stage) in section 4.3. Such is the case, for example, of Lanford's infinite billiard (Earman, 1986), of Norton's infinite domino cascade (Norton, 2021) or of the infinite chain of masses and springs (Norton, 1999). In contrast, the fourth stage of section 4.4 seems to be exclusively confined to unmoved mover models. This singles them out and, albeit summarily, to some extent allows them to be accurately framed within the broader context of indeterministic infinite systems. Finally, in order to complete the picture, it should be noted out that cases of indeterminism such as Laraudogoitia, 1997 (involving creation ex nihilo, i.e., the evolution of a vacuum and not a system of particles) are not amenable to analysis under the Limit Postulate (LP) for obvious reasons.

energy and momentum).[7] The other rings present remain forever at rest. Everything stays the same from here on, and the conclusion (after applying the LP) is also similar. Now based on B and an infinite set of rings $\{a_1, a_2, a_3, ..., a_n, ...\}$ with radiuses $r_n = 1/n$, each of unit mass. $a_1, a_2, a_3, ..., a_n, ...$ are initially at rest on the YZ plane centered at point (0,0,0) while B approaches them at unit velocity from the left (there is no "last ring", a_∞, located outside the YZ plane being approached by B). Since $\lim_{n \to \infty} [D + (1/n)\sqrt{(n^2-1)})] = D + 1$, it is clear that, when its geometric center reaches point (D + 1,0,0), B will not continue to move further to the right. At that instant it will undergo an interaction and stop. That is all. Since D can be any non-negative real number, B's sudden stop is once again an indeterministic process (and, again, a novel example of indeterminism). Moreover, we are also dealing with a mysterious interaction in this case, which does not appear to be a collision. The same conclusion is reached if it is argued that the contact condition presented at the end of subsection 4.2 ["at instant t, D is in contact with C (i.e. at zero distance from C)"] is a necessary condition for collision at t between D and C provided that the velocities of the material bodies present in joint system D + C are uniformly bounded. It is eminently plausible that what is termed contact condition is a necessary condition for collision between D and C, when the velocities in D + C are uniformly bounded. Besides being plausible, it is also relevant to the situation at hand because D and C are now B (the ball) and A (the set of rings) respectively, and all velocities in B + A are uniformly bounded in the model. Having acknowledged this, it is then clear that B is not suddenly halted by A by means of a collision. Of course, if the contact condition is not considered a necessary condition for collision (not even in the specific circumstances of uniformly bounded velocities mentioned above), in principle, the possibility remains open that the latter model is still a collision model nonetheless. It would certainly be a qualitatively different collision to usual collisions. The fact that B stops would not be associated with "reciprocal thrust" relating to the impenetrable nature of matter. The collision would be completely unrelated here to interpenetration and, therefore, to contact. It could be termed a collision at a distance.

6 Conclusion

Besides leading to a novel form of indeterminism, the discussion in this paper also shows how the unmoved mover model reveals a conflict between our most basic intuitions concerning the concepts of collision and interaction and their mutual relationships. In this respect, I would like to end with a brief programmatic statement. All the discussions developed around this model of unmoved mover have been brought about by the formal going-to-the-limit operation, whose close study is imperative. As mentioned above, the limit postulate (LP) introduced in section 3 is intended to be just a first step in this respect, sufficient for the needs of this paper, but which would require

[7] Fig. 4 in Appendix I is helpful here. It can be considered to show the instant B is stopped by ring a_n. That is, the point S coordinates are (D, 0, 0). Since $PS = 2 - a = 1 + (1/n)\sqrt{(n^2 - 1)}$, the point P coordinates are $(D + 1 + (1/n)\sqrt{(n^2 - 1)}, 0, 0)$. Finally, since B has unit radius, the coordinates of its geometric center will be $(D + (1/n)\sqrt{(n^2 - 1)}, 0, 0)$.

further development when considering more general situations. In view of the results obtained here, this project seems to have borne fruit.

APPENDIX I

CALCULATING a

The situation at the instant B collides with a_n is shown in Fig. 4 below, where one looks from the positive Z-axis (the Figure thus shows the B and a_n projections on the XY plane):

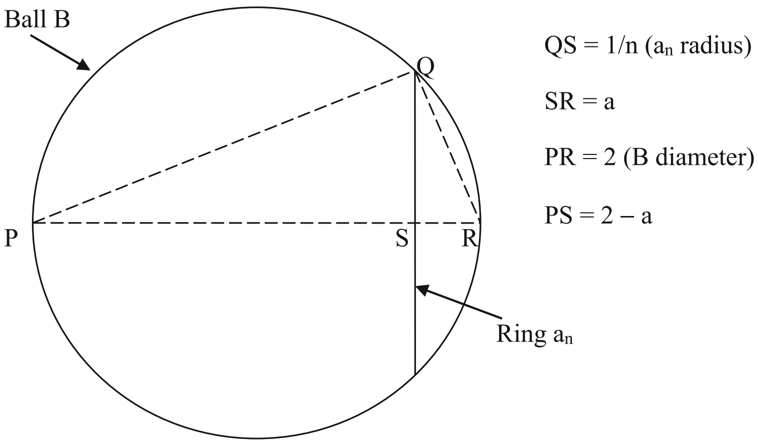

Fig. 4 A rigid ball colliding with a ring

As Thales already knew in the sixth century BC, angle PQR is a right angle, so triangles PQS and SQR are similar. Hence, QS/SP = SR/QS, that is, $1/[n(2 - a)] = n \cdot a$. This yields the quadratic equation in a: $a^2 - 2a + (1/n^2) = 0$. Of its two roots, it is clear that the appropriate root for this situation is that smaller than 1, worth $a = 1 - (1/n)\sqrt{(n^2 - 1)}$.

APPENDIX II

BOUNCING VERSUS STOPPING

Let us consider the initial state in Fig. 1, subsection 4.2, in more detail. First, it will be assumed that the rings are rigidly bound to each other (however, for the sake of simplicity, their rigid connections will be considered massless). The system formed by the rigidly bound rings will be called A*. What is B's collision with A*? It is a binary collision (the "particles" involved are ball B and A*) where B is known to bounce at the same unit velocity at which it collided with A*. This result is clearly not postulated by dynamics. Rather, it is deduced from the analysis of B's collision (known to be at initial unit velocity) with a body of finite mass n, A^*_n (formed only by n rings of unit mass rigidly bound to each other) at rest. Let v be B's final velocity and w, A^*_n's final velocity after collision B-A^*_n. Momentum and energy conservation respectively lead to: $1 = v + nw$ and $1 = v^2 + nw^2$ where B's post-collision velocity is:

$$v = (1-n)/(1+n) \tag{1}$$

Where the target is A*, of infinite mass, the limit in (1) is only to be taken when $n \to \infty$, resulting in $\lim_{n \to \infty} v = \lim_{n \to \infty} (1 - n)/(1 + n) = -1$ (ball B bounces with no energy loss).

What happens if the rings are not rigidly bound to each other but, in principle, are able to move independently of one another? This is the system called A in section 4.2. Since A has infinite mass (like A*), one might think that the final result of collision B-A* could be simply transferred ("extrapolated") to case B-A: B should bounce with no energy loss. However, the fact that collision B-A* is a collision between rigid bodies, while collision B-A is not (system A of rigid rings is clearly not rigid), should be a warning sign against such a reckless leap (which shall be termed "asystematic intuition", AI). Thus, the only way to justify this extrapolation from B-A* (i.e. to justify asystematic intuition, AI) is to check whether the going-to-the-limit process applied to case B-A* can also be applied to case B-A with an identical result. It turns out, however, that the results are not identical! This can be seen by considering B's collision (initial velocity unity) with a body of finite mass n, A_n (formed only by n rings not bound to each other, e.g. $a_1, a_2, a_3, ..., a_n$, as in Fig. 1, section 4.2) at rest. Collision B-A_n is now binary collision B-a_n. Let v be B's final velocity and w, a_n's final velocity. Momentum and energy conservation respectively lead to:

$1 = v + w$ and $1 = v^2 + w^2$ where B's post-collision velocity is:

$$v = 0 \tag{2}$$

Where the target is A, of infinite mass, the limit in (2) is only to be taken when $n \to \infty$, trivially resulting in $\lim_{n \to \infty} v = 0$ (ball B is halted). This shows that asystematic intuition (AI) lacks justification. As mentioned at the beginning of section 5, LP implies that the ball can simply be stopped (which would not be possible if the collision was with A*). In this precise sense, B's behavior depends on the target of rings' characteristics.

Finally, what if (in response to my arguments against its justification) IA is simply postulated with no justification whatsoever? There is then yet another powerful argument against it: an argument that reveals its logical weakness, its deductive poverty. This can be seen by considering the B-A collision anew, the only difference being that B's mass is not 1 but $1/\alpha$ ($\alpha \geq 1$). According to AI, this does not make any significant difference: B will bounce off A while conserving its energy (as it would do so if it collided with A*). However, by predicting such an event shows its logical weakness compared to LP, because LP allows far more detailed analysis of the situation, and is able to distinguish B's possible post-collision states depending on its mass. This can be seen by considering B's collision (initial velocity unity, mass $1/\alpha$, $\alpha \geq 1$) with finite mass body n, A_n, at rest. Collision B-A_n is once again binary collision B-a_n. Let v be B's final velocity and w, a_n's final velocity. Momentum and energy conservation respectively lead to: $1/\alpha = (v/\alpha) + w$ and $1/\alpha = (v^2/\alpha) + w^2$, that is, $1 = v + \alpha w$ and $1 = v^2 + \alpha w^2$, where B's post-collision velocity is:

$$v = (1-\alpha)/(1+\alpha) \tag{3}$$

Where the target is A, of infinite mass, the limit in (3) should only be taken with $n \to \infty$, trivially resulting in $\lim_{n \to \infty} v = (1 - \alpha)/(1 + \alpha)$. That is, ball B bounces if $\alpha > 1$ (and does not if $\alpha = 1$), yet (as can be immediately seen) after the collision, kinetic energy of magnitude $2/(1 + \alpha)^2$ is lost. In other words, all initial energy (1/2) is lost if $\alpha = 1$, not all if $\alpha > 1$ is finite, and no energy if α is infinite. Besides being unjustified, AI is completely refractory and blind to all these possibilities of evolution opened up by LP.

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