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Exceedance Counts and GOD's Order Statistics

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Abstract

In this paper we derive a characterization of the distribution of the number of exceedances among the components of a random vector in terms of order statistics of generators of D-norms (GOD). The computation of the fragility index is an immediate consequence.

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1 Introduction

Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a random vector that realizes in $\mathbb{R}^d$. We are interested in the number of exceedances among the components $X_1, \dots, X_d$ above high thresholds. Precisely, choose $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $k \in \{1, \dots, d\}$ and put $N_{\boldsymbol{x}} := \sum_{i=1}^d 1_{(x_i,\infty)}(X_i)$. We want to analyze in this paper the probability

$$P(X_i > x_i \text{for at least } k \text{ of the components } i = 1, \dots, d) = P\left(N_{\boldsymbol{x}} \geq k\right)$$

for a large $\boldsymbol{x}$, i.e., each component x_i of $\boldsymbol{x}$ is large.

Suppose the vector $\boldsymbol{x} = (x, \dots, x) \in \mathbb{R}^d$ has constant entry $x \in \mathbb{R}$ and put $N_x := N_{(x,\dots,x)}$. The *fragility index* (FI) corresponding to $\boldsymbol{X}$ is the asymptotic conditional expected number of exceedances, given that there is at least one exceedance, i.e., $\mathrm{FI} = \lim_{x \nearrow} E(N_x \mid N_x > 0)$.

The FI was introduced by Geluk et al. (2007) to measure the stability of the stochastic system $\{X_1, \dots, X_d\}$. The system is called *stable* if FI = 1, otherwise it is called *fragile*.

In Falk and Tichy (2012) the asymptotic conditional distribution $p_k := \lim_{x \nearrow} P(N_x = k \mid N_x > 0)$ was investigated under the condition that the components $X_1, \dots, X_d$ are identically distributed.

It turned out that this *asymptotic conditional distribution of exceedance counts* (ACDEC) exists, if the copula C, which corresponds to $\boldsymbol{X}$ by Sklar's theorem (c.f. Sklar (1959), Sklar (1996)), is *in the max-domain of attraction* of a multivariate extreme value distribution (EVD) G. This means that, for any $\boldsymbol{x} = (x_1, \dots, x_d) \le \boldsymbol{0} \in \mathbb{R}^d$,

$$C^n\left(1 + \frac{x_1}{n}, \dots, 1 + \frac{x_d}{n}\right) \to_{n\to\infty} G(\boldsymbol{x}), \tag{1.1}$$

where G is a non degenerate distribution function (df) on $\mathbb{R}^d$. This is quite a mild condition, satisfied by almost every textbook copula C, c.f. Section 3.3 in Falk (2019).

Falk and Tichy (2011) investigated the ACDEC, dropping the assumption that the margins $X_1, \dots, X_d$ are identically distributed.

Recent results on D-norms (Falk (2019), Falk and Fuller (2021)) enable in the present paper the representation of the distribution of $N_{\boldsymbol{x}}$ in terms of order statistics corresponding to the generator of a D-norm, introduced below.

Let $Z_1, \dots, Z_d$ be random variables with the properties

$$Z_i \ge 0,\ E(Z_i) = 1, \qquad 1 \le i \le d.$$

Then

$$\|\boldsymbol{x}\|_D := E\left(\max_{1\le i\le d}(|x_i|\, Z_i)\right), \qquad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

defines a norm on $\mathbb{R}^d$, called *D-norm*, and $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ is a *generator* of the D-norm (GOD).

D-norms are the skeleton of multivariate extreme value theory (MEVT) as elaborated in detail in Falk (2019). Suppose, for example, that the random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ follows a *standard multivariate Generalized Pareto distribution* (GPD). In this case there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that

$$P(\boldsymbol{X} \le \boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D\,,$$

for all $\boldsymbol{x}$ in a left neighborhood of $\boldsymbol{0} = (0, \dots, 0) \in \mathbb{R}^d$, i.e., for all $\boldsymbol{x} \in [\varepsilon, 0]^d$ with some $\varepsilon < 0$. The characteristic property of a GPD is its *excursion stability* or *exceedance stability*, see Proposition 3.1.2 and Remark 3.1.3 in Falk (2019). This stability explains the crucial role of GPD in MEVT. In what follows, all operations on vectors such as $\boldsymbol{x} \le \boldsymbol{y}$ etc. are always meant componentwise.

According to equation (2.16) in Falk (2019), the survival function of a standard GPD is given by

$$P(\boldsymbol{X} > \boldsymbol{x}) = P(\boldsymbol{X} \geq \boldsymbol{x}) = \wr\, \boldsymbol{x} \,\wr_D := E\left(\min_{1\leq i\leq d}(|x_i|\, Z_i)\right), \qquad \boldsymbol{x} \in [\varepsilon, 0]^d,$$

where $\wr\, \boldsymbol{x} \,\wr_D$ is the *dual D-norm function* pertaining to $\|\cdot\|_D$. Note that the df of $\boldsymbol{X}$ is continuous on $[\varepsilon, 0]^d$ and, thus, $P(\boldsymbol{X} \geq \boldsymbol{x}) = P(\boldsymbol{X} > \boldsymbol{x})$ for $\boldsymbol{x} \in [\varepsilon, 0]^d$.

The generator $\boldsymbol{Z}$ of $\|\cdot\|_D$ is not uniquely determined. But we know from Corollary 1.6.3 in Falk (2019) that the dual D-norm function $\wr\, \boldsymbol{x} \,\wr_D$, which corresponds to $\|\cdot\|_D$, is independent of the particular generator $\boldsymbol{Z}$ of $\|\cdot\|_D$.

So far we have considered the maximum $\max_{1\leq i\leq d}(|x_i|\, Z_i)$ and the minimum $\min_{1\leq i\leq d}(|x_i|\, Z_i)$, leading to the D-norm $\|\boldsymbol{x}\|_D$ and dual D-norm $\wr\, \boldsymbol{x} \,\wr_D$ by taking expectations. But we can clearly order the set $\{|x_1|\, Z_1, \dots, |x_d|\, Z_d\}$ completely

$$\begin{aligned}
\min_{1\leq i\leq d}(|x_i|\, Z_i) &\leq \text{2nd smallest value among } \{|x_1|\, Z_1, \dots, |x_d|\, Z_d\}\\
&\leq \text{3rd smallest value among } \{|x_1|\, Z_1, \dots, |x_d|\, Z_d\}\\
&\ \ \vdots\\
&\leq \max_{1\leq i\leq d}(|x_i|\, Z_i)
\end{aligned}$$

and put, for $k = 1, \dots, d$,

$$\|\boldsymbol{x}\|_{D,(k)} := E\left(k\text{-th smallest value among } \{|x_1|\, Z_1, \dots, |x_d|\, Z_d\}\right).$$

We call this sequence $\|\boldsymbol{x}\|_{D,(1)} \leq \|\boldsymbol{x}\|_{D,(2)} \leq \dots \leq \|\boldsymbol{x}\|_{D,(d)}$ of increasing functions *ordered D-norms*. In particular

$$\|\boldsymbol{x}\|_{D,(1)} = \wr\, \boldsymbol{x} \,\wr_D$$

is the dual D-norm function, and

$$\|\boldsymbol{x}\|_{D,(d)} = \|\boldsymbol{x}\|_D$$

is the D-norm. Note that only $\|\boldsymbol{x}\|_{D,(k)}$ with $k = d$ is actually a norm.

We have, for each $k = 1, \dots, d$,

$$\|t\boldsymbol{x}\|_{D,(k)} = t\, \|\boldsymbol{x}\|_{D,(k)}, \qquad t \geq 0,\ \boldsymbol{x} \in \mathbb{R}^d,$$

i.e., $\|\cdot\|_{D,(k)}$ is homogeneous of order 1, and it is a continuous function on $\mathbb{R}^d$, with $\|\mathbf{0}\|_{D,(k)} = 0$ for each $k = 1, \dots, d$. The following result is, therefore, an immediate consequence of Theorem 3.5 in Falk and Fuller (2021).

LEMMA 1.1. *For each $k = 1, \dots, d$, the value $\|\boldsymbol{x}\|_{D,(k)}$ does not depend on the choice of the particular generator $\boldsymbol{Z}$ of $\|\cdot\|_D$.*

For a vector with constant entry $\boldsymbol{x} = (x, \dots, x) \in \mathbb{R}^d$ we obtain

$$\begin{aligned} \|\boldsymbol{x}\|_{D,(k)} &= |x|\, E(k\text{-th smallest value among } \{Z_1, \dots, Z_d\}) \\ &= |x|\, E\left(Z_{k:d}\right), \end{aligned}$$

where $Z_{1:d} \le \dots \le Z_{d:d}$ denote the ordered values of $Z_1, \dots, Z_d$, i.e., the corresponding usual *order statistics.* This links the analysis of multivariate exceedances with the theory of univariate order statistics.

Though it has received much attention in the literature, the computation of the exact value $E(Z_{k:d})$ of an arbitrary order statistic is by no means obvious, even if $Z_1, \dots, Z_d$ are independent and identically distributed. The exact value is known only in a few cases. Put, for instance, $Z_i := 2U_i$, $1 \le i \le d$, where $U_1, \dots, U_d$ are independent random variables with common uniform distribution on $(0, 1)$. In this case we have $E(Z_{k:d}) = 2k/(d + 1)$, see equation (1.7.3) in Reiss (1989).

In the general case, where $Z_1, \dots, Z_d$ are not necessarily independent and identically distributed, the exact value $E(Z_{k:d})$ is in general inaccessible. But a Monte-Carlo simulation would provide an obvious and easy to implement approximation.

The main result in Section 2 is Theorem 2.1, where we establish, for each $1 \le k \le d$, the equation

$$P(X_i > x_i \text{ for at least } k \text{ of the components } 1 \le i \le d) = \|\boldsymbol{x}\|_{D,(d-k+1)}$$

for $\boldsymbol{x}$ in a left neighborhood of $\boldsymbol{0} \in \mathbb{R}^d$, if $\boldsymbol{X} = (X_1, \dots, X_d)$ follows a standard GPD. From this result we can establish in Eq. 2.1 the exact distribution of the number $N_{\boldsymbol{x}}$ of exceedances. For $\boldsymbol{x} = (x, \dots, x)$ with constant entry x, we obtain in Eq. 2.3 the corresponding FI.

In Section 3 we extend the results of Section 2 to a random vector $\boldsymbol{X}$, whose copula is in a proper neighborhood of a shifted GPD, see Eq. 3.1. The main result is Theorem 3.1. It leads to the representation (3.5) of the probability that $X_i > x_i$ for at least k components under the mild condition that that the copula corresponding to the random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ is in the max-domain of attraction of a multivariate max-stable df G. Note that this condition does not require an absolutely continuously distributed random vector $\boldsymbol{X}$.

2 Main Result for Standard GPD

The following result reveals the particular significance of ordered D-norms concerning multivariate exceedances of standard GPD.

THEOREM 2.1. *Suppose the random vector* $\boldsymbol{X} = (X_1, \dots, X_d)$ *follows a standard GPD with corresponding D-norm* $\|\cdot\|_D$ *on* $\mathbb{R}^d$. *Then we have*

$$\begin{aligned} & P(X_i > x_i \text{ for at least } k \text{ of the components } 1 \le i \le d) \\ = \; & P(N_{\boldsymbol{x}} \ge k) \\ = \; & \|\boldsymbol{x}\|_{D,(d-k+1)}, \end{aligned}$$

for any $\boldsymbol{x} = (x_1, \dots, x_d)$ *in a left neighborhood of* $\mathbf{0} \in \mathbb{R}^d$.

The assertion is obvious for $k = d$, because $\|\cdot\|_{D,(1)} = \wr \cdot \wr_D$. For $k = 1$ we obtain

$$\begin{aligned} & P(X_i > x_i \text{ for at least one of the components } 1 \le i \le d) \\ = \; & 1 - P(\boldsymbol{X} \le \boldsymbol{x}) = \|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_{D,(d)}, \end{aligned}$$

for $\boldsymbol{x}$ in a left neighborhood of $\mathbf{0} \in \mathbb{R}^d$.

PROOF OF THEOREM 2.1. According to equation (2.14) in Falk (2019), we can suppose the representation

$$\boldsymbol{X} = -U\left(\frac{1}{Z_1}, \dots, \frac{1}{Z_d}\right),$$

where the random variable U is on $(0,1)$ uniformly distributed and independent of $\boldsymbol{Z}$. We can also suppose by Theorem 1.7.1 in Falk (2019) that the generator $\boldsymbol{Z}$ is bounded.

Conditioning on $U = u$, we obtain for $\boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d$ close enough to $\mathbf{0}$, by the boundedness of $\boldsymbol{Z}$,

$$\begin{aligned} & P(X_i > x_i \text{ for at least } k \text{ of the components } 1 \le i \le d) \\ = \; & P(U < |x_i| Z_i \text{ for at least } k \text{ of the components } 1 \le i \le d) \\ = \; & \int_0^1 P(u < |x_i| Z_i \text{ for at least } k \text{ of the components } 1 \le i \le d) \, du \\ = \; & \int_0^1 P\big((d-k+1)\text{-th smallest value among } \{|x_1| Z_1, \dots, |x_d| Z_d\} \ge u\big) \, du \\ = \; & \int_0^\infty P\big((d-k+1)\text{-th smallest value among } \{|x_1| Z_1, \dots, |x_d| Z_d\} \ge u\big) \, du \\ = \; & E((d-k+1)\text{-th smallest value among } \{|x_1| Z_1, \dots, |x_d| Z_d\}) \end{aligned}$$

$$= \quad \|\boldsymbol{x}\|_{D,(d-k+1)}\,,$$

using the general equation $E(Y) = \int_0^\infty P(Y > u)\,du$ with a random variable $Y \geq 0$.

We obtain from the previous result the *exact* distribution of the number of exceedances for $\boldsymbol{x}$ in a left neighborhood of $\mathbf{0} \in \mathbb{R}^d$ if $\boldsymbol{X} = (X_1, \dots, X_d)$ follows a standard GPD:

$$\begin{aligned} P(N_{\boldsymbol{x}} = k) &= P(N_{\boldsymbol{x}} \geq k) - P(N_{\boldsymbol{x}} \geq k-1) \\ &= \begin{cases} \|\boldsymbol{x}\|_{D,(d-k+1)} - \|\boldsymbol{x}\|_{D,(d-k)}\,, & k = 1, \dots, d-1, \\ \|\boldsymbol{x}\|_{D,(1)} = \wr\, \boldsymbol{x} \,\wr_D, & k = d, \end{cases} \\ &=: p_{\boldsymbol{x},D}(k). \end{aligned} \tag{2.1}$$

With

$$p_{\boldsymbol{x},D}(0) := P(N_{\boldsymbol{x}} = 0) = P(X_i \leq x_i \text{ for } 1 \leq i \leq d) = 1 - \|\boldsymbol{x}\|_D\,,$$

the numbers $p_{\boldsymbol{x},D}(k)$, $k = 0, \dots, d$, define the distribution of the exceedance counts $N_{\boldsymbol{x}}$ on $\{0, 1, \dots, d\}$.

In case of a vector $\boldsymbol{x} = (x, \dots, x) \in \mathbb{R}^d$ with constant entries, we obtain, with $Z_{0:d} := 0$,

$$\begin{aligned} p_{x,D}(k) &:= p_{(x,\dots,x),D}(k) \\ &= \begin{cases} |x|\,\big(E(Z_{d-k+1:d}) - E(Z_{d-k:d})\big), & 1 \leq k \leq d, \\ 1 - |x|\,E(Z_{d:d}), & k = 0, \end{cases}. \end{aligned} \tag{2.2}$$

This reveals a crucial role of the *spacings* $Z_{d-k+1:d} - Z_{k-d:d}$ in the analysis of multivariate exceedances.

The expectation of N_x is by Eq. 2.2

$$\begin{aligned} E(N_x) &= \sum_{k=1}^d k p_{x,D}(k) \\ &= |x| \sum_{k=1}^d k\big(E(Z_{d-k+1:d}) - E(Z_{d-k:d})\big) \\ &= |x| \sum_{k=1}^d E(Z_{d-(k-1):d}) \\ &= |x|\, E\left(\sum_{k=1}^d Z_{d-(k-1):d}\right) \end{aligned}$$

$$= \quad |x|\, E\left(\sum_{k=1}^{d} Z_k\right) = d\,|x|$$

by the fact that $E(Z_k) = 1$, $1 \le k \le d$. The fragility index is, therefore, with $x < 0$ close enough to zero,

$$\begin{aligned} \mathrm{FI} &= E(N_x \mid N_x > 0) \\ &= \frac{E(N_x)}{1 - P(N_x = 0)} \\ &= \frac{d\,|x|}{1 - p_{x,D}(0)} \\ &= \frac{d}{E\left(Z_{d:d}\right)} = \frac{d}{\|\mathbf{1}\|_D}. \end{aligned} \tag{2.3}$$

Note that in this case of a standard GPD, the number $E(N_x \mid N_x > 0)$ does not depend on x, i.e., the FI does not require a limit. The number $\|\mathbf{1}\|_D$ is known as the *extremal coefficient*, introduced by Smith (1990). It measures the tail dependence of the components $X_1, \dots, X_d$ by just one number. If we have $\|\mathbf{1}\|_D = d$, then there is tail independence, and in case $\|\mathbf{1}\|_D = 1$ we have complete dependence, see Falk (2019), equation (2.28).

Example 2.1. Take, for example, the Dirichlet D-norm $\|\cdot\|_{D(\alpha)}$ with parameter $\alpha = 1$, see Example 1.7.4 in Falk (2019). Its generator is $\boldsymbol{Z} = (Z_1, \dots, Z_d)$, where $Z_1, \dots, Z_d$ are independent and identically standard exponential distributed random variable. We have

$$E\left(Z_{k:d}\right) = \sum_{j=d-k+1}^{d} \frac{1}{j}, \qquad 1 \le k \le d,$$

see equation (1.7.7) in Reiss (1989) and, thus, we obtain for a vector with constant entry $\boldsymbol{x} = (x, \dots, x) \in \mathbb{R}^d$

$$p_{x,D}(k) = \begin{cases} \frac{|x|}{k}, & 1 \le k \le d \\ 1 - |x| \sum_{k=1}^{d} \frac{1}{k}, & k = 0. \end{cases}$$

Recall that the previous considerations require that the vector $\boldsymbol{x}$ is in a left neighborhood of $\mathbf{0} \in \mathbb{R}^d$. Otherwise, with arbitrary $x \neq 0$, the preceding equation could not be true.

The fragility index is consequently,

$$\mathrm{FI} = \frac{d}{E(Z_{d:d})} = \frac{d}{\sum_{k=1}^{d} \frac{1}{k}}$$

with the extremal coefficient

$$\|\mathbf{1}\|_D = \sum_{k=1}^{d} \frac{1}{k}.$$

EXAMPLE 2.2. Let $U_1, \dots, U_d$ be independent and on $(0,1)$ uniformly distributed random variables, i.e., $P(U_i \le u) = u$, $u \in [0,1]$, $1 \le i \le d$. Then $\boldsymbol{Z} = (Z_1, \dots, Z_d) := 2(U_1, \dots, U_d)$ is a generator of a D-norm $\|\cdot\|_D$. From the fact that $E(U_{k:d}) = k/(d+1)$, $1 \le k \le d$, see equation (1.7.3) in Reiss (1989), we obtain $E(Z_{k:d}) = 2k/(d+1)$ and, thus, by Eq. 2.2,

$$p_{x,D}(k) = \begin{cases} \frac{2|x|}{d+1}, & 1 \le k \le d, \\ 1 - 2\,|x|\,\frac{d}{d+1}, & k = 0. \end{cases}$$

In this case, the conditional probability that we have exactly k exceedances above x among $X_1, \dots, X_d$, given that there is at least one, is by Eq. 2.1,

$$\begin{aligned} P(N_x = k \mid N_x \ge 1) &= \frac{P(N_x = k)}{P(N_x \ge 1)} \\ &= \frac{p_{x,D}(k)}{1 - p_{x,D}(0)} = \frac{1}{d}, \qquad 1 \le k \le d, \end{aligned}$$

which is the uniform distribution on the set of integers $\{1, \dots, d\}$. The fragility index is $\mathrm{FI} = (d+1)/2$ and the extremal coefficient is $\|\mathbf{1}\|_D = 1/E(Z_{d:d}) = (d+1)/2d \to_{d\to\infty} 1/2$.

Suppose that the copula C of the random vector $\boldsymbol{Y} = (Y_1, \dots, Y_d)$ is a *generalized Pareto copula* (GPC), i.e., there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that

$$C(\boldsymbol{u}) = 1 - \|\mathbf{1} - \boldsymbol{u}\|_D\,, \qquad \boldsymbol{u}_0 \le \boldsymbol{u} \le \mathbf{1} \in \mathbb{R}^d,$$

for some $\boldsymbol{u}_0 < \mathbf{1} \in \mathbb{R}^d$, see Section 3.1 in Falk (2019). Then we have equality in distribution

$$\boldsymbol{Y} =_D \left(F_1^{-1}(U_1), \dots, F_d^{-1}(U_d)\right),$$

where F_i is the df of Y_i, $1 \le i \le d$, and the random vector $\boldsymbol{U} = (U_1, \dots, U_d)$ follows the GPC C. Note that the random vector $\boldsymbol{X} := \boldsymbol{U} - \mathbf{1}$ then follows a GPD with D-norm $\|\cdot\|_D$.

From the general equivalence $F^{-1}(q) > t \iff q > F(t)$, valid for $q \in (0,1)$, $t \in \mathbb{R}$ and an arbitrary univariate df F, we obtain

$$\begin{aligned} &P(Y_i > y_i \text{ for at least } k \text{ of the components } 1 \le i \le d) \\ =\ &P(U_i > F_i(y_i) \text{ for at least } k \text{ of the components } 1 \le i \le d) \end{aligned}$$

$$\begin{aligned} &= P(U_i - 1 > F_i(y_i) - 1 \text{ for at least } k \text{ of the components } 1 \le i \le d) \\ &= P(X_i > F_i(y_i) - 1 \text{ for at least } k \text{ of the components } 1 \le i \le d) \\ &= \|(1 - F_1(y_1), \dots, 1 - F_d(y_d))\|_{D,(d-k+1)}, \end{aligned}$$

for $\boldsymbol{y}$ large enough.

3 Extension to Multivariate Max-Domain of Attraction

In the next step we extend the considerations in the previous section to a copula C, which satisfies the max-domain of attraction condition (1.1). In this case there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ with

$$\lim_{t \downarrow 0} \frac{1 - C(1 + t\boldsymbol{x})}{t} = \|\boldsymbol{x}\|_D, \qquad \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d, \tag{3.1}$$

and the EVD G is given by

$$G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right), \qquad \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d,$$

see Corollary 3.1.6 in Falk (2019).

EXAMPLE 3.1. Take an arbitrary *Archimedean copula* on $\mathbb{R}^d$

$$C_\varphi(\boldsymbol{u}) = \varphi^{-1}\big(\varphi(u_1) + \dots + \varphi(u_d)\big), \qquad \boldsymbol{u} = (u_1, \dots, u_d) \in (0,1)^d,$$

where φ is a continuous and strictly increasing function from $(0,1]$ to $[0,\infty)$ with $\varphi(1) = 0$, see, for example, McNeil and Nešlehová (2009), Theorem 2.2. Suppose that

$$p := -\lim_{s \downarrow 0} \frac{s\varphi'(1-s)}{\varphi(1-s)}$$

exists in $[1,\infty]$. Then C_φ satisfies condition (3.1) with the D-norm $\|\cdot\|_D$ being the logistic one $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_p = \left(\sum_{i=1}^p |x_i|^p\right)^{1/p}$, with parameter $p \in [1,\infty]$ and the convention $\|\boldsymbol{x}\|_\infty = \max_{1 \le i \le d} |x_i|$, see Corollary 3.1.15 in Falk (2019).

THEOREM 3.1. *Suppose that the random vector $\boldsymbol{U} = (U_1, \dots, U_d)$ follows a copula C, which satisfies* Eq. 3.1. *Then we have, for each $k = 1, \dots, d$,*

$$\begin{aligned} &P\big(U_i > 1 + tx_i \text{ for at least } k \text{ of the components } 1 \le i \le d\big) \\ &= t\,\|\boldsymbol{x}\|_{D,(d-k+1)} + o(t) \end{aligned}$$

as $t \downarrow 0$, for each $\boldsymbol{x} = (x_1, \dots, x_d) \le \boldsymbol{0} \in \mathbb{R}^d$.

The preceding result implies for a random vector $\boldsymbol{U}$, whose copula satisfies Eq. 3.1,

$$\begin{aligned} &P\big(U_i > 1 + x_i \text{ for at least } k \text{ of the components } 1 \le i \le d\big) \\ =\ & \|\boldsymbol{x}\| \left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right\|_{D,(d-k+1)} + o(\|\boldsymbol{x}\|), \end{aligned} \tag{3.2}$$

$\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, $\boldsymbol{x} \ne \boldsymbol{0}$, as $\|\boldsymbol{x}\| \to 0$, with an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^d$.

By repeating the arguments in Section 2, we obtain for the FI corresponding to $N_{tx} = \sum_{i=1}^d 1_{(1+tx,1]}(U_i)$, $x < 0$, if the copula C of $\boldsymbol{U} = (U_1, \dots, U_d)$ satisfies condition (3.1),

$$\mathrm{FI} = \lim_{t \downarrow 0} \frac{E(N_{tx})}{1 - P(N_{tx} = 0)} = \frac{d}{\|\boldsymbol{1}\|_d}. \tag{3.3}$$

This was already observed in Falk and Tichy (2012), Theorem 4.1.

The following representation of $\|\boldsymbol{x}\|_{D,(d-k+1)}$ will be a crucial tool in the derivation of Theorem 3.1. We briefly explain its notation. By $\boldsymbol{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$ we denote the i-th unit vector in $\mathbb{R}^d$, $i = 1, \dots, d$. Any $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ can be represented as $\boldsymbol{x} = \sum_{i=1}^d x_i \boldsymbol{e}_i$. Choose a subset $A \subset \{1, \dots, d\}$. Then we set

$$\boldsymbol{x}_A := \sum_{i \in A} x_i \boldsymbol{e}_i \in \mathbb{R}^d,$$

with the convention $\boldsymbol{x}_\emptyset = \boldsymbol{0} \in \mathbb{R}^d$. By $|A|$ we denote the number of elements in a set A.

Lemma 3.1. *We have for any D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ and each $k = 1, \dots, d$*

$$\begin{aligned} &\|\boldsymbol{x}\|_{D,(d-k+1)} \\ =\ & \sum_{m=k}^{d} \sum_{T \subset \{1,\dots,d\},\, |T| = m} \sum_{S \subset T} (-1)^{|S|-1} \left\| \boldsymbol{x}_{S \cup T^{\complement}} \right\|_D, \quad \boldsymbol{x} \in \mathbb{R}^d. \end{aligned} \tag{3.4}$$

The preceding probabilistic result entails the following non probabilistic representation of the $(d-k+1)$-th smallest value among arbitrary nonnegative numbers $x_1, \dots, x_d$ in terms of maxima of subsets of $\{x_1, \dots, x_d\}$.

Choosing the particular D-norm $\|\cdot\|_D = \|\cdot\|_\infty$, with constant generator $\boldsymbol{Z} = (1, \dots, 1)$, Lemma 3.1 implies, with $\boldsymbol{x} = (x_1, \dots, x_d) \ge \boldsymbol{0} \in \mathbb{R}^d$,

the $(d-k+1)$-th smallest value among $x_1, \dots, x_d$

$$= \sum_{m=k}^{d} \sum_{T\subset\{1,\dots,d\},\,|T|=m} \sum_{S\subset T} (-1)^{|S|-1} \max_{S\cup T^{\complement}} x_i, \qquad k=1,\dots,d.$$

In particular for $k=d$ we obtain

$$\min_{1\le i\le d} x_i = \sum_{S\subset\{1,\dots,d\}} (-1)^{|S|-1} \max_{i\in S} x_i,$$

which is a well known representation of a minimum of nonnegative numbers in terms of maxima, c.f. Lemma 1.6.1 in Falk (2019).

PROOF OF LEMMA 3.1. We present a probabilistic proof of Lemma 3.1. Let $\boldsymbol{X}=(X_1,\dots,X_d)$ follow a standard GPD with D-norm $\|\cdot\|_D$, i.e.,

$$P(\boldsymbol{X}\le\boldsymbol{x}) = 1-\|\boldsymbol{x}\|_D$$

for all $\boldsymbol{x}\in[\varepsilon,0]^d$, with some $\varepsilon<0$. For such $\boldsymbol{x}=(x_1,\dots,x_d)$ we obtain from Theorem 2.1, with $k\in\{1,\dots,d\}$,

$$\begin{aligned}
\|\boldsymbol{x}\|_{D,(d-k+1)} &= P\left(\sum_{i=1}^{d} 1_{(x_i,0]}(X_i)\ge k\right)\\
&= \sum_{m=k}^{d} P\left(\sum_{i=1}^{d} 1_{(x_i,0]}(X_i)=m\right)\\
&= \sum_{m=1}^{d} \sum_{T\subset\{1,\dots,d\}\,|T|=m} P\left(X_i>x_i, i\in T;\, X_j\le x_j, j\in T^{\complement}\right).
\end{aligned}$$

The Inclusion-Exclusion Principle (see Corollary 1.6.2 in Falk (2019)) implies

$$\begin{aligned}
&P\left(X_i>x_i, i\in T;\, X_j\le x_j, j\in T^{\complement}\right)\\
&= P\left(X_i>x_i, i\in T \mid X_j\le x_j, j\in T^{\complement}\right) P\left(X_j\le x_j, j\in T^{\complement}\right)\\
&= \left(1-P\left(\bigcup_{i\in T}\{X_i\le x_i\} \,\middle|\, X_j\le x_j, j\in T^{\complement}\right)\right) P\left(X_j\le x_j, j\in T^{\complement}\right)\\
&= \left(1-\sum_{\emptyset\ne S\subset T} (-1)^{|S|-1} P\left(X_i\le x_i, i\in S \mid X_j\le x_j, j\in T^{\complement}\right)\right)\\
&\quad\times P\left(X_j\le x_j, j\in T^{\complement}\right)\\
&= P\left(X_j\le x_j, j\in T^{\complement}\right) - \sum_{\emptyset\ne S\subset T} (-1)^{|S|-1} P\left(X_i\le x_i, i\in S;\, X_j\le x_j, j\in T^{\complement}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{S\subset T}(-1)^{|S|}P\left(X_i\le x_i, i\in S;\ X_j\le x_j, j\in T^{\complement}\right)\\
&= \sum_{S\subset T}(-1)^{|S|}\left(1-\left\|\boldsymbol{x}_{S\cup T^{\complement}}\right\|_D\right)\\
&= \sum_{S\subset T}(-1)^{|S|-1}\left\|\boldsymbol{x}_{S\cup T^{\complement}}\right\|_D,
\end{aligned}
$$

where the final equation is due to the fact that $\sum_{S\subset T}(-1)^{|S|}=0$, see equation (1.10) in Falk (2019).

Altogether we have shown that, for $\boldsymbol{x}\in[\varepsilon,0]^d$,

$$
\begin{aligned}
&\|\boldsymbol{x}\|_{D,(d-k+1)}\\
&= \sum_{m=k}^{d}\ \sum_{T\subset\{1,\dots,d\},\,|T|=m}\ \sum_{S\subset T}(-1)^{|S|-1}\left\|\boldsymbol{x}_{S\cup T^{\complement}}\right\|_D, \qquad \boldsymbol{x}\in\mathbb{R}^d.
\end{aligned}
$$

The fact that $\|t\boldsymbol{x}\|_{D,(d-k+1)}=t\,\|\boldsymbol{x}\|_{D,(d-k+1)}$ and $\left\|t\boldsymbol{x}_{S\cup T^{\complement}}\right\|_D=t\left\|\boldsymbol{x}_{S\cup T^{\complement}}\right\|_D$ for $t\ge 0$ implies that the above equation is true for each $\boldsymbol{x}\in\mathbb{R}^d$.

Now we can prove Theorem 3.1 in a straightforward way.

Proof of Theorem 3.1. Let the random vector $\boldsymbol{U}=(U_1,\dots,U_d)$ follow a copula C, which satisfies Eq. 3.1. Choose $k\in\{1,\dots,d\}$ and $\boldsymbol{x}=(x_1,\dots,x_d)\le\mathbf{0}\in\mathbb{R}^d$. By repeating arguments in the proof of Lemma 3.1 we obtain

$$
\begin{aligned}
&P\left(\sum_{i=1}^d 1_{(1+tx_i,1]}(U_i)\ge k\right)\\
&= \sum_{m=k}^d P\left(\sum_{i=1}^d 1_{(1+tx_i,1]}(U_i)=m\right)\\
&= \sum_{m=k}^d\ \sum_{T\subset\{1,\dots,d\},|T|=m} P\left(U_i>1+tx_i, i\in T;\ U_j\le 1+tx_j, j\in T^{\complement}\right)\\
&= \sum_{m=k}^d\ \sum_{T\subset\{1,\dots,d\},|T|=m}\left\{P\left(U_j\le 1+tx_j, j\in T^{\complement}\right)\right.\\
&\qquad\left.-\sum_{\emptyset\ne S\subset T}(-1)^{|S|-1}P\left(U_i\le 1+tx_i, i\in S;\ U_j\le 1+tx_j, j\in T^{\complement}\right)\right\}\\
&= \sum_{m=k}^d\ \sum_{T\subset\{1,\dots,d\},|T|=m}\left\{1-t\left\|\boldsymbol{x}_{T^{\complement}}\right\|_D\right.
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{\emptyset \neq S \subset T} (-1)^{|S|-1} \left(1 - t \left\| \boldsymbol{x}_{S \cup T^{\complement}} \right\|_D\right) \Big\} + o(t) \\
= & \sum_{m=k}^{d} \sum_{T \subset \{1,\dots,d\}, |T|=m} \Big\{ 1 - \sum_{\emptyset \neq S \subset T} (-1)^{|S|-1} \\
& + t \sum_{S \subset T} (-1)^{|S|-1} \left\| \boldsymbol{x}_{S \cup T^{\complement}} \right\|_D \Big\} + o(t) \\
= & \; t \sum_{m=k}^{d} \sum_{T \subset \{1,\dots,d\}, |T|=m} \sum_{S \subset T} (-1)^{|S|-1} \left\| \boldsymbol{x}_{S \cup T^{\complement}} \right\|_D \Big\} + o(t) \\
= & \; t \left\| \boldsymbol{x} \right\|_{D,(d-k+1)} + o(t)
\end{aligned}
$$

by Lemma 3.1. This completes the proof of Theorem 3.1.

Consider next a random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ that is in the max-domain of attraction of a multivariate max-stable df G. This is equivalent with the condition that the copula C corresponding to $\boldsymbol{X}$ satisfies Eq. 3.1, together with the condition that, for each $i = 1, \dots, d$, the (univariate) df F_i of X_i is in the max-domain of attraction of a univariate max-stable df G_i; see, e.g. Proposition 3.1.10 in Falk (2019).

Then we obtain from Eq. 3.2, with $\boldsymbol{U} = (U_1, \dots, U_d)$ following the copula C, so that $\boldsymbol{X} = (X_1, \dots, X_d) =_{\mathcal{D}} \left(F_1^{-1}(U_1), \dots, F_d^{-1}(U_d)\right)$

$$
\begin{aligned}
& P\left(X_i > y_i \text{ for at least } k \text{ of the components } i = 1, \dots, d\right) \\
= & \; P\left(U_i > 1 + (F_i(y_i) - 1) \text{ for at least } k \text{ of the components } i = 1, \dots, d\right) \\
= & \left\| (1 - F_i(y_i))_{i=1}^d \right\| \times \left\| \frac{(1 - F_i(y_i))_{i=1}^d}{\left\| (1 - F_i(y_i))_{i=1}^d \right\|} \right\|_{D,(d-k+1)} \\
& + o\left(\left\| (1 - F_i(y_i))_{i=1}^d \right\| \right), \qquad \boldsymbol{y} = (y_1, \dots, y_d) \in \mathbb{R}^d, \qquad (3.5)
\end{aligned}
$$

with an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^d$. Note that we actually do not have to require in Eq. 3.5 that each univariate margin F_i is in the max-domain of attraction of a univariate max-stable df. The preceding equation is, therefore, true if the copula C corresponding to $\boldsymbol{X}$ satisfies condition (3.1). Note that this condition on the copula of $\boldsymbol{X}$ does not require that $\boldsymbol{X}$ is absolutely continuous, i.e., Eq. 3.5 is true for *arbitrary* univariate df $F_1, \dots, F_d$, not necesscarily continuous ones.

But, if each F_i is in the max-domain of attraction of a max-stable df G_i, then there exist constants $a_{it} > 0$, $b_{it} \in \mathbb{R}$ for $t > 0$, with

$$t\big(1 - F_i(a_{it}y + b_{it})\big) \to_{t\to\infty} -\log(G_i(y)), \qquad y \in \mathbb{R};$$

see, for example, Falk (2019), equation (2.3). As a consequence we obtain from Eq. 3.5

$$\begin{aligned} & P\left(X_i > a_{it}y_i + b_{it} \text{ for at least } k \text{ of the components } i = 1, \dots, d\right) \\ = \ & \frac{1}{t}\left\| t\left(1 - F_i(a_{it}y_i + b_{it})\right)_{i=1}^d \right\|_{D,(d-k+1)} + o\left(\frac{1}{t}\right) \\ = \ & \frac{1}{t}\left\| \big(\log(G_i(y_i))\big)_{i=1}^d \right\|_{D,(d-k+1)} + o\left(\frac{1}{t}\right) \end{aligned}$$

if $G_i(y_i) \in (0, 1]$, $1 \le i \le d$.

Suppose identical distributions of the components of $\boldsymbol{X}$, i.e., $F_1 = \cdots = F_d =: F$ and identical entries of $\boldsymbol{y}$, i.e., $y_1 = \cdots = y_d =: y$. Then we can repeat the arguments in Section 2, with $\boldsymbol{x} := (F(y) - 1, \dots, F(y) - 1)$, and obtain, with $N_y := \sum_{i=1}^d 1_{(y,\infty)}(X_i)$,

$$\begin{aligned} p_{y,D}(k) &= P(N_y = k) \\ &= \begin{cases} (1 - F(y))\big(E(Z_{d-k+1:d}) - E(Z_{d-k:d})\big), & 1 \le k \le d, \\ 1 - (1 - F(y))E(Z_{d:d}) \end{cases} \\ &\quad + o(1 - F(y)) \end{aligned}$$

as $y \uparrow \omega(F) := \sup\{t \in \mathbb{R} : F(t) < 1\}$.

In particular we obtain for the FI

$$\begin{aligned} \text{FI} &= \lim_{y\uparrow\omega(F)} E(N_y \mid N_y > 0) \\ &= \frac{E(N_y)}{1 - P(N_y = 0)} = \frac{d}{\|\mathbf{1}\|_D}, \end{aligned}$$

as already observed in (Falk and Tichy, 2012, Theorem 5.1).

If, for example, the underlying D-norm is a logistic one, $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_p := \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$, with parameter $p \in [1, \infty]$, and the convention $\|\boldsymbol{x}\|_\infty = \max_{1\le i\le d} |x_i|$, then the FI is

$$\text{FI} = \frac{d}{\|\mathbf{1}\|_p} = \begin{cases} 1, & \text{if } p = 1, \\ d^{1-\frac{1}{p}}, & \text{if } p \in (1, \infty) \\ d, & \text{if } p = \infty. \end{cases}$$

If $p = 1$, the components $X_1, \ldots, X_d$ are tail independent with FI $= 1$, i.e., the system $\{X_1, \ldots, X_d\}$ is stable. If $p = \infty$, the components $X_1, \ldots, X_d$ are completely tail dependent and FI $= d$, i.e., the system is extremely fragile.

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