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Loops on schemes and the algebraic fundamental group

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Abstract
In this note we give a re-interpretation of the algebraic fundamental group for proper schemes that is rather close to the original definition of the fundamental group for topological spaces. The idea is to replace the standard interval from topology by what we call interval schemes. This leads to an algebraic version of continuous loops, and the homotopy relation is defined in terms of the monodromy action. Our main results hinge on Macaulayfication for proper schemes and Lefschetz type results.

Keywords Algebraic fundamental group · Interval scheme · Lefschetz type theorem

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Contents

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Introduction

The fundamental group $\pi_1(X, x_0)$ of a topological space X with respect to a base point x_0 is an invariant of great significance, even more so as its definition is elementary and intuitive: the elements are loops up to homotopy, where a loop is a continuous morphism $I \to X$ from the standard interval $I = [0, 1]$, such that the end points are mapped to the base point x_0. Roughly speaking, two loops are homotopic if one can be deformed to the other, respecting the base point. For connected and locally simply-connected spaces X, one may interpret the fundamental group also as the group of deck transformations of the universal covering $\tilde{X} \to X$.

In the realm of algebraic topology, where one works with schemes rather than topological spaces, the first construction above makes little sense. However, Grothendieck [9] realized that the second description has an analog in algebraic geometry. He introduced the notion of a *Galois category* $\mathcal{C}$, the objects of which should be considered as abstract finite coverings, which admits a fiber functor $\Phi : \mathcal{C} \to (\text{FinSet})$ to the category of finite sets satisfying certain properties. These properties ensure that the automorphism group of Φ is equal to the opposite group of the automorphisms of an abstract pro-finite universal covering. The algebraic fundamental group $\pi_1^{\text{alg}}(X, x_0)$ of a connected scheme X with respect to a geometric point x_0 is then defined by applying this general construction to the category (FinEt/X) of finite étale coverings of X, with fiber functor given by base-changing along x_0.

If X is of finite type over the complex numbers, the group $\pi_1^{\text{alg}}(X, x_0)$ equals the pro-finite completion of the classical fundamental group of $X(\mathbb{C})$, endowed with the classical topology. If $X = \text{Spec}(K)$ is the spectrum of a field and x_0 is given by some separable closure, $\pi_1^{\text{alg}}(X, x_0)$ gives back the corresponding Galois group.

In this note we observe that the original construction of the fundamental group using loops has a meaningful analogy for schemes, once the notions of intervals and loops are interpreted in an algebraic manner. More precisely, the crucial properties of the interval $I = [0, 1]$ in the construction above are: I is connected, quasi-compact, one dimensional, with no non-trivial coverings, and is endowed with two distinguished points. Translating these properties to algebraic geometry we define in Sect. 3 *interval schemes* as reduced, connected, affine, one-dimensional schemes I, which have no non-trivial finite étale coverings and contain two distinguished closed points with separably closed residue fields. The latter are called *end points*. An *algebraic loop* on a scheme X based at a geometric point x_0 is a morphism of schemes $I \to X$ mapping the end points to the base point.

Algebraic loops define monodromy transformations. We call two algebraic loops *homotopic* if the resulting monodromies agree. The *algebraic loop group* $\pi_0\Omega^{\text{alg}}(X, x_0)$ is defined as the set of homotopy classes of algebraic loops; the group structure is induced by concatenating algebraic loops, see Sect. 4.

Interval schemes are very often non-noetherian. One example of an interval scheme is the universal Galois covering (introduced by Grothendieck as a pro-object) of a noetherian, connected, affine, reduced, and one dimensional scheme. Such universal Galois coverings were systematically studied in [24], where Vakil and Wickelgren define the fundamental group scheme using universal coverings, which are certain pro-finite étale maps. The notion of interval scheme introduced above is also inspired by their work. But there are many other examples of interval schemes, which are more direct to obtain. For example, if R is an integral noetherian one-dimensional ring and A is its integral closure in the separable closure of $\mathrm{Frac}(R)$, then the choice of two closed geometric points in $\mathrm{Spec}(A)$ turns it into an interval scheme.

By construction, the monodromy induces an injective homomorphism

$$\pi_0\Omega^{\mathrm{alg}}(X, x_0)^{\mathrm{op}} \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0), \tag{1}$$

of groups, where "op" refers to the opposite group structure. The main result of this note is the following, see Theorem 4.4:

Theorem *Let X be a connected scheme that is separated and of finite type over a ground field k, endowed with a geometric point $x_0 : \mathrm{Spec}(k^{\mathrm{sep}}) \to X$. Then the injection (1) has dense image. It is actually bijective, provided that X is proper.*

The main step in the proof of the above theorem for proper X is a Lefschetz type result saying that for a proper and connected k-scheme X we find a closed connected curve $C \subset X$ such that the algebraic fundamental group of C surjects to the one of X, see Proposition 5.5. This is well-known in the case where X is Cohen–Macaulay and projective over a field, see [10], Exposé XII. We reduce the general situation to this using a van-Kampen-like argument and Macaulayfication, which was in a special case constructed by Faltings [4] and in full generality by Kawasaki [18]. For further results on Macaulayfication, see the recent work of Česnavičius [2]. The proof of Theorem 4.4 is given in Sect. 7. We do not expect the map (1) to be an isomorphism for affine schemes of finite type over a field in general.

1 Monodromy

Let Y be a scheme, and write $(\mathrm{FinEt}\,/Y)$ for the category of Y-schemes X whose structure morphism $f : X \to Y$ is finite and étale. Note that such an f is proper, affine, flat, of finite presentation, and for each point $b \in Y$ the fiber $f^{-1}(b)$ is the spectrum of some étale algebra over the residue field $k = \kappa(b)$. For the following result, see for example [20], Theorem 5.10 and Exercise 5.21.

Proposition 1.1 *Suppose Y is connected, and $f : X \to Y$ finite and étale. Then there is a surjective finite étale morphism $Y' \to Y$ such that $X' = X \times_Y Y'$ is isomorphic over Y' to the disjoint union $\coprod_{i=1}^{r} Y'$ for some integer $r \geqslant 0$.*

This has an immediate consequence:

Corollary 1.2 *Suppose Y is connected, and $f : X \to Y$ finite and étale. Then X has only finitely many connected components $U \subset X$, each of which is open-and-closed. Moreover, the induced morphism $U \to Y$ is finite and étale.*

Proof Take $Y' \to Y$ as in Proposition 1.1. Since the projection $X' \to X$ is surjective, the connected components of X are images of the connected components of X', hence there are only finitely many. This implies that the connected components U of X are open and closed and hence the composition $U \hookrightarrow X \to Y$ is étale and finite. □

The proposition tells us that the Y-scheme X is a *twisted form* of the disjoint union $\coprod_{i=1}^r Y$, with respect to the étale topology. It thus corresponds to a class in the non-abelian cohomology set $H^1(Y, S_r)$, with coefficients in the symmetric group S_r on $r \geqslant 0$ letters ([8], Chapter III, Section 2.3). To summarize:

Proposition 1.3 *If Y is connected, the following are equivalent:*

(i) Every finite étale Y-scheme is isomorphic to some $\coprod_{i=1}^r Y$, $r \geqslant 0$.
(ii) We have $H^1(Y, S_r) = \{*\}$ for all integers $r \geqslant 0$.
(iii) Each finite étale morphism $X \to Y$ from a non-empty connected scheme X is an isomorphism.

Definition 1.4 We say a connected scheme Y is *simply connected*, if it satisfies the equivalent conditions of Proposition 1.3.

Let X be a Y-scheme, with structure morphism $f : X \to Y$, and $a : A \to Y$ be some other morphism. To simplify notation, we write

$$X(A) = \mathrm{Hom}_Y(A, X) = \{a' : A \longrightarrow X \mid f \circ a' = a\}$$

of liftings of $a : A \to Y$ with respect to $f : X \to Y$.

Proposition 1.5 *Suppose that Y is connected, with $H^1(Y, S_r) = \{*\}$ for all $r \geqslant 0$, and $f : X \to Y$ is finite and étale. Let $a : A \to Y$ and $b : B \to Y$ be morphisms with connected and non-empty domains. Then the sets $X(A)$ and $X(B)$ are finite, and for each $a' \in X(A)$ there is a unique $b' \in X(B)$ such that the images $a'(A)$ and $b'(B)$ lie in the same connected component of X.*

Proof By Proposition 1.3, we may assume $X = \coprod_{i=1}^r Y$, for some $r \geqslant 0$. The set $X(A)$ is in bijection with the set of sections of $\coprod_{i=1}^r A = X \times_Y A \to A$, whence is finite, and we see that every $a' \in X(A)$ corresponds uniquely to one of the maps

$$A \hookrightarrow \coprod_i A \xrightarrow{\coprod a} \coprod_i Y = X$$

given by including A into one of the r summands. This implies the statement. □

1.6. In the situation of Proposition 1.5 we obtain a mapping

$$\mu_X : X(A) \longrightarrow X(B), \quad a' \longmapsto b',$$

which is called the *monodromy*, and will play a crucial role throughout. We regard it as a natural transformation between $X \mapsto X(A)$ and $X \mapsto X(B)$, viewed as functors $(\mathrm{FinEt}/Y) \to (\mathrm{FinSet})$. By Proposition 1.5, the monodromy μ_X is a natural isomorphism, and is given as the composition of the following natural bijections

$$\mu_X : X(A) \longrightarrow \pi_0(X \times_Y A) \longrightarrow \pi_0(X) \longrightarrow \pi_0(X \times_Y B) \longrightarrow X(B),$$

where $\pi_0(X)$ denotes the set of connected components of X.

2 Galois categories

In this section we recall the notion of Galois categories, which were introduced by Grothendieck to unify Galois theory from algebra and covering space theory from topology ([9], Exposé V).

2.1. Recall that a category $\mathcal{C}$ is called a *Galois category* if there exists a functor $\Phi : \mathcal{C} \to (\text{FinSet})$ such that the following six axioms hold:

(G1) Fiber products and final objects exist in $\mathcal{C}$.
(G2) Finite sums and quotients by finite group actions exist in $\mathcal{C}$.
(G3) Every morphism $X' \to X$ in $\mathcal{C}$ factors into a strict epimorphism $X' \to U$ and the inclusion of a direct summand $U \subset X$.
(G4) The functor Φ commutes with fiber products and final objects.
(G5) It also commutes with finite direct sums and forming quotients by finite group actions, and transforms strict epimorphisms into surjections.
(G6) If for a morphism $u : X' \to X$ in $\mathcal{C}$ the resulting map $\Phi(u)$ is bijective, then u is an isomorphism.

One calls Φ a *fundamental functor* or *fiber functor* for the Galois category $\mathcal{C}$, and denotes by $\pi = \text{Aut}(\Phi)$ the group of natural isomorphisms of the fundamental functor to itself. In turn, we have an inclusion

$$\pi \subset \prod_{X \in \mathcal{C}} S_{\Phi(X)}$$

inside a product of symmetric groups $\text{Aut}(\Phi(X)) = S_{\Phi(X)}$. These groups are finite. We endow them with the discrete topology, and the product with the product topology. The latter becomes a topological group that is compact and totally disconnected. Such topological groups are also called *pro-finite groups*. One easily checks that the subgroup π is closed, and thus inherits the structure of a pro-finite group. Every fiber functor on $\mathcal{C}$ is (non-canonically) isomorphic to Φ and hence, up to a uncanonical isomorphism, the pro-finite group π depends only on the Galois category $\mathcal{C}$, and not on the choice of fiber functor Φ.

Now write (π-FinSet) for the category of finite sets F endowed with a π-action from the left, where the kernel of the canonical homomorphism $\pi \to S_F$ is closed. In other words, the action $\pi \times F \to F$ is continuous, when the finite set F is endowed with the discrete topology. With respect to the natural π-action on the $\Phi(X)$, $X \in \mathcal{C}$, the fundamental functor becomes a functor

$$\Phi : \mathcal{C} \longrightarrow (\pi\text{-FinSet}),$$

and Grothendieck deduced from the axioms (G1)–(G6) that the above is an equivalence of categories. Conversely, if G is a pro-finite group, the category (G-FinSet) is a Galois category: The functor Φ that forgets the G-action is a fundamental functor, and the resulting $\pi = \text{Aut}(\Phi)$ becomes identified with G.

One should see $\pi = \mathrm{Aut}(\Phi)$ as a common generalization of the opposite Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)^{\mathrm{op}}$ for fields F, and the pro-finite completion $\widehat{\pi}_1(Y, y_0)$ of the fundamental group, say for connected and locally simply-connected topological spaces Y.

2.2. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact covariant functor between Galois categories. By [9, Exposé V, Proposition 6.1] the exactness is equivalent to the statement, that the composition $\Phi' \circ H$ is a fiber functor for $\mathcal{C}$, whenever Φ' is a fiber functor for $\mathcal{C}'$. An exact functor H induces a morphism $h : \pi' \to \pi$ between the corresponding fundamental groups with reversed direction, which is well-defined up to conjugation. The following statements are equivalent (see [9, Exposé V, Proposition 6.9])

(1) $H : \mathcal{C} \to \mathcal{C}'$ is fully faithful.
(2) $h : \pi' \to \pi$ is surjective.
(3) For each connected object $X \in \mathcal{C}$, the object $H(X)$ is connected.

2.3. Let Y be a connected scheme. Then $\mathcal{C} = (\mathrm{FinEt}/Y)$ becomes a Galois category: For each morphism $y_0 : \mathrm{Spec}(K) \to Y$, where K is a separably closed field, we get a fiber functor

$$\Phi_{y_0} : (\mathrm{FinEt}/Y) \longrightarrow (\mathrm{FinSet}), \quad X \longmapsto X(K)$$

given by the set of morphisms $b' : \mathrm{Spec}(K) \to X$ lifting the given $b : \mathrm{Spec}(K) \to Y$. The resulting group

$$\pi_1^{\mathrm{alg}}(Y, y_0) = \pi = \mathrm{Aut}(\Phi_{y_0})$$

is called the *algebraic fundamental group* of the connected scheme Y with respect to y_0. As in topology, the latter is called *base point*.

3 Interval schemes

In algebraic topology, the *standard interval* $I = [0, 1]$ plays a central role. From our perspective, the following are the crucial properties:

(i) The topological space I is connected and quasi-compact.
(ii) There are two distinguished points $0, 1 \in I$.
(iii) The universal covering $\tilde{I} \to I$ is a homeomorphism.
(iv) The interval I is one-dimensional.

In this section we introduce a class of schemes with analogous properties. Fix a separably closed field K.

Definition 3.1 An *interval scheme with K-valued endpoints* is a triple (I, a_0, a_1), where I is a reduced, connected, simply connected, affine, and one-dimensional scheme and $a_i : \mathrm{Spec}(K) \to X$ are two closed embeddings.

The point a_0 is called the *left endpoint*, whereas a_1 is the *right endpoint*. By abuse of notation, we usually write I for the interval scheme (I, a_0, a_1). The simplest example of an interval scheme is $I = \mathbb{A}^1_k$ where k is an algebraically closed field of characteristic zero, and the end points are the rational points $a_0 = 0$ and $a_1 = 1$. However, if k is not algebraically closed or of positive characteristic, this will not be an interval scheme. In fact, interval schemes are very often non-noetherian. The following gives the most basic class of interval schemes:

Proposition 3.2 *Let A be a one-dimensional integral ring that is normal and whose field of fractions $F = \operatorname{Frac}(A)$ is separably closed, and suppose there are two surjections $\varphi_i : A \to K$. Then $I = \operatorname{Spec}(A)$ becomes an interval scheme, where the endpoints a_i correspond to the homomorphisms φ_i.*

Proof Let $X \to I$ be a connected finite étale covering. It follows that X is an affine connected normal scheme, and thus is integral, see [14], Proposition (17.5.7). In particular the function field L of X is a finite separable field extension $L/\operatorname{Frac}(A)$. Since $\operatorname{Frac}(A)$ is separably closed we find $L \cong \operatorname{Frac}(A)$. Hence $X \to I$ is an isomorphism. Thus I has no non-trivial finite étale covering and therefore defines an interval scheme. □

Example 3.3 Rings as in Proposition 3.2 easily occur as follows: Suppose that R is a one-dimensional noetherian ring, endowed with two integral homomorphisms $\psi_i : R \to K$. The latter means that each $\lambda \in K$ is the root of a monic polynomial with coefficients from R. Choose a separable closure F^{sep} for the field of fractions $F = \operatorname{Frac}(R)$, and write $A = R^{\text{sep}}$ for the integral closure of $R \subset F^{\text{sep}}$. By construction A is integral and normal, with field of fractions F^{sep}, and the ring extension $R \subset A$ is integral. According to the Going-Up Theorem, the map $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ is surjective, thus the $\psi_i : R \to K$ extend to some homomorphisms $\varphi_i : R^{\text{sep}} \to K$. Proposition 3.2 yields that the scheme $I = \operatorname{Spec}(R^{\text{sep}})$ in an interval scheme, where the endpoints a_i correspond to the homomorphisms φ_i.

Recall that a local ring R is called *strictly henselian* if each factorization $P \equiv P_1 P_2$ into coprime polynomials over the residue field $k = R/\mathfrak{m}_R$ of a monic polynomial $P \in R[T]$ is induced by a factorization over R, and moreover the residue field is separably closed. See [14], Proposition (18.8.1) for the next example of an interval scheme, for which the image points of the two end points agree.

Proposition 3.4 *Let A be a one-dimensional local ring that is strictly henselian, and whose residue field is isomorphic to K, and let $\varphi_i : A/\mathfrak{m}_A \to K$ be two isomorphisms. Then $I = \operatorname{Spec}(A)$ becomes an interval scheme, where the endpoints a_i correspond to the homomorphisms φ_i.*

3.5. Let I, J be two interval schemes, with K-valued endpoints a_0, a_1 and b_0, b_1, respectively. We write $I * J$ for the concatenation of I and J with respect to the right endpoint on I and the left endpoint on J. In other words, we have a cocartesian square in the category of schemes

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{b_0} & J \\ {\scriptstyle a_1}\downarrow & & \downarrow \\ I & \longrightarrow & I * J. \end{array} \tag{3.1}$$

Note that the cocartesian square above exits in the category of schemes by [6], Théorème 5.4, and is in fact also cartesian. The scheme $I * J$ comes with closed embeddings of I and J, and we take $a_0 \in I \subset I \star J$ as new left endpoint, and $b_1 \in J \subset I * J$ as new right endpoint.

Lemma 3.6 *In the above situation, the concatenation $I * J$ is an interval scheme, with endpoints a_0 and b_1.*

Proof Write $I = \mathrm{Spec}(A)$ and $J = \mathrm{Spec}(B)$. Then $I * J = \mathrm{Spec}(A \times_K B)$ is affine and reduced. Here the fiber product is formed with respect to the homomorphisms $A \to K \leftarrow B$ corresponding to the morphisms a_1 and b_0. By construction we have $I * J = I \cup J$ and $I \cap J = \mathrm{Spec}(K)$, where the latter can be viewed as the image of both a_1 and b_0. Hence $I * J$ is also one-dimensional and connected. Let $X \to I * J$ be a finite étale covering. Set $X(Z) = \mathrm{Hom}_{I*J}(Z, X)$, where Z is an $I * J$-scheme. By [14], Corollaire (17.9.4), the set $X(I * J)$ is in bijection with the connected components of X. By construction of $I * J$ the set $X(I * J)$ is the pushout of the maps $X(I) \leftarrow X(I \cap J) \to X(J)$ induced by the morphisms a_1 and b_0. Since I, J, and $\mathrm{Spec}(K)$ are simply connected the monodromy (see 1.6) induces bijections

$$X(I) \cong X(a_1) \cong X(I \cap J) \cong X(b_0) \cong X(J).$$

Hence $|X(I * J)| = |X(I \cap J)| = \deg(X_{I\cap J} \to I \cap J) = \deg(X \to I * J)$, where $X_{I\cap J} \to I \cap J$ denotes the base change of $X \to I * J$ along $I \cap J \hookrightarrow I * J$.

Thus any finite étale covering of $I * J$ is trivial and hence $I * J$ is an interval scheme. □

4 The algebraic loop group

Fix some separably closed field K and let Y be a scheme, endowed with two K-valued points $y_i : \mathrm{Spec}(K) \to Y$. We may regard this as an object in the category (K^2/Sch) of schemes endowed with two K-valued points.

Definition 4.1 An *algebraic path* in Y starting at y_0 and ending in y_1 is an interval scheme (I, a_0, a_1) with K-valued endpoints, together with a morphism

$$w : (I, a_0, a_1) \longrightarrow (Y, y_0, y_1)$$

of schemes endowed with two K-valued points. An algebraic path w is called *algebraic loop* if $y_0 = y_1$.

By abuse of notation, we often write $w : I \to Y$ for the algebraic path $w : (I, a_0, a_1) \to (Y, b_0, b_1)$. For each finite étale map $X \to Y$, the base change induces a finite étale map $X \times_Y I \to I$, which takes the form $\coprod_{i=1}^r I$, for some $r \geqslant 0$, and we have an identification $X(y_i) = (X \times_Y I)(a_i)$ of fiber sets. In turn, the monodromy gives a transformation

$$\mu_w : X(y_0) = (X \times_Y I)(a_0) \longrightarrow (X \times_Y I)(a_1) = X(y_1)$$

that is bijective, and natural in the objects $X \in (\mathrm{FinEt}\,/Y)$. In other words, the monodromy μ_w attached to the path $w : I \to Y$ from y_0 to y_1 is a bijective natural transformation between fiber functors

$$\Phi_{y_0}, \Phi_{y_1} : (\mathrm{FinEt}\,/Y) \longrightarrow (\mathrm{FinSet}).$$

We now use this monodromy to give an algebraic version of homotopy:

Definition 4.2 We say that two algebraic paths $w : I \to Y$ and $v : J \to Y$ from y_0 to y_1 are *homotopic* if $\mu_w = \mu_v$ as natural transformations from Φ_{y_0} to Φ_{y_1}.

4.3. We denote the class of algebraic loops in Y based at the geometric point y_0 by

$$\Omega^{\mathrm{alg}}(Y, y_0) = \{w : (I, a_0, a_1) \longrightarrow (Y, y_0, y_0) \mid (I, a_0, a_1) \text{ interval scheme }\}.$$

Let $w : (I, a_0, a_1) \to (Y, y_0, y_0)$ and $v : (J, b_0, b_1) \to (Y, y_0, y_0)$ be two loops based at y_0. It follows from the pushout diagram (3.1) and Lemma 3.6 that we can concatenate these loops to get a new loop

$$w * v : (I * J, a_0, b_1) \longrightarrow (Y, y_0, y_0).$$

The monodromy transformation corresponding to $w * v$ can be factored, for $X \in (\mathrm{FinEt}\,/Y)$ as

$$\begin{array}{ccccccc}
X(y_0) & & & \xrightarrow{\quad\mu_{w*v}\quad} & & & X(y_0) \\
\uparrow \simeq & & & & & & \uparrow \simeq \\
(X)_{I*J}(a_0) & & & \xrightarrow{\quad\simeq\quad} & & & (X)_{I*J}(b_1) \\
\uparrow \simeq & & & & & & \uparrow \simeq \\
X_I(a_0) & \xrightarrow{\simeq} & X_I(a_1) \cong (X)_{I*J}(a_1) & = & (X)_{I*J}(b_0) \cong X_J(b_0) & \xrightarrow{\simeq} & X_J(b_1),
\end{array}$$

where we use the notation $X_Z = X \times_Y Z$, the vertical maps in the upper square are induced by base change of $w * v$, and the isomorphisms $X_I(a_i) \cong X_{I*J}(a_i)$ and $X_J(b_i) \cong X_{I*J}(b_i)$, for $i = 0, 1$, are induced by the natural closed immersions $I \hookrightarrow I * J$ and $J \hookrightarrow I * J$. The upper square commutes by the definition of μ_{w*v}, the lower square commutes by the construction of the isomorphisms, see the proof of Proposition 1.5. Thus the whole square commutes and by definition of the monodromy we obtain the equality

$$\mu_{w*v} = \mu_v \circ \mu_w \tag{4.1}$$

of automorphisms of the fiber functor $\Phi_{y_0} : (\mathrm{FinEt}\,/Y) \to (\mathrm{FinSet})$. Furthermore if $u : (I, a_0, a_1) \to (Y, y_0, y_0)$, $v : (J, b_0, b_1) \to (Y, y_0, y_0)$, and $w : (L, c_0, c_1) \to (Y, y_0, y_0)$ are three loops based at y_0, then the universal property of pushout diagrams yields a canonical isomorphism $\tau : I * (J * L) \to (I * J) * L$ and an equality

$$(u * v) * w \circ \tau = u * (v * w) \quad \text{in } (\mathrm{Sch}/Y). \tag{4.2}$$

Clearly, homotopy between paths defines an equivalence relation $w \sim v$ and we denote by

$$\pi_0\Omega^{\mathrm{alg}}(Y, y_0)$$

the set of homotopy classes of algebraic loops based at y_0. We denote by $[w]$ the homotopy class of a loop $w : I \to Y$ at y_0. According to (4.1) we obtain a well defined operation

$$* : \pi_0\Omega^{\mathrm{alg}}(Y, y_0) \times \pi_0\Omega^{\mathrm{alg}}(Y, y_0) \longrightarrow \pi_0\Omega^{\mathrm{alg}}(Y, y_0), \quad ([w], [v]) \longmapsto [w * v].$$

This operation is associative by (4.2) and clearly any constant loop $I \to y_0 \to Y$ has the same homotopy class denoted by e, which defines a neutral element for $*$. Moreover, for a loop $w : (I, a_0, a_1) \to (Y, y_0, y_0)$ we define the loop $w' : (I, a_1, a_0) \to (Y, y_0, y_0)$ by switching the end points of I. Clearly $[w]*[w'] = e$. Hence concatenation of algebraic loops defines a group structure on $\pi_0\Omega^{\mathrm{alg}}(Y, y_0)$, which we therefore call the *algebraic loop group*.

By definition of the algebraic fundamental group, our homotopy relation, and the relation (4.1) the algebraic loop space $\Omega^{\mathrm{alg}}(Y, y_0)$ induces an injective homomorphism

$$\pi_0\Omega^{\mathrm{alg}}(Y, y_0)^{\mathrm{op}} \longrightarrow \pi_1^{\mathrm{alg}}(Y, y_0), \quad w \longmapsto \mu_w \tag{4.3}$$

of groups, where we use the opposite group structure on the left hand side. We regard this as an inclusion of groups.

The following is the main result of this note.

Theorem 4.4 *Let X be a connected scheme that is separated and of finite type over a field k, endowed with a geometric point $x_0 : \mathrm{Spec}(k^{\mathrm{sep}}) \to X$. Then the canonical injection (4.3) has dense image. It is actually bijective, provided X is proper.*

The proof of Theorem 4.4 requires some preparations and will be given in Sect. 7. We remark that we do not expect (4.3) to be an isomorphism for non-proper schemes.

5 A Lefschetz type theorem

The *Lefschetz Hyperplane Theorem* gives a strong relation between the homology of a projective complex manifold X of dimension $n \geqslant 2$ and the homology of an ample divisor $D \subset X$. The original arguments appear in [19], Chapter V, Section III. Analogous statements for fundamental groups were first obtained by Bott [1]: The induced map $\pi_1(D, x_0) \to \pi_1(X, x_0)$ is bijective provided $n \geqslant 3$, and at least surjective if $n \geqslant 2$.

The latter statement extends to projective schemes X over arbitrary ground fields k: According to [10], Exposé XII, Corollary 3.5 the map $\pi_1^{\mathrm{alg}}(D, x_0) \to \pi_1^{\mathrm{alg}}(X, x_0)$ is surjective provided that $\mathrm{depth}(\mathscr{O}_{X,a}) \geqslant 2$ for each closed point $a \in X$. If X is additionally *Cohen–Macaulay*, i.e., at every point $a \in X$ the equality $\dim(\mathscr{O}_{X,a}) =$

depth($\mathscr{O}_{X,a}$) holds, the above can be iterated and one finds a connected curve $C \subset X$ such that $\pi_1^{\text{alg}}(C, x_0) \to \pi_1^{\text{alg}}(X, x_0)$ is surjective.

In this section we generalize the latter statement to arbitrary proper k-schemes.

5.1. Let X be a non-empty connected noetherian scheme. We consider the following property:

(C) For each closed subscheme $Z \subset X$ with $\dim(Z) \leqslant 0$, there is a connected closed subscheme $C \subset X$ with $0 \leqslant \dim(C) \leqslant 1$ and $Z \subset C$ such that, for each finite étale covering $U \to X$ with connected total space, the restriction $C_U = C \times_X U$ remains connected.

Remark 5.2 We remark that property (C) does not hold for affine schemes in general, as the following simple example shows (confer Lemma 5.4 in [3]): Let k be a ground field of characteristic $p > 0$, and C be an connected affine plane curve inside $\mathbb{A}^2 = \operatorname{Spec}(R)$, defined by some non-constant polynomial $f = f(x, y)$ inside the ring $R = k[x, y]$. Then there exists a connected finite étale covering $X \to \mathbb{A}^2$ whose restriction to C becomes trivial. To see this, take any $h \in (f)$ not of the form $g^p - g$ with $g \in R$. Via the identification

$$H^1_{\text{et}}(\mathbb{A}^2_k, \mathbb{Z}/p) = R/\{g^p - g \mid g \in R\}$$

coming from the Artin–Schreier sequence, the polynomial h corresponds to a non-trivial $\mathbb{Z}/p$-torsor $X \to \mathbb{A}^2_k$, in particular it is a connected finite étale covering of $\mathbb{A}^2_k$. On the other hand h maps to zero in $\bar{R}/\{a^p - a \mid a \in \bar{R}\}$, where $\bar{R} = R/(f)$. In other words, the restriction of X to C is trivial.

Lemma 5.3 *Let X be a non-empty connected noetherian scheme, $f : X' \to X$ be a proper and surjective morphism, and $X'_v \subset X'$ be the connected components. If property (C) holds for all X'_v, then it also holds for X.*

The proof of this lemma is inspired by the Seifert–van Kampen Theorem from [23], but is more elementary. Note that even if one wants to use property (C) on X with $Z = \varnothing$, the proof for the lemma relies in an essential way on property (C) on X'_v with non-empty Z'. We first gather some basic material on graphs.

5.4. Let $f : X' \to X$ be as in the statement of Lemma 5.3. Set $X'' = X' \times_X X'$ and write $\text{pr}_1, \text{pr}_2 : X'' \to X'$ for the two projections. The schemes X' and X'' are noetherian, hence the sets of connected components $\pi_0(X')$ and $\pi_0(X'')$ are finite. Consider the induced maps

$$\text{pr}_1 \times \text{pr}_2 : \pi_0(X'') \longrightarrow \pi_0(X') \times \pi_0(X').$$

This defines an *oriented graph* $\Gamma = (E, V, \text{pr}_1 \times \text{pr}_2)$ in the sense of Serre [21], Section 2.1: the set of *vertices* is $V = \pi_0(X')$, and the set of *oriented edges* is $E = \pi_0(X'')$. The endpoints of an edge $e \in E$ are the images $v_i = \text{pr}_i(e)$. The orientation is given by declaring v_1 as the initial vertex, and v_2 as the terminal vertex. We usually write $X'_v \subset X'$ and $X''_e \subset X''$ for the connected components corresponding to a vertex v and an edge e.

Note that edges could have the same initial and terminal vertices, and several edges could share their initial and terminal vertices. By abuse of notation, we also write $v \in \Gamma$ and $e \in \Gamma$ to denote vertices and edges of the graph, if there is no risk of confusion. A morphism $f : \Gamma \to \Gamma'$ between oriented graphs comprises compatible maps $V \to V'$ and $E \to E'$. We simply say that f is a *map of oriented graphs*. Also note that the graph Γ constructed above is connected, since the scheme X is connected.

Proof of Lemma 5.3 Since $f : X' \to X$ is proper, the image $f(Z)$ of a closed subscheme $Z \subset X'$ is closed and satisfies $\dim f(Z) \leqslant \dim(Z)$. In particular, closed points are mapped to closed points. The assertion is trivial for $\dim(X) \leqslant 1$. We now assume $\dim X \geqslant 2$. We use the notation from 5.4. Let $Z \subset X$ be a zero-dimensional closed subscheme or the empty set. For each edge $e \in \Gamma$ choose a closed point $x_e \in X''_e$. Set

$$x_{e,i} := \mathrm{pr}_i(x_e) \in X'_{v_i}, \quad \text{where } v_i = \mathrm{pr}_i(e) \in \Gamma,\ i = 1, 2.$$

Since Γ is finite we find for each vertex $v \in \Gamma$ a 0-dimensional closed subset $Z'_v \subset X'_v$, such that

(a) $Z \cap f(X'_v) \subset f(Z'_v)$ and
(b) $x_{e,i} \in Z'_v$, for all edges $e \in \Gamma$ with $v = \mathrm{pr}_i(e)$ for $i = 1$ or 2.

Condition (b) is immediate, and one can achieve (a) by picking a closed point in each of the finitely many schemes $f^{-1}(z) \cap X'_v$, with $z \in Z \cap f(X'_v)$. By the surjectivity of f we have

$$Z \subset \bigcup_v f(Z'_v). \tag{5.1}$$

Applying (C) to X'_v and Z'_v we find an at most 1-dimensional closed subscheme $C'_v \subset X'_v$ containing Z'_v, such that the pullback of any connected finite étale covering of X'_v to C'_v stays connected. Then $C = \bigcup_v f(C'_v)$ is closed, at most 1-dimensional, and contains Z, by (5.1). It remains to show that for each connected finite étale covering $U \to X$ the pullback $U \times_X C$ remains connected (then C is connected as well).

To this end fix such a finite étale covering $u : U \to X$, with U non-empty and connected. Denote by Γ_U the graph defined by $\mathrm{pr}_1 \times \mathrm{pr}_2 : \pi_0(U'') \to \pi_0(U') \times \pi_0(U')$, where $U' = U \times_X X'$ and $U'' = U \times_X X'' = U' \times_U U'$. We obtain a surjection of graphs $u : \Gamma_U \to \Gamma$ and for each edge $\epsilon \in \Gamma_U$ we obtain a finite and étale morphism between connected schemes $U''_\epsilon \to X''_{u(\epsilon)}$ which therefore is surjective. Thus for each edge $e \in \Gamma$ and edge $\epsilon \in \Gamma_U$ mapping to e we can choose a closed point $x_{U,\epsilon} \in U''_\epsilon$ with $u(x_{U,\epsilon}) = x_e$.

For a vertex $w \in \Gamma_U$ mapping to $v \in \Gamma$ denote by I_w the image of $U'_w \times_{X'_v} C'_v$ under the map

$$U' \times_{X'} C'_v = U \times_X C'_v \longrightarrow U \times_X C_v, \quad \text{where } C_v = f(C'_v).$$

The map is induced by the base change with the composition $C'_v \hookrightarrow X'_v \hookrightarrow X' \xrightarrow{f} X$, and therefore is closed and surjective. Hence

$$U \times_X C_v = \bigcup_{w \in u^{-1}(v)} I_w \quad \text{and} \quad U \times_X C = \bigcup_{w \text{ edge in } \Gamma_U} I_w, \tag{5.2}$$

where each I_w is closed. By our choice of C'_v the pullback of the connected étale covering $U'_w \to X'_v$ over C'_v remains connected. Thus I_w is the image of a connected scheme and is hence connected. Let w_1 and w_2 be the initial and the terminal vertices of an edge $\epsilon \in \Gamma_U$, then $x_{U,\epsilon} \in U''_\epsilon$ maps via the ith projection to points $\mathrm{pr}_i(x_{U,\epsilon})$ in $U'_{w_i} \times_{X'_{u(w_i)}} C'_{u(w_i)}$, $i = 1, 2$, and these points map to same point in U. Thus the intersection $I_{w_1} \cap I_{w_2}$ is non-empty, if w_1 and w_2 are linked by an edge in Γ_U. Since the graph Γ_U is connected so is the scheme $U \times_X C$. This completes the proof. □

Proposition 5.5 *Let X be a connected scheme that is proper over a field k. Then X has property* (C).

Proof We proceed by induction on $n = \dim X$. There is nothing to prove for $n \leqslant 1$. Assume $n \geqslant 2$ and that (C) holds for all connected schemes that are proper over k and have dimension $\leqslant n - 1$. Using Lemma 5.3 we can make the following reductions:

(i) X reduced (using the proper bijection $X_{\mathrm{red}} \to X$);
(ii) X projective over k (using Chow's Lemma);
(iii) X integral (using the proper surjection $\coprod X_i \to X$, with X_i the irreducible components of X).

According to Kawasaki's result ([18], Theorem 1.1) there is a proper birational $X' \to X$ such that the scheme X' is Cohen–Macaulay. Moreover, this Macaulayfication arises as a sequence of blowing-ups. Hence applying Lemma 5.3 one more time, we are reduced to consider the case that X is projective, integral, and Cohen–Macaulay over a field k. Let $Z \subset X$ be a closed subscheme with $\dim(Z) \leqslant 0$. By, e.g., [7], Theorem 5.1, we find an effective ample divisor $D \subset X$ containing Z. By induction the following claim implies that X satisfies (C):

Claim 5.6 Let $X' \to X$ be an étale covering whose total space is connected. Then the restriction $D' = X' \times_X D$ remains connected.

The argument to prove the claim is similar to [15], Chapter II, Corollary 6.2. Let us recall it for the sake of completeness: Consider the ample invertible sheaf $\mathscr{L} = \mathscr{O}_X(D)$. Since $X' \to X$ is finite and surjective, the inclusion $D' \subset X'$ remains an effective Cartier divisor, and the corresponding invertible sheaf is the pullback $\mathscr{L}' = \mathscr{L}|X'$, which is still ample. Since X is projective and Cohen–Macaulay, so is X'. Let $\omega_{X'}$ be the dualizing sheaf over k. Then $h^1(\mathscr{L}'^{\otimes -t}) = h^{n-1}(\omega_{X'} \otimes \mathscr{L}'^{\otimes t})$ for every integer t. The right hand side vanishes for t sufficiently large, because $\mathscr{L}'$ is ample and $n - 1 \geqslant 1$. Replacing D by tD, we may assume $H^1(X', \mathscr{L}'^{\otimes -1}) = 0$. The short exact sequence $0 \to \mathscr{L}^{\otimes -1} \to \mathscr{O}_{X'} \to \mathscr{O}_{D'} \to 0$ thus gives a surjection of rings $H^0(X', \mathscr{O}_{X'}) \to H^0(D', \mathscr{O}_{D'})$. The term on the left is a finite extension of the ground field k because X' is integral and proper. Hence the above map is bijective, and D' must be connected. □

In view of 2.2 we obtain the following corollary.

Corollary 5.7 *Let k be a field and set $K = k^{\mathrm{sep}}$. Let X be a connected scheme which is proper over k and let $x_0 : \mathrm{Spec}(K) \to X$ be a geometric point. Then there exists a connected, reduced, affine, and 1-dimensional scheme C of finite type over k and a k-morphism $C \to X$, such that x_0 factors via C and the induced map*

$$\pi_1^{\mathrm{alg}}(C, x_0) \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0)$$

is surjective.

Proof By Proposition 5.5 and 2.2 we find a connected closed subscheme $C_1 \subset X$ of dimension at most 1, such that x_0 factors via C_1 and the induced map

$$\pi_1^{\mathrm{alg}}(C_1, x_0) \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0)$$

is surjective. Since passing to the reduced subscheme does not change the fundamental group, we may assume C_1 reduced. If $\dim(C_1) = 0$, then $C_1 = \mathrm{Spec}(L)$ with L a subfield of K. In this case we can take $C := \mathbb{A}^1_L$ with map $\mathbb{A}^1_L \to \mathrm{Spec}(L) = C_1 \to X$ and factorization of x_0 given by the composition of $\mathrm{Spec}(K) \to \mathrm{Spec}(L)$ with the inclusion of the zero section $\mathrm{Spec}(L) \hookrightarrow \mathbb{A}^1_L$.

Assume $\dim(C_1) = 1$. Note that C_1 is quasi-projective, hence we find an affine open $U \subset C_1$ which is connected and contains the singular locus of C_1 and the image point of x_0. In particular we remove from C_1 only finitely many regular closed points, whose local rings are therefore discrete valuation rings. Hence it follows from [22, Tag 0BSC] that $\pi_1^{\mathrm{alg}}(U, x_0) \to \pi_1^{\mathrm{alg}}(C_1, x_0)$ is surjective and we can take $C = U$. □

6 The non-proper case

In view of Remark 5.2, we consider the following weaker variant of condition (C) in this section.

6.1. Let X be a non-empty connected noetherian scheme. We consider the following property:

(C*) For each closed subscheme $Z \subset X$ with $\dim(Z) \leqslant 0$ and each finite étale covering $U \to X$ with connected total space there exists a connected closed subscheme $C \subset X$ with $0 \leqslant \dim(C) \leqslant 1$ and $Z \subset C$, such that $C_U = C \times_X U$ remains connected.

Lemma 6.2 *Let X be a non-empty connected noetherian scheme. Let $f : X' \to X$ be a universally closed and surjective morphism from a noetherian scheme X', with connected components X'_v. If property (C*) holds for all X'_v, then it also holds for X.*

We remark, that besides proper maps, all integral morphisms are universally closed, see [11, Proposition (6.1.10)]. Thus the above lemma applies to the normalization map $X' \to X$ of an integral scheme, and also to the projection from the base-change $X' = X \otimes_k k'$ with respect to any algebraic ground field extension, provided that X' stays noetherian.

Since in Lemma 6.2 the map $f : X' \to X$ is not assumed to be of finite type, the product $X' \times_X X'$ may not be noetherian and might have infinitely many connected components. Thus the graph constructed in 5.4 might have an infinite set of edges. To deal with this we record the following lemma.

Lemma 6.3 *Let Γ be a connected oriented graph with a finite set of vertices. Then there exists a finite oriented subgraph $\Gamma' \subset \Gamma$, which has the same set of vertices as Γ and which is connected.*

Proof Choose for each pair of vertices $v, w \in \Gamma$ a path $p_{v,w}$ connecting them. Take Γ' to be the graph whose set of vertices is equal to the set of vertices of Γ and whose edges are given by all the finitely many edges appearing in the paths $p_{v,w}$, for all pairs of vertices (v, w). □

Proof Lemma 6.2 Since f is closed and surjective the image $f(Z)$ of a closed subscheme $Z \subset X'$ is closed and satisfies $\dim f(Z) \leqslant \dim(Z)$, see [12, Proposition (5.4.1)]. In particular, closed points are mapped to closed points.

We assume $\dim X \geqslant 2$. Let $Z \subset X$ be a closed subset with $\dim(Z) \leqslant 0$. Set $X'' = X' \times_X X'$. Let $u : U \to X$ be a finite étale covering with connected total space. We want to find a closed connected at most one dimensional subscheme C which contains Z and such that the pullback of U over C stays connected. To this end we may assume that $u : U \to X$ is a finite étale Galois covering with Galois group G. We denote by $U' = U \times_X X'$ and $U'' = U \times_X X''$ the base changes and by Γ_X and Γ_U the oriented graphs defined as in 5.4 by $\mathrm{pr}_1 \times \mathrm{pr}_2 : \pi_0(X'') \to \pi_0(X') \times \pi_0(X')$ and $\mathrm{pr}_1 \times \mathrm{pr}_2 : \pi_0(U'') \to \pi_0(U') \times \pi_0(U')$, respectively. These graphs are connected, since X and U are, and have finite sets of vertices. The map u induces a surjective map of graphs $\Gamma_U \to \Gamma_X$ again denoted by u. For any vertex $w \in \Gamma_U$ with $v = u(w) \in \Gamma_X$, the morphism u induces a finite étale morphism $U'_w \to X'_v$.

Let $\Gamma'_U \subset \Gamma_U$ be a finite connected subgraph with the same vertices as Γ_U, see Lemma 6.3. Choose closed points $x_{U,e} \in U''_e$ for any edge $e \in \Gamma'_U$. Set

$$x_{e,i} := u(\mathrm{pr}_i(x_{U,e})) \in X'_{v_i}, \quad \text{where } v_i = u(\mathrm{pr}_i(e)) \in \Gamma,\ i = 1, 2.$$

Since Γ'_U is finite we find as in the proof of Lemma 5.3 for each vertex $v \in \Gamma_X$ a 0-dimensional closed subset $Z'_v \subset X'_v$ such that

(a) $Z \cap f(X'_v) \subset f(Z'_v)$ and
(b) $x_{e,i} \in Z'_v$, for all edges $e \in \Gamma'_U$ with $v = u(\mathrm{pr}_i(e))$ for $i = 1$ or 2.

By the surjectivity of f we have

$$Z \subset \bigcup_v f(Z'_v). \tag{6.1}$$

Fix a vertex $v \in \Gamma$ and choose $w_0 \in \Gamma'_U$ mapping to v. Applying (C*) to the finite étale covering $U'_{w_0} \to X'_v$ we find an at most 1-dimensional connected closed subscheme

$C'_v \subset X'_v$ containing Z'_v, such that the restriction $U'_{w_0} \times_{X'_v} C'_v$ remains connected. The base change $U \times_X C'_v \to C'_v$ is a Galois covering with Galois group G. In particular G acts transitively on the connected components of $U \times_X C'_v$ and we obtain isomorphisms

$$U'_w \times_{X'_v} C'_v \cong U'_{w_0} \times_{X'_v} C'_v, \quad \text{for all } w \in \Gamma_U \text{ mapping to } v, \tag{6.2}$$

in particular all these schemes are connected. Set

$$C_v := f(C'_v) \quad \text{and} \quad C := \bigcup_v C_v.$$

It follows that $C \subset X$ is closed, non-empty and at most 1-dimensional. By (6.1) and $Z'_v \subset C'_v$ we have $Z \subset C$. Moreover, using the choice of C'_v together with (6.2) we can argue in the same way as in the last paragraph of the proof of Lemma 5.3 with Γ_U there replaced by Γ'_U here to deduce that $U \times_X C$ is connected. □

Lemma 6.4 *Let k be an algebraically closed field and X an integral quasi-projective k-scheme of* $\dim X \geqslant 2$. *Let $Z \subset X$ be a finite (possibly empty) set of closed points and $X' \to X$ a finite and surjective morphism with X' irreducible. Then there is an integral closed subscheme $H \subset X$ of codimension* 1 *containing Z, such that $X' \times_X H$ is irreducible.*

Proof This follows from a classical Bertini theorem, where we use a trick of Mumford to ensure that the hyperplane contains Z, see the proof of the Lemma on p. 56 in [17]: Let $f : Y \to X$ be the blowing up with center Z. Then $\dim f^{-1}(z) \geqslant 1$, for all $z \in Z$. We fix an embedding $Y \hookrightarrow \mathbb{P}^n_k$. Denote by $Y' \subset Y \times_X X'$ an irreducible component which maps birationally onto X'. By [16, I, Corollary 6.11, 3)] applied to the quasi-finite morphism $Y' \to Y \to \mathbb{P}^n_k$ we find a hyperplane $H_1 \subset \mathbb{P}^n_k$ which is not contained in the exceptional locus of f and such that its pullback to Y' is irreducible. (Here we use k algebraically closed, since in *loc. cit.* Y' is required to be geometrically irreducible and k to be infinite.) Since $f^{-1}(z)$ is closed in $\mathbb{P}^n_k$ and $\dim f^{-1}(z) + \dim H_1 \geqslant n$ we find $f^{-1}(z) \cap H_1 \neq \emptyset$, for all $z \in Z$. Set $H := f(H_1 \cap Y)_{\mathrm{red}}$. Then $Z \subset H$ and the pullback of H to X' is birationally dominated by $H_1 \times_{\mathbb{P}^n} Y'$ and hence is irreducible. □

Proposition 6.5 *Let X be a connected scheme which is separated and of finite type over a field k. Then X has property (C*).*

Proof We proceed by induction on $d = \dim X$. There is nothing to prove for $d \leqslant 1$. Assume $d \geqslant 2$ and that (C*) holds for all connected schemes that are separated and of finite type over k and have dimension $\leqslant d-1$. Since X is connected all its irreducible components have dimension $\geqslant 1$. Using Lemma 6.2 we can therefore make the following reductions:

(i) k algebraically closed (since for $\bar{k}$ the algebraic closure of k the morphism $X \otimes_k \bar{k} \to X$ is integral, whence universally closed);
(ii) X reduced (by considering the proper morphism $X_{\mathrm{red}} \to X$);

(iii) X quasi-projective over k (Chow's Lemma);
(iv) X integral (by considering the proper and surjective morphism $\sqcup X_i \to X$, with X_i the irreducible components of X);
(v) X normal (by considering the normalization $\widetilde{X} \to X$).

Assume k and X satisfy the conditions above. Let $U \to X$ be a finite étale map with U connected. By the normality of X the scheme U is normal as well. Thus U is irreducible. Hence the existence of a curve for $U \to X$ as in (C*) follows directly from Lemma 6.4 and the induction hypothesis. □

Corollary 6.6 *Let k be a field and set $K = k^{\text{sep}}$. Let X be a connected scheme which is separated and of finite type over k and let $x_0 : \operatorname{Spec}(K) \to X$ be a geometric point. Let $X' \to X$ be a finite étale Galois covering. Then there exists a connected, reduced, affine, and 1-dimensional scheme C of finite type over k and a k-morphism $C \to X$, such that x_0 factors via C and the composite map*

$$\pi_1^{\text{alg}}(C, x_0) \longrightarrow \pi_1^{\text{alg}}(X, x_0) \longrightarrow \operatorname{Aut}(X'/X)^{\text{op}}, \tag{6.3}$$

is surjective. Here the second map is the natural surjection from [9, Exp. V, 4, h)].

Proof In general the composition (6.3) is surjective if the pullback of X' over C stays connected. Hence the statement follows from Proposition 6.5 the same way Corollary 5.7 follows from Proposition 5.5. □

7 Proof of the main theorem

We prove Theorem 4.4. We start by proving the second statement, i.e., for a *proper* and connected scheme X over a field k with geometric point $x_0 : \operatorname{Spec}(k) \to X$ we want to show that the natural injective group homomorphism $\pi_0\Omega^{\text{alg}}(X, x_0) \to \pi_1^{\text{alg}}(X, x_0)$ is surjective as well.

Set $K := k^{\text{sep}}$. By Corollary 5.7 we find a connected, affine, reduced and 1-dimensional k-scheme of finite type C with a morphism $C \to X$ such that x_0 factors via C and the natural $\pi_1^{\text{alg}}(C, x_0) \twoheadrightarrow \pi_1^{\text{alg}}(X, x_0)$ is surjective. We obtain a commutative diagram

$$\begin{array}{ccc} \pi_0\Omega^{\text{alg}}(C, x_0)^{\text{op}} & \longrightarrow & \pi_1^{\text{alg}}(C, x_0) \\ \downarrow & & \downarrow\!\!\downarrow \\ \pi_0\Omega^{\text{alg}}(X, x_0)^{\text{op}} & \longrightarrow & \pi_1^{\text{alg}}(X, x_0), \end{array} \tag{7.1}$$

in which the vertical arrow on the right is surjective. It therefore remains to show that the top horizontal arrow is surjective. To this end we choose some pro-object $(C_i)_{i\in I}$ in $(\text{FinEt}\,/C)$ representing Φ_{x_0}, see [9], Exposé V, 4. We write $\text{Pro}(\text{FinEt}\,/C)$ for pro-objects in $(\text{FinEt}\,/C)$. Since I is a filtered set and the transition maps $C_i \to C_j$ are finite étale, and therefore affine, we may form the projective limit $\widetilde{C} = \varprojlim_{i\in I} C_i$

in the category of C-schemes, see [13], Proposition (8.2.3). For a C-scheme T we obtain a functorial isomorphism

$$\mathrm{Hom}_C(T, \widetilde{C}) = \varprojlim_i \mathrm{Hom}_C(T, C_i).$$

In particular an element $a_0 \in \varprojlim \Phi_{x_0}(C_i)$ is a morphism $a_0 : \mathrm{Spec}(K) \to \widetilde{C}$ over x_0. Furthermore, by [9], Exposé V, 4, h), we have

$$\begin{aligned}\mathrm{Aut}_{\mathrm{Pro(FinEt}\,/C)}((C_i)_i) &= \mathrm{Hom}_{\mathrm{Pro(FinEt}\,/C)}((C_i)_i, (C_j)_j)\\ &= \varprojlim_j \mathrm{Hom}_{\mathrm{Pro(FinEt}\,/C)}((C_i)_i, C_j)\\ &= \varprojlim_j \mathrm{Hom}_C(C_j, C_j) = \varprojlim_j \mathrm{Hom}_C(\widetilde{C}, C_j)\\ &= \mathrm{Hom}_C(\widetilde{C}, \widetilde{C}) = \mathrm{Aut}_C(\widetilde{C}).\end{aligned}$$

By *loc. cit.*, the choice of $a_0 : \mathrm{Spec}(K) \to \widetilde{C}$ over x_0 yields a functorial isomorphism

$$\mathrm{Hom}_C(\widetilde{C}, U) \xrightarrow{\simeq} \Phi_{x_0}(U), \quad f \longmapsto f \circ a_0, \qquad U \in (\mathrm{FinEt}\,/C).$$

We obtain an isomorphism

$$\theta : \mathrm{Aut}_C(\widetilde{C})^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Aut}(\Phi_{x_0}) = \pi_1^{\mathrm{alg}}(C, x_0),$$

which sends a C-automorphism $\sigma : \widetilde{C} \to \widetilde{C}$ to the automorphism $\theta(\sigma)$ of Φ_{x_0}, which on $U \in (\mathrm{FinEt}\,/C)$ is given by

$$\Phi_{x_0}(U) \ni f \circ a_0 \longmapsto f \circ (\sigma \circ a_0) \in \Phi_{x_0}(U), \quad f \in \mathrm{Hom}_C(\widetilde{C}, U).$$

We claim that for any σ the automorphism $\theta(\sigma)$ is in the image of $\pi_0\Omega^{\mathrm{alg}}(C, x_0)^{\mathrm{op}}$. Indeed, by construction $\widetilde{C}$ is affine, reduced, connected and 1-dimensional and is simply connected We have the K-rational point $a_0 : \mathrm{Spec}(K) \to \widetilde{C}$ over $x_0 : \mathrm{Spec}(K) \to C$. We note that a_0 is a closed immersion. Indeed, for any finite separable field extension L/k a connected component C_0 of $C \otimes_k L$ is a finite étale covering of C; hence we have a map $\widetilde{C} \to C_0$. It follows that the algebraic closure of k in $H^0(\widetilde{C}, \mathscr{O}_{\widetilde{C}})$ is equal to $K = k^{\mathrm{sep}}$, which implies that a_0 is a closed immersion. Thus $(\widetilde{C}, a_0, \sigma \circ a_0)$ is an interval scheme in the sense of Definition 3.1 and the map $\widetilde{C} \to C$ induces an algebraic loop $w : (\widetilde{C}, a_0, \sigma \circ a_0) \to (C, x_0, x_0)$. The map (4.3) sends the loop w to the monodromy μ_w which by construction is equal to $\theta(\sigma)$. This completes the proof of the second part of the theorem.

It remains to show that assuming X is connected and only separated and of finite type over k, then the image of $\pi_0\Omega^{\mathrm{alg}}(X, x_0)^{\mathrm{op}} \to \pi_1^{\mathrm{alg}}(X, x_0)$ is dense. It suffices to show that for any finite étale Galois covering $X' \to X$ the composition

$$\pi_0\Omega^{\mathrm{alg}}(X, x_0)^{\mathrm{op}} \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0) \longrightarrow \mathrm{Aut}(X'/X)^{\mathrm{op}}$$

is surjective. By Corollary 6.6 we find a curve C as above such that the composition

$$\pi_1^{\mathrm{alg}}(C, x_0) \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0) \longrightarrow \mathrm{Aut}(X'/X)^{\mathrm{op}}$$

is surjective. Thus the statement follows from the surjectivity of top horizontal map in (7.1) proved above. □

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Declarations

Conflict of interests The authors have no conflict of interest to declare that are relevant to the content of this article.

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