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Fractional nonlinear heat equations and characterizations of some function spaces in terms of fractional Gauss–Weierstrass semi–groups

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Abstract
We present a new proof of the caloric smoothing related to the fractional Gauss–Weierstrass semi–group in Triebel-Lizorkin spaces. This property will be used to prove existence and uniqueness of mild and strong solutions of the Cauchy problem for a fractional nonlinear heat equation.

Keywords Fractional nonlinear heat equation · Function spaces · Fractional Gauss-Weierstrass semi–group

Mathematics Subject Classification 35K30 · 35K05 · 46E35

1 Introduction

The aim of the paper is twofold. First we justify the smoothing property

$$t^{\frac{d}{2\alpha}} \, \| W_t^\alpha w \,|A^{s+d}_{p,q}(\mathbb{R}^n)\| \le c \, \|w \,|A^s_{p,q}(\mathbb{R}^n)\|, \qquad 0 < t \le 1, \quad d \ge 0, \tag{1}$$

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of the fractional Gauss–Weierstrass semi–group W_t^α,

$$W_t^\alpha w(x) = \left(e^{-t|\xi|^{2\alpha}}\,\widehat{w}\right)^\vee(x), \qquad w \in A^s_{p,q}(\mathbb{R}^n), \quad \alpha > 0, \tag{2}$$

in Besov and Triebel-Lizorkin spaces (see Definition 1)

$$A^s_{p,q}(\mathbb{R}^n), \qquad A \in \{B, F\}, \quad s \in \mathbb{R} \quad \text{and} \quad 1 \le p, q \le \infty. \tag{3}$$

Here $\wedge$ and $\vee$ stand for the Fourier transform and its inverse, respectively. It will be a straightforward consequence of the characterization of some of these spaces in terms of the semi–group W_t^α. Based on these observations we deal secondly with the Cauchy problem

$$\partial_t u(x,t) + (-\Delta)^\alpha u(x,t) - \sum_{j=1}^n \partial_j u^2(x,t) = 0, \qquad x \in \mathbb{R}^n, 0 < t < T \tag{4}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{5}$$

where $0 < T \le \infty$ and $2 \le n \in \mathbb{N}$ in the context of the semi–group W_t^α in (2) and the fractional Laplacian

$$(-\Delta)^\alpha w = \left(|\xi|^{2\alpha}\widehat{w}\right)^\vee. \tag{6}$$

Here as usual $\partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$. In particular, $(-\Delta)^\alpha = (-\Delta_x)^\alpha$ refers to the space variables. If $\alpha = 1$ then (4) refers to Burgers equation. The peculiar nonlinearity $Du^2 = \sum_{j=1}^n \partial_j u^2$ is considered as the scalar counterpart of the related (vector–valued) non–linearity in the Navier–Stokes equations. In general the fractional Laplacian $(-\Delta)^\alpha$ is modelling dissipation (hyperdissipation if $\alpha > 1$, hyperviscousity if $\alpha = 2$). In this respect the case $\alpha = \frac{n+2}{4}$ attracted special attention. We refer to [7] and [13]. The fractional Burgers equation (4) with $1/2 \le \alpha < 1$ has been considered in [6]. For investigations of solutions of Cauchy problems for fractional dissipative heat equations with several types of nonlinearities we refer to [12]. It turns out that both in generalized Navier–Stokes equations and in other generalized equations of physical and biological relevance (such as quasi-geostrophic equations, Keller-Segel equations, chemotaxis equations) the suggestion is to replace the Laplace operator by a (fractional) power $(-\Delta)^\alpha$ in order to achieve adequate mathematical models. The motivation to study the smoothing property (1) for fractional Gauss-Weierstrass semi–groups comes from our interest in so–called mild solutions of (4), (5) being fixed points of the operator $\mathcal{T}_{u_0}$,

$$\mathcal{T}_{u_0} u(x,t) = W_t^\alpha u_0(x) + \int_0^t W^\alpha_{t-\tau} Du^2(x,\tau)\,\mathrm{d}\tau, \qquad x \in \mathbb{R}^n, \quad 0 < t < T \tag{7}$$

in suitable weighted Lebesgue spaces with respect to the Bochner integral $L_v((0,T),b,X)$, where $X = A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B,F\}$, $s \in \mathbb{R}$, $1 \le p \le \infty$, $1 \le q \le \infty$, stands for a Besov or Triebel-Lizorkin space. This means that for some $b \in \mathbb{R}$ and v our solution satisfies

$$\| f | L_v((0,T),b,X) \| = \left(\int_0^T t^{bv} \| f(\cdot,t) | X \|^v \mathrm{d}t \right)^{1/v} < \infty \tag{8}$$

if $1 \le v < \infty$ and

$$\| f | L_\infty((0,T),\ b,\ X) \| = \sup_{0<t<T} t^b \| f(\cdot,t) | X \| < \infty \tag{9}$$

if $v = \infty$. Note that (after extension from $\mathbb{R}^n \times (0,T)$ to $\mathbb{R}^{n+1}$ by zero)

$$L_v((0,T),b,X) \subset S'(\mathbb{R}^{n+1}) \quad \text{if} \quad b + \frac{1}{v} < 1 \tag{10}$$

(see [18, formulae (4.17), (4.21)]) for details and an appropriate interpretation). Here $S'(\mathbb{R}^{n+1})$ stands for the space of tempered distributions. As far as the smoothing property (1) is concerned we refer to [18, Theorem 4.1] ($\alpha = 1$, classical case), [2, Proposition 3.4] ($\alpha \in \mathbb{N}$) as well as [8, Corollary 5.4] and [9, Example 4.7] ($\alpha > 0$). We present a different proof in Theorem 3 based on characterizations of $A^s_{p,q}(\mathbb{R}^n)$ ($s > 0$) in terms of the fractional Gauss-Weierstrass semi–group W^α_t provided in Theorem 2. As far as the Cauchy problem (4), (5) is concerned we follow the method developed in [18, Subsection 4.5] and [1] ($\alpha = 1$) as well as [2] and [3] ($\alpha \in \mathbb{N}$). The main results are contained in Theorem 6 (existence of mild and strong solutions) and Theorem 7 (locally well-posedness of the Cauchy problem). Our approach allows us to deal with the above Cauchy problem for initial data u_0 belonging to spaces $A^{s_0}_{p,q}(\mathbb{R}^n)$ in the so-called supercritical case $s_0 > \frac{n}{p} - 2\alpha + 1$, where $\alpha > 1/2$, $1 \le p \le \infty$, $1 \le q \le \infty$. Apart from the smoothing property (1) a key ingredient in the proof turns out to be the mapping property of the nonlinearity $Du^2 = \sum_{j=0}^n \partial_j u^2$ in (4) considered in Proposition 4 and Corollary 1 which require the condition $s > \left(\frac{n}{p} - \frac{n}{2}\right)_+$ for spatial solution spaces $A^s_{p,q}(\mathbb{R}^n)$.

The paper is organized as follows. Section 2 is concerned with the characterization of spaces $A^s_{p,q}(\mathbb{R}^n)$ in terms of fractional Gauss–Weierstrass semi–groups and the proof of the smoothing property (1). The existence, uniqueness and stability of mild and strong solutions of the Cauchy problem (4), (5) are treated in Sect. 3. In the final Sect. 4 we illustrate different cases of solution spaces depending on the choice of $\alpha > 0$, dimension n and integrability p. In particular we consider how close the spaces of admitted initial data approach to the so-called critical line $s_0 = \frac{n}{p} - 2\alpha + 1$ (for more details see explanations below). Moreover, we discuss the special case when initial data u_0 belong to $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Finally, we investigate how our results fit in the current literature and compare them with related results.

2 Function spaces

2.1 Definitions and basic ingredients

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$, is the standard complex quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f\,|L_p(\mathbb{R}^n)\| = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \right)^{1/p} \tag{11}$$

with the usual modification if $p = \infty$. Similarly $L_p(M)$ where M is a Lebesgue-measurable subset of $\mathbb{R}^n$. As usual, $\mathbb{Z}$ is the collection of all integers and $\mathbb{Z}^n$ where $n \in \mathbb{N}$ denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_k \in \mathbb{Z}$. Let $Q_{j,m} = 2^{-j}m + 2^{-j}(0,1)^n$ with $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$ be the usual dyadic cubes in $\mathbb{R}^n$, $n \in \mathbb{N}$, with sides of length 2^{-j} parallel to the axes of coordinates and $2^{-j}m$ as the lower left corner.

If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) \,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n, \tag{12}$$

denote the Fourier transform of φ. As usual, $\mathcal{F}^{-1}\varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (12) with i in place of $-i$. Here $x\xi$ stands for the scalar product in $\mathbb{R}^n$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(x) = 0 \text{ if } |x| \ge 3/2. \tag{13}$$

We define the sequences

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \qquad x \in \mathbb{R}^n, \quad j \in \mathbb{N} \tag{14}$$

and

$$\varphi^j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \qquad x \in \mathbb{R}^n, \quad j \in \mathbb{Z}. \tag{15}$$

Since

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1,\ x \in \mathbb{R}^n \text{ and } \sum_{j=-\infty}^{\infty} \varphi^j(x) = 1,\ x \in \mathbb{R}^n \setminus \{0\}, \tag{16}$$

$\varphi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ and $\varphi = \{\varphi^j\}_{j\in\mathbb{Z}}$ form a dyadic resolution of unity, respectively. The entire analytic functions $(\varphi_j \widehat{f})^\vee(x)$ $(j \in \mathbb{N}_0)$ and $(\varphi^j \widehat{f})^\vee(x)$ $(j \in \mathbb{Z})$ make sense pointwise in $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$.

We are interested in inhomogeneous Besov and Triebel-Lizorkin spaces $A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ with $s \in \mathbb{R}$ and $0 < p, q \le \infty$. The standard norms of these spaces and their homogeneous counterparts are given as follows

Definition 1 Let $0 < p \le \infty$ ($p < \infty$ if $A = F$), $0 < q \le \infty$ and $s \in \mathbb{R}$.
(i) Let $\varphi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ be the above dyadic resolution of unity. Then $A^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\,|A^s_{p,q}(\mathbb{R}^n)\|_\varphi = \begin{cases} \left(\sum\limits_{j=0}^{\infty} 2^{jsq} \left\|(\varphi_j \widehat{f})^\vee\,|L_p(\mathbb{R}^n)\right\|^q\right)^{1/q}, & \text{if } A = B \\ \left\|\left(\sum\limits_{j=0}^{\infty} 2^{jsq}\left|(\varphi_j \widehat{f})^\vee(\cdot)\right|^q\right)^{1/q} \Big|L_p(\mathbb{R}^n)\right\|, & \text{if } A = F \end{cases} \tag{17}$$

is finite (with the usual modification if $q = \infty$).
(ii) Let $\varphi = \{\varphi^j\}_{j\in\mathbb{Z}}$ be the above homogeneous dyadic resolution of unity in $\mathbb{R}^n \setminus \{0\}$. Then $\dot{A}^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\,|\dot{A}^s_{p,q}(\mathbb{R}^n)\|_\varphi = \begin{cases} \left(\sum\limits_{j=-\infty}^{\infty} 2^{jsq} \left\|(\varphi^j \widehat{f})^\vee\,|L_p(\mathbb{R}^n)\right\|^q\right)^{1/q}, & \text{if } A = B \\ \left\|\left(\sum\limits_{j=-\infty}^{\infty} 2^{jsq}\left|(\varphi^j \widehat{f})^\vee(\cdot)\right|^q\right)^{1/q} \Big|L_p(\mathbb{R}^n)\right\|, & \text{if } A = F \end{cases} \tag{18}$$

is finite (with the usual modification if $q = \infty$).
(iii) Let $0 < q < \infty$ and $s \in \mathbb{R}$. Then $F^s_{\infty,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \mid F^s_{\infty,q}(\mathbb{R}^n)\|_\varphi = \sup_{J\in\mathbb{N}_0, M\in\mathbb{Z}^n} 2^{Jn/q} \left(\int_{Q_{J,M}} \sum_{j\ge J} 2^{jsq}\left|(\varphi_j \widehat{f})^\vee(x)\right|^q \mathrm{d}x\right)^{1/q} \tag{19}$$

with $\varphi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ as in (i) is finite.
(iv) Let $0 < q < \infty$ and $s \in \mathbb{R}$. Then $\dot{F}^s_{\infty,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \mid \dot{F}^s_{\infty,q}(\mathbb{R}^n)\|_\varphi = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} 2^{Jn/q} \left(\int_{Q_{J,M}} \sum_{j\ge J} 2^{jsq}\left|(\varphi^j \widehat{f})^\vee(x)\right|^q \mathrm{d}x\right)^{1/q} \tag{20}$$

with $\varphi = \{\varphi^j\}_{j\in\mathbb{Z}}$ as in (ii) is finite.

Remark 1 We recall that all spaces defined above are independent of the respective resolution of unity φ according to (13)–(16) (equivalent quasi-norms). This justifies the omission of the subscript φ in (17)–(20) in the sequel (and any other marks in connection with equivalent quasi–norms). Note that the spaces $A^s_{p,q}(\mathbb{R}^n)$ are translation invariant. This follows easily from elementary properties of the Fourier transform and the translation invariance of L_p – spaces. The theory of *inhomogeneous* spaces, including special cases and their history may be found in [14–16] and [20]. As far as *homogeneous* spaces are concerned we refer to [14, Chapter 5] as well as to [19, Definition 2.8] as far as (20) is concerned. We will use these spaces only in the context of norm equivalences. Especially for our purposes it is not necessary to discuss the usual ambiguity of *homogeneous* spaces. Finally, $F^s_{\infty,\infty}(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$ and $\dot{F}^s_{\infty,\infty}(\mathbb{R}^n) = \dot{B}^s_{\infty,\infty}(\mathbb{R}^n)$ as discussed in [20, Definition 1.1, Remark 1.2, pp. 2–3 and p.116].

We need a few specific properties of the above defined spaces. Let φ_0 and $\varphi = \{\varphi^j\}_{j\in\mathbb{Z}}$ be as in (13) and (15). Let $1 \le p, q \le \infty$ ($p < \infty$ for F–spaces) and $s > 0$. Then

$$\|f\,|B^s_{p,q}(\mathbb{R}^n)\| \sim \|f\,|L_p(\mathbb{R}^n)\| + \|f\,|\,\dot{B}^s_{p,q}(\mathbb{R}^n)\| \sim \|(\varphi_0\widehat{f})^\vee|L_p(\mathbb{R}^n)\| + \|f\,|\,\dot{B}^s_{p,q}(\mathbb{R}^n)\| \tag{21}$$

are equivalent norms in $B^s_{p,q}(\mathbb{R}^n)$ and

$$\|f\,|F^s_{p,q}(\mathbb{R}^n)\| \sim \|f\,|L_p(\mathbb{R}^n)\| + \|f\,|\,\dot{F}^s_{p,q}(\mathbb{R}^n)\| \sim \|(\varphi_0\widehat{f})^\vee|L_p(\mathbb{R}^n)\| + \|f\,|\,\dot{F}^s_{p,q}(\mathbb{R}^n)\| \tag{22}$$

are equivalent norms in $F^s_{p,q}(\mathbb{R}^n)$ (with the usual modification if $q = \infty$). This is a special case of corresponding assertions in [15, Section 2.3.3, pp. 97–100] where one finds also continuous versions with $t > 0$ in place of 2^{-j}, $j \in \mathbb{Z}$, which are nearer to what follows. These norms are characterizing what means that $f \in S'(\mathbb{R}^n)$ belongs to $A^s_{p,q}(\mathbb{R}^n)$ if, and only if, the corresponding norm is finite. We need an extension of the above norms to a wider class of functions $\varphi^j(x) = \varphi^0(2^{-j}x)$ and $\varphi^0(tx)$.

Let $h \in S(\mathbb{R}^n)$ and $H \in S(\mathbb{R}^n)$ with

$$h(x) = 1 \text{ if } |x| \le 1, \qquad \operatorname{supp} h \subset \{x : |x| \le 2\} \tag{23}$$

and

$$H(x) = 1 \text{ if } 1/2 \le |x| \le 2, \qquad \operatorname{supp} H \subset \{x : 1/4 \le |x| \le 4\}. \tag{24}$$

Proposition 1 *Let φ_0 be as in (13) and $\varphi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ with $|\varphi(x)| > 0$ if $1/2 \le |x| \le 2$.*

(i) *Let* $1 \le p, q \le \infty$ *and* $0 < s < \sigma$. *Let*

$$\int_{\mathbb{R}^n} \left| \left(\frac{\varphi(z)\, h(z)}{|z|^\sigma} \right)^\vee (y) \right| \mathrm{d}y < \infty \tag{25}$$

and

$$\sup_{m \in \mathbb{N}} \int_{\mathbb{R}^n} \left| \left(\varphi\left(2^m \cdot\right) H(\cdot) \right)^\vee (y) \right| \mathrm{d}y < \infty. \tag{26}$$

Then

$$\begin{aligned} \| f \,| B^s_{p,q}(\mathbb{R}^n)\| &\sim \| f \,| L_p(\mathbb{R}^n)\| + \left(\int_0^\infty t^{-sq} \left\| \left(\varphi(t\cdot) \widehat{f} \right)^\vee | L_p(\mathbb{R}^n) \right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\ &\sim \left\| (\varphi_0 \widehat{f})^\vee | L_p(\mathbb{R}^n) \right\| + \left(\int_0^\infty t^{-sq} \left\| \left(\varphi(t\cdot) \widehat{f} \right)^\vee | L_p(\mathbb{R}^n) \right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \end{aligned} \tag{27}$$

(usual modification if $q = \infty$*) are equivalent norms in* $B^s_{p,q}(\mathbb{R}^n)$.
(ii) *Let* $1 \le p < \infty$, $1 \le q \le \infty$, $0 < s < \sigma$ *and* $a > n$. *Let*

$$\int_{\mathbb{R}^n} \left| \left(\frac{\varphi(z)\, h(z)}{|z|^\sigma} \right)^\vee (y) \right| (1 + |y|)^a \, \mathrm{d}y < \infty \tag{28}$$

and

$$\sup_{m \in \mathbb{N}} \int_{\mathbb{R}^n} \left| \left(\varphi(2^m \cdot)\, H(\cdot) \right)^\vee (y) \right| (1 + |y|)^a \, \mathrm{d}y < \infty. \tag{29}$$

Then

$$\begin{aligned} &\| f \,| F^s_{p,q}(\mathbb{R}^n)\| \\ &\quad \sim \| f \,| L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^\infty t^{-sq} \left| \left(\varphi(t\cdot) \widehat{f} \right)^\vee (\cdot) \right|^q \frac{\mathrm{d}t}{t} \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \\ &\quad \sim \left\| (\varphi_0 \widehat{f})^\vee | L_p(\mathbb{R}^n) \right\| + \left\| \left(\int_0^\infty t^{-sq} \left| \left(\varphi(t\cdot) \widehat{f} \right)^\vee (\cdot) \right|^q \frac{\mathrm{d}t}{t} \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \end{aligned} \tag{30}$$

(usual modification if $q = \infty$*) are equivalent norms in* $F^s_{p,q}(\mathbb{R}^n)$.

Proof The extension of (21), (22) from $\varphi^0(2^{-j}x)$ and its continuous counterpart $\varphi^0(tx)$ to the above assertion follows from [19, Proposition 2.10, pp. 18–19] and the references given there specified to the above values of the parameters s, p, q. Again these norms are characterizations as explained above. □

Remark 2 Note that by [19, Proposition 2.10] the second summands on right-hand-sides in (27) and (30) are equivalent norms in $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$, respectively, for all admitted parameters.

Let us shortly comment on the conditions with respect to φ_0 and φ supposed in Proposition 1. The system $(\varphi_j)_{j=1}^{\infty}$ introduced in order to define the spaces $A^s_{p,q}(\mathbb{R}^n)$ (see Definition 1) can be rewritten as

$$\varphi_j(x) = \varrho(2^{-j}x), \quad \text{where} \quad \varrho(x) = \varphi_0(x) - \varphi_0(2x)$$

and φ_0 has the meaning of (13). The function ϱ has compact support in $\{x : \frac{1}{2} \le |x| \le \frac{3}{2}\}$ and satisfies the Tauberian condition $|\varrho(x)| > 0$ on $\{x : \frac{3}{4} \le |x| \le 1\}$. The characterization of spaces $A^s_{p,q}(\mathbb{R}^n)$ in Proposition 1 can be considered as a continuous extension and generalization of Definition 1, where the generating function ϱ is replaced by φ. In contrast to the properties of ϱ it is not assumed that φ has compact support in a subset of $\mathbb{R}^n \setminus \{0\}$. Conditions (25) and (28) ensure sufficiently strong decay to 0 near the origin, whereas (26) and (29) are responsible for decay if $|x| \to \infty$. For example, it follows from (25) and (28) that $|\varphi(x)| \lesssim |x|^\sigma$ in a neighbourhood of the origin. Moreover, the condition $|\varphi(x)| > 0$ if $\frac{1}{2} \le |x| \le 2$ corresponds to the Tauberian condition with respect to ϱ. Let us also mention that the condition with respect to $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ can be weakened. For a more detailed discussion we refer to [15, Corollary 2.4.1/1, Remark 2.4.1/3]. Relevant examples will be treated in the next subsection.

2.2 Characterizations of some function spaces in terms of fractional Gauss–Weierstrass semi–groups

We wish to apply Proposition 1 to

$$\varphi(\xi) = |\xi|^\delta e^{-|\xi|^{2\alpha}}, \qquad \xi \in \mathbb{R}, \quad \alpha > 0, \quad \delta > 0. \tag{31}$$

If $0 < \alpha \notin \mathbb{N}$ then $\varphi(\xi)$ is not smooth at $\xi = 0$ and some extra care is needed. This is just the reason why we prefer now Proposition 1 (under the indicated restrictions for the underlying spaces) compared with the original inhomogeneous versions according to [15, Theorems 2.4.1, 2.5.1, pp. 100, 101, 132] (which apply to all spaces $A^s_{p,q}(\mathbb{R}^n)$ with exception of $F^s_{\infty,q}(\mathbb{R}^n)$). Rescue comes from the following observations in [12]. Let

$$K^\alpha(x) = \left(e^{-|\xi|^{2\alpha}}\right)^\vee(x), \qquad x \in \mathbb{R}^n, \quad \alpha > 0, \tag{32}$$

and according to (6)

$$K^{\alpha,\sigma}(x) = (-\Delta)^{\sigma/2} K^\alpha(x) = \left(|\xi|^\sigma e^{-|\xi|^{2\alpha}}\right)^\vee(x), \qquad x \in \mathbb{R}^n, \quad \sigma > 0, \quad \alpha > 0. \tag{33}$$

Then the estimates

$$|K^{\alpha}(x)| \le c\,(1+|x|)^{-n-2\alpha}, \qquad x \in \mathbb{R}^n, \quad \alpha > 0, \tag{34}$$

and

$$|K^{\alpha,\sigma}(x)| \le c\,(1+|x|)^{-n-\sigma}, \qquad x \in \mathbb{R}^n, \quad \alpha > 0, \quad \sigma > 0, \tag{35}$$

are covered by [12, Lemma 2.1, Lemma 2.2, pp. 463, 465]. With W_t^{α} as in (2) one has for $k \in \mathbb{N}_0$,

$$\partial_t^k W_t^{\alpha} w(x) = (-1)^k \left(|\xi|^{2k\alpha}\, e^{-t|\xi|^{2\alpha}}\, \widehat{w} \right)^{\vee}(x), \qquad x \in \mathbb{R}^n, \quad t > 0. \tag{36}$$

In the distinguished case $\alpha = 1$ one has now final characterizations of all spaces $A^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $0 < p, q \le \infty$ in terms of $\partial_t^k W_t w$ for the classical Gauss–Weierstrass semi–group $W_t = W_t^1$. This may be found in [20, Section 3.2.7, pp. 106–109] and the references given there. We extend now these assertions to the fractional Gauss–Weierstrass semi–group W_t^{α} under the same restrictions for the spaces $A^s_{p,q}(\mathbb{R}^n)$ as in Proposition 1.

Theorem 2 *Let φ_0 be as in (13) and W_t^{α} be as in (2) with $\alpha > 0$.*
(i) *Let $1 \le p, q \le \infty$, $s > 0$ and $k \in \mathbb{N}$ such that $2\alpha k > s$. Then*

$$\begin{aligned} \|f\,|B^s_{p,q}(\mathbb{R}^n)\| &\sim \|f\,|L_p(\mathbb{R}^n)\| + \left(\int_0^{\infty} t^{(k-\frac{s}{2\alpha})q} \left\| \partial_t^k W_t^{\alpha} f\,|L_p(\mathbb{R}^n) \right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\ &\sim \|(\varphi_0 \widehat{f})^{\vee} | L_p(\mathbb{R}^n)\| + \left(\int_0^{\infty} t^{(k-\frac{s}{2\alpha})q} \left\| \partial_t^k W_t^{\alpha} f\,|L_p(\mathbb{R}^n) \right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \end{aligned} \tag{37}$$

(equivalent norms), usual modification if $q = \infty$.
(ii) *Let $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$ and $k \in \mathbb{N}$ such that $2\alpha k > s + n$. Then*

$$\begin{aligned} &\|f\,|F^s_{p,q}(\mathbb{R}^n)\| \\ &\quad \sim \|f\,|L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^{\infty} t^{(k-\frac{s}{2\alpha})q} \left| \partial_t^k W_t^{\alpha} f(\cdot) \right|^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\ &\quad \sim \|(\varphi_0 \widehat{f})^{\vee} | L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^{\infty} t^{(k-\frac{s}{2\alpha})q} \left| \partial_t^k W_t^{\alpha} f(\cdot) \right|^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \end{aligned} \tag{38}$$

(equivalent norms), usual modification if $q = \infty$.

Proof *Step 1.* We rely on part (i) of Proposition 1 choosing there

$$\varphi(\xi) = |\xi|^{\sigma} e^{-|\xi|^{2\alpha}}, \qquad \xi \in \mathbb{R}^n, \quad \sigma = 2\alpha k > s. \tag{39}$$

Then (25) follows from (34) and

$$\int_{\mathbb{R}^n} \left| \left(e^{-|\cdot|^{2\alpha}} h(\cdot)\right)^\vee (y)\right| \mathrm{d}y \le \int_{\mathbb{R}^n} \left| \left(e^{-|\cdot|^{2\alpha}}\right)^\vee (y)\right| \mathrm{d}y \cdot \int_{\mathbb{R}^n} |h^\vee(y)| \,\mathrm{d}y < \infty. \tag{40}$$

Secondly we have to justify (26) with φ as in (39). But this follows from

$$\int_{\mathbb{R}^n} |g^\vee(x)| \,\mathrm{d}x \le c \left(\int_{\mathbb{R}^n} \left|(1+|x|^2)^{l/2} g^\vee(x)\right|^2 \mathrm{d}x \right)^{1/2} \sim \|g \,|W_2^l(\mathbb{R}^n)\|, \tag{41}$$

$n/2 < l \in \mathbb{N}$, where $W_2^l(\mathbb{R}^n)$ are the classical Sobolev spaces. Then the second terms in (27) are equivalent to

$$\begin{aligned} &\sim \left(\int_0^\infty \tau^{-sq} \left\| \left(\varphi(\tau\cdot)\widehat{f}\right)^\vee |L_p(\mathbb{R}^n)\right\|^q \frac{\mathrm{d}\tau}{\tau} \right)^{1/q} \\ &\quad \sim \left(\int_0^\infty t^{-\frac{s}{2\alpha}q} \left\| \left(\varphi(t^{\frac{1}{2\alpha}}\cdot)\widehat{f}\right)^\vee |L_p(\mathbb{R}^n)\right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \end{aligned} \tag{42}$$

again with φ as in (39) and $\tau = t^{\frac{1}{2\alpha}}$. One has by

$$\varphi\left(t^{\frac{1}{2\alpha}}\xi\right) = t^k |\xi|^{2k\alpha} e^{-t|\xi|^{2\alpha}}, \quad t > 0, \quad \xi \in \mathbb{R}^n, \tag{43}$$

and (2) that

$$\left(\varphi(t^{\frac{1}{2\alpha}}\cdot)\widehat{f}\right)^\vee (x) = (-1)^k t^k \partial_t^k W_t^\alpha f(x). \tag{44}$$

Inserted in (42) one obtains (37).
Step 2. For the proof of part (ii) we rely on part (ii) of Proposition 1 choosing

$$\varphi(\xi) = |\xi|^\delta e^{-|\xi|^{2\alpha}}, \qquad \xi \in \mathbb{R}^n, \quad \delta = 2\alpha k > s+n. \tag{45}$$

Using $(1+|y|)^a \le (1+|y-z|)^a(1+|z|)^a$, $a > n$, one obtains similarly as in (40) that the expression (28) can be estimated from above by

$$c \int_{\mathbb{R}^n} \left| \left(\frac{\varphi(z)}{|z|^\sigma}\right)^\vee (1+|y|)^a \,\mathrm{d}y = c \int_{\mathbb{R}^n} \left| \left(|z|^{\delta-\sigma} e^{-|z|^{2\alpha}}\right)^\vee (y)\right| (1+|y|)^a \,\mathrm{d}y \tag{46}$$

with $\sigma > s$ such that also $\delta - \sigma > a > n$ and some $c > 0$. Now (28) follows from (33), (35). As far as the terms (29) are concerned one argues as in (41) incorporating the factor $(1+|x|)^a$. Afterwards one is in the same position as in (42)–(44) now based on (30). This proves (38). □

Remark 3 As already mentioned above the equivalent norms in (37) and (38) are characterizations. This means in our case that $A^s_{p,q}(\mathbb{R}^n)$ collects all $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, such that the corresponding norm is finite. In particular it follows from the above considerations immediately that always $\partial^k_t W^\alpha_t f \in L_p(\mathbb{R}^n)$ if $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$. But we will not stress this point in the sequel.

2.3 Smoothing properties

We justify (1)–(3). As already mentioned in the Introduction assertions of this type are not new. A proof of (1) with $\alpha = 1$ for the classical Gauss–Weierstrass semi–group $W_t w = W^1_t w$ covering all spaces $A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$, $s \in \mathbb{R}$ and $0 < p, q \le \infty$ may be found in [20, Theorem 3.35, p. 110]. It relies on characterization of these spaces in terms of $W_t w$ using in a decisive way that the underlying kernel $e^{-|\xi|^2} \in S(\mathbb{R}^n)$ is smooth at the origin $\xi = 0$. This is no longer the case in general if one steps from W_t to W^α_t, $\alpha > 0$, this means from $e^{-|\xi|^2}$ to $e^{-|\xi|^{2\alpha}}$. On the other hand, (1) for the classical Gauss–Weierstrass semi–group $W_t = W^1_t$, restricted to $A \in \{B, F\}$, $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ ($p < \infty$ for F–spaces) is also a special case of a corresponding assertion for related hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$. This may be found in [18, Theorem 4.1, p. 114] including related references and comments. The extension of (1) from $\alpha = 1$ to $\alpha \in \mathbb{N}$ for the spaces $A \in \{B, F\}$, $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ ($p < \infty$ for F–spaces) goes back to [2, Theorem 3.5, p. 2123]. The arguments both in [18] (including underlying references) and [2] rely on the elaborated machinery of (caloric) wavelet expansions. The step from $\alpha \in \mathbb{N}$ to $\alpha > 0$ in (1) for $A \in \{B, F\}$, $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ is covered by the recent paper [9] in the larger context of convolution inequalities in these spaces. What follows may be considered as a surprising simple proof of these assertions relying on Theorem 2 and a few well–known properties of the spaces $A^s_{p,q}(\mathbb{R}^n)$ as introduced in Definition 1.

Theorem 3 *Let W^α_t be as in (2). Let $A \in \{B, F\}$, $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Let $d \ge 0$. Then there is a constant $c > 0$ such that for all t with $0 < t \le 1$ and all $w \in A^s_{p,q}(\mathbb{R}^n)$,*

$$t^{\frac{d}{2\alpha}} \, \| W^\alpha_t w \,|A^{s+d}_{p,q}(\mathbb{R}^n)\| \le c \|w\,|A^s_{p,q}(\mathbb{R}^n)\|. \tag{47}$$

Proof *Step 1.* Let $s > 0$ and let $w \in B^s_{p,q}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$. We put

$$\|w\,|\overset{*}{B}{}^s_{p,q}(\mathbb{R}^n)\| = \left(\int_0^\infty t^{(k-\frac{s}{2\alpha})q} \left\| \partial^k_t W^\alpha_t w \,|L_p(\mathbb{R}^n)\right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{48}$$

for the second summand on the right–hand side of (37). According to (2) and (34) we have $f = W^\alpha_\tau w \in L_p(\mathbb{R}^n)$ (see also Remark 3) and it holds

$$\partial^k_t W^\alpha_t f(x) = \partial^k_t \left(e^{-t|\xi|^{2\alpha}} \widehat{f} \right)^\vee (x) = (-1)^k \left(|\xi|^{2\alpha k} e^{-t|\xi|^{2\alpha}} \widehat{f} \right)^\vee (x)\,. \tag{49}$$

Inserting

$$f = W_\tau^\alpha w = \left(e^{-\tau|\xi|^{2\alpha}}\widehat{w}\right)^\vee$$

we get

$$\begin{aligned}\partial_t^k W_t^\alpha(W_\tau^\alpha w)(x) &= (-1)^k\left(|\xi|^{2\alpha k}e^{-t|\xi|^{2\alpha}}e^{-\tau|\xi|^{2\alpha}}\widehat{w}(\xi)\right)^\vee(x)\\ &= (-1)^k\left(|\xi|^{2\alpha k}e^{-(t+\tau)|\xi|^{2\alpha}}\widehat{w}(\xi)\right)^\vee(x) = \left(\partial_t^k e^{-(t+\tau)|\xi|^{2\alpha}}\widehat{w}(\xi)\right)^\vee(x)\\ &= \partial_t^k W_{t+\tau}^\alpha w(x)\,. \end{aligned} \tag{50}$$

Note that (49) is well defined due to (34). Combining (48) and (50) one obtains

$$\tau^{\frac{d}{2\alpha}}\|W_\tau^\alpha w\,|\overset{*}{B}{}^{s+d}_{p,q}(\mathbb{R}^n)\| = \left(\int_0^\infty \tau^{\frac{d}{2\alpha}q}t^{(k-\frac{s+d}{2\alpha})q}\left\|\partial_t^k W_{t+\tau}^\alpha w\,|L_p(\mathbb{R}^n)\right\|^q\frac{\mathrm{d}t}{t}\right)^{1/q}. \tag{51}$$

Let $d \geq 0$ and let $\frac{s+d}{2\alpha}+\frac{1}{q} < k \in \mathbb{N}$. Then $a = k-\frac{s}{2\alpha}-\frac{1}{q} > \frac{d}{2\alpha}$,

$$0 \leq \varkappa = \frac{d}{2\alpha a} < 1 \quad \text{and} \quad \left(k-\frac{s+d}{2\alpha}-\frac{1}{q}\right)\frac{1}{a} = 1-\varkappa. \tag{52}$$

Then it follows from $\tau^\varkappa t^{1-\varkappa} \leq \tau + t$ that for $1 \leq q < \infty$

$$\tau^{\frac{d}{2\alpha}q}\,t^{(k-\frac{s+d}{2\alpha})q-1} \leq (\tau+t)^{(k-\frac{s}{2\alpha})q-1} \tag{53}$$

(modification if $q = \infty$). Inserted in (51) one obtains

$$\tau^{\frac{d}{2\alpha}}\|W_\tau^\alpha w\,|\overset{*}{B}{}^{s+d}_{p,q}(\mathbb{R}^n)\| \leq \|w\,|\overset{*}{B}{}^{s}_{p,q}(\mathbb{R}^n)\|. \tag{54}$$

As far as the first terms on the right–hand side of (37) are concerned it is sufficient to justify

$$\left\|\left(e^{-\tau|\xi|^{2\alpha}}\widehat{w}\right)^\vee|L_p(\mathbb{R}^n)\right\| \leq c\,\|w\,|L_p(\mathbb{R}^n)\| \tag{55}$$

for some $c > 0$ and all $0 < \tau \leq 1$. Recall that $1 \leq p \leq \infty$. Then (55) follows from

$$\int_{\mathbb{R}^n}\left|\left(e^{-|\lambda\xi|^{2\alpha}}\right)^\vee(x)\right|\mathrm{d}x \leq C \tag{56}$$

for some $C > 0$ and all $0 < \lambda < \infty$ what in turn can be obtained from (32), (34) and

$$\int\limits_{\mathbb{R}^n} \left| \left(e^{-|\lambda\xi|^{2\alpha}} \right)^\vee (x) \right| \mathrm{d}x = \lambda^{-n} \int\limits_{\mathbb{R}^n} \left| \left(e^{-|\xi|^{2\alpha}} \right)^\vee (\lambda^{-1}x) \right| \mathrm{d}x. \tag{57}$$

Now (47) can be obtained for $B^s_{p,q}(\mathbb{R}^n)$ with $s > 0$ and $1 \le p, q \le \infty$ from (37), (54) and (55).
Next we consider the case of F–spaces. Let $s > 0$ and let $\omega \in F^s_{p,q}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$, where $1 \le p < \infty$. We put

$$\|w \,|\overset{*}{F}{}^s_{p,q}(\mathbb{R}^n)\| = \left\| \left(\int_0^\infty t^{(k-\frac{s}{2\alpha})q} \left| \partial_t^k W_t^\alpha w(\cdot) \right|^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{58}$$

Again it holds (50). The counterpart of (51) reads as

$$\tau^{\frac{d}{2\alpha}} \|W_t^\alpha w \,|\overset{*}{F}{}^{s+d}_{p,q}(\mathbb{R}^n)\| = \left\| \left(\int\limits_0^\infty \tau^{\frac{d}{2\alpha}q} t^{(k-\frac{s+d}{2\alpha})q} \left| \partial_t^k W_t^\alpha w(\cdot) \right|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \middle| L_p(\mathbb{R}^n) \right\|, \tag{59}$$

where $d \ge 0$ and $\frac{s+d}{2\alpha} + \frac{1}{q} < k \in \mathbb{N}$. By the same arguments as in the proof of (54) one obtains

$$\tau^{\frac{d}{2\alpha}} \|W_t^\alpha w \,|\overset{*}{F}{}^{s+d}_{p,q}(\mathbb{R}^n)\| \le \|w \,|\overset{*}{F}{}^s_{p,q}(\mathbb{R}^n)\|. \tag{60}$$

Now (47) for $A = F$ is a consequence of (38), (55) and (60).
Step 2. Recall

$$I_\sigma A^s_{p,q}(\mathbb{R}^n) = A^{s+\sigma}_{p,q}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad \sigma \in \mathbb{R} \quad \text{and} \quad 0 < p, q \le \infty, \tag{61}$$

$A \in \{B, F\}$, where

$$I_\sigma f = \left(w_{-\sigma} \widehat{f} \right)^\vee, \qquad f \in S'(\mathbb{R}^n), \tag{62}$$

is the well–known lift based on $w_\delta(x) = (1 + |x|^2)^{\delta/2}$, $x \in \mathbb{R}^n$, $\delta \in \mathbb{R}$, [20, Section 1.3.2, p. 16] and the references given there. By definition of I_σ and W_t^α it is not difficult to see that

$$W_t^\alpha f = I_{-\sigma} \left(W_t^\alpha (I_\sigma f) \right) \tag{63}$$

if $f \in A^s_{p,q}(\mathbb{R}^n)$, $1 \le p < \infty$, $s > 0$ and $\sigma > 0$ (see Remark 4 below). If $s \le 0$ and $f \in A^s_{p,q}(\mathbb{R}^n)$ then we take (63) as definition of $W_t^\alpha f$, where σ is chosen such that $s + \sigma > 0$. Then one can extend (47) from the spaces $A^s_{p,q}(\mathbb{R}^n)$, $s > 0$ and $1 \le p, q \le \infty$ ($p < \infty$ for F–spaces) treated in Step 1 to their counterparts with $s \le 0$.

This covers all spaces in the above theorem with exception of $F^s_{\infty,q}(\mathbb{R}^n)$, $1 \le q < \infty$.

Step 3. Let $\overset{\circ}{F}{}^s_{p,q}(\mathbb{R}^n)$, $1 \le q \le \infty$, be the completion of $S(\mathbb{R}^n)$ in $F^s_{p,q}(\mathbb{R}^n)$. Then one has

$$\overset{\circ}{F}{}^s_{1,q}(\mathbb{R}^n)' = F^{-s}_{\infty,q'}(\mathbb{R}^n), \qquad 1 \le q \le \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1 \quad \text{and} \quad s \in \mathbb{R}, \tag{64}$$

for the related dual spaces in the framework of the dual pairing $\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$. This is a special case of [20, (1.25), p. 5] with a reference to [10, Theorem 4, p. 87] as far as the case $q = \infty$ is concerned (if $q < \infty$ then $S(\mathbb{R}^n)$ is already dense in $F^s_{1,q}(\mathbb{R}^n)$ and the related duality is well known, [20, p. 5] and the references there). If $\varphi \in S(\mathbb{R}^n)$ then $W^\alpha_t \varphi$ can be approximated in, say, $F^{s+1}_{1,1}(\mathbb{R}^n)$ by functions belonging to $S(\mathbb{R}^n)$ for any s. But then it follows by embedding that this is also an approximation in $F^s_{1,\infty}(\mathbb{R}^n)$ for any s. In particular one has by Step 2

$$W^\alpha_t : \quad \overset{\circ}{F}{}^s_{1,q}(\mathbb{R}^n) \hookrightarrow \overset{\circ}{F}{}^{s+d}_{1,q}(\mathbb{R}^n), \qquad 1 \le q \le \infty, \quad s \in \mathbb{R}, \quad d > 0. \tag{65}$$

The operator W^α_t is self–dual, $(W^\alpha_t)' = W^\alpha_t$. Then (47) with $A^s_{p,q}(\mathbb{R}^n) = F^s_{\infty,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 \le q < \infty$, follows from (64), (65). □

Remark 4 We justify (63). Let $f \in A^s_{p,q}(\mathbb{R}^n)$, $1 \le p < \infty$, $s > 0$ and $\sigma > 0$. Without loss of generality we may assume $\sigma < \infty$ (otherwise one replaces s by $s - \varepsilon$ with $0 < \varepsilon < s$). If $f \in S(\mathbb{R}^n)$ then

$$\mathcal{F}(W^\alpha_t I_\sigma f)(\xi) = e^{-t|\xi|^{2\alpha}}(1 + |\xi|^2)^{-\sigma/2}(\mathcal{F}f)(\xi) \tag{66}$$

is well defined pointwise and belongs to $S'(\mathbb{R}^n)$. Hence,

$$I_{-\sigma}(W^\alpha_t I_\sigma f) = \mathcal{F}^{-1} e^{-t|\xi|^{2\alpha}} \mathcal{F} f = W^\alpha_t f. \tag{67}$$

for all $f \in S(\mathbb{R}^n)$. Then it follows (63) for all $f \in A^s_{p,q}(\mathbb{R}^n)$ by (47) (with $d = 0$), the lift property (47) and density of $S(\mathbb{R}^n)$ in $A^s_{p,q}(\mathbb{R}^n)$.

Remark 5 We observe that $\| f | \overset{*}{A}{}^s_{p,q}(\mathbb{R}^n)\|$ is an equivalent norm in $\dot{A}^s_{p,q}(\mathbb{R}^n)$ if $2\alpha k > s$ for Besov spaces and $2\alpha k > s + n$ for Triebel-Lizorkin spaces. This is a direct consequence of Remark 1 and Theorem 2.

3 Nonlinear fractional heat equations

In [3] we dealt with the Cauchy problem (4), (5) where $\alpha > 0$ is a natural number. The case $\alpha = 1$ corresponds to a classical non-linear heat equation. We established mild and strong solutions in appropriate function spaces $L_v\,((0, T), b, X) \cap C^\infty(\mathbb{R}^n \times (0, T))$ being fixed points of the operator T_{u_0} defined in (7).

The aim of this section is to extend some of these results to the case of fractional powers $\alpha > 1/2$. This restriction results from the mapping properties of the nonlinearity Du^2 in $A^s_{p,q}$–spaces, see Proposition 5 below. In particular, we make use of the smoothing properties formulated in Theorem 3.

For later purposes we recall multiplication properties in the respective spaces $A^s_{p,q}(\mathbb{R}^n)$ derived in [3] including $p = \infty$ for F-spaces.

Proposition 4 *Let $1 \le p, q \le \infty$ and $(\frac{n}{p} - \frac{n}{2})_+ < s < \infty$. Let $A \in \{B, F\}$. Then it holds*

$$\| f \cdot g | A^{s-(\frac{n}{p}-s)_+-\varepsilon}_{p,q}(\mathbb{R}^n)\| \le c\, \| f | A^s_{p,q}(\mathbb{R}^n)\| \cdot \| g | A^s_{p,q}(\mathbb{R}^n)\| \tag{68}$$

for all $f, g \in A^s_{p,q}(\mathbb{R}^n)$ and all $\varepsilon > 0$.

Proof Spaces $A^s_{p,q}(\mathbb{R}^n)$ ($p < \infty$ for F-spaces) are multiplication algebras if $s > \frac{n}{p}$. For $F^s_{\infty,q}(\mathbb{R}^n)$, $s > 0$ this property follows from [20, Thm. 2.41]. Then (68) holds with $\varepsilon \ge 0$. If $s = \frac{n}{p}$ assertion (68) holds due to [3, Prop. 2.3] and embedding (2.24) in [20]. Finally, if $s < \frac{n}{p}$ then (68) is a consequence of [3, Prop. 2.1] as well as the Sobolev-type embeddings in [3, Prop. 2.2]. □

Corollary 1 *Let $1 \le p, q \le \infty$, $(\frac{n}{p} - \frac{n}{2})_+ < s < \infty$ and $\sigma < s - 1 - (\frac{n}{p} - s)_+$. Let $A \in \{B, F\}$. Then it holds*

$$\| D(f \cdot g) | A^\sigma_{p,q}(\mathbb{R}^n)\| \le c\, \| f | A^s_{p,q}(\mathbb{R}^n)\| \cdot \| g | A^s_{p,q}(\mathbb{R}^n)\| \tag{69}$$

for all $f, g \in A^s_{p,q}(\mathbb{R}^n)$.

Proof Clearly, we have

$$\| D(f \cdot g) | A^\sigma_{p,q}(\mathbb{R}^n)\| \le c\, \| f \cdot g | A^{\sigma+1}_{p,q}(\mathbb{R}^n)\|.$$

Thus, (69) follows from Proposition 4 and $\sigma + 1 < s - \left(\frac{n}{p} - s\right)_+$. □

Next we derive an estimate of $\mathcal{T}_{u_0}$ as defined in (7) for fixed $t > 0$ in appropriate function spaces $A^s_{p,q}(\mathbb{R}^n)$.

Proposition 5 *Let $2 \le n \in \mathbb{N}$, $1 \le p \le \infty$, $1 \le q \le \infty$, $\left(\frac{n}{p} - \frac{n}{2}\right)_+ < s < \infty$ and let $\alpha > 1/2$. Let $T > 0$ and let $a,\ v,\ d$ such that*

$$\frac{1}{\alpha} < v \le \infty, \quad -\infty < a + \frac{1}{v} < \alpha, \quad 1 + \left(\frac{n}{p} - s\right)_+ < d < 2\left(\alpha - \frac{1}{v}\right). \tag{70}$$

If

$$u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n) \ \ \text{with} \ \ s_0 \le s \ \ \text{and} \ \ u \in L_{2\alpha v}\left((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right) \tag{71}$$

then there exists a constant $c > 0$, independent of u_0 and u, such that

$$\|\mathcal{T}_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \leq c\,t^{-\frac{s-s_0}{2\alpha}}\|u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| \\ +c\,t^{1-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}\|u|L_{2\alpha v}\left((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)\|^2. \tag{72}$$

for all t with $0 < t < T$ (with $\frac{1}{v} = 0$ and the modification (9) if $v = \infty$).

Proof Note that condition (70) with respect to d implies $\alpha > \frac{1}{2}$. Using Theorem 3 and Corollary 1 with $s-d$ in place of σ we can estimate as follows

$$\begin{aligned}
&\|\mathcal{T}_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\
&\leq \|W^\alpha_t u_0|A^s_{p,q}(\mathbb{R}^n)\| + \int_0^t \|W^\alpha_{t-\tau}Du^2(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau \\
&\lesssim t^{-\frac{s-s_0}{2\alpha}}\|u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| + \int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\|Du^2(\cdot,\tau)|A^{s-d}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau \\
&\lesssim t^{-\frac{s-s_0}{2\alpha}}\|u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| + \int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\|u(\cdot,\tau)|A^{s}_{p,q}(\mathbb{R}^n)\|^2\,\mathrm{d}\tau.
\end{aligned} \tag{73}$$

Here we used that

$$\sigma = s - d < s - 1 - \left(\frac{n}{p} - s\right)_+$$

according to (70). By means of Hölder's inequality with exponent $\alpha v > 1$ we obtain

$$\int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\tau^{-\frac{a}{\alpha}}\tau^{+\frac{a}{\alpha}}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2\,\mathrm{d}\tau \\ \lesssim t^{1-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}\|u|L_{2\alpha v}\left((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)\|^2. \tag{74}$$

Here we used the conditions $a + \frac{1}{v} < \alpha$ as well as $d < 2(\alpha - \frac{1}{v})$ to ensure that the integral is finite. □

Theorem 6 *Let $2 \leq n \in \mathbb{N}$, $\frac{1}{2} < \alpha < \infty$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and let $A \in \{B, F\}$. Let*

$$\frac{n}{p} - 2\alpha + 1 < s_0 \tag{75}$$

and

$$\left(\frac{n}{p}-\frac{n}{2}\right)_+ < s_0+\alpha. \tag{76}$$

Let $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$.
(i) *Then there exists a number* $T > 0$ *such that the Cauchy problem (4),(5) has a unique mild solution u belonging to*

$$L_{2\alpha v}\left((0,T),\ \tfrac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right),$$

for all s satisfying

$$s_0 \le s < s_0+\min(\alpha,\ 2\alpha-1) \tag{77}$$

and

$$s > \max\left(\left(\frac{n}{p}-\frac{n}{2}\right)_+,\ \left(\frac{n}{p}-2\alpha+1\right)_+\right), \tag{78}$$

where $a,\ v$ *such that*

$$0 \le \frac{1}{v} < \frac{1}{2}\left(2\alpha-1-\left(\frac{n}{p}-s\right)_+\right) \tag{79}$$

and

$$s-s_0 < a+\frac{1}{v} < \min\left(\alpha,\ 2\alpha-1-\left(\frac{n}{p}-s\right)_+\right). \tag{80}$$

(ii) *The mild solution u obtained in part (i) also belongs to the space* $L_\infty((0,T),$ $A^{s_0}_{p,q}(\mathbb{R}^n))$. *Moreover, if, in addition,* $\max(p,q) < \infty$ *then the above solution* $u(\cdot)$ *converges to* u_0 *with respect to the norm in* $A^{s_0}_{p,q}(\mathbb{R}^n)$ *if* $t \to 0+$.

Proof First we observe that assumptions (75) and (77) imply that

$$0 \le s-s_0 < \min\left(\alpha,\ 2\alpha-1-\left(\frac{n}{p}-s\right)_+\right).$$

Hence, conditions (79) and (80) make sense. *Step 1.* We choose d such that

$$1+\left(\frac{n}{p}-s\right)_+ < d < \min\left(2\alpha-\left(a+\frac{1}{v}\right),\ 2\left(\alpha-\frac{1}{v}\right)\right)$$

which is possible due to conditions (79) and (80). Then it follows from Proposition 5 that

$$\|\mathcal{T}_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \le c\,t^{-\frac{s-s_0}{2\alpha}}\,\|u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| \\ + c\,t^{1-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}\,\|u|L_{2\alpha v}\left((0,T),\ \tfrac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)\|^2.$$

for $0 < t < T$. We multiply both sides with $t^{\frac{a}{2\alpha}}$. Raising to the power of $2\alpha v$ and integrating over $(0,T)$ yield

$$\int_0^T t^{av}\|\mathcal{T}_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}t \\ \le c\,T^{\delta}\,\|u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\|^{2\alpha v} + T^{\varkappa}\,\|u|L_{2\alpha v}\left((0,T),\ \tfrac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)\|^{4\alpha v} \tag{81}$$

with

$$\delta = (a - s + s_0)v + 1 > 0, \quad \text{since } s - s_0 < a + \frac{1}{v} \tag{82}$$

and

$$\varkappa = \left(2\alpha - \frac{1}{v} - d - a\right)v > 0, \quad \text{since } d < 2\alpha - \left(a + \frac{1}{v}\right). \tag{83}$$

Thus, $\mathcal{T}_{u_0}$ maps the unit ball U_T in $L_{2\alpha v}\left((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)$ into itself if T is sufficiently small.

As for the contraction property consider $u, v \in U_T$. A similar calculation with d as above (cf. also (73) and (74)) yields

$$\|\mathcal{T}_{u_0}u(\cdot,t) - \mathcal{T}_{u_0}v(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\ \le c\,t^{1-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t \tau^{\alpha v}\|u^2(\cdot,\tau) - v^2(\cdot,\tau)|A^{s-d+1}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau\right)^{1/\alpha v}.$$

Application of Proposition 4 leads in combination with Hölder's inequality to

$$\|\mathcal{T}_{u_0}u(\cdot,t) - \mathcal{T}_{u_0}v(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \le c\,t^{1-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}} \\ \times\left(\int_0^t \tau^{av}\|u(\cdot,\tau) - v(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/2\alpha v} \\ \times\left(\int_0^t \tau^{av}\|u(\cdot,\tau) + v(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{\alpha v}\,\mathrm{d}\tau\right)^{1/2\alpha v} \tag{84}$$

Let temporarily $X_T^s = L_{2\alpha v}\left((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)$, then it follows from (84)

$$\|\mathcal{T}_{u_0}u - \mathcal{T}_{u_0}v|X_T^s\| \le c\, T^{\frac{\varkappa}{2\alpha v}} \|u+v|X_T^s\|\,\|u-v|X_T^s\| \tag{85}$$

with the same $\varkappa$ as in (83). If $T > 0$ is small enough, then $\mathcal{T}_{u_0} : U_T \mapsto U_T$ is a contraction. Since we deal with Banach spaces we have shown that Tu has a unique fixed point in U_T and hence a mild solution of the Cauchy problem (4), (5).

To extend the uniqueness to the whole space we proceed similarly to e.g. [11, 21]. Let $u \in U_T$ be the above solution and $v \in X_T^s$ a second solution. We observe that (85) holds for any $0 < t \le T_0 \le T$. With $u \in U_T$ we obtain

$$\|u - v|X^s_{T_0}\| \le c\, T_0^{\varkappa}\,(1 + \|v|X_T^s\|)\|u - v|X^s_{T_0}\|. \tag{86}$$

If we choose $T_0 > 0$ small enough such that $c\,T_0^{\varkappa}\,(1+\|v|X_T^s\|) < 1$ it follows that $u(\cdot,t) = v(\cdot,t)$ for any $t \in (0,T_0]$. Now we take $u(\cdot,T_0) \in A^s_{p,q}(\mathbb{R}^n) \hookrightarrow A^{s_0}_{p,q}$ as new initial value and proceed as in the previous steps until (86) inclusively. There exists a unique solution $\widetilde{u}$ in a neighbourhood $U_\delta(T_0)$ with $\widetilde{u}(\cdot,T_0) = u(\cdot,T_0)$. Since it holds that $\widetilde{u}(\cdot,t) = u(\cdot,t)$ for all $t \in (0,T_0] \cap U_\delta(T_0)$ we have extended u to some interval $(0,T_1]$ with $T_0 < T_1$. Thus, we have prolongated $u(\cdot,t) - v(\cdot,t) = 0$ to some interval $(0,T_1]$ where $T_0 < T_1 \le T$ By iteration it follows the uniqueness in $L_{2\alpha v}\left((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)$.

Step 2. We show part (ii) of the theorem. To this end we first prove that the mild solution obtained in Step 1 belongs to $L_\infty((0,T), A^{s_0}_{p,q}(\mathbb{R}^n))$.

Let $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$ and let $u \in L_\infty\left((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\right)$ be the corresponding solution, where s and a satisfy (77), (78) and (80), where $\frac{1}{v}$ is replaced by 0. Let $0 < t < T$. It holds

$$\|u(\cdot,t)|A^{s_0}_{p,q}(\mathbb{R}^n)\| \le \|W^\alpha_t u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| + \int_0^t \|W^\alpha_{t-\tau} Du^2(\cdot,\tau)|A^{s_0}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau.$$

Taking into account (63) and the lift property (61) we may assume $s_0 > 0$. Concerning the first summand we obtain

$$\begin{aligned}
&\|W^\alpha_t u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| \\
&\quad\lesssim \Big\|\int_{\mathbb{R}^n} \left(e^{-t|\xi|^{2\alpha}}\right)^\vee (x-y)\, u_0(y)\,\mathrm{d}y|A^{s_0}_{p,q}(\mathbb{R}^n)\Big\| \\
&\quad= \Big\|\int_{\mathbb{R}^n} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z)\, u_0(x - t^{1/2\alpha}z)\,\mathrm{d}z|A^{s_0}_{p,q}(\mathbb{R}^n)\Big\| \\
&\quad\le \int_{\mathbb{R}^n} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z)\|u_0(x - t^{1/2\alpha}z)|A^{s_0}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}z
\end{aligned} \tag{87}$$

$$\lesssim \|u_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| \int\limits_{\mathbb{R}^n} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z)\,\mathrm{d}z < \infty \tag{88}$$

independent of t, where (87) follows from the generalized Minkowski inequality for Banach spaces and (88) from the translation invariance of $A^s_{p,q}(\mathbb{R}^n)$– spaces (see also Remark 1) and (34).

In order to estimate the second summand we first consider the case that $s - s_0 \leq 1 + \left(\frac{n}{p} - s\right)_+$ and we put

$$d := 1 + \left(\frac{n}{p} - s\right)_+ - (s - s_0) + \varepsilon,$$

where ε is chosen such that $0 < \varepsilon < 2\alpha - 1 - (\frac{n}{p} - s)_+ - (s - s_0)$ according to (80). Then, $d > 0,\ s - s_0 < \alpha - \frac{d}{2}$, and we may choose a such that

$$0 < s - s_0 < a < \min\left(\alpha - \frac{d}{2},\ 2\alpha - 1 - \left(\frac{n}{p} - s\right)_+\right).$$

Applying Theorem 3 with $s_0 - d$ in place of s and Corollary 1 with $\sigma = s_0 - d < s - 1 - \left(\frac{n}{p} - s\right)_+$ we obtain

$$\begin{aligned}
&\int\limits_0^t \|W^\alpha_{t-\tau} Du^2(\cdot,\tau)|A^{s_0}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau \\
&\quad\leq c \int\limits_0^t (t-\tau)^{-\frac{d}{2\alpha}} \|Du^2(\cdot,\tau)|A^{s_0-d}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau \\
&\quad\leq c \int\limits_0^t (t-\tau)^{-\frac{d}{2\alpha}} \|u(\cdot,\tau)|A^{s}_{p,q}(\mathbb{R}^n)\|^2\mathrm{d}\tau \\
&\quad\leq c\, t^{-\frac{d}{2\alpha}-\frac{a}{\alpha}+1} \left(\sup_{0<\tau<T} \tau^{\frac{a}{2\alpha}} \|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\right)^2
\end{aligned} \tag{89}$$

because of $0 < d < 2\alpha$ and $a < \alpha$.

If $s - s_0 > 1 + \left(\frac{n}{p} - s\right)_+$ then we can apply Theorem 3 with $d = 0$ and Corollary 1 with s_0 in place of σ to get

$$\begin{aligned}
\int\limits_0^t \|W^\alpha_{t-\tau} Du^2(\cdot,\tau)|A^{s_0}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau &\leq c \sup_{0<t<T} \int\limits_0^t \|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2\,\mathrm{d}\tau \\
&\leq c\, t^{-\frac{a}{\alpha}+1} \left(\sup_{0<\tau<T} \tau^{\frac{a}{2\alpha}} \|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\right)^2.
\end{aligned} \tag{90}$$

The boundedness of $u(t,\cdot)$ on $(0,T)$ in $A^{s_0}_{p,q}(\mathbb{R}^n)$ follows from (88), (89) (due to $a < \alpha - \frac{d}{2}$), and (90) (because of $a < \alpha$).

Next we consider the limit of $u(t,\cdot)$ in $A^{s_0}_{p,q}(\mathbb{R}^n)$ if $t \to 0+$. It holds

$$\begin{aligned} &\|u(\cdot,t) - u_0 | A^{s_0}_{p,q}(\mathbb{R}^n)\| \\ &\quad \le \|W^\alpha_t u_0 - u_0 | A^{s_0}_{p,q}(\mathbb{R}^n)\| + \int_0^t \|W^\alpha_{t-\tau} D u^2(\cdot,\tau) | A^{s_0}_{p,q}(\mathbb{R}^n)\| \mathrm{d}\tau. \end{aligned}$$

The second summand on the right-hand side tends to zero if $t \to 0+$ as a consequence of (89), (90) and the conditions with respect to $a,\ \alpha$, and d. Using the identity

$$u_0(x) = u_0(x) \cdot \left(\left(e^{-t|\xi|^{2\alpha}}\right)^\vee\right)^\wedge(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(e^{-|t\xi|^{2\alpha}}\right)^\vee (x-y) u_0(x)\,\mathrm{d}y$$

we obtain the estimate

$$\begin{aligned} &\|W^\alpha_t u_0 - u_0 | A^{s_0}_{p,q}(\mathbb{R}^n)\| \\ &\quad \lesssim \int_{|z|>N} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z) \|u_0(x - t^{1/2\alpha} z) - u_0(x) | A^{s_0}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}z \qquad (91) \\ &\quad + \int_{|z|\le N} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z) \|u_0(x - t^{1/2\alpha} z) - u_0(x) | A^{s_0}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}z. \qquad (92) \end{aligned}$$

The first summand is lower than ε if we choose N large enough. Fixing this N the second summand tends to zero for t tending to zero. This follows from the fact that the Schwartz space $S(\mathbb{R}^n)$ is dense in $A^s_{p,q}(\mathbb{R}^n)$ if $\max(p,q) < \infty$ and the continuity of the translation (See also [5, Subsection 1.2.d] for more details with respect to approximate identities). This completes the proof. □

In addition to the results of the previous part one may ask for well-posedness of the Cauchy problem. The notation well-posedness is not totally fixed in the literature (see the comments in [17, Subsection 6.2.5]) We adapt the standard notation, see e.g. [4]. The Cauchy problem is called locally well-posed if there exists a unique mild and strong solution according to Theorem 6. In addition it is required continuous dependence of the solutions with respect to initial data. This means that for solutions u_1 and u_2 of (4), (5) according to Theorem 6 with respect to initial data u_0^1 and u_0^2, respectively, for any $\varepsilon > 0$ there exists a $\delta > 0$ and a time $T > 0$ such that for all $0 < t < T$

$$\|u_1(\cdot,t) - u_2(\cdot,t) | A^{s_0}_{p,q}(\mathbb{R}^n)\| \le \varepsilon \qquad (93)$$

holds if

$$\|u_0^1 - u_0^2 | A^{s_0}_{p,q}(\mathbb{R}^n)\| \le \delta\,. \qquad (94)$$

It is sufficient to consider solutions $u \in L_\infty\Big((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\Big)$. Recall that by construction of the solution as a fixed point of $\mathcal{T}_{u_0}$ we may assume $\|u|L_\infty\Big((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\Big)\| \le 1$.

Theorem 7 *Let $u_i \in L_\infty((0,T_i),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n))$ $(i=1,2)$ be solutions of (4), (5) obtained in Theorem 6 with initial data $u_0^i \in A^{s_0}_{p,q}(\mathbb{R}^n)$ in the corresponding time interval $(0,T_i)$. Let $\max(p,q) < \infty$. Then under the conditions of Theorem 6 the Cauchy problem (4), (5) is locally well-posed.*

Proof Let u_1, u_2 be two solutions of (4), (5) with corresponding initial data u_0^1, u_0^2. We have

$$\begin{aligned}&\|u_1(\cdot,t)-u_2(\cdot,t)|A^{s_0}_{p,q}(\mathbb{R}^n)\|\\&\quad\le \|W^\alpha_t(u_0^1-u_0^2)|A^{s_0}_{p,q}(\mathbb{R}^n)\| + \int_0^t \|W^\alpha_{t-\tau}\, D(u_1^2-u_2^2)(\cdot,\tau)|A^{s_0}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau\end{aligned} \tag{95}$$

To estimate the first summand of the right-hand side we use again Theorem 3 with $d=0$. The second summand can be treated in the same way as in Step 2 of the proof of Theorem 6 with u^1-u^2 in place of u. Note, that $u^1 \in L_\infty((0,T_1),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n))$ and $u^2 \in L_\infty((0,T_2),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n))$. Hence, application of Minkowski's inequality leads to

$$\|u_1(\cdot,t)-u_2(\cdot,t)|A^{s_0}_{p,q}(\mathbb{R}^n)\| \le c\,\|u_0^1-u_0^2|A^{s_0}_{p,q}(\mathbb{R}^n)\| + c\,t^{-\frac{d}{2\alpha}-\frac{a}{\alpha}+1} \tag{96}$$

with the same choice of $d \ge 0$ as in step 2 of Theorem 6 and for all $0 < t < T$ with $T \le \min(T_1,T_2)$. Then the right hand side in (96) is lower the the given ε if this T is chosen small enough. □

4 Comments and special cases

As already mentioned in the introduction our approach allows to deal with the Cauchy problem (4), (5) for initial data u_0 belonging to spaces $A^{s_0}_{p,q}(\mathbb{R}^n)$ with smoothness s_0 satisfying the a-priori condition (75), i. e. $s_0 > \frac{n}{p} - 2\alpha + 1$.

It refers to the so-called supercritical case, where the existence of local (small T) solutions can be expected. For a detailed discussion of (sub/super)critical spaces in the context of Navier–Stokes and the related scalar nonlinear heat equation (4) we refer to [18, Subsection 5.5] ($\alpha = 1$), [3, Subsection 3.2] ($\alpha \in \mathbb{N}$) and the references given there. The arguments can be adapted to the case of fractional $\alpha > 1/2$.

The second a-priori condition (76), i. e. $\left(\frac{n}{p}-\frac{n}{2}\right)_+ < s_0+\alpha$ is due to the mapping properties of the nonlinearity Du^2 in (4) and relevant if $\alpha > 1$. A breaking point is $\alpha = \frac{n}{2}+1$. If $1 < \alpha \le \frac{n}{2}+1$ then we obtain a dependence on the parameter p whether or not all supercritical spaces are admitted. If $\alpha > \frac{n}{2} + 1$ then the supercritical case can never be completely covered by our method.

A further notable exponent is $\alpha = \frac{n+2}{4}$ ($\alpha = \frac{5}{4}$ if $n = 3$). In this case we have the coincidence $\frac{n}{p} - 2\alpha + 1 = \frac{n}{p} - \frac{n}{2}$ resulting in consequences with respect to spatial smoothness s of our solution spaces $L_{2\alpha v}\left((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)\right)$. All these aspects will be discussed in more detail in the following.

We always assume $n \in \mathbb{N}$, $n \geq 2$, $\alpha > \frac{1}{2}$, $1 \leq p, q \leq \infty$, $A \in \{B,\ F\}$ and $a,\ v$ as in (79) and (80). In the figures below the area of admitted s_0 is shaded, that one of s is hatched.

Remark 6 The case $\frac{1}{2} < \alpha \leq 1$. Let $s_0 > \frac{n}{p} - 2\alpha + 1$ and let $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$. Then (76) is satisfied and (77) and (78) read as

$$s_0 \leq s < s_0 + 2\alpha - 1 \quad \text{and} \quad s > \left(\frac{n}{p} - 2\alpha + 1\right)_+ ,$$

respectively.

(i) If $\frac{n}{p} - 2\alpha + 1 < s_0 \leq 0$ and $\frac{n}{2\alpha-1} < p \leq \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)\right)$, where $0 < s < s_0 + 2\alpha - 1$.

(ii) If $s_0 > \left(\frac{n}{p} - 2\alpha + 1\right)_+$ and $1 \leq p \leq \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)\right)$, where $s_0 \leq s < s_0 + 2\alpha - 1$.

This is well-known in the case $\alpha = 1$ (see, for example, [1, 2], and [18, Subsection 4.4]). Supercritical spaces $A^{s_0}_{p,q}(\mathbb{R}^n)$ are completely covered for all p, $1 \leq p \leq \infty$ (see Fig. 1). In particular, initial data $u_0 \in A^0_{p,q}(\mathbb{R}^n)$ are admitted if $\frac{n}{2\alpha-1} < p \leq \infty$.

Remark 7 The case $1 < \alpha \leq \frac{n+2}{4}$.

This case implies $n > 2$ and (75) as well as (76) are satisfied if

$$s_0 > \left(\frac{n}{p} - \alpha + 1\right)_+ - \alpha.$$

Conditions (77) and (78) read as

$$s_0 \leq s < s_0 + \alpha \quad \text{and} \quad s > \left(\frac{n}{p} - 2\alpha + 1\right)_+ ,$$

respectively.

(i) If $\left(\frac{n}{p} - \alpha + 1\right)_+ - \alpha < s_0 \leq 0$ and $\frac{n}{2\alpha-1} < p \leq \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)\right)$, where $0 < s < s_0 + \alpha$.

(ii) If $s_0 > \left(\frac{n}{p} - 2\alpha + 1\right)_+$ and $1 \leq p \leq \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)\right)$, where $s_0 \leq s < s_0 + \alpha$.

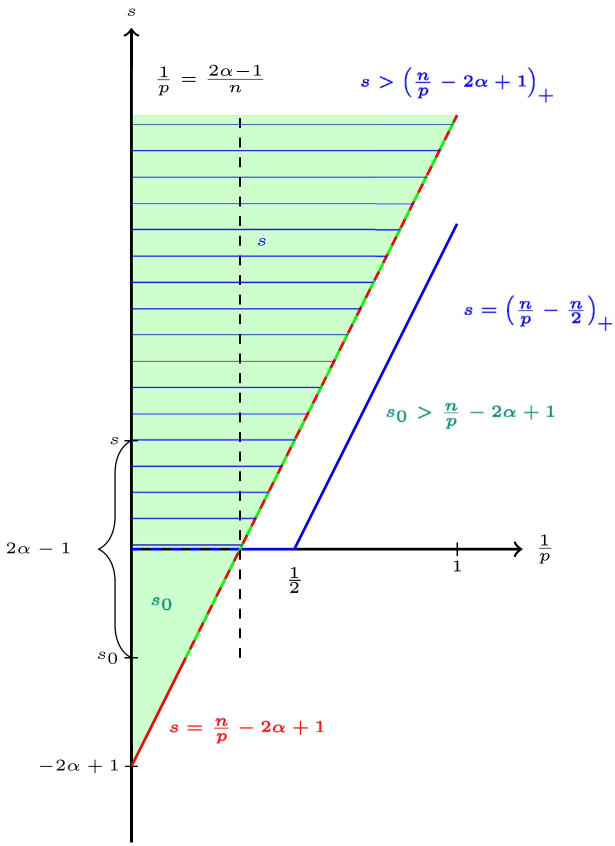

Fig. 1 $\frac{1}{2} < \alpha \leq 1$

Supercritical spaces $A^{s_0}_{p,q}(\mathbb{R}^n)$ are completely covered if $1 \leq p \leq \frac{n}{\alpha-1}$ (see Fig. 2).. In particular, initial data $u_0 \in A^0_{p,q}(\mathbb{R}^n)$ are admitted if $\frac{n}{2\alpha-1} < p \leq \infty$.

Remark 8 The case $\frac{n+2}{4} < \alpha \leq \frac{n}{2} + 1$.

As in the previous case (75) as well as (76) are satisfied if

$$s_0 > \left(\frac{n}{p} - \alpha + 1\right)_+ - \alpha.$$

Now, conditions (77) and (78) read as

$$s_0 \leq s < s_0 + \alpha \quad \text{and} \quad s > \left(\frac{n}{p} - \frac{n}{2}\right)_+,$$

respectively.

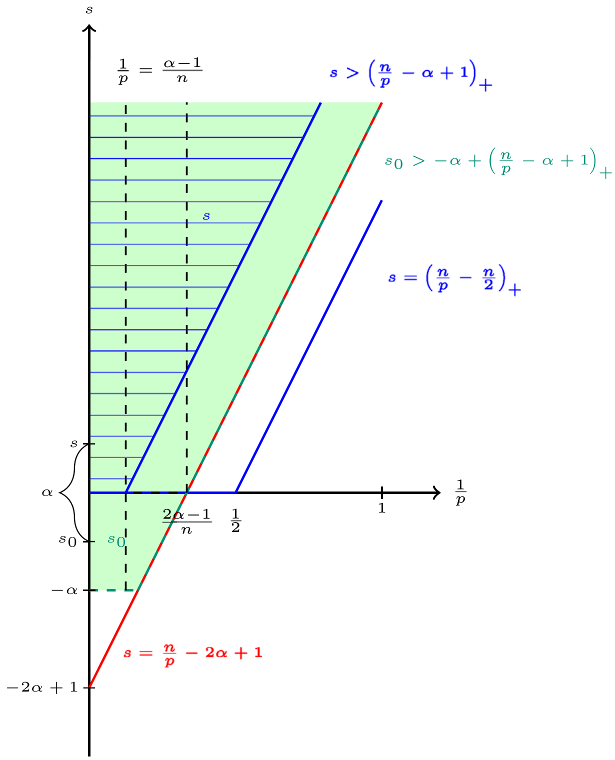

Fig. 2 $1 < \alpha \le \frac{n+2}{4}$

(i) If $\left(\frac{n}{p} - \alpha + 1\right)_+ - \alpha < s_0 \le \left(\frac{n}{p} - \frac{n}{2}\right)_+$ and $1 \le p \le \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0, T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)\right)$, where $\left(\frac{n}{p} - \frac{n}{2}\right)_+ < s < s_0 + \alpha$.

(ii) If $s_0 > \left(\frac{n}{p} - 2\alpha + 1\right)_+$ and $1 \le p \le \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0, T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)\right)$, where $s_0 \le s < s_0 + \alpha$.

Supercritical spaces $A^{s_0}_{p,q}(\mathbb{R}^n)$ are completely covered if $1 \le p \le \frac{n}{\alpha-1}$ (see Fig. 3). Initial data $u_0 \in A^0_{p,q}(\mathbb{R}^n)$ are admitted provided that $\frac{n}{2\alpha-1} < p \le \infty$ if $\frac{n}{2\alpha-1} \ge 1$ or $1 \le p \le \infty$ if $\frac{n}{2\alpha-1} < 1$.

Remark 9 The case $\alpha > \frac{n}{2} + 1$.

In this case (76) implies (75). We have

$$s_0 > \left(\frac{n}{p} - \frac{n}{2}\right)_+ - \alpha > \frac{n}{p} - 2\alpha + 1.$$

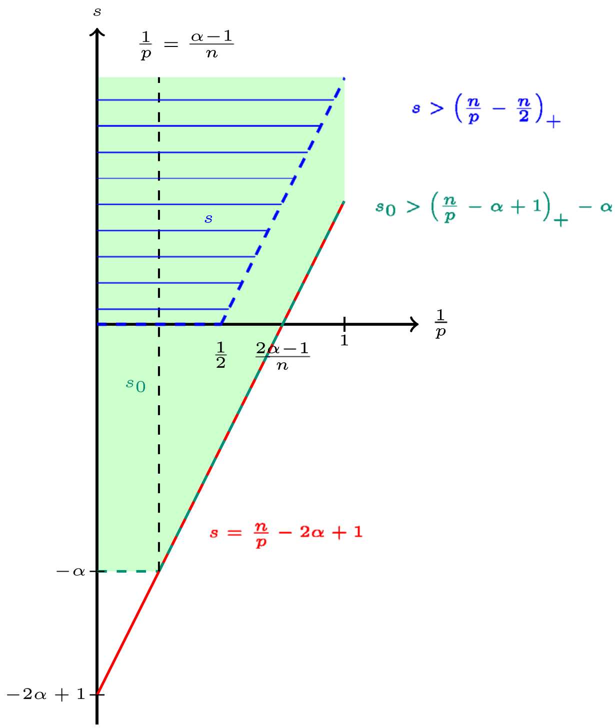

Fig. 3 $\frac{n+2}{4} < \alpha \leq \frac{n}{2} + 1$

Conditions (77) and (78) read as

$$s_0 \leq s < s_0 + \alpha \quad \text{and} \quad s > \left(\frac{n}{p} - \frac{n}{2}\right)_+,$$

respectively.

(i) If $\left(\frac{n}{p} - \frac{n}{2}\right)_+ - \alpha < s_0 \leq \left(\frac{n}{p} - \frac{n}{2}\right)_+$ and $1 \leq p \leq \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)\right)$, where $\left(\frac{n}{p} - \frac{n}{2}\right)_+ < s < s_0 + \alpha$.

(ii) If $s_0 > \left(\frac{n}{p} - \frac{n}{2}\right)_+$ and $1 \leq p \leq \infty$ then there exists a unique mild solution $u \in L_{2\alpha v}\left((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)\right)$, where $s_0 \leq s < s_0 + \alpha$.

Supercritical spaces $A^{s_0}_{p,q}(\mathbb{R}^n)$ can never be completely covered for given p, $1 \leq p \leq \infty$ (see Fig. 4). Initial data $u_0 \in A^0_{p,q}(\mathbb{R}^n)$ are admitted for all p, $1 \leq p \leq \infty$.

Remark 10 Some attention attracted Cauchy problems of type (4), (5) for fractional power dissipative equations with initial data belonging to spaces $L_p(\mathbb{R}^n)$, $1 < p < \infty$ (see, for example, [12]). Let us suppose that initial data $u_0 \in A^0_{p,q}(\mathbb{R}^n)$, where

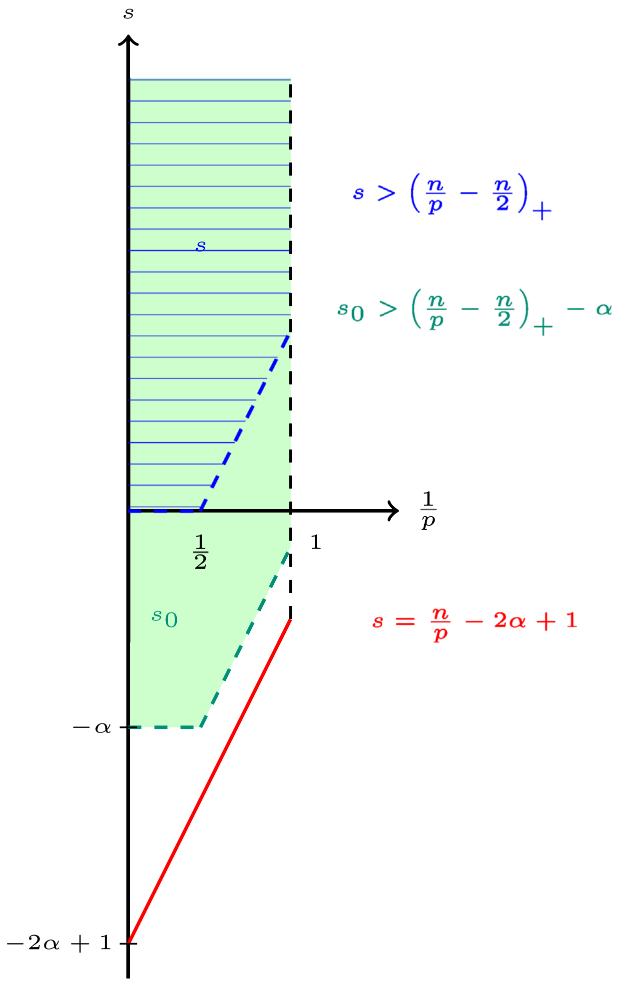

Fig. 4 $\alpha > \frac{n}{2} + 1$

$1 < p < \infty$, $1 \le q \le \infty$. In particular, this applies to $u_0 \in L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$. We follow our approach and recall (see the preceding remarks) that $u_0 \in A^0_{p,q}(\mathbb{R}^n)$ is admitted if $0 < \frac{1}{p} < \frac{2\alpha-1}{n}$ $\left(0 < \frac{1}{p} \le 1 \text{ if } \frac{2\alpha-1}{n} > 1\right)$ for given $\alpha > \frac{1}{2}$ and $n \in \mathbb{N}$ $(n \ge 2)$. Introducing a new parameter $\mu = a + \frac{1}{v}$ we can reformulate

$$u \in L_{2\alpha v}\Big((0,T),\ \tfrac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n)\Big)$$

in Theorem 6 as

$$\int_0^T t^{\mu v} \| u(\cdot,t) | A^s_{p,q}(\mathbb{R}^n) \|^{2\alpha v} \frac{\mathrm{d}t}{t} < \infty \tag{97}$$

(with modification $\sup_{0<t<T} t^{\frac{\mu}{2\alpha}} \| \dots \| < \infty$ if $v = \infty$), where

$$\left(\frac{n}{p} - \frac{n}{2}\right)_+ < s < \min(\alpha, 2\alpha - 1),$$
$$\mu > s \quad \text{and} \quad 0 \le \frac{2}{v} < \left(2\alpha - 1 - \left(\frac{n}{p} - s\right)_+\right).$$

In the following we shall make use of the embeddings

$$A^{\lambda + \frac{n}{p}}_{p,q}(\mathbb{R}^n) \hookrightarrow A^{\lambda}_{\infty,q}(\mathbb{R}^n) \tag{98}$$

for $\lambda \ge 0$ and

$$A^{\frac{n}{p} - \frac{n}{r}}_{p,q}(\mathbb{R}^n) \hookrightarrow A^0_{r,q}(\mathbb{R}^n) \tag{99}$$

for $0 < \frac{1}{r} < \frac{1}{p}$. We always assume $u_0 \in A^0_{p,q}(\mathbb{R}^n)$ and u stands for the unique mild solution according to Theorem 6. We distinguish the following cases:
Case 1: Let $\frac{1}{2} < \alpha < 1$ and let $0 < \frac{1}{p} < \frac{2\alpha - 1}{n}$. It follows from (97) and (98) that

$$\int_0^T t^{\mu v} \| u(\cdot,t) | A^{\lambda}_{\infty,q}(\mathbb{R}^n) \|^{2\alpha v} \frac{\mathrm{d}t}{t} < \infty, \tag{100}$$

where $0 \le \lambda < 2\alpha - 1 - \frac{n}{p}$, $\mu > \lambda + \frac{n}{p}$, and $0 \le \frac{2}{v} < 2\alpha - 1$.
Combining (97) and (99) we get

$$\int_0^T t^{\mu v} \| u(\cdot,t) | A^0_{r,q}(\mathbb{R}^n) \|^{2\alpha v} \frac{\mathrm{d}t}{t} < \infty, \tag{101}$$

for $0 < \frac{1}{r} < \frac{1}{p}$, $\mu > \frac{n}{p} - \frac{n}{r}$, and $0 \le \frac{2}{v} < 2\alpha - 1 - \frac{n}{r}$.
Case 2: Let $\alpha \ge 1$ and let $0 < \frac{1}{p} < \frac{\alpha}{n}$ ($\frac{1}{p} \le 1$ if $\alpha > n$). According to (97) and (98) we find that (100) holds for $0 \le \lambda < \alpha - \frac{n}{p}$, $\mu > \lambda + \frac{n}{p}$, and $0 \le \frac{2}{v} < 2\alpha - 1$. Moreover, it holds (101) for $0 < \frac{1}{r} < \min\left(\frac{1}{p}, \frac{1}{2}\right)$,
$\mu > \frac{n}{p} - \frac{n}{r} > \left(\frac{n}{p} - \frac{n}{2}\right)_+$, and $0 \le \frac{2}{v} < 2\alpha - 1 - \frac{n}{r}$.

Case 3: Let $\alpha \geq 1$ and let $\frac{\alpha}{n} \leq \frac{1}{p} < \frac{2\alpha-1}{n}$ ($\frac{1}{p} \leq 1$ if $\alpha > n$). Then u satisfies (101) for $0 < \frac{1}{p} - \frac{\alpha}{n} < \frac{1}{r} < \min\left(\frac{1}{p}, \frac{1}{2}\right)$, $\mu > \frac{n}{p} - \frac{n}{r} > \left(\frac{n}{p} - \frac{n}{2}\right)_+$, and $0 \leq \frac{2}{v} < 2\alpha - 1 - \frac{n}{r}$.
We may choose $\mu = \frac{1}{v}$ in (101) if

$$\frac{n}{p} - \frac{n}{r} < \frac{1}{v} < \frac{1}{2}\left(2\alpha - 1 - \frac{n}{r}\right), \quad \frac{1}{p} < \frac{2\alpha - 1}{n} < 1, \quad 0 < \frac{1}{r} < \min\left(\frac{1}{p}, \frac{1}{2}\right).$$

For example, this is the case if $\frac{1}{p} \leq \frac{2\alpha-1}{2n} \leq \frac{1}{2}$. Then

$$\frac{n}{p} - \frac{n}{r} < \frac{1}{v} < \frac{1}{2}(2\alpha - 1) - \frac{n}{r} < \frac{1}{2}\left(2\alpha - 1 - \frac{n}{r}\right) \text{ and } \frac{1}{p} < \frac{2\alpha - 1}{2n} < \frac{\alpha}{n}.$$

Thus, it follows from Case 2 and (101) that

$$\int_0^T \| u(\cdot, t) | A^0_{r,q}(\mathbb{R}^n) \|^{2\alpha v} \, dt \; < \infty, \tag{102}$$

if $u_0 \in A^0_{p,q}(\mathbb{R}^n)$, $0 < \frac{1}{r} < \frac{1}{p} < \frac{2\alpha-1}{2n} \leq \frac{1}{2}$, and $\frac{1}{v} > \frac{n}{p} - \frac{n}{r}$. Results of type (101) (in the case $v = \infty$) and (102) can be found in [12, Theorems 4.3 and 4.4]. In a certain sense Theorem 6 is an extension of their investigations (in the case b=d=1).

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