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Linear representations of fundamental groups of Klein surfaces derived from spinor representations of Clifford algebras

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Abstract
We study actions of multiplicative subgroups of Clifford algebras on Riemann surfaces. We show that every Klein surface of algebraic genus greater than 1 is isomorphic to the orbit space of such an action. We obtain linear representations of fundamental groups of Klein surfaces by using the spinor representations of Clifford algebras.

Keywords Riemann and Klein surfaces · NEC groups · Clifford algebras · Spin representations

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1 Introduction

Let $\mathrm{Cl}_{n-t,t}$ be a Clifford algebra associated with a vector space $V = K^n$ for $K = \mathbb{R}$ or $K = \mathbb{C}$ and a quadratic form Q_t defined by

$$Q(x_1, \ldots, x_n) = (x_1^2 + \ldots + x_t^2) - (x_{t+1}^2 + \ldots + x_n^2) \text{ for } (x_1, \ldots, x_n) \in V.$$

We will identify vectors of V with their images in $\mathrm{Cl}_{n-t,t}$. A multiplicative subgroup $M_{n-t,t}$ of $\mathrm{Cl}_{n-t,t}$ generated by an ortogonal basis of V is called a base group. Let $M^+_{n-t,t} = M_{n-t,n} \bigcap \mathrm{Cl}^+_{n-t,t}$ for the subalgebra $\mathrm{Cl}^+_{n-t,t}$ preserved by an automorphism of $\mathrm{Cl}_{n-t,t}$ which maps v to $-v$ for all $v \in V$.

We prove that for any Klein surface Y of algebraic genus $d \geq 2$ there are actions of base groups $G_t = M_{d+1-t,t}$ for $t = 0, 1$ on a Riemann surface X of genus $g = 1 + 2^{d+1}(d-1)$ such that the orbit space X/G_t is isomorphic to Y. The surface Y has a double cover Y^+ being a Riemann surface. We show that for a proper Klein

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surface Y an action of G_t can be defined in such a way that Y^+ is isomorphic to the orbit space X/G_t^+ for $G_t^+ = M_{d+1-t,t}^+$. Let π_Y and π_{Y^+} be the fundamental groups of Y and Y^+. Using the spinor representation of a complex Clifford algebra $\mathrm{Cl}_{d,1}$ for an odd d and spinor representation of complex algebra $\mathrm{Cl}_{d+1,0}^+$ for an even d we obtain linear representations $\rho : \pi_Y \to \mathrm{Gl}(2^m, \mathbb{C})$ and $\rho : \pi_{Y^+} \to \mathrm{Gl}(2^m, \mathbb{C})$, respectively, for $m = \frac{d+d_{(2)}}{2}$ and $d_{(2)} \in \{0, 1\}$ such that $d_{(2)} \equiv d \bmod 2$.

2 Preliminaries

This chapter contains elementary information needed to understand the paper. Sections on tensor algebras, exterior algebras, Clifford algebras and their spinor representations are based on books [6, 7]; sections on NEC groups and Klein surfaces are based on the book [3].

2.1 Tensor algebras and external algebras

Let V and W be vector spaces over a field K. The *tensor product* of V and W is a K-vector space $V \otimes W$ with a bilinear map $j : V \times W \to V \otimes W$ such that for every K-vector space Z and every bilinear map $f : V \times W \to Z$ there is a unique linear map $\bar{f} : V \otimes W \to Z$ for which $f = \bar{f} \circ j$.

Vectors $v \otimes w = j(v, w)$ for $v \in V$ and $w \in W$ are called *elementary tensors*. If V and W are finite dimensional and $\{v_i\}_1^n$ and $\{w_i\}_1^m$ are their basis, respectively, then the set

$$\{v_i \otimes w_j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$$

is a basis of $V \otimes W$. So $\dim V \otimes W = \dim V \cdot \dim W$.

The tensor product of a finite family of K-vector spaces $\{V_i\}_1^n$ is defined as

$$\otimes_{i=1}^n V_i = (\ldots((V_1 \otimes V_2) \otimes V_3) \otimes \ldots) \otimes V_n.$$

In the case when $V_1 = \ldots = V_n = V$ the tensor product $\otimes_{i=1}^n V_i$ is denoted by $V^{\otimes n}$ and $V^{\otimes 0} = K$.

A *multiplication of tensors* is a bilinear map $V^{\otimes n} \times V^{\otimes m} \to V^{\otimes n+m}$ which to each pair of tensors $t_1 = v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ and $t_2 = w_1 \otimes \ldots \otimes w_m \in V^{\otimes m}$ assigns tensor

$$t_1 \cdot t_2 = v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m \in V^{\otimes n+m}.$$

The *tensor algebra* of a K-vector space V is a direct sum $T(V) = \oplus_{n \geq 0} V^{\oplus n}$ with the multiplication defined by

$$\left(\sum_p t_p^1\right) \cdot \left(\sum_q t_q^2\right) = \sum_s \left(\sum_{p+q=s} t_p^1 \cdot t_q^2\right)$$

for $\sum_p t_p^1,\ \sum_q t_q^2 \in T(V)$.

An endomorphism $A \in \mathrm{End}(V^{\otimes n})$ defined by the formula

$$A(v_1 \otimes \ldots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}\, \sigma\, v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

for $t = v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ is called *antisymmetrization*. The vector space $\bigwedge^n V = A(V^{\otimes n})$ is called the *n-th exterior power* of V and $\bigwedge^0 V$ denotes the field K.

Vector $t = A(v_1 \otimes \ldots \otimes v_n)$ is denoted by $v_1 \wedge \ldots \wedge v_n$. If $v_i = v_j$ for some $i \neq j$, then $t = 0$. Thus $\dim \bigwedge^n V = 0$ for $n > \dim V$. Moreover, for any permutation $\tau \in S_n$,

$$v_{\tau(1)} \wedge \ldots \wedge v_{\tau(n)} = \mathrm{sgn} \tau\, v_1 \wedge \ldots \wedge v_n.$$

Consequently, if $\dim V = k$ and $\{v_i\}_1^k$ is a basis of V, then the set

$$\{1\} \cup \{v_{i_1} \wedge \ldots \wedge v_{i_n} :\ 1 \le i_1 < \ldots < i_n \le k\}$$

is a basis of $\bigwedge^n V$ and

$$\dim \bigwedge^n V = \frac{k!}{n!(k-n)!}.$$

The *exterior algebra* of a vector space V of dimension k is the direct sum $\bigwedge V = \oplus_{n=0}^k \bigwedge^n V$ with the multiplication defined by

$$t \wedge t' = A(t \otimes t') \text{ for } t, t' \in \bigwedge V.$$

This algebra is the quotient of the tensor algebra $T(V)$ modulo the ideal generated by tensors $v \otimes v$ for all $v \in V$. Moreover, $\dim \bigwedge V = \sum_{n=0}^k \binom{k}{n} = 2^k$.

2.2 Clifford algebras

Let V be a finite-dimensional vector space over a field K. A *quadratic form* is a function $Q : V \to K$ such that

$$Q(\alpha v) = \alpha^2 Q(v) \text{ for all } v \in V \text{ and } \alpha \in K$$

and the mapping $B : V \times V \to K$ defined by the formula

$$B(v_1, v_2) = \frac{1}{2}[Q(v_1 + v_2) - Q(v_1) - Q(v_2)] \tag{1}$$

is a bilinear form. It is said that the quadratic form Q is *nondegenerate*, if for every $0 \neq v \in V$ there exists $w \in V$ such that $B(v, w) \neq 0$.

The *Clifford algebra* associated with V and Q is an associative algebra $Cl(V, Q)$ over K together with a linear map $j : V \to Cl(V, Q)$ such that

$$(j(v))^2 = Q(v) \cdot 1 \text{ for all } v \in V \tag{2}$$

and for every algebra A over K and every linear map $f : V \to A$ with

$$(f(v))^2 = Q(v) \cdot 1_A \text{ for all } v \in V, \tag{3}$$

there is a unique algebra homomorphism $\bar{f} : Cl(V, Q) \to A$ for which $f = \bar{f} \circ j$. Linear maps $f : V \to A$ satisfying the condition (3) are called *Clifford maps*. It is said that the Clifford algebra is *universal* for Clifford maps.

Any two Clifford algebras associated with V and Q are isomorphic. The Clifford algebra $Cl(V, Q)$ can be seen as the quotient of the tensor algebra $T(V)$ modulo the ideal generated by elements of the form $v \otimes v - Q(v) \cdot 1$ for all $v \in V$.

To simplify the notation, from now on we will write v instead of $j(v)$ for $v \in V$. By Eqs. (1) and (2) we have

$$2B(v, w) = Q(v + w) - Q(v) - Q(w) = (v + w)^2 - v^2 - w^2 = vw + wv$$

what implies that

$$vw + wv = 2B(v, w). \tag{4}$$

Let $\{v_i\}_1^n$ be an ortogonal basis of V with $B(v_i, v_j) = 0$ for $i \neq j$. Then the set

$$\{1\} \cup \{v_{i_1} v_{i_2} \cdots v_{i_l} : 1 \leq i_1 < \ldots < i_l \leq n\} \tag{5}$$

is a basis of $Cl(V, Q)$, where 1 denotes the multiplicative unit of $\mathrm{Cl}(V, Q)$ being the image of $1 \in K$ under the projection $TV \to Cl(V, Q)$.

The subalgebra of $Cl(V, Q)$ generated by 1 and all elements $v_{i_1} v_{i_2} \cdots v_{i_l}$ of above basis with even l is denoted by $Cl(V, Q)^+$. This subalgebra is fixed by an automorphism of $Cl(V, Q)$ which maps v to $-v$ for all $v \in V$. We have $\dim Cl(V, Q) = 2^n$ and $\dim Cl(V, Q)^+ = 2^{n-1}$. If $Q \equiv 0$, then $Cl(V, Q)$ is isomorphic to the exterior algebra $\bigwedge V$. Otherwise, $Cl(V, Q)$ and $\bigwedge V$ are isomorphic only as vector spaces.

2.3 Spinor representation of a Clifford algebra

Let V be a finite-dimensional vector space over a field K and let Q be a non-degenerate quadratic form on V. A subspace W of V is called a *totally isotropic subspace*, if $Q(w) = 0$ for all vectors $w \in W$. In the case when W does not contain any non-zero vector w with $Q(w) = 0$, it is said that W is an *anisotropic subspace*. A subspace W' of V is called *ortogonal* to W, if $B(w, w') = 0$ for all $w \in W$ and $w' \in W'$, where B is a bilinear form associated with Q by the formula (1).

There exists a Witt decomposition of V into a direct sum of three subspaces $V = W \oplus U \oplus T$ such that W and U are maximal totally isotropic subspaces of the same dimension m, while T is anisotropic and orthogonal to $W \oplus U$. Moreover, for any basis $\{w_1, \ldots, w_m\}$ of W, there exists a basis $\{u_1, \ldots, u_m\}$ of U such that $B(w_i, u_j) = \delta_{ij}$, where δ_{ij} are the Kronecker symbols. If the field K is algebraically closed, then T has dimension 1 or 0 according to whether $n = \dim V$ is odd or even, respectively.

First assume that n is even and $V = W \oplus U$. The set

$$\mathrm{Cl}(V, Q)f = \{af : a \in \mathrm{Cl}(V, Q)\} \text{ for } f = w_1 \cdots w_m$$

is a subalgebra of $\mathrm{Cl}(V, Q)$ generated by

$$\{f\} \bigcup \{u_{i_1} \cdots u_{i_k} f : 1 \leq i_1 < \ldots < i_k \leq m\}.$$

Since $Q|_U \equiv 0$, it follows that the algebra $\mathrm{Cl}(U, Q|_U)$ is isomorphic to the external algebra $\bigwedge U$ which is spanned by the set

$$\mathcal{B} = \{1\} \bigcup \{u_{i_1} \wedge \ldots \wedge u_{i_k} : 1 \leq i_1 < \ldots < i_k \leq m\}.$$

Let $\varphi : \bigwedge U \to \mathrm{Cl}(V, Q)f$ and $\psi : \mathrm{Cl}(V, Q) \to \mathrm{End}(\mathrm{Cl}(V, Q)f)$ be given by

$$\varphi(u_{i_1} \wedge \ldots \wedge u_{i_k}) = u_{i_1} \cdots u_{i_k} f \text{ for } u_{i_1} \wedge \ldots \wedge u_{i_k} \in \mathcal{B}$$

and

$$\psi(a)(bf) = abf, \text{ for } a, b \in \mathrm{Cl}(V, Q),$$

respectively. Then the assignments

$$a \to \eta(a) = \varphi^{-1} \circ \psi(a) \circ \varphi \text{ for } a \in \mathrm{Cl}(V, Q) \tag{6}$$

define an isomorphism $\eta : \mathrm{Cl}(V, Q) \to \mathrm{End}(\bigwedge U)$ called the *spinor representation* of the Clifford algebra $\mathrm{Cl}(V, Q)$.

In the case when K is algebraically closed and $n = \dim V = 2m + 1$, the Witt decomposition of V has the form $V = W \oplus U \oplus \mathrm{Lin}(v_0)$ for a subspace $\mathrm{Lin}(v_0)$ spanned by a vector $v_0 \in V$ ortogonal to $V' = W \oplus U$ with $Q(v_0) \neq 0$. The mapping $Q' : V' \to K$ defined by the formula

$$Q'(v) = -Q(v_0)Q(v) \text{ for } v \in V'$$

is a nondegenerate quadratic form on a vector space V' of even dimension $2m$. Thus by the previus case there exists an isomorphism $\eta : \mathrm{Cl}(V', Q') \to \mathrm{End}(\bigwedge U)$. A linear map $f : V' \to \mathrm{Cl}(V, Q)^+$ defined by

$$f(v) = v_0 v \text{ for } v \in V'$$

is a Clifford map because $f(v)^2 = -Q(v_0)Q(v) \cdot 1 = Q'(v) \cdot 1$. Since algebra $\mathrm{Cl}(V', Q')$ is universal for Clifford maps, there exists a unique algebra homomorphism

$\bar{f} : \mathrm{Cl}(V', Q') \to \mathrm{Cl}(V, Q)^+$ such that $f = \bar{f} \circ j$ for $j : V' \to \mathrm{Cl}(V', Q')$. The homomorphism $\bar{f}$ is an isomorphism because it is injective and

$$\dim \mathrm{Cl}(V, Q)^+ = 2^{\dim V - 1} = 2^{\dim V'} = \dim \mathrm{Cl}(V', Q').$$

An isomorphism $\mu = \eta \circ \bar{f}^{-1} : \mathrm{Cl}(V, Q)^+ \to \mathrm{End}(\bigwedge U)$ is called the *spinor representation* of the algebra $\mathrm{Cl}(V, Q)^+$.

2.4 Non-Euclidean-crystallographic groups (NEC groups)

An NEC group is a discrete in the topology of $\mathbb{R}^4$ subgroup Λ of the group $\mathrm{Aut}(\mathcal{H})$ of isometries of the hyperbolic plane $\mathcal{H}$ with compact orbit space $\mathcal{H}/\Lambda$. If an NEC group Λ is contained in the group $\mathrm{Aut}^+(\mathcal{H})$ of orientation preserving isometries, then it is called a *Fuchsian group*. Otherwise, it is said that Λ is a *proper* NEC group and $\Lambda^+ = \Lambda \cap \mathrm{Aut}^+(\mathcal{H})$ is called the *canonical Fuchsian subgroup* of Λ.

The basics of NEC group theory were developed by Wilkie [15], Macbeath [8] and Natanzon [10, 11]. The algebraic structure of an NEC group Λ is given by the so-called *signature* which has the form

$$\sigma(\Lambda) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k})\}). \tag{7}$$

The number g is called the *orbit genus*, the integers m_i are said to be *proper periods*, the brackets $(n_{i,1}, \ldots, n_{i,s_i})$ are called *period cycles* and the integers $n_{i,j}$ are the *link periods* of Λ. The set of proper periods may be empty as well as the set of period cycles. In addition, an individual period-cycle may be empty too. For example, the signature $(g; +; [-]; \{-\})$ has no proper periods and no period cycle; the signature $(g; -; [m]; \{(-), (-)\})$ has one proper period m and two empty period cycles. A Fuchsian group can be regarded as an NEC group with the signature $(h; +; [m_1, \ldots, m_t]; \{-\})$, usually shortened to $(h; m_1, \ldots, m_t)$.

If there is a sign $+$ in the signature $\sigma(\Lambda)$, then the presentation of Λ consists of generators a_i, b_i $(i = 1, \ldots, g)$, x_i $(i = 1, \ldots, r)$, c_{ij}, e_i $(i = 1, \ldots, k, j = 1, \ldots, s_i)$ and the relations

$$\begin{aligned}
&x_i^{m_i} = 1, && i = 1, \ldots, r,\\
&c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1, && i = 1, \ldots, k, j = 1, \ldots, s_i,\\
&e_i c_{i0} e_i^{-1} = c_{is_i}, && i = 1, \ldots, k\\
&x_1 \cdots x_r e_1 \cdots e_k [a_1, b_1] \cdots [a_g, b_g] = 1.
\end{aligned}$$

If there is a sign $-$ in the signature $\sigma(\Lambda)$, then we just replace the generators a_i, b_i by d_i $(i = 1, \ldots, g)$ and the last relation by

$$x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1.$$

The last relation in the presentation of Λ will be called *long relation*.

The generators a_i, b_i are hyperbolic, d_i are glide reflections, x_i are elliptic, e_i are hyperbolic or elliptic, and c_{ij} are reflections.

Any generators of an NEC group satisfying the above relations are called *canonical generators*.

In [8] Macbeath proved the following

Theorem 2.1 *Let* Λ *and* Λ' *be* NEC *groups with signatures* (7) *and*

$$\sigma(\Lambda') = (g'; \pm; [m'_1, \ldots, m'_{r'}]; \{(n'_{1,1}, \ldots, n'_{1,s'_1}), \ldots, (n'_{k',1}, \ldots, n'_{k',s'_{k'}})\}),$$

respectively. Let $P_i = (n_{i,1}, \ldots, n_{i,s_i})$ *and* $P'_i = (n'_{i,1}, \ldots, n'_{i,s'_i})$ *be the period cycles in* $\sigma(\Lambda)$ *and* $\sigma(\Lambda')$*, respectively. Then* Λ *and* Λ' *are isomorphic if and only if the following conditions are satisfied:*

(i) *the sign in* $\sigma(\Lambda)$ *is the same as in* $\sigma(\Lambda')$*,*
(ii) $g' = g$*,* $r = r'$ *and* $k = k'$*,*
(iii) $(m'_1, \ldots, m'_{r'})$ *is a permutation of* $(m_1, \ldots, m_r)$*,*
(iv) *there is a permutation* φ *of* $\{1, \ldots, k\}$ *such that* $s_i = s'_{\varphi(i)}$ *for* $i = 1, \ldots, k$*.*
(v) *if the sign is "* $+$ *" then either for each* i*,* P'_i *is a cyclic permutation of* $P_{\varphi(i)}$ *or for each* i*,* P'_i *is a cyclic permutation of the inverse of* $P_{\varphi(i)}$*; if the sign is "* $-$ *", then for each* i *either* P'_i *is a cyclic permutation of* $P_{\varphi(i)}$ *or is a cyclic permutation of the inverse of* $P_{\varphi(i)}$*.*

There is a closed subset $E \subset \mathcal{H}$ associated with an NEC group Λ, called a *fundamental region* of Λ. It has the property that for every $z \in \mathcal{H}$ there exists $\lambda \in \Lambda$ such that $\lambda(z) \in E$ and this λ is unique if $\lambda(z) \in \operatorname{Int} E$. The hyperbolic area $\mu(\Lambda)$ of E depends only on the signature of Λ. If $\sigma(\Lambda)$ is given by (7), then

$$\mu(\Lambda) = 2\pi \left[\alpha g + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{i,j}}\right) \right], \tag{8}$$

where $\alpha = 1$ or $\alpha = 2$ according to whether the sign in $\sigma(\Lambda)$ is $-$ or $+$, respectively (e.g. [5, 14]). An abstract group with an algebraic structure determined by a signature (7) is an NEC group if and only if the right hand of (8) is positive.

If Γ is a finite index subgroup of an NEC group Λ, then it is an NEC group itself and there is a Hurwitz-Riemann formula, which says that

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}. \tag{9}$$

2.5 Klein surfaces

An NEC group with a signature

$$\sigma = (\gamma; \pm; [-]; \{(-), \overset{k}{\ldots}, (-)\}) \tag{10}$$

is called a *surface NEC group*. The orbit space of the hyperbolic plane $\mathcal{H}$ under the action of this group is a surface of topological genus γ with k boundary components which is orientable or non-orientable according to whether the sign $+$ or $-$ occurs in the signature. The integer $\alpha\gamma + k - 1$ is called the *algebraic genus* of the surface, where $\alpha = 2$ in the orientable case and $\alpha = 1$ in the non-orientable case.

A Klein surface is a compact topological surface equipped with a dianalitic structure. A Riemann surface can be seen as an orientable Klein surface without boundary. Preston proved in [12] that any Klein surface Y of algebraic genus $d \geq 2$ is an orbit space $\mathcal{H}/\Gamma$ for some surface NEC group Γ with the signature (10).

Alling and Greenleaf [1] constructed certain double cover Y^+ of Y being a Riemann surface. If Γ is a proper NEC group, then $Y^+ \simeq \mathcal{H}/\Gamma^+$ for the canonical Fuchsian subgroup $\Gamma^+ < \Gamma$ with the signature $(d; +; [-]; \{-\})$ which consists of all preserving automorphisms of Λ. If Y is an orientable surface without boundary, then Y^+ consists of two connected components Y_1 and Y_2 with different analytic structures each one homeomorphic to Y and there is an anticonformal isomorphism from Y_1 to Y_2.

By Proposition 3 of May in [9], an automorphism group of $Y = \mathcal{H}/\Gamma$ is isomorphic to the quotient group Λ/Γ for some NEC group Λ containing Γ as a normal subgroup. So an action of a finite group G on a Klein surface of algebraic genus $d \geq 2$ is associated with a short exact sequence of homomorphisms

$$1 \to \Gamma \to \Lambda \stackrel{\theta}{\to} G \to 1, \tag{11}$$

in which Λ is an NEC group and Γ is a surface NEC group isomorphic to the fundamental group of the surface. This action is denoted by (Λ, θ, G).

If there does not exist another NEC group containing Λ properly, then Λ is called a *maximal NEC group* and $G = \Lambda/\Gamma$ is the full automorphism group of the Klein surface. The detailed exposition of maximality can be found in [4].

A signature σ is called *maximal*, if for every NEC group Λ' with a signature σ' containing an NEC group Λ with the signature σ and having the same Teichmüller dimension, the equality $\Lambda = \Lambda'$ holds. For any maximal signature σ, there exists a maximal NEC group with the signature σ. The lists of non-maximal signatures are given in [2, 4].

3 Clifford actions on Riemann surfaces

3.1 The maximal base subgroups of Clifford algebras

We start this chapter by introducing a notation that will be valid throughout the entire paper. Let t and n be two integers such that $0 \leq t < n$. By $\mathrm{Cl}(n-t, t)$ we denote the Clifford algebra $\mathrm{Cl}(V, Q)$ associated with a vector space $V = K^n$ for $K = \mathbb{R}$ or $K = \mathbb{C}$ and a quadratic form Q_t defined by

$$Q_t(v) = (x_1^2 + \dots x_t^2) - (x_{t+1}^2 \dots + x_n^2) \text{ for } v = (x_1, \dots, x_n) \in V. \tag{12}$$

We use the same symbols for vectors of V and their images in the algebra $\mathrm{Cl}(n-t,t)$. There is an automorphism of $\mathrm{Cl}(n-t,t)$ which maps v to $-v$ for all $v \in V$. The subalgebra of $\mathrm{Cl}(n-t,t)$ fixed by this automorphism will be denoted by $\mathrm{Cl}(n-t,t)^{+}$.

Let $\mathcal{B} = \{v_p\}_1^n$ be the canonical basis of V such that $v_p = (x_1, \ldots, x_n)$, where $x_p = 1$ and $x_i = 0$ for $i \neq p$. By the *base group* of the algebra $\mathrm{Cl}(n-t,t)$ we mean its multiplicative subgroup $M_{n-t,t}$ generated by $v_1, \ldots, v_n$.

Lemma 3.1 *For $n > 1$, the base group $M_{n-t,t}$ of the algebra $\mathrm{Cl}(n-t,t)$ has order 2^{n+1}.*

Proof In $\mathrm{Cl}(n-t,t)$ an element v_p satisfies the relation $v_p^2 = Q_t(v_p) \cdot 1$ what implies that $v_p^2 = 1$ for $p = 1, \ldots, t$ and $v_p^2 = -1$ for $p > t$. Moreover, for $p \neq r$, we have $v_p v_r + v_r v_p = 2B_t(v_p, v_r) = 0$, where B_t is the bilinear form associated with Q_t. Thus $v_p v_r v_p^{-1} = -v_r$. The element -1 is central in the group $M_{n-t,t}$ and the quotient $M_{n-t,t}/\langle -1 \rangle$ is an abelian group generated by n elements of order 2 what implies that $|M_{n-t,t}| = 2^{n+1}$. □

By the proof of Lemma 3.1 we get the relations in the group $M_{n-t,t}$ which will be used later in the paper.

Corollary 3.2 *The generators $v_1, \ldots, v_n$ of the group $M_{n-t,t}$ for $t = 0, 1$ satisfy the following relations:*
$t = 1 : v_1^2 = 1,\ v_p^4 = 1,\ v_p v_1 v_p^{-1} = v_1 v_p^2$ *for* $p > 1$,
$v_q^2 = v_p^2$ *for* $1 < p, q \leq n$ *and* $v_q v_p v_q^{-1} = v_p^{-1}$ *for* $p \neq q$,
$t = 0 : v_p^4 = 1,\ v_p^2 = v_q^2$ *for* $1 \leq p, q \leq n$ *and* $v_q v_p v_q^{-1} = v_p^{-1}$ *for* $p \neq q$.

3.2 Clifford actions defining Klein surfaces

An action of the base group $M_{n-t,t}$ on a Riemann surface X of genus $g \geq 2$ will be called a *(n-t,t,g)-Clifford action*. This action *is full* if $M_{n-t,t}$ is the group of all automorphisms of X. We restrict our attention to Clifford actions for which the orbit space $X/M_{n-t,t}$ is isomorphic to a Klein surface Y of algebraic genus $d > 1$. In this case we will say that Y is *definable* by the $(n-t,t,g)$-Clifford action.

Let $M_{n-t,t}^{+} = \mathrm{Cl}(n-t,t)^{+} \bigcap M_{n-t,t}$. The orbit space $Y' = X/M_{n-t,t}^{+}$ is a double cover of Y which will be called the *Clifford double cover* defined by the (n-t,t,g)-Clifford action.

Theorem 3.3 *Every Klein surface Y of algebraic genus $d \geq 2$ is definable by a $(n-t,t,g)$-Clifford action for $g = 1 + 2^{d+1}(d-1)$, $n = d+1$ and $t = 0, 1$. If Y is a proper Klein surface except a sphere with three boundary components, then the Clifford double cover defined by this Clifford action is isomorphic to the canonical double cover of Y and in the exceptional case this is true only for $t = 0$. Moreover, for any Klein surface of genus $d > 3$, there exists a full $(n-t,t,g)$-Clifford action defining a Klein surface homeomorphic to Y.*

Proof Assume that Y is a Klein surface of algebraic genus $d \geq 2$ with k boundary components. Then Y is isomorphic to the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of a surface NEC group Λ with a signature σ given by (10) in which the sign is $+$ or $-$ according to whether Y is orientable or not, respectively. To simplify the notation we will write "$sign(\sigma) = +$" or "$sign(\sigma) = -$". In the first case Λ is generated by hyperbolic elements $a_1, b_1, \ldots, a_\gamma, b_\gamma$ and reflections $c_{1,0}, \ldots, c_{k,0}$ and connecting generators $e_1, \ldots, e_k$ which satisfy the relations $[e_i, c_{i,0}] = 1$ for $i = 1, \ldots, k$ and

$$[a_1, b_1] \cdots [a_\gamma, b_\gamma] e_1 \cdots e_k = 1.$$

In the second case there are generating glide reflections $d_1, \ldots, d_\gamma$ instead of hyperbolic generators and the long relation has the form

$$d_1^2 \cdots d_\gamma^2 e_1 \cdots e_k = 1.$$

Let $M_{n-t,t}$ be the base group of the algebra $\mathrm{Cl}_{n-t,t}$ for $t = 0, 1$ and $n = d + 1 = \alpha\gamma + k$, where $\alpha = 2$ or $\alpha = 1$ according to whether "$sign(\sigma) = +$" or "$sign(\sigma) = -$", respectively. In order to find an action of the group $M_{n-t,t}$ on a Riemann surface for which the orbit space is isomorphic to Y we need to find a smooth epimorphism $\theta : \Lambda \to M_{n-t,t}$ for which $\Gamma = \ker\theta$ is a torsion free Fuchsian group. Then $M_{n-t,t} \simeq \Lambda/\Gamma$ is an automorphism group of the Riemann surface $X = \mathcal{H}/\Gamma$. By the Hurwitz–Riemann formula, X has genus $g = 1 + 2^{d+1}(d-1)$ and

$$X/M_{n-t,t} \simeq (\mathcal{H}/\Gamma)/(\Lambda/\Gamma) \simeq \mathcal{H}/\Lambda \simeq Y.$$

The preimage $\Lambda' = \theta^{-1}(M^+_{n-t,t})$ is a subgroup of Λ with index 2 and the orbit space $X/M^+_{n-t,t} \simeq \mathcal{H}/\Lambda'$ is a double cover of Y.

Using the relations listed in Corollary 3.2, we will define an epimorphism $\theta : \Lambda \to M_{n-t,t}$ for which the long relation is preserved and all generating reflections of Λ are mapped to elements of order 2 and none product of generators of the group Λ containing an odd number of anti-conformal elements is mapped to 1. These conditions guarantee that kernel of θ is a surface Fuchsian group.

We start with the case when Y is a Riemann surface uniformized by a surface Fuchsian group Λ with a signature $\sigma(\Lambda) = (\gamma; +; [-]; \{-\})$ for some $\gamma \geq 2$. For the base group $M_{n-t,t}$ with $n = 2\gamma$, let $\theta : \Lambda \to M_{n-t,t}$ be induced by:

$$\theta(a_i) = v_n v_{2i-1}, \quad \theta(b_i) = v_n v_{2i} \text{ for } 1 \leq i \leq \gamma - 1 \tag{13}$$

and $\theta(a_\gamma) = v_{n-1}$, $\theta(b_\gamma) = v_n$ if γ is even or $\theta(a_\gamma) = v_{n-2}v_{n-1}$ and $\theta(b_\gamma) = v_n$ if γ is odd. Thanks the relations listed in Corollary 3.2 we have $\prod_{i=1}^{\gamma}[\theta(a_i), \theta(b_i)] = 1$. So $\Gamma = \ker\theta$ is a surface Fuchsian group and the Riemann surface $X = \mathcal{H}/\Gamma$ has an automorphism group $\Lambda/\Gamma = M_{n-t,t}$ such that $X/M_{n-t,t} \simeq \mathcal{H}/\Lambda \simeq Y$.

The pre-image $\Lambda' = \theta^{-1}(M^+_{n-t,t})$ is a surface Fuchsian group which by the Hurwitz–Riemann formula has signature $(2\gamma - 1; +; [-]; \{-\})$. We can choose elements $A_1, B_1, \ldots A_{2\gamma-1}, B_{2\gamma-1}$ as the canonical generators of Λ', where for even γ:

$$\begin{array}{lll} A_i = a_i, & B_i = b_i & \text{for } 1 \le i \le \gamma - 1 \\ A_{\gamma-1+i} = A_{2\gamma-1} b_{\gamma-i} A_{2\gamma-1}^{-1}, & B_{\gamma-1+i} = A_{2\gamma-1} a_{\gamma-i} A_{2\gamma-1}^{-1} & \text{for } 1 \le i \le \gamma - 1 \\ A_{2\gamma-1} = a_\gamma b_\gamma, & B_{2\gamma-1} = b_\gamma a_\gamma. & \end{array}$$

and for odd γ:

$$\begin{array}{lll} A_i = b_\gamma a_i b_\gamma^{-1}, & B_i = b_\gamma b_i b_\gamma^{-1} & \text{for } 1 \le i \le \gamma - 1 \\ A_{\gamma-1+i} = a_i, & B_{\gamma-1+i} = b_i & \text{for } 1 \le i \le \gamma - 1 \\ A_{2\gamma-1} = a_\gamma, & B_{2\gamma-1} = b_\gamma^2. & \end{array}$$

We leave to a reader checking that $\prod_{i=1}^{2\gamma-1}[A_i, B_i] = 1$ and that θ-images of $A_1, B_1, \dots A_{2\gamma-1}, B_{2\gamma-1}$ generate the group $M_{n-t,t}^+$. The orbit space $Y' = X/M_{n-t,t}^+$ is isomorphic to Riemann surface $\mathcal{H}/\Lambda'$. The quotient group $\mathbb{Z}_2 \simeq \Lambda/\Lambda'$ acts in natural way on Y' and the orbit space is isomorphic to Y.

Next, we assume that Λ is a proper NEC group with a signature (10) and Λ^+ is its canonical Fuchsian subgroup consisting of all conformal elements in Λ. If there exists a smooth epimorphism $\theta : \Lambda \to M_{n-t,t}$ which maps all conformal generators of Λ to elements of the group $M_{n-t,t}^+$ and maps all anti-conformal generators to elements outside $M_{n-t,t}^+$, then $\theta(\Lambda^+) \subseteq M_{n-t,t}^+$ because any element of Λ^+ is a product of the canonical generators of Λ containing an even number of anti-conformal elements. If additionally, the generators $v_1, \dots, v_n$ of $M_{n-t,t}$ are θ-images of anti-conformal elements of the group Λ then they are themselves anti-conformal and since any element of $M_{n-t,t}^+$ is a product of even number of these generators, it follows that $M_{n-t,t}^+ \subseteq \mathrm{Aut}^+(X) = \theta(\Lambda^+)$. Consequently, $M_{n-t,t}^+ = \theta(\Lambda^+) \simeq \Lambda^+/\Gamma$ for $\Gamma = \ker\theta$ which means that the group $M_{n-t,t}^+$ acts on the Riemann space $X \simeq \mathcal{H}/\Gamma$ and the orbit space is isomorphic to the canonical double cover $Y^+ = \mathcal{H}/\Lambda^+$ of $Y = \mathcal{H}/\Lambda$. So in order to prove that Y is definable by a $(n-t, t, g)$-Clifford action for which the Clifford cover is the canonical double cover of Y, we need to define a smooth epimorphism $\theta : \Lambda \to M_{n-t,t}$ such that the image $\theta(\lambda)$ of any generator λ of Λ belongs to $M_{n-t,t}^+$ if and only if λ is conformal, and all generators v_p of $M_{n-t,t}$ are θ-images of anti-conformal elements of Λ. Using the relations given in Corollary 3.2, it is easy to check that θ defined below is a smooth epimorphism which satisfies the above conditions. The definition is divided into a few cases depending on parameters γ, k and ε, where $\varepsilon = 1$ is γ is odd and $\varepsilon = 0$ if γ is even.

A smooth epimorphism $\theta : \Lambda \to M_{n,0}$ can be defined as follows:

(1) $\mathrm{sign}(\sigma) = -, \gamma = 1$ and $k \ge 2$
$\theta(d_1) = v_1, \theta(e_1) = \theta(e_2) = v_1 v_{k+1}, \theta(e_j) = 1$ for $3 \le j \le k$ and $\theta(c_{i0}) = v_1 v_{k+1} v_{i+1}$ for $i = 1, \dots, k-1$, $\theta(c_{k0}) = \theta(c_{k-10})$.

(2) $\mathrm{sign}(\sigma) = -, \gamma > 1, \gamma + k \ge 3$
$\theta(c_{j0}) = v_1 v_2 v_{\gamma+j}$ for $1 \le j \le k$, $\theta(d_i) = v_i$ for $1 \le i \le \gamma - 1$ and $\theta(e_j) = 1$ for $2 \le j \le k$. If $k > 0$, then $\theta(d_\gamma) = v_\gamma$ and $\theta(e_1) = v_\gamma^{2\varepsilon}$. If $k = 0$, then $\theta(d_\gamma) = (v_1 v_2)^\varepsilon v_\gamma$.

(3) $\mathrm{sign}(\sigma) = +, \gamma > 0, k > 0$ and $2\gamma + k \ge 3$
$\theta(a_i) = v_{2i-1} v_{2\gamma+k}$, $\theta(b_i) = v_{2i} v_{2\gamma+k}$, for $1 \le i \le \gamma$, $\theta(c_{j0}) = v_1 v_2 v_{2\gamma+j}$ for $j = 1, \dots, k$, $\theta(e_1) = v_1^{2\varepsilon}$ and $e_j = 1$ for $j > 1$.

(4) $\mathrm{sign}(\sigma) = +$, $\gamma = 0$ and $k \geq 3$
$\theta(e_1) = v_1 v_k, \theta(e_2) = v_2 v_k, \theta(e_3) = v_2 v_1$ and $\theta(e_j) = 1$ for $j > 3$, $\theta(c_{k-10}) = \theta(c_{k0}) = v_1 v_2 v_k$, and $\theta(c_{j0}) = v_1 v_2 v_{j+2}$ for $j = 1, \ldots, k-2$.

Next, we define a smooth epimorphism $\theta : \Lambda \to M_{n-1,1}$ as follows:

(1) $\gamma > 0, k > 0$ and $\alpha\gamma + k \geq 3$
$\theta(c_{i0}) = v_1$ for $1 \leq i \leq k$,
$\theta(e_i) = v_{k+1} v_{i+1}$ for $1 \leq i < k, \theta(e_k) = (\sqcap_{i=1}^{k-1} v_{k+1} v_{i+1})^{-1} v_{k+1}^{2\varepsilon}$,
if sign $= -$, then $\theta(d_j) = v_{k+j}$ for $1 \leq j \leq \gamma$,
if sign $= +$, then $\theta(a_i) = v_{k+2i-1} v_1, \theta(b_i) = v_{k+2i} v_1$ for $1 \leq i \leq \gamma$,
(2) $\gamma = 0$ and $k \geq 3$:
if $k = 3$ then $\theta(c_{10}) = v_1, \theta(c_{20}) = v_1 v_2, \theta(c_{30}) = v_1 v_3$ and $\theta(e_1) = \theta(e_2) = \theta(e_3) = 1$(*the exceptional case in which Y is not double definable)*,
if $k > 3$ then $\theta(c_{10}) = v_2 v_3 v_4, \theta(c_{i0}) = v_1$ for $2 \leq i \leq k, \theta(e_1) = v_3 v_4$, $\theta(e_i) = v_2 v_{i+1}$ for $2 \leq i \leq k-1, \theta(e_k) = (\theta(e_1) \cdots \theta(e_{k-1}))^{-1}$.
(3) $\gamma \geq 3, k = 0$ and $\mathrm{sign}(\sigma) = -$
$\theta(d_j) = v_j$ for $j = 1, \ldots, \gamma - 1$ and $\theta(d_\gamma) = (v_2 v_3)^{1-\varepsilon} v_\gamma$,

By browsing the lists of non-maximal NEC signatures we check that the signature (10) is non-maximal only in few cases listed in Table below.

non-maximal surface signatures	d
$(2; +; [-]; \{-\})$	3
$(1; +; [-]; \{(-)\})$	2
$(0; +; [-]; \{(-), (-), (-)\})$	2
$(1; -; [-]; \{(-), (-)\})$	2
$(2; -; [-]; \{(-)\})$	2
$(3; -; [-]; \{-\})$	2

For $d > 3$, the signature (10) is maximal. There exists a maximal NEC group with any given maximal signature which is not contained properly in any other NEC group. Let Λ' be a maximal NEC group with a signature (10) for $d > 3$. By Theorem 2.1, NEC groups with the same signatures are isomorphic. Thus there is an isomorphism $\tau : \Lambda' \to \Lambda$. Composing τ with θ we get an epimorphism $\theta' : \Lambda' \to M_{n-t,t}$ with kernel Γ which defines a full action $(\Lambda', \theta', M_{n-t,t})$ on the Riemann surface $X = \mathcal{H}/\Gamma$. It means that $M_{n-t,t}$ is the group of all automorphisms of X. Otherwise, there would be an NEC group Λ'' containing Λ' as a proper subgroup, against the assumption that Λ' is maximal.

Corollary 3.4 *For a given integer $n \geq 3$, let $g = 1 + 2^n(n-2)$ and let $n_{(2)} \in \{0, 1\}$ such that $n_{(2)} \equiv n \bmod 2$. Then for any $t \in \{0, 1\}$ there are at least n $(n-t, t, g)$-Clifford actions defining non-homeomorphic non-orientable Klein surfaces of algebraic genus $n-1$ and there are at least $1 + \frac{n-n_{(2)}}{2}$ such actions defining non-homeomorphic orientable Klein surfaces of algebraic genus $n-1$.*

Proof Let (γ, k, α) be a triple of nonnegative integers such that $n = \alpha\gamma + k, \alpha \in \{1, 2\}$, and $\gamma \neq 0$ if $\alpha = 1$. For $\alpha = 1$, there are n such triples because γ can be any integer in the range $1 \leq \gamma \leq n$ and k is uniquely determined for a fixed γ. For $\alpha = 2$, γ can be any integer in the range $0 \leq \gamma \leq \frac{n-n_{(2)}}{2}$ and we get $1 + \frac{n-n_{(2)}}{2}$ different triples. Let σ be a signature (10) corresponding to a given triple (γ, k, α), where $\mathrm{sign}(\sigma) = -$ for $\alpha = 1$ and $\mathrm{sign}(\sigma) = +$ for $\alpha = 2$. There exists an NEC group Λ with the signature σ and the orbit space $Y = \mathcal{H}/\Lambda$ is a Klein surface of algebraic genus $d = n - 1 \geq 2$ which is non-orientable or orientable according to whether $\alpha = 1$ or $\alpha = 2$, respectively. According to Theorem 3.3, the surface Y is definable by a $(n - t, t, g)$-Clifford action for $g = 1 + 2^n(n - 2)$ and $t \in \{1, 0\}$. □

4 Linear representations of surface NEC groups

In the previous chapter we proved that every Klein surface Y of algebraic genus $d \geq 2$ is the orbit space of a Riemann surface under the action of a base group of some Clifford algebra. Using the spinor representation of this algebra, described in section 2.3, we will get a linear representation of the fundamental group of Y or linear representation of the fundamental group of the double Clifford cover of Y depending on whether d is odd or even. For this purpose we need Clifford algebras under an algebraically closed field. Therefore in this chapter we assume that $\mathrm{Cl}(n - t, t)$ for $t = 1, 0$ is the Clifford algebra associated with a complex vector space $V = \mathbb{C}^n$ and a quadratic form Q_t defined by (12) and $M_{n-t,t}$ is the base group of $\mathrm{Cl}(n - t, t)$ generated by images of vectors of the canonical basis $\{v_i\}_{i=1}^n$ of V. Vectors of this basis and their images in $\mathrm{Cl}(n - t, t)$ will be denoted with the same symbols. By σ_1, σ_2 and σ_3 we denote the following Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \text{ and } \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Theorem 4.1 *Let $\eta_m : \mathrm{Cl}(2m - 1, 1) \to \mathrm{End}(Z)$ be the spinor representation of the Clifford algebra $\mathrm{Cl}(2m - 1, 1)$ for a vector space Z of dimension 2^m. Then in some basis of Z the endomorphism $\eta_m(v_p)$ is represented by a matrix $A_{m,p}$, where $A_{1,1} = \sigma_1, \ A_{1,2} = -i\sigma_2$ and for $m \geq 2$ the matrix $A_{m,p}$ is defined as follows:*

$$A_{m,p} = \begin{bmatrix} A_{m-1,p} & 0 \\ 0 & A_{m-1,p} \end{bmatrix}, \quad p = 1, \ldots, m-1, \tag{14}$$

$$A_{m,p} = \begin{bmatrix} A_{m-1,p-2} & 0 \\ 0 & A_{m-1,p-2} \end{bmatrix}, \quad p = m+2, \ldots, 2m, \tag{15}$$

$$A_{m,m} = \begin{bmatrix} 0 & -D_m \\ D_m & 0 \end{bmatrix} \text{ and } A_{m,m+1} = \begin{bmatrix} 0 & D_m(-i) \\ D_m(-i) & 0 \end{bmatrix}, \tag{16}$$

with

$$D_2 = \sigma_3 \text{ and } D_m = \begin{bmatrix} D_{m-1} & 0 \\ 0 & -D_{m-1} \end{bmatrix} \text{ for } m > 2.$$

Proof Let $\{v_p\}_{p=1}^n$ be the canonical basis of a vector space $V = \mathbb{C}^n$ for $n = 2m$. Then the maximal totally isotropic subspaces W and U of dimension m in Witt decomposition $V = W \oplus U$ can be spanned by sets $\{w_p\}_1^m$ and $\{u_p\}_1^m$, respectively, where

$$w_1 = \frac{1}{2}(v_1 - v_n), \quad u_1 = \frac{1}{2}(v_1 + v_n),$$
$$w_p = \frac{1}{2}(iv_{n+1-p} - v_p) \text{ and } u_p = \frac{1}{2}(iv_{n+1-p} + v_p) \text{ for } 2 \le p \le m.$$

The images of these vectors in $\mathrm{Cl}(n-1, 1)$ satisfy the relations:

$$w_p^2 = 0, u_p^2 = 0 \text{ and } w_p u_j + u_j w_p = \delta_{p,j} \text{ for } \quad p, j = 1, \ldots, m. \tag{17}$$

For $f = w_1 \cdots w_m$, the subalgebra $\mathrm{Cl}(n-1, 1)f = \{af : a \in \mathrm{Cl}(n-1, 1)\}$ of $\mathrm{Cl}(n-1, 1)$ is generated by

$$\mathcal{B} = \{f\} \cup \{u_{j_1} \cdots u_{j_k} f : 1 \le j_1 < \ldots < j_k \le m\}.$$

The external algebra $Z = \bigwedge U$ of dimension 2^m with basis

$$\{1\} \cup \{u_{j_1} \wedge \ldots \wedge u_{j_k} : 1 \le j_1 < \ldots < j_k \le m\} \tag{18}$$

is isomorphic to the Clifford algebra associated with vector space U and the zero quadratic form $Q_1|_U$. Let $\varphi : Z \to \mathrm{Cl}(n-1, 1)f$ be given by

$$\varphi(1) = f, \ \varphi(u_{j_1} \wedge u_{j_2} \wedge \ldots \wedge u_{j_k}) = u_{j_1} u_{j_2} \cdots u_{j_k} f.$$

The spinor representation $\eta_m : \mathrm{Cl}(n-1, 1) \to \mathrm{End}(Z)$ of the algebra $\mathrm{Cl}(n-1, 1)$ is defined by

$$\eta_m(a)(u) = \varphi^{-1}(a\varphi(u)) \text{ for } a \in \mathrm{Cl}(n-1, 1) \text{ and } u \in Z.$$

For an ordered subset $S = \{z_1, \ldots, z_k\} \subset Z$ and $z \in Z$, let $Z \wedge z$ denote an ordered set $\{z_1 \wedge z, \ldots, z_k \wedge z\}$. There is a sequence

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots \subset \mathcal{B}_m$$

of ordered subsets of $\bigwedge U$ such that $\mathcal{B}_1 = \{1, u_1\}$ and $\mathcal{B}_{j+1} = \mathcal{B}_j \cup (\mathcal{B}_j \wedge u_{j+1})$ for $j = 1, \ldots, m-1$. So $\mathcal{B}_2 = \{1, u_1, u_2, u_1 \wedge u_2\}$, $\mathcal{B}_3 = \{1, u_1, u_2, u_1 \wedge u_2, u_3, u_1 \wedge u_3, u_2 \wedge u_3, u_1 \wedge u_2 \wedge u_3\}$ and so on. In particular, the set $\mathcal{B}_m$ is the basis of Z given by (18). Let $A_{m,p} \in M_{2^m \times 2^m}(\mathbb{C})$ denote the matrix of $\eta_m(v_p)$ in this basis.

For $m = 1$, the canonical basis of V consists of two vectors

$$v_1 = u_1 + w_1 \text{ and } v_2 = u_1 - w_1. \tag{19}$$

Thus by the relations (17) we have

$$\eta_1(v_1)(1) = \varphi^{-1}((u_1 + w_1)w_1) = u_1$$

and

$$\eta_1(v_1)(u_1) = \varphi^{-1}((u_1 + w_1)u_1w_1) = \varphi^{-1}(w_1u_1w_1) = \varphi^{-1}((1 - u_1w_1)w_1) = 1$$

what implies that $A_{1,1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1$. By similar calculation we get $A_{1,2} = -i\sigma_2$.

Now let $P = \{1, \ldots, m-1\}$ and $R = \{m+2, \ldots n\}$ for $m > 1$. Vectors v_1 and v_2 are given by (19) while for $p = 2, \ldots, m$ we have

$$v_p = (u_p - w_p), \quad v_{n+1-p} = (u_p + w_p)(-i). \tag{20}$$

Since for all $p \in P \cup R$ vector v_p is a linear combination of vectors u_i and w_i with $1 \le i \le m-1$, it follows that $\eta_m(v_p)(u) \in \mathrm{Lin}(\mathcal{B}_{m-1})$ and $\eta_m(v_p)(u \wedge u_m) \in \mathrm{Lin}(\mathcal{B}_{m-1} \wedge u_m)$ for any $u \in \mathcal{B}_{m-1}$. It means that there are zero square matrices of dimension 2^{m-1} in the right upper corner and in the left lower corner of the matrix $M_{m,p}$ for all $p \in P \cup R$. Moreover, if $p \in P$, then $\eta_m(v_p)(u) = \eta_{m-1}(v_p)(u)$ and $\eta_m(v_p)(u \wedge u_m) = (\eta_{m-1}(v_p)(u)) \wedge u_m$ what implies that there is the matrix $A_{m-1,p}$ in the left upper corner and in the right lower corner of the matrix $A_{m,p}$. Thus the endomorphism $\eta_m(v_p)$ in basis $\mathcal{B}_m = \mathcal{B}_{m-1} \cup (\mathcal{B}_{m-1} \wedge u_m)$ has the matrix

$$A_{m,p} = \begin{bmatrix} A_{m-1,p} & 0 \\ 0 & A_{m-1,p} \end{bmatrix} \text{ for } p \in P.$$

Since $\eta_m(v_{n+1-j})(u) = \eta_{m-1}(v_{n-1-j})(u)$ for $j = 1, \ldots, m-1$ and $u \in \mathcal{B}_{m-1}$, it follows that $\eta_m(v_p)$ has the matrix

$$A_{m,p} = \begin{bmatrix} A_{m-1,p-2} & 0 \\ 0 & A_{m-1,p-2} \end{bmatrix} \text{ for } p \in R.$$

It remains to determine the matrices $M_{m,m}$ and $M_{m,m+1}$. If $u = u_{i_1} \wedge \ldots \wedge u_{ik} \in \mathcal{B}_{m-1}$, then by the relations (17), we have $\eta_m(w_m)(u) = 0$ and $\eta_m(u_m)(u \wedge u_m) = 0$. Thus

$$\eta_m(v_m)(u) = \eta_m(u_m - w_m)(u) = \eta_m(u_m)(u) = (-1)^k u \wedge u_m \tag{21}$$

and

$$\eta_m(v_m)(u \wedge u_m) = \eta(-w_m)(u \wedge u_m) = (-1)^{k+1} u \tag{22}$$

what implies that $A_{m,m} = \begin{bmatrix} 0 & -D_m \\ D_m & 0 \end{bmatrix}$ for some diagonal matrix $D_m \in M_{2^{m-1}\times 2^{m-1}}(\mathbb{C})$. Similarly, for $v_{m+1} = (u_m + w_m)(-i)$ we have

$$\eta_m(v_{m+1})(u) = (-1)^k(-i)(u \wedge u_m) \text{ and } \eta(v_{m+1})(u \wedge u_m) = (-1)^k(-i)u$$

what implies that $A_{m,m+1} = \begin{bmatrix} 0 & D_m(-i) \\ D_m(-i) & 0 \end{bmatrix}$. By simple calculation for $m = 2$ we get

$$A_{m,m} = A_{2,2} = \begin{bmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{bmatrix} \text{ and } A_{m,m+1} = A_{2,3} = \begin{bmatrix} 0 & \sigma_3(-i) \\ \sigma_3(-i) & 0 \end{bmatrix}.$$

For $m = 3$ we have

$$A_{m,m} = \begin{bmatrix} 0 & 0 & -\sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_3 \\ \sigma_3 & 0 & 0 & 0 \\ 0 & -\sigma_3 & 0 & 0 \end{bmatrix} \text{ and } A_{m,m+1} = \begin{bmatrix} 0 & 0 & -\sigma_3 i & 0 \\ 0 & 0 & 0 & \sigma_3 i \\ -\sigma_3 i & 0 & 0 & 0 \\ 0 & \sigma_3 i & 0 & 0 \end{bmatrix}.$$

These two examples help us to notice that

$$D_m = \begin{bmatrix} D_{m-1} & 0 \\ 0 & -D_{m-1} \end{bmatrix}$$

for any $m > 2$. This is a consequence of relations (21) and (22) and the fact that $\mathcal{B}_m = \mathcal{B}_{m-1} \cup (\mathcal{B}_{m-1} \wedge z_m)$.

Theorem 4.2 *Let* $\mu_m : \mathrm{Cl}(n, 0)^+ \to \mathrm{End}(Z)$ *be the spinor representation of the Clifford algebra* $\mathrm{Cl}(n, 0)^+$ *for* $n = 2m + 1$. *Then the homomorphisms* $\mu_m(v_n v_p)$ *with* $1 \le p \le 2m$ *are represented in some basis of* Z *by* $A_{1,1} = (-i)\sigma_2,\ A_{1,2} = (-i)\sigma_1$ *for* $m = 1$ *and by matrices* $A_{m,p}$ *given by* (14), (15) *and* (16) *for* $m > 1$.

Proof Let $\{v_i\}_1^n$ be the canonical basis of a vector space $V = \mathbb{C}^n$ for $n = 2m + 1$. There exists Witt decomposition $V = W \oplus U \oplus \mathrm{Lin}(v_n)$ for W and U generated by

$$\left\{ w_p = \frac{1}{2}(i v_{n-p} - v_p) \right\}_{p=1}^{m} \text{ and } \left\{ u_p = \frac{1}{2}(i v_{n-p} + v_p) \right\}_{p=1}^{m},$$

respectively. Here W and U are two maximal totally isotropic subspaces of V and v_n is a non-isotropic vector ortogonal to $V' = W \oplus U$ with respect to the bilinear form B_0 associated with Q_0. The subspace V' is spanned by the set $\{v_i\}_1^{n-1}$ and it has a non-degenerate quadratic form Q' defined by

$$Q'(v') = -Q_0(v_n)Q(v') \text{ for } v' \in V'.$$

Let us notice that $Q' = Q_0|_{V'}$ because $Q_0(v_n) = -1$. A linear map $f : V' \to \mathrm{Cl}(n, 0)^+$ given by $f(v') = v_n v'$ satisfies the condition $f(v')^2 = Q'(v') \cdot 1$ for all $v' \in$

V'. Thus by universality of the Clifford algebra $\mathrm{Cl}(V', Q')$, there is an isomorphism $\bar{f} : \mathrm{Cl}(V', Q') \to \mathrm{Cl}(n, 0)^+$ such that $\bar{f} \circ j = f$ for the canonical map $j : V' \to \mathrm{Cl}(V', Q')$. It is induced by

$$\bar{f}(v_p) = v_n v_p \text{ for } p = 1, \ldots, n-1.$$

Let $\eta_m : \mathrm{Cl}(V', Q') \to \mathrm{End}(\bigwedge U)$ be the spinor representation of the algebra $\mathrm{Cl}(V', Q')$. Then $\mu_m = \eta_m \circ \bar{f}^{-1} : \mathrm{Cl}(V, Q)^+ \to \mathrm{End}(\bigwedge U)$ is an isomorphism such that

$$\mu_m(v_n v_p) = \eta_m(v_p) \text{ for } p = 1, \ldots, n-1.$$

In order to determine the matrices $A_{m,p}$ of endomorphisms $\mu_m(v_n v_p)$ in basis $\mathcal{B}_m$ of $\bigwedge U$ we can use formulas

$$v_p = u_p - w_p \text{ and } v_{n-p} = (-i)(u_p + w_p) \text{ for } p = 1, \ldots, m$$

and the relations (17) which are satisfied in the algebra $\mathrm{Cl}(V', Q')$ by vectors w_p and u_p. By repeating the argumentation from the proof of Theorem 4.1 we get that matrices $A_{m,p}$ are defined for $m > 1$ by (14), (15) and (16); and for $m = 1$ we have $A_{1,1} = (-i)\sigma_2$, and $A_{1,2} = (-i)\sigma_1$.

Theorem 4.3 *Let π_Y be the fundamental group of a proper Klein surface Y of algebraic genus $d \geq 2$, π_{Y^+}-the fundamental group of the Riemann surface Y^+ being a double cover of Y and let $m = (d + d_{(2)})/2$ for $d_{(2)} \in \{0, 1\}$ such that $d_{(2)} \equiv d \bmod 2$. If d is odd, then there is a linear representation $\rho : \pi_Y \to \mathrm{Gl}(2^m, \mathbb{C})$ with image generated by the matrices $A_{m,1} \ldots A_{m,2m}$ defined in Theorem 4.1. If d is even, then there is a linear representation $\rho : \pi_{Y^+} \to \mathrm{Gl}(2^m, \mathbb{C})$ with image generated by the matrices $A_{m,1} \ldots A_{m,2m}$ defined in Theorem 4.2.*

Proof According to Theorem 3.3, any proper Klein surface Y of algebraic genus $d \geq 2$ is definable by a $(n-t, t, g)$-Clifford action $(\Lambda, \theta, M_{n-t,t})$ for $n = d + 1$, $g = 1 + 2^{d+1}(d-1)$ and $t \in \{0, 1\}$ and the Clifford cover defined by this action is isomorphic to the canonical double cover Y^+ of Y. Here Λ is a surface NEC group isomorphic to the fundamental group of Y and $\theta : \Lambda \to M_{n-t,t}$ is a smooth epimorphism. By composing θ with the spinor representation of the algebra $\mathrm{Cl}(n-t, t)$ for even n or with the spinor representation of the algebra $\mathrm{Cl}(n-t, t)^+$ for odd n, we get linear representations of the fundamental groups of Y or Y^+, respectively.

Let $m = \frac{d+d_{(2)}}{2}$ and let $t = d \bmod 2$. For an odd d, the spinor representation η_m of the algebra $\mathrm{Cl}_{d,1}$ associates with every generator v_p of the group $M_{d,1}$ an isomorphism of a vector space Z of dimension 2^m. By Theorem 4.1, there is a basis $\mathcal{B}$ of Z in which endomorphisms $\eta_m(v_p)$ are represented by matrices $A_{m,p} \in \mathrm{Gl}(2^m, \mathbb{C})$, where $A_{1,1} = \sigma_1$ and $A_{1,2} = -i\sigma_2$ for $m = 1$ and $A_{m,p}$ are given by (14), (15) and (16) for $m > 1$. Thus we get an epimorphism $\rho = \eta \circ \theta : \Lambda \to G \subset \mathrm{Gl}(2^m, \mathbb{C})$ onto the group generated by matrices $A_{m,1}, \ldots A_{m,2m}$.

If d is even, then there is a spinor representation $\eta_m : \mathrm{Cl}^+_{n,0} \to \mathrm{Gl}(2^m, \mathbb{C})$ such that the generators $v_1 v_n, \ldots, v_{n-1} v_n$ of the group $M^+_{n,0}$ are represented by matrices given in Theorem 4.2. Thus composing $\theta|_{\Lambda^+}$ with η_m we get an epimorphism $\rho : \Lambda^+ \to G$ onto the group generated by these matrices.

Corollary 4.4 *For any odd $d \geq 3$ and $m = \frac{d+1}{2}$, there exist a Klein surface $Y \simeq \mathcal{H}/\Lambda$ of algebraic genus d and a linear representation $\rho : \Lambda \to \mathrm{Gl}(2^m, \mathbb{C})$ which maps bijectively canonical generators of a canonical presentation of Λ to matrices $A_{m,1}, \ldots, A_{m,2m}$ defined in Theorem 4.1.*

Proof Let Λ be an NEC group with the signature $(\gamma; -; [-]; \{(-)\})$ for $\gamma = d \geq 3$. Then $Y = \mathcal{H}/\Lambda$ is a Klein surface of algebraic genus d. Let $d_1, \ldots, d_\gamma$ be generating glide reflections of Λ and let c_{10} and e_1 be generators of Λ associated with the only period cycle. Then $e_1 = (d_1^2 \cdot \ldots \cdot d_\gamma^2)^{-1}$, $c_{10}^2 = 1$ and $e_1 c_{10} e_1^{-1} = c_{10}$. There is a homomorphism $\theta : \Lambda \to M_{d,1}$ induced by $\theta(d_i) = v_{1+i}$ for $i = 1, \ldots, \gamma$, $\theta(c_{10}) = v_1$ and $\theta(e_1) = v_{d+1}^2$. The generator e_i is redundant because it can be expressed by $d_1, \ldots, d_\gamma$. So generators of Λ correspond bijectively to generators $v_1, \ldots, v_{d+1}$ which according to Theorem 4.1 are represented by matrices $M_{m,1}, \ldots, A_{m,d+1}$ by the spinor representation of $\mathrm{Cl}_{d,1}$.

Corollary 4.5 *For any even $d \geq 2$ and $m = \frac{d}{2}$, there exist a Klein surface $Y \simeq \mathcal{H}/\Lambda$ of algebraic genus d and a linear representation $\rho : \Lambda \to \mathrm{Gl}(2^m, \mathbb{C})$ which maps bijectively canonical conformal generators of a canonical presentation of Λ to matrices $A_{m,1}, \ldots, A_{m,2m}$ defined in Theorem 4.2.*

Proof An NEC group Λ with the signature $(\gamma; +; [-]; \{(-)\})$ for $\gamma = \frac{d}{2}$ is generated by elements $a_1, b_1, \ldots, a_\gamma, b_\gamma$ and c_{10} such that $c_{10}^2 = 1$ and $e_1 c_{10} e_1^{-1} = c_{10}$ for $e_1 = ([a_1, b_1] \cdots [a_\gamma, b_\gamma])^{-1}$. There is a smooth epimorphism $\theta : \Lambda \to M_{n,0}$ for $n = d + 1$ induced by $\theta(a_i) = v_{2i-1} v_n$, $\theta(b_n) = v_{2i} v_n$ for $i = 1, \ldots, \gamma$, $\theta(c_{10}) = v_1 v_2 v_n$ and $\theta(e_1) = v_n^\varepsilon$, where $\varepsilon = \gamma \mod (2)$. The generator e_1 is redundant and the other conformal generators of Λ are mapped to generators $v_1 v_n, \ldots, v_{n-1} v_n$ of the group $M_{n,0}^+$ which according to Theorem 4.2 are represented by matrices $A_{m,1}, \ldots, A_{m,2m+1}$ for $m = \frac{d}{2}$ by the spinor representation of the algebra $\mathrm{Cl}_{n,0}^+$

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