On some Fournier–Gagliardo type inequalities

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Abstract

In this paper we consider nonnegative functions f on $\mathbb{R}^n$ which are defined either by $f(x) = \min(f_1(x_1), \dots, f_n(x_n))$ or by $f(x) = \min(f_1(\hat{x}_1), \dots, f_n(\hat{x}_n))$. Such minimum-functions are useful, in particular, in embedding theorems. We prove sharp estimates of rearrangements and Lorentz type norms for these functions, and we find the link between their Lorentz norms and geometric properties of their level sets.

Keywords Rearrangements · Lorentz spaces · Embeddings · Mixed norms

Mathematics Subject Classification Primary 46E30; Secondary 46E35 · 42B35

1 Introduction

Let $x = (x_1, \dots, x_n)$. Denote by $\hat{x}_k$ the $(n-1)$ -dimensional vector obtained from the n -tuple x by removal of its k -th coordinate. We shall write $x = (x_k, \hat{x}_k)$.

Let $\psi_1, \dots, \psi_n$ be nonnegative measurable functions on $\mathbb{R}^{n-1}$ ($n \geq 2$). We consider the geometric mean

$$\mathcal{G}(x) = \left(\prod_{k=1}^n \psi_k(\hat{x}_k) \right)^{1/n}, \quad x \in \mathbb{R}^n.$$

Gagliardo [12] proved the following theorem.

Theorem 1.1 Assume that $n \geq 2$ and $\psi_k \in L^1(\mathbb{R}^{n-1})$ ($k = 1, \dots, n$). Then $\mathcal{G} \in L^{n'}(\mathbb{R}^n)$ ($n' = n/(n-1)$) and

$$\|\mathcal{G}\|_{L^{n'}(\mathbb{R}^n)} \leq \left(\prod_{k=1}^n \|\psi_k\|_{L^1(\mathbb{R}^{n-1})} \right)^{1/n}. \quad (1.1)$$

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This theorem yields the embedding of the Sobolev space $W_1^1(\mathbb{R}^n)$ into $L^{n'}(\mathbb{R}^n)$ (see [12]). Later on, it was shown in [2] (see also [10, 11, 25]) that the space $W_1^1(\mathbb{R}^n)$ is embedded into the Lorentz space $L^{n',1}(\mathbb{R}^n)$ which is strictly smaller than $L^{n'}(\mathbb{R}^n)$. We emphasize that a similar refinement of Theorem 1.1 is not true; that is, the function $\mathcal{G}$ may not belong to $L^{n',1}(\mathbb{R}^n)$ (see [19, Remark 3.12]). However, set

$$\Psi(x) = \min_{1 \leq k \leq n} \psi_k(\hat{x}_k), \quad x \in \mathbb{R}^n. \quad (1.2)$$

Clearly, $\Psi(x) \leq \mathcal{G}(x)$. The following theorem holds [11].

Theorem 1.2 *Assume that $n \geq 2$ and $\psi_k \in L^1(\mathbb{R}^{n-1})$ ($\psi_k \geq 0$, $k = 1, \dots, n$). Then $\Psi \in L^{n',1}(\mathbb{R}^n)$ and*

$$\|\Psi\|_{L^{n',1}(\mathbb{R}^n)} \leq \left(\prod_{k=1}^n \|\psi_k\|_{L^1(\mathbb{R}^{n-1})} \right)^{1/n}. \quad (1.3)$$

It is important to note that there are normalizing factors in the definitions of Lorentz norms given in Sect. 2.

Theorem 1.2 implies the embedding of $W_1^1(\mathbb{R}^n)$ into the Lorentz space $L^{n',1}(\mathbb{R}^n)$. Observe that this theorem was obtained by Fournier [11] in an equivalent form in terms of mixed norm spaces $L_{\hat{x}_k}^1[L_{x_k}^\infty]$ (see Sect. 3).

Different extensions of Theorem 1.2 and their applications have been studied in the works [1, 7, 17, 19, 23].

As in [1, 19], in this paper we consider Lorentz spaces defined in terms of *iterated* rearrangements. Let Ω_n be the collection of all permutations of the set $\{1, \dots, n\}$. For each $\sigma \in \Omega_n$, and $0 < p, r < \infty$, we define a Lorentz space $\mathcal{L}_\sigma^{p,r}(\mathbb{R}^n)$ (see Sect. 2). The relations between $\mathcal{L}_\sigma^{p,r}$ -spaces and the classical Lorentz $L^{p,r}$ -spaces are the following:

$$\mathcal{L}_\sigma^{p,r} \subset L^{p,r} \quad (r \leq p), \quad L^{p,r} \subset \mathcal{L}_\sigma^{p,r} \quad (p \leq r), \quad (1.4)$$

and for $p \neq r$ these embeddings are strict (see [26]).

We proved in [16] that Sobolev embeddings can be strengthened using the $\mathcal{L}$ -norms. In particular, we have that for any $f \in W_1^1(\mathbb{R}^n)$

$$\|f\|_{\mathcal{L}_\sigma^{n',1}} \leq c \left(\prod_{k=1}^n \|D_k f\|_1 \right)^{1/n} \quad (n \geq 2, \quad \sigma \in \Omega_n). \quad (1.5)$$

By virtue of (1.4), (1.5) implies embedding $W_1^1(\mathbb{R}^n) \subset L^{n',1}(\mathbb{R}^n)$.

In [1], there were proved estimates of generalized Lorentz norms in terms of mixed norm spaces $L_{\hat{x}_k}^1[L_{x_k}^\infty]$. These estimates provided a strengthening of Fournier's estimates of classical Lorentz norms [11], but the constants were not optimal. Optimal constants were obtained in [19], where we proved the following extension of Theorem 1.2.

Theorem 1.3 Assume that $\psi_k \in L^1(\mathbb{R}^{n-1})$ ($\psi_k \geq 0$, $k = 1, \dots, n$). Let Ψ be defined by (1.2). Then $\Psi \in \mathcal{L}_{\sigma}^{n',1}(\mathbb{R}^n)$ and

$$\|\Psi\|_{\mathcal{L}_{\sigma}^{n',1}} \leq \left(\prod_{k=1}^n \|\psi_k\|_{L^1(\mathbb{R}^{n-1})} \right)^{1/n} \quad (1.6)$$

for any $\sigma \in \Omega_n$.

Since $\|f\|_{L^{n',1}} \leq \|f\|_{\mathcal{L}_{\sigma}^{n',1}}$ (see (2.5) below), inequality (1.6) gives a refinement of inequality (1.3). Inequality (1.6) cannot be improved; it becomes equality if ψ_k are equal to the characteristic function of the unit cube $[0, 1]^{n-1}$.

It follows from (1.6) that inequality (1.5) holds with $c = 1/2$. This constant is optimal (see [19]).

We see that estimates of functions defined as *minimum* provide sharp embeddings of Sobolev spaces $W_1^1(\mathbb{R}^n)$. The main objective of this paper is to study another type of such functions. Let f_j be nonnegative rearrangeable functions on $\mathbb{R}$. Set

$$f(x) = \min_{1 \leq j \leq n} f_j(x_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (1.7)$$

We observe that in the case, when the functions f_j vanish outside $\mathbb{R}_+$ and decrease on $\mathbb{R}_+$, this definition and the expression for the rearrangement $f^*(t)$ (see Sect. 4) are closely related to the concept of *average modulus of continuity* (see [15], [18, p. 51]). This concept was useful in the study of embeddings of some anisotropic function classes.

We prove that

$$\|f\|_{L^{n,1}} \leq \left(\prod_{k=1}^n \|f_k\|_{L^1} \right)^{1/n}. \quad (1.8)$$

Further, we show that

$$\Psi^*(t) \leq f^*(t^{n-1}) \quad \text{for all } t > 0 \quad (1.9)$$

(where Ψ is defined by (1.2) and $f_j = \psi_j^*$ in (1.7)). This implies that estimate (1.8) (with $f_j = \psi_j^*$) gives a strengthening of the estimate (1.3).

Our main results are related to generalized Lorentz norms (defined in terms of iterated rearrangements). A crucial role here is played by the following equality involving iterated rearrangements of the function (1.7): for any $\alpha_j > 0$ and any $\sigma \in \Omega_n$

$$\int_{\mathbb{R}_+^n} \prod_{j=1}^n x_j^{\alpha_j-1} \mathcal{R}_{\sigma} f(x) dx = \left(\prod_{j=1}^n \alpha_j \right)^{-1} \int_0^\infty \prod_{j=1}^n \lambda_{f_j}(y)^{\alpha_j} dy. \quad (1.10)$$

First, this equality immediately implies that

$$\|f\|_{\mathcal{L}_{\sigma}^{p,1}} = \|f\|_{L^{p,1}} \quad \text{for any } \sigma \in \Omega_n \text{ and any } p > 0. \quad (1.11)$$

The phenomenon exhibited in (1.11) is closely connected with an important geometric property of functions f defined by (1.7). Denote by $\mathcal{P}_n$ the class of all measurable nonnegative functions on $\mathbb{R}^n$ such that for any $y > 0$ the Lebesgue set $\{x \in \mathbb{R}^n : f(x) > y\}$ is a cartesian product of measurable sets $E_i(y) \subset \mathbb{R}$. It is obvious that any function f defined by (1.7) belongs to $\mathcal{P}_n$. We prove that the inverse statement also is true, that is, any function $f \in \mathcal{P}_n$ can be represented as in (1.7). Thus, for any function $f \in \mathcal{P}_n$ its classical and generalized Lorentz norms coincide.

Another immediate consequence of (1.10) is the following inequality: if $\sum_{k=1}^n \alpha_k = 1$ and $f_j \in L^1(\mathbb{R})$, then for any $\sigma \in \Omega_n$

$$\int_{\mathbb{R}^n} \prod_{j=1}^n x_j^{\alpha_j-1} \mathcal{R}_\sigma f(x) dx \leq \left(\prod_{j=1}^n \alpha_j \right)^{-1} \prod_{j=1}^n \|f_j\|_1^{\alpha_j}. \quad (1.12)$$

Taking $\alpha_j = 1/n$ ($j = 1, \dots, n$) we obtain the inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^n x_j^{1/n-1} \mathcal{R}_\sigma f(x) dx \leq n^n \left(\prod_{j=1}^n \|f_j\|_1 \right)^{1/n}. \quad (1.13)$$

By (1.11), this inequality is the same as inequality (1.8). We show that inequality (1.13) (with $f_j = \psi_j^*$) is stronger than inequality (1.6).

Note also that (1.13) has an equivalent form in terms of mixed norm spaces $L_{x_k}^1 [L_{\hat{x}_k}^\infty]$ (see Sect. 3).

We observe that our interest to these questions is inspired by their connections with embedding theorems and Loomis–Whitney type inequalities. Recall that Loomis–Whitney inequality [20] states that for any F_σ -set $E \subset \mathbb{R}^n$

$$\text{mes}_n E \leq \prod_{k=1}^n (\text{mes}_{n-1} E_k)^{1/(n-1)}, \quad (1.14)$$

where E_k is the orthogonal projection of E onto the coordinate hyperplane $x_k = 0$. Actually (1.14) can be immediately derived from inequality (1.1). Successively applying (1.14), one can obtain a more general inequality (see [14, Chapter 4, 4.4.2])

$$\text{mes}_n E \leq \prod_{1 \leq i_1 < \dots < i_m \leq n} (\text{mes}_{n-m} E_{i_1, \dots, i_m})^{1/\binom{n-1}{m}}, \quad (1.15)$$

where $E_{i_1, \dots, i_m}$ is the projection of E onto $(n-m)$ -dimensional coordinate plane $x_{i_1} = \dots = x_{i_m} = 0$ ($1 \leq m \leq n-1$). The case $m = 1$ corresponds to the function (1.2). The case $m = n-1$ (in which (1.15) is obvious) corresponds to the function (1.7). Probably, it would be interesting to consider similar *minimum*-functions for any $1 < m < n-1$.

The paper is organized as follows. In Sect. 2 we give the basic definitions of rearrangements and Lorentz spaces. In Sect. 3 we define mixed norm spaces and study

some of their properties. In Sect. 4 we obtain some simple results concerning estimates of rearrangements and classical Lorentz norms of functions (1.2) and (1.7). Section 5 is devoted to the main results of this paper related to generalized Lorentz norms of the function (1.7). In Sect. 6 we show that $\mathcal{L}_{\sigma}^{n',1}$ -norms of the function (1.2) are majorized by $\mathcal{L}_{\sigma}^{n,1}$ -norms of the function (1.7) (where $f_j = \psi_j^*$).

2 Rearrangements and Lorentz spaces

A measurable and almost everywhere finite real-valued function f on $\mathbb{R}^n$ is said to be *rearrangeable* if

$$\lambda_f(y) = |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty \quad \text{for each } y > 0.$$

A *nonincreasing rearrangement* of a rearrangeable function f defined on $\mathbb{R}^n$ is a non-negative and nonincreasing function f^* on $\mathbb{R}_+ \equiv (0, +\infty)$ which is equimeasurable with $|f|$, that is, $\lambda_{f^*} = \lambda_f$. We shall assume in addition that the rearrangement f^* is left-continuous on $\mathbb{R}_+$. Under this condition it is defined uniquely by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) < t\}, \quad 0 < t < \infty. \quad (2.1)$$

Besides, we have the equality (see [9, p. 32])

$$f^*(t) = \sup_{|E|=t} \inf_{x \in E} |f(x)|, \quad 0 < t < \infty. \quad (2.2)$$

Let $0 < p, r < \infty$. A rearrangeable function f on $\mathbb{R}^n$ belongs to the Lorentz space $L^{p,r}(\mathbb{R}^n)$ if

$$\|f\|_{L^{p,r}} = \|f\|_{p,r} = \left(\frac{r}{p} \int_0^\infty \left(t^{1/p} f^*(t) \right)^r \frac{dt}{t} \right)^{1/r} < \infty.$$

We emphasize that the latter integral is multiplied by r/p (as in the original definitions of Lorentz [21, 22]).

We have that $\|f\|_{p,p} = \|f\|_p$. For a fixed p , the Lorentz spaces $L^{p,r}$ strictly increase as the secondary index r increases; that is, the strict embedding $L^{p,r} \subset L^{p,s}$ ($r < s$) holds (see [5, Ch. 4]).

The Lorentz quasinorm can be given in an alternative form. Namely,

$$\|f\|_{p,r} = \left(r \int_0^\infty y^{r-1} \lambda_f(y)^{r/p} dy \right)^{1/r} \quad (2.3)$$

(see [13, Proposition 1.4.9]).

Let f be a rearrangeable function on $\mathbb{R}^n$ and let $1 \leq k \leq n$. We fix $\hat{x}_k \in \mathbb{R}^{n-1}$ and consider the $\hat{x}_k$ -section of the function f

$$f_{\hat{x}_k}(x_k) = f(x_k, \hat{x}_k), \quad x_k \in \mathbb{R}.$$

For almost all $\hat{x}_k \in \mathbb{R}^{n-1}$ the function $f_{\hat{x}_k}$ is rearrangeable on $\mathbb{R}$. We set

$$\mathcal{R}_k f(u, \hat{x}_k) = f_{\hat{x}_k}^*(u), \quad u \in \mathbb{R}_+.$$

Stress that the k -th argument of the function $\mathcal{R}_k f$ is equal to u . The function $\mathcal{R}_k f$ is defined almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$; we call it *the rearrangement of f with respect to the k -th variable*. It is easy to show that $\mathcal{R}_k f$ is a measurable function equimeasurable with $|f|$. As above, let Ω_n be the collection of all permutations of the set $\{1, \dots, n\}$. For each $\sigma = \{k_1, \dots, k_n\} \in \Omega_n$ we call the function

$$\mathcal{R}_\sigma f(t) = \mathcal{R}_{k_n} \cdots \mathcal{R}_{k_1} f(t), \quad t \in \mathbb{R}_+^n,$$

the $\mathcal{R}_\sigma$ -rearrangement of f . Thus, we obtain $\mathcal{R}_\sigma f$ by “rearranging” f in non-increasing order successively with respect to the variables $x_{k_1}, \dots, x_{k_n}$. The rearrangement $\mathcal{R}_\sigma f$ is defined on $\mathbb{R}_+^n$. It is nonnegative, nonincreasing in each variable, and equimeasurable with $|f|$ (see [3, 4, 8, 16, 18, 19]).

Let $0 < p, r < \infty$ and let $\sigma \in \Omega_n$ ($n \geq 2$). We denote by $\mathcal{L}_\sigma^{p,r}(\mathbb{R}^n)$ the class of all rearrangeable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{L}_\sigma^{p,r}} = \|f\|_{p,r;\sigma} = \left(\left(\frac{r}{p} \right)^n \int_{\mathbb{R}_+^n} \left(\prod_{k=1}^n t_k \right)^{r/p-1} \mathcal{R}_\sigma f(t)^r dt \right)^{1/r} < \infty \quad (2.4)$$

(see [8]). The choice of a permutation σ is essential.

The relations between $L^{p,r}$ - and $\mathcal{L}_\sigma^{p,r}$ -norms are described by embeddings (1.4). Moreover, the following proposition gives sharp constants in these relations (see [3, 19]).

Proposition 2.1 *Let f be a rearrangeable function on $\mathbb{R}^n$, $n \geq 2$. Then for any $\sigma \in \Omega_n$*

$$\|f\|_{p,r} \leq \|f\|_{p,r;\sigma} \quad \text{if } 0 < r \leq p < \infty \quad (2.5)$$

and

$$\|f\|_{p,r;\sigma} \leq \|f\|_{p,r} \quad \text{if } 0 < p \leq r < \infty. \quad (2.6)$$

These inequalities are optimal.

We stress again that for $p \neq r$ the norms $\|\cdot\|_{p,r}$ and $\|\cdot\|_{p,r;\sigma}$ may be essentially different. We consider the following simple example which will be also used in the sequel.

As usual, by χ_E we denote the characteristic function of a set E , which is equal to 1 on E and 0 outside E .

Example 2.2 Let $N \geq 1$. Set

$$g_N(x) = \min \left(\frac{1}{x(\ln(e/x))^2}, N \right), \quad 0 < x \leq 1, \\ E_N = \{(x, y) : 0 < x \leq 1, 0 \leq y \leq g_N(x)\}, \quad \text{and} \quad f_N(x, y) = \chi_{E_N}(x, y).$$

Then $|E_N| \leq 1$ and

$$\|f_N\|_{2,1} = \frac{1}{2} \int_0^{|E_N|} t^{-1/2} dt \leq 1.$$

On the other hand,

$$\begin{aligned} \int_0^\infty \int_0^\infty (xy)^{-1/2} f_N(x, y) dx dy &= \int_0^1 x^{-1/2} dx \int_0^{g_N(x)} y^{-1/2} dy \\ &= 2 \int_0^1 x^{-1/2} g_N(x)^{1/2} dx \geq 2 \int_{\alpha_n}^1 \frac{1}{x \ln(e/x)} dx, \end{aligned}$$

where α_N is determined by the equation $\alpha_N (\ln(e/\alpha_N))^2 = 1/N$ and $\alpha_N \rightarrow 0$ as $N \rightarrow \infty$. Thus, $\|f_N\|_{2,1;\sigma} \rightarrow \infty$.

3 Mixed norm spaces

For a measurable set $E \subset \mathbb{R}^d$, we denote by $\text{mes}_d E$ the Lebesgue measure of E in $\mathbb{R}^d$.

Let f be a measurable function on $\mathbb{R}^n$ and let $1 \leq k \leq n$. By Fubini's theorem, for almost all $\hat{x}_k \in \mathbb{R}^{n-1}$ the sections $f_{\hat{x}_k}$ are measurable functions on $\mathbb{R}$. Moreover, the function

$$\psi_k(\hat{x}_k) = \|f_{\hat{x}_k}\|_{L^\infty(\mathbb{R})} = \text{ess sup}_{x_k \in \mathbb{R}} |f(x_k, \hat{x}_k)| \quad (3.1)$$

(defined a.e. on $\mathbb{R}^{n-1}$) is measurable. It suffices to prove the latter statement in the case when f is a bounded function with compact support. In this case we have

$$\psi_k(\hat{x}_k) = \lim_{v \rightarrow \infty} \|f_{\hat{x}_k}\|_{L^v(\mathbb{R})},$$

and the functions $\hat{x}_k \mapsto \|f_{\hat{x}_k}\|_{L^v(\mathbb{R})}$ are measurable by Fubini's theorem.

Denote by $\mathcal{M}_{dec}(\mathbb{R}_+^n)$ the class of all nonnegative functions on $\mathbb{R}^n$ which vanish off $\mathbb{R}_+^n$ and are nonincreasing in each variable on $\mathbb{R}_+^n$.

Let $f \in \mathcal{M}_{dec}(\mathbb{R}_+^n)$. Assume that for almost all $\hat{x}_k \in \mathbb{R}^{n-1}$ the function f is bounded with respect to x_k . Then

$$\psi_k(\hat{x}_k) = \lim_{x_k \rightarrow 0+} f(x_k, \hat{x}_k), \quad \hat{x}_k \in \mathbb{R}^{n-1}.$$

Thus, in this case ψ_k is the *trace* of f on the hyperplane $x_k = 0$ in the sense of almost everywhere convergence (see [24, Chapter 6]).

Let

$$\mathcal{V}_k(\mathbb{R}^n) = L_{\hat{x}_k}^1(\mathbb{R}^{n-1})[L_{x_k}^\infty(\mathbb{R})], \quad 1 \leq k \leq n,$$

be the space of measurable functions on $\mathbb{R}^n$ with the finite mixed norm

$$\|f\|_{\mathcal{V}_k} = \|\psi_k\|_{L^1(\mathbb{R}^{n-1})}, \quad (3.2)$$

where ψ_k is defined by (3.1). As it was observed above, ψ_k is measurable on $\mathbb{R}^{n-1}$, and thus this definition is correct. Denote also

$$\mathcal{V}(\mathbb{R}^n) = \bigcap_{k=1}^n \mathcal{V}_k(\mathbb{R}^n). \quad (3.3)$$

An equivalent form of Theorem 1.2 proved by Fournier [11, Theorem 4.1] is the following.

Theorem 3.1 *If $f \in \mathcal{V}(\mathbb{R}^n)$ ($n \geq 2$), then $f \in L^{n',1}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{n',1}} \leq \left(\prod_{k=1}^n \|f\|_{\mathcal{V}_k} \right)^{1/n}. \quad (3.4)$$

The constant is optimal.

Theorem 3.1 implies the embedding of $W_1^1(\mathbb{R}^n)$ into the Lorentz space $L^{n',1}(\mathbb{R}^n)$. Indeed, let $f \in W_1^1(\mathbb{R}^n)$ and let $D_k f$ denote the first order weak partial derivative of f with respect to x_k . Then

$$\|f\|_{\mathcal{V}_k} \leq \frac{1}{2} \|D_k f\|_{L^1} \quad (k = 1, \dots, n) \quad (3.5)$$

(see [11, p. 57]). Thus, by (3.4),

$$\|f\|_{L^{n',1}} \leq \frac{1}{2} \left(\prod_{k=1}^n \|D_k f\|_{L^1} \right)^{1/n}. \quad (3.6)$$

The following equivalent form of Theorem 1.3 was proved in [19].

Theorem 3.2 *If $f \in \mathcal{V}(\mathbb{R}^n)$ ($n \geq 2$), then $f \in \mathcal{L}_{\sigma}^{n',1}(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{L}_{\sigma}^{n',1}(\mathbb{R}^n)} \leq \left(\prod_{k=1}^n \|f\|_{\mathcal{V}_k} \right)^{1/n}$$

for any $\sigma \in \Omega_n$. The constant is optimal.

Now we introduce another type of mixed norm spaces (related to the traces on coordinate straight lines). Let

$$\mathcal{U}_k(\mathbb{R}^n) = L_{x_k}^1(\mathbb{R})[L_{\hat{x}_k}^\infty(\mathbb{R}^{n-1})], \quad 1 \leq k \leq n, \quad (3.7)$$

be the space of measurable functions on $\mathbb{R}^n$ with the finite mixed norm

$$\|f\|_{\mathcal{U}_k} = \|f_k\|_{L^1(\mathbb{R})}, \quad \text{where } f_k(x_k) = \operatorname{ess\,sup}_{\hat{x}_k \in \mathbb{R}^{n-1}} |f(x)|. \quad (3.8)$$

Denote also

$$\mathcal{U}(\mathbb{R}^n) = \bigcap_{k=1}^n \mathcal{U}_k(\mathbb{R}^n).$$

As above, it is easy to see that these definitions are correct.

Let $f \in \mathcal{M}_{dec}(\mathbb{R}_+^n)$. Assume that for almost all $x_k \in \mathbb{R}$ the function f is bounded with respect to $\hat{x}_k$. Then

$$f_k(x_k) = \lim_{\hat{x}_k \rightarrow 0} f(x_k, \hat{x}_k), \quad x_k \in \mathbb{R}.$$

Thus, in this case f_k is the *trace* of f on the coordinate straight line $\hat{x}_k = 0$ in the sense of almost everywhere convergence.

Remark 3.3 Clearly, for $n = 2$ the spaces $\mathcal{U}$ and $\mathcal{V}$ coincide. It is also easy to see that for $n \geq 3$ neither of the spaces $\mathcal{U}$ and $\mathcal{V}$ is contained in the other. Indeed, let $n = 3$. The function

$$f(x, y, z) = \frac{1}{(x + y + z)^2} \chi_{[1, \infty)^3}(x, y, z)$$

belongs to $\mathcal{U}$ but does not belong to $\mathcal{V}$. On the other hand, the function

$$f(x, y, z) = \frac{1}{x + y + z} \chi_{(0, 1)^3}(x, y, z)$$

belongs to $\mathcal{V}$ but does not belong to $\mathcal{U}$.

By (3.5), $\mathcal{V}_k$ -norms are estimated by L^1 -norms of the first order weak partial derivatives. Similarly, $\mathcal{U}_k$ -norms can be estimated by L^1 -norms of the pure (non-mixed) partial derivatives of the order $n - 1$.

Assume that for a function $f \in L^1(\mathbb{R}^m)$ all pure weak partial derivatives

$$D_k^m f = \frac{\partial^m f}{\partial x_k^m} \quad (k = 1, \dots, m)$$

of the order m exist and belong to $L^1(\mathbb{R}^m)$. Sobolev's theorem asserts that in this case the function f can be modified on a set of measure zero so as to become uniformly continuous and bounded on $\mathbb{R}^m$ and

$$\|f\|_\infty \leq c \sum_{k=1}^m \|D_k^m f\|_1 \quad (3.9)$$

(see [6, §10]).

Assume that a function $f \in L^1(\mathbb{R}^n)$ ($n \geq 2$) has all pure weak partial derivatives of the order $n - 1$. Then for any $1 \leq j \leq n$ and almost every fixed $x_j \in \mathbb{R}$ the function $f_{x_j}(\hat{x}_j) = f(x)$ satisfies the conditions of Sobolev's theorem for $m = n - 1$ and therefore by (3.9)

$$\operatorname{ess\,sup}_{\hat{x}_j \in \mathbb{R}^{n-1}} |f(x)| \leq c \sum_{k \neq j} \int_{\mathbb{R}^{n-1}} |D_k^{n-1} f(x_j, u)| du.$$

This implies that

$$\|f\|_{\mathcal{U}_j} \leq c \sum_{k \neq j} \|D_k^{n-1} f\|_1 \quad (j = 1, \dots, n).$$

4 Embeddings into classical Lorentz spaces

We begin with the following simple theorem.

Theorem 4.1 *Let f_j ($j = 1, \dots, n$) be nonnegative rearrangeable functions on $\mathbb{R}$ and let*

$$f(x) = \min_{1 \leq j \leq n} f_j(x_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (4.1)$$

Then

$$\lambda_f(y) = \prod_{j=1}^n \lambda_{f_j}(y) \quad \text{for any } y > 0. \quad (4.2)$$

If, in addition, $f_j \in L^1(\mathbb{R})$, then $f \in L^{n,1}(\mathbb{R}^n)$ and

$$\|f\|_{n,1} \leq \left(\prod_{j=1}^n \|f_j\|_1 \right)^{1/n}. \quad (4.3)$$

The constant in (4.3) is optimal.

Proof Let $y > 0$. Then $f(x) > y$ if and only if $f_j(x_j) > y$ for all $j = 1, \dots, n$. That is,

$$\{x \in \mathbb{R}^n : f(x) > y\} = \prod_{j=1}^n \{x_j \in \mathbb{R} : f_j(x_j) > y\}. \quad (4.4)$$

This implies (4.2).

Now, applying (2.3), (4.2), and using Hölder's inequality, we get

$$\begin{aligned} \|f\|_{n,1} &= \int_0^\infty \lambda_f(y)^{1/n} dy = \int_0^\infty \left(\prod_{j=1}^n \lambda_{f_j}(y) \right)^{1/n} dy \\ &\leq \left(\prod_{j=1}^n \int_0^\infty \lambda_{f_j}(y) dy \right)^{1/n} = \left(\prod_{j=1}^n \|f_j\|_1 \right)^{1/n}. \end{aligned}$$

If $f_j(u) = \chi_{[0,1]}(u)$, then $f(x) = \chi_{[0,1]^n}(x)$, $f^*(t) = \chi_{[0,1]}(t)$, and we have equality in (4.3). Thus, the constant is optimal. $\square$

Similarly to Theorems 3.1, 4.1 can be expressed in terms of mixed norms defined by (3.8). Namely, the following theorem is equivalent to Theorem 4.1.

Theorem 4.2 *If $f \in \mathcal{U}(\mathbb{R}^n)$ ($n \geq 2$), then $f \in L^{n,1}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{n,1}(\mathbb{R}^n)} \leq \left(\prod_{k=1}^n \|f\|_{\mathcal{U}_k} \right)^{1/n}. \quad (4.5)$$

To show the equivalence, we first assume that $f \in \mathcal{U}(\mathbb{R}^n)$. Let f_k be defined by (3.8). Then

$$|f(x)| \leq \min_{1 \leq k \leq n} f_k(x_k) \quad \text{a.e. on } \mathbb{R}^n.$$

Since $\|f\|_{\mathcal{U}_k} = \|f_k\|_1$, (4.5) follows immediately from (4.3). Conversely, assume that $f_j \in L^1(\mathbb{R})$ ($j = 1, \dots, n$) are nonnegative functions on $\mathbb{R}$ and f is defined by (4.1). Then

$$\operatorname{ess\,sup}_{\hat{x}_k \in \mathbb{R}^{n-1}} f(x) \leq f_k(x_k) \quad \text{and} \quad \|f\|_{\mathcal{U}_k} \leq \|f_k\|_1 \quad (k = 1, \dots, n).$$

Thus, (4.5) implies (4.3).

We observe that without loss of generality we can assume that in (4.1) f_j are nonincreasing functions on $\mathbb{R}_+$ with $f_j(+\infty) = 0$ that vanish off $\mathbb{R}_+$ ($j = 1, \dots, n$). Indeed, let f be defined by (4.1). Let $\sigma \in \Omega_n$ (recall that by Ω_n we denote the collection of all permutations of the set $\{1, \dots, n\}$). We consider the iterated rearrangement $\mathcal{R}_\sigma f(t)$, $t \in \mathbb{R}_+^n$. We have

$$\mathcal{R}_\sigma f(t) = \min_{1 \leq j \leq n} f_j^*(t_j), \quad t = (t_1, \dots, t_n) \in \mathbb{R}_+^n \quad (4.6)$$

for any $\sigma \in \Omega_n$. To show this, it is sufficient to use the following observation: if $g(x) \geq 0$ ($x \in \mathbb{R}$), $a \geq 0$, and $h(x) = \min(g(x), a)$, then $h^*(t) = \min(g^*(t), a)$.

It is easy to obtain the rearrangement of the function $\Phi = \mathcal{R}_\sigma f$ in an explicit form.

Theorem 4.3 Let φ_j be nonnegative nonincreasing and left-continuous functions on $\mathbb{R}_+$ with $\varphi_j(+\infty) = 0$ ($j = 1, \dots, n$). Set

$$\Phi(x) = \min_{1 \leq j \leq n} \varphi_j(x_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}_+^n. \quad (4.7)$$

Then

$$\Phi^*(t) = \sup\{\Phi(u) : \prod_{j=1}^n u_j = t, \quad u_j > 0\} \quad (4.8)$$

for any $t > 0$.

Proof Fix $t > 0$. Assume that

$$\prod_{j=1}^n u_j = t, \quad u_j > 0.$$

Since the function Φ decreases in each variable $x_j \in \mathbb{R}_+$, we have $\Phi(x) \geq \Phi(u)$ for any $x \in (0, u_1] \times \dots \times (0, u_n]$, that is, for any x from a set of measure t . By (2.2), this implies that $\Phi^*(t) \geq \Phi(u)$. Denoting

$$\omega(t) = \sup\{\Phi(u) : \prod_{j=1}^n u_j = t, \quad u_j > 0\},$$

we have that $\Phi^*(t) \geq \omega(t)$. Let $\varepsilon > 0$. Denote $y = \Phi^*(t) - \varepsilon$. Then $\lambda_\Phi(y) \geq t$. Thus, by (4.2),

$$\prod_{j=1}^n \lambda_{\varphi_j}(y) \geq t.$$

Therefore, there exist $\tilde{u}_j \leq \lambda_{\varphi_j}(y)$ ($j = 1, \dots, n$) such that

$$\prod_{j=1}^n \tilde{u}_j = t. \quad (4.9)$$

Since the functions φ_j are nonincreasing and left-continuous, we have

$$\varphi_j(\tilde{u}_j) \geq \varphi_j(\lambda_{\varphi_j}(y)) \geq y, \quad j = 1, \dots, n.$$

This implies that $\Phi(\tilde{u}) \geq y$ ($\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)$) and, by (4.9), we obtain that $\omega(t) \geq y = \Phi^*(t) - \varepsilon$. It follows that $\omega(t) \geq \Phi^*(t)$. Thus, $\Phi^*(t) = \omega(t)$. $\square$

Remark 4.4 Assume, in addition, that all functions φ_j are continuous and strictly decreasing on $\mathbb{R}_+$. Then there exist functions u_j on $\mathbb{R}_+$ such that for any $t > 0$

$$\Phi^*(t) = \varphi_j(u_j(t)), \quad j = 1, \dots, n, \quad (4.10)$$

and

$$\prod_{j=1}^n u_j(t) = t. \quad (4.11)$$

Indeed, taking into account that $\Phi^*(t) < \varphi_j(+0)$ for any $t > 0$ and any $j = 1, \dots, n$, define

$$u_j(t) = \varphi_j^{-1}(\Phi^*(t)).$$

Then we have (4.10). Further, $(\Phi^*)^{-1}(y) = \lambda_\Phi(y)$ and $\varphi_j^{-1}(y) = \lambda_{\varphi_j}(y)$. Thus, by (4.2),

$$\prod_{j=1}^n \varphi_j^{-1}(y) = (\Phi^*)^{-1}(y).$$

Setting $y = \Phi^*(t)$, we obtain that

$$\prod_{j=1}^n u_j(t) = \prod_{j=1}^n \varphi_j^{-1}(\Phi^*(t)) = (\Phi^*)^{-1}(\Phi^*(t)) = t,$$

and (4.11) holds.

Now we show that inequality (4.3) implies Fournier's inequality (1.3). First, we have the following theorem.

Theorem 4.5 *Let ψ_j ($j = 1, \dots, n$, $n \geq 2$) be nonnegative rearrangeable functions on $\mathbb{R}^{n-1}$. Set*

$$\Psi(x) = \min_{1 \leq j \leq n} \psi_j(\hat{x}_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (4.12)$$

Let $\varphi_j = \psi_j^$ and*

$$\Phi(u) = \min_{1 \leq j \leq n} \varphi_j(u_j), \quad u \in \mathbb{R}_+^n. \quad (4.13)$$

Then

$$\Psi^*(t) \leq \Phi^*(t^{n-1}) \quad \text{for all } t > 0 \quad (4.14)$$

and

$$\|\Psi\|_{n',1} \leq \|\Phi\|_{n,1}. \quad (4.15)$$

Proof Let $t > 0$ and let $E \subset \mathbb{R}^n$ be an F_σ -set with $\text{mes}_n E = t$. Let E_j be the projection of E onto the hyperplane $x_j = 0$ ($j = 1, \dots, n$). Then for any $x \in E$ and any $j = 1, \dots, n$

$$\inf_{\hat{x}_j \in E_j} \psi_j(\hat{x}_j) \geq \inf_{x \in E} \Psi(x). \quad (4.16)$$

Set

$$u_j = \text{mes}_{n-1} E_j \quad \text{and} \quad \mu = \prod_{j=1}^n u_j.$$

By (2.2) and (4.16),

$$\varphi_j(u_j) = \psi_j^*(u_j) \geq \inf_{x \in E} \Psi(x) \quad (j = 1, \dots, n).$$

Further, the functions φ_j are left-continuous because they are defined as nonincreasing rearrangements (see (2.1)). Therefore, by Theorem 4.3, for all $t > 0$ we have equality (4.8). Thus,

$$\Phi^*(\mu) \geq \inf_{x \in E} \Psi(x). \quad (4.17)$$

By the Loomis–Whitney inequality (1.14),

$$\mu = \prod_{j=1}^n \text{mes}_{n-1} E_j \geq (\text{mes}_n E)^{n-1} = t^{n-1}.$$

Hence, by (4.17),

$$\Phi^*(t^{n-1}) \geq \inf_{x \in E} \Psi(x).$$

Here $E \subset \mathbb{R}^n$ is an arbitrary F_σ -set with $\text{mes}_n E = t$. Thus, by (2.2), the latter inequality implies (4.14).

Now, applying inequality (4.14), we obtain

$$\begin{aligned} \int_0^\infty t^{1/n'-1} \Psi^*(t) dt &\leq \int_0^\infty t^{1/n'-1} \Phi^*(t^{n-1}) dt \\ &= \frac{1}{n-1} \int_0^\infty u^{1/n-1} \Phi^*(u) du. \end{aligned}$$

This implies inequality (4.15). $\square$

Inequalities (4.15) and (4.3) yield the following strengthening of inequality (1.3).

Corollary 4.6 *Let $\psi_j \in L^1(\mathbb{R}^{n-1})$ ($j = 1, \dots, n$, $n \geq 2$) be nonnegative functions. Set*

$$\Psi(x) = \min_{1 \leq j \leq n} \psi_j(\hat{x}_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

and

$$\Phi(u) = \min_{1 \leq j \leq n} \psi_j^*(u_j), \quad u \in \mathbb{R}_+^n.$$

Then

$$\|\Psi\|_{n',1} \leq \|\Phi\|_{n,1} \leq \left(\prod_{k=1}^n \|\psi_k\|_{L^1} \right)^{1/n}. \quad (4.18)$$

.

In comparison with (1.3), inequality (4.18) contains the intermediate term $\|\Phi\|_{n,1}$. We shall see that this term can be essentially greater than $\|\Psi\|_{n',1}$.

Remark 4.7 First, we observe that for $n = 2$ we have $\Psi^*(t) = \Phi^*(t)$ for all $t > 0$. Indeed, let $t > 0$. Assume that $u_1, u_2 \in \mathbb{R}_+$, $u_1 u_2 = t$. There exist F_σ -sets $E_1, E_2 \subset \mathbb{R}$ such that

$$\text{mes}_1 E_j = u_j \quad \text{and} \quad \varphi_j(u_j) = \psi_j^*(u_j) = \inf_{x_j \in E_j} \psi_j(x_j) \quad (j = 1, 2).$$

Let $E = E_1 \times E_2$. Then $\text{mes}_2 E = t$ and

$$\min(\varphi_1(u_1), \varphi_2(u_2)) = \inf_{(x_1, x_2) \in E} \Psi(x_1, x_2) \leq \Psi^*(t).$$

By (4.8), this implies that $\Phi^*(t) \leq \Psi^*(t)$, and we obtain that $\Phi^*(t) = \Psi^*(t)$.

Similar arguments fail for $n \geq 3$. Indeed, let $n = 3$ and assume that $0 < \varepsilon < 1$. Let $I = [0, \varepsilon] \times [0, 1/\varepsilon]$. Set $\psi_j = \chi_I$ ($j = 1, 2, 3$). Then

$$\begin{aligned} \Psi(x_1, x_2, x_3) &= \min(\psi_1(x_2, x_3), \psi_2(x_1, x_3), \psi_3(x_1, x_2)) \\ &= \chi_{P_\varepsilon}(x_1, x_2, x_3), \end{aligned}$$

where $P_\varepsilon = [0, \varepsilon] \times [0, \varepsilon] \times [0, 1/\varepsilon]$. Thus, $\Psi^*(t) = \chi_{[0, \varepsilon]}(t)$. On the other hand, $\psi_j^*(u) = \chi_{[0, 1]}(u)$ ($j = 1, 2, 3$), and

$$\Phi(u_1, u_2, u_3) = \chi_{[0, 1]^3}(u_1, u_2, u_3), \quad \Phi^*(t) = \chi_{[0, 1]}(t).$$

We have $\|\Phi\|_{3,1} = 1$ and $\|\Psi\|_{3/2,1} = \varepsilon^{2/3}$. Thus, for $n \geq 3$ inequality (4.3) (with $f_j = \psi_j^*$) is essentially stronger than (1.3).

5 Estimates of generalized Lorentz norms

In this section we obtain our main results.

As above, by Ω_n we denote the collection of all permutations of the set $\{1, \dots, n\}$.

First, we prove the following theorem.

Theorem 5.1 *Let $f_j \in L^1(\mathbb{R})$ ($j = 1, \dots, n$) be nonnegative rearrangeable functions on $\mathbb{R}$ and let*

$$f(x) = \min_{1 \leq j \leq n} f_j(x_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (5.1)$$

Let $\alpha_j > 0$ ($j = 1, \dots, n$). Then for any $\sigma \in \Omega_n$

$$\int_{\mathbb{R}_+^n} \prod_{j=1}^n x_j^{\alpha_j-1} \mathcal{R}_\sigma f(x) dx = \left(\prod_{j=1}^n \alpha_j \right)^{-1} \int_0^\infty \prod_{j=1}^n \lambda_{f_j}(y)^{\alpha_j} dy. \quad (5.2)$$

Proof Set

$$\varphi_j = f_j^* \quad \text{and} \quad \Phi(x) = \min_{1 \leq j \leq n} \varphi_j(x_j), \quad x \in \mathbb{R}_+^n. \quad (5.3)$$

By (4.6), for any $\sigma \in \Omega_n$

$$\mathcal{R}_\sigma f(x) dx = \Phi(x) \quad \text{for all } x \in \mathbb{R}_+^n. \quad (5.4)$$

Thus, (5.2) is equivalent to the equality

$$\int_{\mathbb{R}_+^n} \prod_{j=1}^n x_j^{\alpha_j-1} \Phi(x) dx = \left(\prod_{j=1}^n \alpha_j \right)^{-1} \int_0^\infty \prod_{j=1}^n \lambda_{\varphi_j}(y)^{\alpha_j} dy. \quad (5.5)$$

We may assume that all functions φ_j are continuously differentiable, $\varphi_j(+0) = +\infty$, and

$$\varphi_j(t) > 0, \quad \varphi_j'(t) < 0 \quad \text{for all } t > 0 \quad \text{and all } j = 1, \dots, n.$$

Then we have $\varphi_j(\mathbb{R}_+) = \mathbb{R}_+$ ($j = 1, \dots, n$). Denote

$$\eta_j(y) = \varphi_j^{-1}(y) = \lambda_{\varphi_j}(y), \quad y > 0, \quad j = 1, \dots, n.$$

Let A_j be the set of all $x \in \mathbb{R}_+^n$ such that $\Phi(x) = \varphi_j(x_j)$, $j = 1, \dots, n$. Then

$$\bigcup_{j=1}^n A_j = \mathbb{R}_+^n \quad \text{and} \quad \text{mes}_n(A_i \cap A_j) = 0 \quad \text{if } i \neq j.$$

Indeed, the first equality is obvious. Further, let $i \neq j$. Then $x \in A_i \cap A_j$ if and only if $x_j = \eta_j(\varphi_i(x_i))$. Thus, the projection of the set $A_i \cap A_j$ onto the 2-dimensional plane (x_i, x_j) has the 2-dimensional measure zero and therefore $\text{mes}_n(A_i \cap A_j) = 0$.

Denote $\gamma_{j,k}(x_k) = \eta_j(\varphi_k(x_k))$, $x_k \in \mathbb{R}_+$. For a fixed $1 \leq j \leq n$ we have

$$\begin{aligned} Q_j &\equiv \int_{A_j} \prod_{k=1}^n x_k^{\alpha_k-1} \Phi(x) dx \\ &= \int_{\mathbb{R}_+^{n-1}} \prod_{k \neq j} x_k^{\alpha_k-1} \int_{\sigma_j(\hat{x}_j)}^\infty x_j^{\alpha_j-1} \varphi_j(x_j) dx_j d\hat{x}_j, \end{aligned}$$

where $\sigma_j(\hat{x}_j) = \max_{k \neq j} \gamma_{j,k}(x_k)$. Indeed, $x \in A_j$ if and only if $\hat{x}_j \in \mathbb{R}_+^{n-1}$ and $x_j \geq \gamma_{j,k}(x_k)$ for any $k \neq j$. Thus, by Fubini's theorem, we have

$$\begin{aligned} Q_j &= \int_0^\infty x_j^{\alpha_j-1} \varphi_j(x_j) \prod_{k \neq j} \int_0^{\gamma_{k,j}(x_j)} x_k^{\alpha_k-1} dx_k dx_j \\ &= \left(\prod_{k \neq j} \alpha_k \right)^{-1} \int_0^\infty x_j^{\alpha_j-1} \varphi_j(x_j) \prod_{k \neq j} \eta_k(\varphi_j(x_j))^{\alpha_k} dx_j. \end{aligned}$$

Change of variable $y = \varphi_j(x_j)$ gives

$$\begin{aligned} \mathcal{Q}_j &= - \left(\prod_{k \neq j} \alpha_k \right)^{-1} \int_0^\infty \eta_j(y)^{\alpha_j-1} \prod_{k \neq j} \eta_k(y)^{\alpha_k} y \, d\eta_j(y) \\ &= - \left(\prod_{k \neq j} \alpha_k \right)^{-1} \int_0^\infty \prod_{k=1}^n \eta_k(y)^{\alpha_k} y \, d(\ln \eta_j(y)) \\ &= - \left(\prod_{k=1}^n \alpha_k \right)^{-1} \int_0^\infty \prod_{k=1}^n \eta_k(y)^{\alpha_k} y \, d(\ln[\eta_j(y)^{\alpha_j}]). \end{aligned}$$

From here, setting

$$c_n = \left(\prod_{k=1}^n \alpha_k \right)^{-1} \quad \text{and} \quad \zeta(y) = \prod_{k=1}^n \eta_k(y)^{\alpha_k},$$

we obtain

$$\begin{aligned} \int_0^\infty \prod_{k=1}^n x_k^{\alpha_k-1} \Phi(x) \, dx &= \sum_{j=1}^n \mathcal{Q}_j \\ &= -c_n \int_0^\infty \zeta(y) y \, d \left(\sum_{j=1}^n \ln[\eta_j(y)^{\alpha_j}] \right) \\ &= -c_n \int_0^\infty \zeta(y) y \, d(\ln \zeta(y)) \\ &= -c_n \int_0^\infty y \, d\zeta(y) = c_n \int_0^\infty \zeta(y) \, dy. \end{aligned}$$

Since $\eta_k(y) = \lambda_{\varphi_k}(y)$, this implies (5.5). $\square$

Corollary 5.2 Assume, in addition, that

$$\sum_{k=1}^n \alpha_k = 1.$$

Then for any $\sigma \in \Omega_n$

$$\int_{\mathbb{R}_+^n} \prod_{k=1}^n x_k^{\alpha_k-1} \mathcal{R}_\sigma f(x) \, dx \leq \left(\prod_{j=1}^n \alpha_j \right)^{-1} \prod_{j=1}^n \|f_j\|_1^{\alpha_j}. \quad (5.6)$$

Indeed, applying Hölder's inequality at the right-hand side of (5.2), we obtain

$$\int_{\mathbb{R}_+^n} \prod_{j=1}^n \lambda_{f_j}(y)^{\alpha_j} dy \leq \prod_{j=1}^n \left(\int_0^\infty \lambda_{f_j}(y) dy \right)^{\alpha_j} = \prod_{j=1}^n \left(\int_0^\infty f_j(t) dt \right)^{\alpha_j}.$$

In the case $f_j(t) = \chi_{[0,1]}(t)$ ($j = 1, \dots, n$) we have equality in (5.6). This shows that the constant in (5.6) is optimal.

Now we suppose that, in Theorem 5.1, $\alpha_1 = \dots = \alpha_n = 1/p$. Then equality (5.2) assumes the form

$$\int_{\mathbb{R}_+^n} \prod_{j=1}^n x_j^{1/p-1} \mathcal{R}_\sigma f(x) dx = p^n \int_0^\infty \prod_{j=1}^n \lambda_{f_j}(y)^{1/p} dy. \quad (5.7)$$

By (5.3), (4.2), and (5.4),

$$\prod_{j=1}^n \lambda_{f_j}(y) = \lambda_\Phi(y) = \lambda_f(y).$$

Hence, using (2.3), we obtain that

$$\frac{1}{p^n} \int_{\mathbb{R}_+^n} \prod_{j=1}^n x_j^{1/p-1} \mathcal{R}_\sigma f(x) dx = \int_0^\infty \lambda_f(y)^{1/p} dy = \|f\|_{p,1} \quad (5.8)$$

for any $\sigma \in \Omega_n$. Thus, we have the following statement.

Theorem 5.3 *Let $f_j \in L^1(\mathbb{R})$ ($j = 1, \dots, n$) be nonnegative functions on $\mathbb{R}$ and let*

$$f(x) = \min_{1 \leq j \leq n} f_j(x_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (5.9)$$

Then for any $p > 0$

$$\|f\|_{\mathcal{L}_\sigma^{p,1}} = \|f\|_{p,1} \quad \text{for any } \sigma \in \Omega_n. \quad (5.10)$$

We have emphasized in Sect. 2 that, in general, the spaces $\mathcal{L}_{p,1;\sigma}(\mathbb{R}^n)$ are strictly smaller than $L_{p,1}(\mathbb{R}^n)$ (see Example 2.2). We shall now analyse the situation that we have in Theorem 5.3.

First, denote by $\mathcal{Q}_n$ the class of all measurable functions on $\mathbb{R}^n$ such that for any $y > 0$ the set

$$E(y) = \{x \in \mathbb{R}^n : |f(x)| > y\} \quad (5.11)$$

is essentially a cube in $\mathbb{R}^n$ (that is, for any $y > 0$ there exists a cube $Q \subset \mathbb{R}^n$ with edges parallel to coordinate axes which differs from $E_f(y)$ by a set of measure 0). It was shown in [19] that for any function $f \in \mathcal{Q}_n$ ($n \geq 2$)

$$\|f\|_{\mathcal{L}_\sigma^{n',1}} = \|f\|_{n',1} \quad \text{for any } \sigma \in \Omega_n. \quad (5.12)$$

We shall show that equality (5.12) is true for a much wider class of functions.

Denote by $\mathcal{P}_n$ the class of all measurable functions on $\mathbb{R}^n$ such that for any $y > 0$ the set (5.11) is a cartesian product of measurable sets $E_i(y) \subset \mathbb{R}$,

$$E(y) = E_1(y) \times \cdots \times E_n(y). \quad (5.13)$$

By virtue of equality (4.4), any function f defined by (4.1) belongs to $\mathcal{P}_n$. We shall prove the inverse statement.

Proposition 5.4 *Let $f \in \mathcal{P}_n$ ($n \geq 2$) be a nonnegative rearrangeable function on $\mathbb{R}^n$. Set*

$$f_k(x_k) = \operatorname{ess\,sup}_{\hat{x}_k \in \mathbb{R}^{n-1}} f(x), \quad k = 1, \dots, n.$$

Then

$$f(x) = \min_{1 \leq k \leq n} f_k(x_k) \quad \text{for almost all } x \in \mathbb{R}^n. \quad (5.14)$$

Proof First, we have for all $x \in \mathbb{R}^n$

$$f(x) \leq \min_{1 \leq k \leq n} f_k(x_k). \quad (5.15)$$

Fix $x \in \mathbb{R}^n$ and denote $y = f(x)$. Let

$$E(y) = \{z \in \mathbb{R}^n : f(z) > y\}.$$

Then $E(y)$ is a cartesian product (5.13), where $E_i(y) \subset \mathbb{R}$. Since $x \notin E(y)$, there exists $j \in \{1, \dots, n\}$ such that $x_j \notin E_j(y)$. Whatever be $u \in \mathbb{R}^{n-1}$, we have $(x_j, u) \notin E_y$ and therefore

$$f_j(x_j) = \operatorname{ess\,sup}_{u \in \mathbb{R}^{n-1}} f(x_j, u) \leq y.$$

Thus, $f_j(x_j) \leq f(x)$. Taking into account (5.15), we obtain (5.14). $\square$

Applying Proposition 5.4 and Theorem 5.3, we get the following result.

Theorem 5.5 *Let $f \in \mathcal{P}_n$ ($n \geq 2$) be a rearrangeable function on $\mathbb{R}^n$. Then for any $p > 0$*

$$\|f\|_{\mathcal{L}_\sigma^{p,1}} = \|f\|_{p,1} \quad \text{for any } \sigma \in \Omega_n. \quad (5.16)$$

Remark 5.6 We observe that the coincidence of the classical and generalized Lorentz norms which holds for any function f defined by (5.9), may not hold for the function Ψ defined by (4.12). Let $N \geq 1$ and let g_N be the function defined in Example 2.2.

Set

$$\begin{aligned}\psi_1(y, z) &= \begin{cases} 2 & \text{if } 0 \leq y \leq N, 0 \leq z \leq 1, \\ 0 & \text{otherwise,} \end{cases} \\ \psi_2(x, z) &= \begin{cases} 2 & \text{if } 0 \leq x \leq 1, 0 \leq z \leq 1, \\ 0 & \text{otherwise,} \end{cases} \\ \psi_3(x, y) &= \begin{cases} 1 & \text{if } 0 \leq x \leq 1, 0 \leq y \leq g_N(x), \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Let

$$\Psi_N(x, y, z) = \min(\psi_1(y, z), \psi_2(x, z), \psi_3(x, y)).$$

Then $\Psi_N = \chi_{E_N}$, where

$$E_N = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq g_N(x), 0 \leq z \leq 1\}.$$

We have $|E_N| \leq 1$ and $\|\Psi_N\|_{2,1} \leq 1$. At the same time,

$$\iiint_{E_N} (xyz)^{-1/2} dx dy dz \geq c \ln \ln N \quad (c > 0).$$

Thus, $\|\Psi_N\|_{2,1;\sigma} \rightarrow \infty$ as $N \rightarrow \infty$.

Finally, we return to Corollary 5.2. Taking $\alpha_k = 1/n$ ($k = 1, \dots, n$), we obtain

Theorem 5.7 *Let $f_k \in L^1(\mathbb{R})$ ($k = 1, \dots, n$) be nonnegative functions on $\mathbb{R}$ and let f be defined by (4.1). Then for any $\sigma \in \Omega_n$*

$$\|f\|_{\mathcal{L}_\sigma^{n,1}} \leq \left(\prod_{j=1}^n \|f_j\|_1 \right)^{1/n}, \quad (5.17)$$

and the constant is optimal.

Similarly to Theorems 4.1, 5.7 can be expressed in terms of mixed norms defined by (3.8). Namely, the following theorem is equivalent to Theorem 5.7.

Theorem 5.8 *Assume that $f \in \mathcal{U}(\mathbb{R}^n)$ ($n \geq 2$). Then $f \in \mathcal{L}_\sigma^{n,1}(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{L}_\sigma^{n,1}} \leq \left(\prod_{k=1}^n \|f\|_{\mathcal{U}_k} \right)^{1/n} \quad (5.18)$$

for any $\sigma \in \Omega_n$.

The equivalence follows by standard arguments. First, assume that $f \in \mathcal{U}(\mathbb{R}^n)$. We have

$$|f(x)| \leq \operatorname{ess\,sup}_{\hat{x}_k \in \mathbb{R}^{n-1}} |f(x_k, \hat{x}_k)| \quad (k = 1, \dots, n)$$

for almost all $x \in \mathbb{R}^n$. Denote the right-hand side of the above inequality by $f_k(x_k)$ and set

$$\tilde{f}(x) = \min_{1 \leq k \leq n} f_k(x_k).$$

We have $\|f\|_{\mathcal{U}_k} = \|f_k\|_1$ and $|f(x)| \leq \tilde{f}(x)$. Thus, (5.18) follows immediately from Theorem 5.7.

Conversely, under the conditions of Theorem 5.7,

$$\operatorname{ess\,sup}_{\hat{x}_k \in \mathbb{R}^{n-1}} f(x_k, \hat{x}_k) \leq f_k(x_k) \quad (k = 1, \dots, n)$$

and therefore $\|f\|_{\mathcal{U}_k} \leq \|f_k\|_1$. Thus, (5.17) follows from Theorem 5.8.

However, there is a difference between inequalities (5.17) and (5.18). Namely, by Theorem 5.3, the left-hand side of (5.17) coincides with $\|f\|_{n,1}$. At the same time, the condition $f \in \mathcal{U}(\mathbb{R}^n)$ doesn't imply such coincidence, and the norm $\|f\|_{\mathcal{L}_{\sigma}^{n,1}}$ at the left-hand side of (5.18) may be much greater than $\|f\|_{n,1}$ (see Example 2.2).

6 Comparison of estimates (1.13) and (1.6)

In Sect. 4 we obtained inequality (4.15) between classical Lorentz norms of functions (4.12) and (4.13). In this section we prove a similar inequality between generalized Lorentz norms of these functions. For this, we apply a special bijection of $\mathbb{R}_+^n$ onto $\mathbb{R}_+^n$.

Assume that $u \in \mathbb{R}_+^m$, $m \geq 2$. Then

$$\prod_{k=1}^m \prod_{j \neq k} u_j = \left(\prod_{j=1}^m u_j \right)^{m-1}. \quad (6.1)$$

In what follows we denote

$$\pi_m(u) = \prod_{k=1}^m u_k \quad \text{for any } u \in \mathbb{R}_+^m. \quad (6.2)$$

Lemma 6.1 *Let $n \geq 2$ and let the mapping $y = \Lambda(x)$ be defined by*

$$y_j = \pi_{n-1}(\hat{x}_j), \quad j = 1, \dots, n, \quad x \in \mathbb{R}_+^n. \quad (6.3)$$

Then Λ is a differentiable bijective mapping from $\mathbb{R}_+^n$ onto $\mathbb{R}_+^n$ with the jacobian

$$|J\Lambda(x)| = (n-1)\pi_n(x)^{n-2}, \quad x \in \mathbb{R}_+^n. \quad (6.4)$$

Proof First we show that Λ is injective. For $n = 2$ it is obvious. Let $n \geq 3$. Assume that there exist $x', x'' \in \mathbb{R}_+^n$ such that

$$x'_1 > x''_1 \quad \text{but} \quad y' = y'', \quad \text{where} \quad y' = \Lambda(x'), \quad y'' = \Lambda(x''). \quad (6.5)$$

By our assumption,

$$x'_1 \prod_{j \neq 1, k} x'_j = x''_1 \prod_{j \neq 1, k} x''_j \quad \text{for all} \quad k = 2, \dots, n.$$

Thus

$$\prod_{j \neq 1, k} x'_j < \prod_{j \neq 1, k} x''_j \quad \text{for all} \quad k = 2, \dots, n.$$

Multiplying these inequalities and using (6.1), we obtain that

$$\left(\prod_{k=2}^n x'_k \right)^{n-2} < \left(\prod_{k=2}^n x''_k \right)^{n-2} \quad \text{and thus} \quad y'_1 < y''_1,$$

which contradicts (6.5).

Next, we show that Λ is surjective. Let $y \in \mathbb{R}_+^n$. Set

$$x_j = \pi_n(y)^{1/(n-1)} / y_j, \quad j = 1, \dots, n.$$

Then

$$\pi_n(x) = \pi_n(y)^{n/(n-1)} / \pi_n(y) = \pi_n(y)^{1/(n-1)}.$$

Thus,

$$y_j = \pi_n(x) / x_j = \pi_{n-1}(\hat{x}_j) \quad (j = 1, \dots, n), \quad \text{and} \quad y = \Lambda(x).$$

It remains to prove equality (6.4). Fix $x \in \mathbb{R}_+^n$ and consider the Jacobi matrix of Λ at the point x . The i -th row of this matrix is formed by the partial derivatives of the function

$$x \mapsto \pi_{n-1}(\hat{x}_i), \quad x \in \mathbb{R}^n.$$

Assume that $x_k \neq 0$ for all $k = 1, \dots, n$. The j -th element of the i -th row is equal to

$$\prod_{k \neq i, j} x_k = \frac{\pi_n(x)}{x_i x_j} \quad \text{if} \quad i \neq j,$$

and it is equal to 0 if $i = j$. It is easily verified that the determinant of this matrix is equal to $\pi_n(x)^{n-2} \det(\gamma_{ij})$, where

$$\gamma_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

Adding to the first row of the matrix (γ_{ij}) all other rows, we obtain the matrix $(\tilde{\gamma}_{ij})$, where $\tilde{\gamma}_{1j} = n - 1$ and $\tilde{\gamma}_{ij} = \gamma_{ij}$ for $i \geq 2$ ($j = 1, \dots, n$). Next, from all rows of the matrix $(\tilde{\gamma}_{ij})$ beginning from the second, we subtract its first row divided by $n - 1$. We get a triangular matrix with the diagonal $n - 1, -1, \dots, -1$. Thus, $\det(\gamma_{ij}) = (-1)^{n-1}(n - 1)$, and we obtain equality (6.4). $\square$

As above, we denote by $\mathcal{M}_{dec}(\mathbb{R}_+^m)$ the class of all nonnegative functions on $\mathbb{R}^m$ which vanish off $\mathbb{R}_+^m$ and are nonincreasing in each variable on $\mathbb{R}_+^m$. If $f \in \mathcal{M}_{dec}(\mathbb{R}_+^m)$, then

$$f(u) \leq f^*(\pi_m(u)), \quad u \in \mathbb{R}_+^m. \quad (6.6)$$

Indeed,

$$f(v) \geq f(u) \quad \text{for any } v \in E_u = (0, u_1] \times \dots \times (0, u_m],$$

and $\text{mes}_m E_u = \pi_m(u)$. Applying (2.2), we obtain (6.6).

Theorem 6.2 Assume that $n \geq 2$ and $\psi_j \in L^1(\mathbb{R}^{n-1})$ ($j = 1, \dots, n$). Set

$$\Psi(x) = \min_{1 \leq j \leq n} \psi_j(\hat{x}_j), \quad x \in \mathbb{R}^n.$$

Let $\varphi_j(t) = \psi_j^*(t)$, $t \in \mathbb{R}_+$, and

$$\Phi(x) = \min_{1 \leq j \leq n} \varphi_j(x_j), \quad x \in \mathbb{R}_+^n.$$

Then for any $\sigma \in \Omega_n$

$$\int_{\mathbb{R}_+^n} \pi_n(x)^{-1/n} \mathcal{R}_\sigma \Psi(x) dx \leq \frac{1}{n-1} \int_{\mathbb{R}_+^n} \pi_n(x)^{-1/n'} \Phi(x) dx. \quad (6.7)$$

Proof Let $\sigma \in \Omega_n$ and let σ_k be obtained from σ by removal of k . Then for any $1 \leq k \leq n$

$$\mathcal{R}_\sigma \Psi(u) \leq \tilde{\psi}_k(\hat{u}_k), \quad \text{where } \tilde{\psi}_k(\hat{u}_k) = \mathcal{R}_{\sigma_k} \psi_k(\hat{u}_k).$$

Set

$$\tilde{\Psi}(u) = \min_{1 \leq k \leq n} \tilde{\psi}_k(\hat{u}_k).$$

Then $\mathcal{R}_\sigma \Psi(u) \leq \tilde{\Psi}(u)$, $u \in \mathbb{R}_+^n$, and $\tilde{\psi}_k^*(t) = \psi_k^*(t)$, $t > 0$. Since $\tilde{\psi}_k \in \mathcal{M}_{dec}(\mathbb{R}_+^{n-1})$ and $\tilde{\psi}_k^* = \varphi_k$, we have by (6.6)

$$\tilde{\psi}_k(\hat{x}_k) \leq \varphi_k(\pi_{n-1}(\hat{x}_k)), \quad k = 1, \dots, n. \quad (6.8)$$

We consider the mapping $y = \Lambda(x)$ from $\mathbb{R}_+^n$ to $\mathbb{R}_+^n$ defined in Lemma 6.1,

$$y_j = \pi_{n-1}(\hat{x}_j), \quad j = 1, \dots, n, \quad x \in \mathbb{R}_+^n. \quad (6.9)$$

By Lemma 6.1, Λ is bijective and $|J\Lambda(x)| = (n-1)\pi_n(x)^{n-2}$. For $y = \Lambda(x)$ we have $\pi_n(y) = \pi_n(x)^{n-1}$ (see (6.1)). Thus, the inverse mapping $G = \Lambda^{-1}$ satisfies the equality

$$|JG(y)| = \frac{1}{|J\Lambda(x)|} = \frac{1}{(n-1)\pi_n(y)^{(n-2)/(n-1)}}, \quad y = \Lambda(x). \quad (6.10)$$

Applying inequality (6.8), performing the change of variables (6.9), and using equality (6.10), we get

$$\begin{aligned} \int_{\mathbb{R}_+^n} \pi_n(x)^{-1/n} \tilde{\Psi}(x) dx &\leq \int_{\mathbb{R}_+^n} \pi_n(x)^{-1/n} \min_{1 \leq j \leq n} \varphi_j(\pi_{n-1}(\hat{x}_j)) dx \\ &= \int_{\mathbb{R}_+^n} \pi_n(y)^{-1/(n(n-1))} \Phi(y) |JG(y)| dy = \frac{1}{n-1} \int_{\mathbb{R}_+^n} \pi_n(y)^{1/n-1} \Phi(y) dy. \end{aligned}$$

Since $\mathcal{R}_\sigma \Psi(x) \leq \tilde{\Psi}(x)$, this yields (6.7). $\square$

Remark 6.3 The example given above in Remark 4.7 shows that inequality (6.7) cannot be reverted, even by inserting an arbitrarily small constant to the right-hand side. Namely, in this example

$$\iiint_{\mathbb{R}_+^3} (x_1 x_2 x_3)^{-1/3} \Psi(x_1, x_2, x_3) dx_1 dx_2 dx_3 = 27\varepsilon^{2/3}/8$$

and

$$\iiint_{\mathbb{R}_+^3} (x_1 x_2 x_3)^{-2/3} \Phi(x_1, x_2, x_3) dx_1 dx_2 dx_3 = 27.$$

Theorem 6.2 and Remark 6.3 show that inequality (1.13) (with $f_j = \psi_j^*$) is stronger than inequality (1.6).

However, we observe that inequality (6.6) which was used in the proof of Theorem 6.2 is rather rough. Apparently, the constant in inequality (6.7) may be improved. Namely, Theorems 1.3 and 5.7 suggest that the optimal constant should be $(n-1)^{-n}$.

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