Caccioppoli-type estimates and Hardy-type inequalities derived from weighted p -harmonic problems

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Abstract

We obtain Caccioppoli-type estimates for nontrivial and nonnegative solutions to anticoercive partial differential inequalities of elliptic type involving weighted p -Laplacian: $-\Delta_{p,a}u := -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) \geq b(x)\Phi(u)\chi_{\{u>0\}}$, where u is defined in a domain Ω . Using Caccioppoli-type estimates, we obtain several variants of Hardy-type inequalities in weighted Sobolev spaces.

Keywords p -Harmonic PDEs · p -Laplacian · Nonlinear eigenvalue problems · Degenerate PDEs · Quasilinear PDEs

Mathematics Subject Classification Primary 26D10; Secondary 35D30 · 35J60 · 35R45

1 Introduction

In this paper we investigate nonnegative solutions $u : \Omega \rightarrow \mathbf{R}$ to the partial differential inequality (PDI):

$$-\Delta_{p,a}u \geq b(x)\Phi(u)\chi_{\{u>0\}}, \quad (1.1)$$

where $\Omega \subseteq \mathbf{R}^n$ is an arbitrary open domain, $p > 1$, the operator $\Delta_{p,a}u = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)$ is the weighted p -Laplacian involving a weight function $a(\cdot)$:

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$\Omega \rightarrow [0, \infty)$ which belongs to B_p -class introduced by Kufner and Opic [45], $b(\cdot)$ is a measurable function defined on Ω , and $\Phi : (0, \infty) \rightarrow (0, \infty)$ is a given continuous function.

One of our two main results, Theorem 4.1, says that if a nonnegative function u is admissible in (1.1), then we can apply it to construct the family of Hardy-type inequalities of the form:

$$\int_{\Omega} |\xi|^p \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_2(dx),$$

where the measures μ_1 and μ_2 involve u and other quantities from (1.1), whereas ξ is an arbitrary Lipschitz compactly supported function defined on Ω . Those inequalities are constructed as a direct consequence of the Caccioppoli-type estimate for solutions to (1.1) derived in Theorem 3.1, our another main result.

Our purpose is to investigate the two following issues: the qualitative theory of solutions to nonlinear problems and derivation of precise Hardy-type inequalities. We contribute to the first of them by obtaining Caccioppoli-type estimate for a priori not known solution, which in general is an important tool in the regularity theory. For the second issue, in most cases we assume that the solution to (1.1) is known and we use it in construction of Hardy-type inequalities. Substituting $a \equiv 1$ in our considerations, we retrieve several results obtained by the third author in [55], where she dealt with the partial differential inequality of the form $-\Delta_p u \geq \Phi$, admitting the function Φ depending on u and x . Some of the inequalities derived in [55] and retrieved here are precise, i.e. they hold with the best constants. These cases are recalled in Remark 4.1. Furthermore, we provide here also new inequalities with the optimal constants, see Theorems 4.3 and 4.2. More new Hardy inequalities constructed using a solution to (1.1) are provided in Theorem 4.5.

The approach presented here and in the papers [55, 56] is a modification of techniques originating in [43]. In all of these papers, the investigations start with derivation of Caccioppoli-type estimates for the solutions to nonlinear problems. The method was inspired by the well-known nonexistence results by Pohozaev and Mitidieri [51].

In contrast to the results from [43, 55], in this paper we admit the weighted p -Laplacian: $\Delta_{p,a}$ instead of the classical one in (1.1). Our main results are the Caccioppoli-type estimate (Theorem 3.1) and the Hardy-type inequality (Theorem 4.1). Some of the results obtained here are new even in the nonweighted case when $a \equiv 1$, see Sect. 4.4 for discussion.

The discussion linking eigenvalue problems with Hardy-type inequalities can be found in the paper by Gurka [39], which generalized earlier results by Beesack [8], Kufner and Triebel [46], Muckenhoupt [52], and Tomaselli [60]. See also related more recent paper by Ghoussoub and Moradifard [37]. Derivation of the Hardy inequalities on the basis of supersolutions to p -harmonic differential problems can be found in papers by D'Ambrosio [22–24] and Barbatis et al. [5, 6]. Other interesting results linking the existence of solutions in elliptic and parabolic PDEs with Hardy type inequalities are presented in [2, 4, 36, 61, 62], see also references therein. We refer also to the contribution by the third author [56], where instead of the nonweighted p -Laplacian in (1.1) one deals with the A -Laplacian: $\Delta_A u = \operatorname{div} \left(\frac{A(|\nabla u|)}{|\nabla u|^2} \nabla u \right)$, involving a function

A from an Orlicz class. Similar estimates in the framework of nonlocal operators can be found e.g. in [12].

Let us present several reasons to investigate the partial differential inequality of the form $-\Delta_{p,a}u \geq b(x)\Phi(u)\chi_{\{u>0\}}$ rather than a simple one $-\Delta_p u \geq \Phi(u)$.

The first inspiration comes from modelling. PDEs of the form

$$\begin{cases} -\operatorname{div}(\Phi(x, u, \nabla u)) + \lambda_0(x)f(u)\chi_{\{u>0\}} = 0, \\ u \geq 0 \quad \text{in } \Omega \subseteq \mathbf{R}^n, \end{cases} \quad (1.2)$$

with $\Phi : \Omega \times \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ having power-type growth are used to describe for example reaction–diffusion and absorption processes observed in the theory of chemical-biological processes, population dynamics or combustion phenomena. In these type of models one can meet the so-called dead-core solutions, that is such that the dead-core zone $\Omega_0 := \{x \in \Omega : u(x) = 0\}$ has positive Lebesgue's n -dimensional measure, see [20,21,47,58]. Dead-core solutions appear when f grows essentially slower close to the origin than u^{p-1} . In our Caccioppoli estimates (Theorem 3.1 and in Lemma 3.1) we take into account the existence of dead-core solutions as well as more generally—supersolutions. On the other hand, another example of an equation related to (1.1) is the Matukuma equation

$$\Delta u + \frac{1}{1+|x|^2}u^q = 0, \quad q > 1,$$

which describes the dynamics of globular clusters of stars [49]. The existence of solutions to the Dirichlet problem generalizing the above equation which reads as follows:

$$\begin{cases} -\operatorname{div}(|x|^\alpha m(|\nabla u|)\nabla u) + \frac{|x|^{s-b}}{(1+|x|^b)^{s/b}}g(u) = 0 \quad \text{in } B(0, R), \\ u = 0 \quad \text{on } \partial B(0, R), \end{cases}$$

has been studied in [32]. Similar PDEs arise often in astrophysics to model several phenomena. For instance, classical models of globular clusters of stars are modeled by Eddington's equation [34]. Similar structure have models of dynamics of elliptic galaxies [3]. Qualitative properties of solutions to the equations inspired by models and their generalizations, are considered e.g. in [3,7,9,19,32,53]. For a collection of various open problems related to investigation on nonlinear PDEs posed in inhomogeneous settings we also refer to [18] and references therein.

The second motivation comes from functional analysis and it concerns the embeddings of weighted Sobolev spaces $W_{a(\cdot)}^{1,p}(\Omega)$ into weighted Lebesgue spaces $L_{b(\cdot)}^s(\Omega)$. The functional setting is introduced in Sect. 2. In such situation the equation

$$-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \gamma b(x)|u|^{s-2}u \quad (1.3)$$

is the Euler–Lagrange equation for the Rayleigh energy functional

$$E(u) = \frac{(\int_{\Omega} |\nabla u(x)|^p a(x) dx)^{\frac{1}{p}}}{(\int_{\Omega} |u(x)|^s b(x) dx)^{\frac{1}{s}}}.$$

The particular case of the embedding $W_{a(\cdot)}^{1,p}(\Omega) \rightarrow L_{b(\cdot)}^s(\Omega)$, where the weights are $a = |x|^\alpha$, $b = |x|^\beta$, is the Caffarelli–Kohn–Nirenberg inequality [17]. Note that this inequality receives recently special attention. The symmetry and the phenomenon of symmetry breaking together with their consequences for optimal constants is studied in [27–31]. The applications of these type of inequalities to investigations on sharp asymptotics for related fast diffusion equations can be found in [14,15], whereas to quantitative regularity results for the related evolution equation in [16].

The third reason to investigate solutions of degenerate PDEs is that, even if we deal with equation of the form (1.3) in the case of $a(x) \equiv 1$ and we know that its solution $u(x) = w(|x|)$ is radial, we can transform Eq. (1.3) into the related degenerate ODE involving two weights. For example, the equation

$$-\operatorname{div} \left(t^{n-1} |v'(t)|^{p-2} v'(t) \right) = \gamma t^{n-1} |v(t)|^{p_\beta^*-2} v(t),$$

where $v(t) = w(r(t))$, $r(t)$ is inverse to $t(r) = \int_0^r s^{-\beta/p} ds = \frac{p}{p-\beta} r^{(p-\beta)/p}$, $p_\beta^* = p \frac{n-\beta}{n-p}$ is the Sobolev exponent in the embedding $W_{|x|^\beta}^{1,p}(\mathbf{R}^n) \rightarrow L^{p_\beta^*}(\mathbf{R}^n)$ given by the Caffarelli–Kohn–Nirenberg inequality [17], is related to the transformation of equation

$$-\Delta_p u = \gamma |x|^{-\beta} |u|^{p_\beta^*-2} u, \quad (1.4)$$

see e.g. [53] and the discussion on page 525 in [54].

In many cases the solutions are known and therefore we can use them to construct Hardy-type inequalities. For example, it has been shown in [54, Theorem 5.1], that the function

$$u(x) = c \left(1 + |x|^{\frac{p-\beta}{p-1}} \right)^{-\frac{(n-p)}{p-\beta}} \quad \text{where } c = \left[\frac{n-\beta}{\gamma} \left(\frac{n-p}{p-1} \right)^{p-1} \right]^{\frac{(n-p)}{p-\beta}}, \quad (1.5)$$

is the solution of the Eq. (1.4) in the case of $\beta < p < n$. When $\beta = 0$, we deal with Talenti extremal profile [59]. This fact was the motivation for the analysis presented in the paper [44], reported in Sect. 4, where the authors, under certain assumptions, obtained the inequality

$$\begin{aligned} & \bar{C}_{\gamma,n,p,r} \int_{\mathbf{R}^n} |\xi|^p \left(1 + r |x|^{\frac{p}{p-1}} \right) \left(1 + |x|^{\frac{p}{p-1}} \right)^{\gamma(p-1)-p} dx \\ & \leq \int_{\mathbf{R}^n} |\nabla \xi|^p \left(1 + |x|^{\frac{p}{p-1}} \right)^{(p-1)\gamma} \end{aligned}$$

in some cases with the best constants. Such inequalities in the case $p = 2$ are of interest in the theory of nonlinear diffusions, where one investigates the asymptotic behavior of solutions of equation $u_t = \Delta u^m$, see [10] and the related works [11, 13, 37].

It might happen that solutions to a PDI or PDE (1.1) are known to exist by some existence theory, but their precise form is not known. In such a situation, under certain assumptions, we are still able to construct the Hardy inequality of the type

$$\int_{\Omega} |\xi|^p b(x) dx \leq \int_{\Omega} |\nabla \xi|^p a(x) dx,$$

which can be applied to study further properties of solutions. For example, the Hardy–Poincaré inequalities as above, where $a(\cdot) = b(\cdot)$, are often equivalent to the solvability of degenerate PDEs of the type

$$\operatorname{div} \left(a(x) |\nabla u(x)|^{p-2} \nabla u(x) \right) = x^*,$$

where x^* is an arbitrary functional on weighted Sobolev space $W_{\varrho,0}^{1,p}(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ in the norm of Sobolev space $W_{\varrho}^{1,p}(\Omega) := \{u \in L_{\varrho}^p(\Omega) : \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \in L_{\varrho}^p(\Omega)\}$, where the derivatives are considered in the weak sense and the weight ϱ belongs to the Kufner–Opic class $B_p(\Omega)$ as in Sect. 2, see [26, Theorem 7.12].

We hope that by the investigation of the qualitative properties of supersolutions to degenerated PDEs, i.e. the Caccioppoli-type estimate, and by constructions of Hardy-type inequalities, we can get deeper insight into the theory of degenerated elliptic PDEs.

2 Preliminaries

Basic notation. In the sequel we assume that $p > 1$, $\Omega \subseteq \mathbf{R}^n$ is an open subset not necessarily bounded. By $a(\cdot)$ - p -harmonic problems we understand those which involve the weighted p -Laplace operator:

$$\Delta_{p,a} u = \operatorname{div}(a(x) |\nabla u|^{p-2} \nabla u), \quad (2.1)$$

with some nonnegative function $a(\cdot)$. The derivatives which appear in (2.1) are understood in a distributional sense. By $D'(\Omega)$ we denote the space of distributions defined on Ω . If f is defined on Ω , then by $f \chi_{\Omega}$ we understand a function defined on $\mathbf{R}^n$ which is equal to f on Ω and which is extended by 0 outside Ω . Negative part of f is denoted by $f^- := \min\{f, 0\}$, while positive one by $f^+ := \max\{f, 0\}$. Moreover, every time when we deal with infimum, we set $\inf \emptyset = +\infty$. Having an arbitrary function $u \in W_{loc}^{1,1}(\Omega)$ (local Sobolev space), we define its value at every point $x \in \Omega$ by the formula

$$u(x) := \limsup_{r \rightarrow 0} \int_{B(x,r)} u(y) dy. \quad (2.2)$$

We use the standard notation $L^p_\varrho(\Omega)$ to denote the weighted L^p space with respect to the measure ϱdx , where ϱ is a weight. We also write $L^p(\Omega, \mu)$, when we want to admit a measure μ which is not absolutely continuous with respect to the Lebesgue measure.

Weighted Beppo Levi and Sobolev spaces. B_p weights: we deal a the special class of measures belonging to the class $B_p(\Omega)$ from [45].

Definition 2.1 (Classes $W(\Omega)$ and $B_p(\Omega)$) Let $\Omega \subseteq \mathbf{R}^n$ be an open set and let $W(\Omega)$ be the set of all Borel measurable real functions ϱ defined on Ω , such that $0 < \varrho(x) < \infty$, for a.e. $x \in \Omega$. We say that a weight $\varrho \in W(\Omega)$ satisfies $B_p(\Omega)$ -condition ($\varrho \in B_p(\Omega)$ for short) if $\varrho^{-1/(p-1)} \in L^1_{loc}(\Omega)$.

The Hölder inequality leads to the following simple observation based on [45, Theorem 1.5]. For readers' convenience we enclose the proof.

Proposition 2.1 Let $\Omega \subset \mathbf{R}^n$ be an open set, $p > 1$ and $\varrho \in B_p(\Omega)$. Then $L^p_{\varrho,loc}(\Omega) \subseteq L^1_{loc}(\Omega)$ and when $u_k \rightarrow u$ locally in $L^p_\varrho(\Omega)$ then also $u_k \rightarrow u$ in $L^1_{loc}(\Omega)$.

Proof For any $\Omega' \subseteq \Omega$ such that $\overline{\Omega'} \subseteq \Omega$ and any $u \in L^p_{\varrho,loc}(\Omega)$

$$\int_{\Omega'} |u| dx = \int_{\Omega'} |u| \varrho^{\frac{1}{p}} \varrho^{-\frac{1}{p}} dx \leq \left(\int_{\Omega'} |u|^p \varrho dx \right)^{\frac{1}{p}} \left(\int_{\Omega'} \varrho^{-\frac{1}{p-1}} dx \right)^{1-\frac{1}{p}} < \infty.$$

The substitution of $u_k - u_l$ instead of u implies second part of the statement. $\square$

Weighted Beppo Levi space. Assume that $\varrho(\cdot) \in B_p(\Omega)$. We deal with the weighted Beppo Levi space

$$\mathcal{L}^{1,p}_\varrho(\Omega) := \left\{ u \in D'(\Omega) : \frac{\partial u}{\partial x_i} \in L^p_\varrho(\Omega) \text{ for } i = 1, \dots, n \right\}.$$

According to the above proposition and [50, Theorem 1, Section 1.1.2], we have $\mathcal{L}^{1,p}_\varrho(\Omega) \subseteq W^{1,1}_{loc}(\Omega)$. We will also consider local variants of Beppo Levi spaces: $\mathcal{L}^{1,p}_{\varrho,loc}(\Omega) := \{u \in D'(\Omega) : \int_{\Omega'} |\nabla u(x)|^p \varrho(x) dx < \infty\}$, whenever $\overline{\Omega'}$ is a compact subset of Ω . As it is also a subset in $W^{1,1}_{loc}(\Omega)$, integration by parts formula applies to elements of $\mathcal{L}^{1,p}_{\varrho,loc}(\Omega)$ in the usual way.

Two weighted Sobolev spaces Let $\varrho_1(\cdot) \in W(\Omega)$, $\varrho_2(\cdot) \in B_p(\Omega)$. We consider the space $W^{1,p}_{(\varrho_1, \varrho_2)}(\Omega) = L^p_{\varrho_1}(\Omega) \cap \mathcal{L}^{1,p}_{\varrho_2}(\Omega)$, i.e.

$$W^{1,p}_{(\varrho_1, \varrho_2)}(\Omega) := \left\{ f \in L^p_{\varrho_1}(\Omega) \cap D'(\Omega) : \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \in L^p_{\varrho_2}(\Omega) \right\}, \quad (2.3)$$

with the norm $\|f\|_{W^{1,p}_{(\varrho_1, \varrho_2)}(\Omega)} := \|f\|_{L^p_{\varrho_1}(\Omega)} + \|\nabla f\|_{L^p_{\varrho_2}(\Omega)}$.

Proposition 2.2 [45] $p > 1$, $\Omega \subseteq \mathbf{R}^n$ be an open set and $\varrho_1(\cdot) \in W(\Omega)$, $\varrho_2(\cdot) \in B_p(\Omega)$. Then $W_{(\varrho_1, \varrho_2)}^{1,p}(\Omega)$ defined by (2.3) equipped with the norm $\|\cdot\|_{W_{(\varrho_1, \varrho_2)}^{1,p}(\Omega)}$ is a Banach space.

When $\varrho_1 \equiv \varrho_2$, we deal with the usual weighted Sobolev space $W_{\varrho_1}^{1,p}(\Omega)$, i.e.

$$W_{\varrho_1}^{1,p}(\Omega) = W_{(\varrho_1, \varrho_1)}^{1,p}(\Omega).$$

By $W_{(\varrho_1, \varrho_2), 0}^{1,p}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the space $W_{(\varrho_1, \varrho_2)}^{1,p}(\Omega)$ and we use the standard notation $W_{(\varrho_1, \varrho_1), 0}^{1,p}(\Omega) = W_{\varrho_1, 0}^{1,p}(\Omega)$ when $\varrho_1 = \varrho_2$.

Weighted p -Laplacian. Assume that $p > 1$, $a \in B_p(\Omega) \cap L_{loc}^1(\Omega)$ (see Definition 2.1), and $u \in \mathcal{L}_{a, loc}^{1,p}(\Omega)$. Then $a|\nabla u|^{p-1} \in L_{loc}^1(\Omega)$ as we have:

$$\int_{\Omega'} a|\nabla u|^{p-1} dx \leq \left(\int_{\Omega'} a dx \right)^{\frac{1}{p}} \left(\int_{\Omega'} |\nabla u|^p a dx \right)^{1-\frac{1}{p}} < \infty,$$

whenever Ω' is a compact subset of Ω . In particular, $a|\nabla u|^{p-2}\nabla u \in L_{loc}^1(\Omega, \mathbf{R}^n)$ and so the weak divergence of $a|\nabla u|^{p-2}\nabla u \in L_{loc}^1(\Omega, \mathbf{R}^n)$ denoted by $\Delta_{p,a}u$ is well defined via the formula

$$\langle \Delta_{p,a}u, w \rangle = \langle \operatorname{div} (a|\nabla u|^{p-2}\nabla u), w \rangle := - \int_{\Omega} a|\nabla u|^{p-2}\nabla u \cdot \nabla w dx \quad (2.4)$$

where $w \in C_0^\infty(\Omega)$. Obviously, in the case $a \equiv 1$ the operator $\Delta_{p,a}u$ reduces to the usual p -Laplacian $\operatorname{div} (|\nabla u|^{p-2}\nabla u)$. It particular, it coincides with the Laplace operator in the case $p = 2$.

Remark 2.1 We have the following observations.

- (i) As $|\nabla u|^{p-2}\nabla u \in L_{a, loc}^{\frac{p}{p-1}}(\Omega, \mathbf{R}^n)$, then the right-hand side in (2.4) is well defined for every $w \in \mathcal{L}_a^{1,p}(\Omega)$ which is compactly supported in Ω .
- (ii) When $u \in \mathcal{L}_a^{1,p}(\Omega)$, formula (2.4) extends for $w \in W_{(b,a), 0}^{1,p}(\Omega)$, whenever $b \in W(\Omega)$. This follows from the estimates

$$\begin{aligned} |\langle \Delta_{p,a}u, w \rangle| &\leq \int_{\Omega} a|\nabla u|^{p-1}|\nabla w| dx = \int_{\Omega} \left(a^{\frac{1}{p'}} |\nabla u|^{p-1} \right) \left(a^{\frac{1}{p}} |\nabla w| \right) dx \\ &\leq \left(\int_{\Omega} |\nabla u|^p a dx \right)^{1-\frac{1}{p}} \left(\int_{\Omega} |\nabla w|^p a dx \right)^{\frac{1}{p}} < \infty. \end{aligned}$$

Therefore in this case $\Delta_{p,a}u$ can be also treated as an element of the dual to the Banach space $W_{(b,a), 0}^{1,p}(\Omega)$ denoted $(W_{(b,a), 0}^{1,p}(\Omega))^*$. We preserve the same notation $\Delta_{p,a}u$ for this functional extension of formula (2.4).

Differential inequality. Our analysis is based on the following differential inequality.

Definition 2.2 Let $a \in B_p(\Omega) \cap L^1_{loc}(\Omega)$ be a given weight, $u \in \mathcal{L}^{1,p}_{a,loc}(\Omega)$ be non-negative, $\Phi : (0, \infty) \rightarrow (0, \infty)$ be a continuous function, $b(\cdot)$ be measurable and $b(\cdot)\Phi(u)\chi_{\{u>0\}} \in L^1_{loc}(\Omega)$. Suppose further that for every nonnegative compactly supported function $w \in \mathcal{L}^{1,p}_a(\Omega)$ one has

$$\int_{\Omega \cap \{u>0\}} b(x)\Phi(u)w \, dx > -\infty.$$

We say that partial differential inequality (PDI for short)

$$-\Delta_{p,a}u \geq b(x)\Phi(u)\chi_{\{u>0\}}, \quad (2.5)$$

holds if for every nonnegative compactly supported function $w \in \mathcal{L}^{1,p}_a(\Omega)$ we have

$$\langle -\Delta_{p,a}u, w \rangle \geq \int_{\Omega \cap \{u>0\}} b(x)\Phi(u)w \, dx, \quad (2.6)$$

where $\langle -\Delta_{p,a}u, w \rangle$ is given by (2.4). In other words, u is a supersolution to the PDE:

$$-\Delta_{p,a}u = b(x)\Phi(u)\chi_{\{u>0\}}. \quad (2.7)$$

Remark 2.2 We have the following observations.

- (i) Inequality (2.5) can be interpreted as a variant of p -superharmonicity condition for the weighted p -Laplacian defined by (2.1).
- (ii) In the case of equation in (2.5): $-\Delta_{p,a}u = b(x)\Phi(u)\chi_{\{u>0\}}$, we deal with the solution of the nonlinear eigenvalue problem. It corresponds to the eigenvalue problem $-\Delta_p u = \lambda|u|^{p-2}u$, but in our case the role of an eigenvalue λ plays the function $b(\cdot)$ and instead of the $|u|^{p-2}u$ we have now the possibly another expression $\Phi(u)\chi_{\{u>0\}}$.

Assumption A

By Assumption A we mean the set of conditions: (a, b) , (Ψ, g) , (u) , and (a)–(c) below.

- (a, b) $a \in L^1_{loc}(\Omega) \cap B_p(\Omega)$, $b(\cdot)$ is measurable;
- (Ψ, g) The couple of continuous functions $(\Psi, g) : (0, \infty) \times (0, \infty) \rightarrow (0, \infty) \times (0, \infty)$, where Ψ is Lipschitz on every closed interval in $(0, \infty)$, satisfy the following compatibility conditions:

- (i) the inequality

$$g(t)\Psi'(t) \leq -C\Psi(t) \quad \text{a.e. in } (0, \infty) \quad (2.8)$$

holds with some constant $C \in \mathbf{R}$ independent of t and Ψ is monotone (not necessarily strictly);

(ii) each of the functions

$$(0, \infty) \ni t \mapsto \Theta(t) := \Psi(t)g^{p-1}(t), \text{ and } t \mapsto \Psi(t)/g(t) \quad (2.9)$$

is nonincreasing or bounded in some neighbourhood of 0.

(u) We assume that $u \in \mathcal{L}_{a,loc}^{1,p}(\Omega)$ is nonnegative, (a, b) holds, $\Phi : (0, \infty) \rightarrow (0, \infty)$ is a continuous function, such that for every nonnegative compactly supported function $w \in \mathcal{L}_a^{1,p}(\Omega)$ one has $\int_{\Omega \cap \{u>0\}} b(x)\Phi(u)w \, dx > -\infty$ and $b\Phi(u)\chi_{\{u>0\}} \in L_{loc}^1(\Omega)$.

Moreover, let us consider the set $\mathcal{A}$ of those $\sigma \in \mathbf{R}$ for which

$$b(x)\Phi(u) + \sigma \frac{a(x)}{g(u)} |\nabla u|^p \geq 0 \quad \text{a.e. in } \Omega \cap \{u > 0\}. \quad (2.10)$$

We suppose that

$$\sigma_0 := \inf \mathcal{A} = \inf \{\sigma \in \mathbf{R} : \sigma \text{ satisfies (2.10)}\} \in \mathbf{R}. \quad (2.11)$$

Since $\inf \emptyset = +\infty$, $\mathcal{A}$ can be neither an empty set nor unbounded from below.

- (a) We suppose that (Ψ, g) and (u) hold. Parameter σ satisfies $\sigma_0 \leq \sigma < C$, where C is given by (2.8) and σ_0 by (2.11).
- (b) We suppose that (u) and (Ψ, g) hold and we assume that for every $R > 0$ we have $b^+(x)(\Phi\Psi)(u)\chi_{0<u\leq R} \in L_{loc}^1(\Omega)$.
- (c) We suppose that (u) and (Ψ, g) hold. We assume that for any compact subset $K \subseteq \Omega$ we have

$$\begin{aligned} \Psi(R) \int_{K \cap \{u \geq R/2\}} |\nabla u(x)|^{p-1} a(x) \, dx &\xrightarrow{R \rightarrow \infty} 0, \\ \Psi(R) \int_{K \cap \{u \geq R/2\}} \Phi(u)b(x) \, dx &\xrightarrow{R \rightarrow \infty} 0. \end{aligned}$$

Comments on assumptions

Remark 2.3 We have the following observations on Condition (Ψ, g) .

- (i) Assume that Condition (Ψ, g) , i) holds and, moreover, $g'(t) \geq -C$. Then $(\Psi/g)' \leq 0$ and $\Psi(t)/g(t)$ is nonincreasing.
- (ii) This condition is satisfied by examples of pairs from Table 1.

The statement below shows that under Assumption A, (u) the function u cannot be constant almost everywhere in Ω . Moreover, in many cases $\mathcal{A}$ is not empty and $\inf \mathcal{A}$ is a real number.

Lemma 2.1 Suppose $u \in \mathcal{L}_{a,loc}^{1,p}(\Omega)$ is a nonnegative solution to the PDI $-\Delta_{p,a} u \geq b(x)\Phi(u)\chi_{\{u>0\}}$ in the sense of Definition 2.2, under all assumptions therein. Moreover, let $b \geq 0$ a.e. in Ω . Then σ_0 given by (2.11) exists and is finite if and only if u is not a constant function a.e. in Ω .

Table 1 Example couples (Ψ, g) which satisfy condition (Ψ, g) from Assumption A

$\Psi(t)$	$g(t)$	C	Remarks
$t^{-\alpha}$	t	α	$\alpha \in \mathbf{R}$
$(t \log(a+t))^{-1}$	$t \log(a+t)$	$\log a$	$a > 1$
e^{-t}	Bounded by $C, g' \geq -C$	C	$C > 0$
e^{-t}/t	$t/(1+t)$	1	–

Proof ($\Leftarrow$) Assume that $u \not\equiv \text{Const}$. Then the set $\mathcal{A}$ is not empty as it contains zero, in particular $\sigma_0 \leq 0$. If $a(\cdot) > 0, b(\cdot) \geq 0$ a.e. in Ω , then the set $\mathcal{A}$ cannot be unbounded from below. Indeed, if $\mathcal{A}$ was unbounded from below, the inequality:

$$b(x)\Phi(u(x)) - \bar{n} \frac{a(x)}{g(u(x))} |\nabla u(x)|^p \geq 0 \quad \text{a.e. in } \Omega \cap \{u > 0\}$$

would hold for every $\bar{n} \in \mathbf{N}$. Consequently we could find $K_1, K_2 > 0$, such that

$$\frac{1}{\bar{n}} b(x)\Phi(u(x)) \geq \frac{a(x)}{g(u(x))} |\nabla u(x)|^p \geq \frac{K_1}{K_2} > 0$$

a.e. in $\{u : |\nabla u|^p a(x) \geq K_1, g(u(x)) \leq K_2\}$, which is the set of positive measure and independent on $\bar{n}$. Taking the limit for $\bar{n} \rightarrow \infty$, we arrive at the contradiction.

($\Rightarrow$) If σ_0 is a finite number, then u cannot be constant. Indeed, for $u \equiv \text{Const} \geq 0$, condition (2.10) implies $\mathcal{A} = (-\infty, \infty)$, which violates (2.11). $\square$

Remark 2.4 Assumption A, (c) is satisfied in each of the following cases:

- (i) When u is locally bounded.
- (ii) When $b \geq 0, u \in \mathcal{L}_{a,loc}^p(\Omega)$ and $\Psi(R)/R$ is bounded at infinity. Indeed, we have from Hölder's inequality

$$\begin{aligned} Z_1(R) &:= \Psi(R) \int_{K \cap \{u \geq R/2\}} |\nabla u(x)|^{p-1} a(x) dx \\ &\leq \Psi(R) \left(\int_{K \cap \{u \geq R/2\}} |\nabla u(x)|^p a(x) dx \right)^{1-\frac{1}{p}} \left(\int_{K \cap \{u \geq R/2\}} a(x) dx \right)^{\frac{1}{p}} \end{aligned}$$

and $Z_2(R) := \left(\int_{K \cap \{u \geq R/2\}} |\nabla u(x)|^p a(x) dx \right)^{1-\frac{1}{p}} \rightarrow 0$ as $R \rightarrow \infty$. On the other hand, by Czebysev's inequality applied to $\mu(x) = a(x)dx$ on K , we get

$$\begin{aligned} \int_{K \cap \{u \geq R/2\}} a(x) dx &= \mu(\{x \in K : u(x) \geq R/2\}) \\ &\leq \frac{2^p}{R^p} \int_K |u|^p a(x) dx =: \frac{1}{R^p} Z_3(R). \end{aligned}$$

Therefore, $Z_1(R) \leq \frac{\Psi(R)}{R} Z_2(R) Z_3(R)^{\frac{1}{p}} \rightarrow 0$ as $R \rightarrow \infty$.

3 Caccioppoli-type estimates

This section provides estimates which we call *Caccioppoli-type inequalities*, see Lemma 3.1 and Theorem 3.1. In the classical setting the Caccioppoli inequality should involve u on the right-hand side only and ∇u exclusively on the left-hand side, see [38]. In our case on the right-hand sides of inequalities (3.2) and (3.3) there is indeed no dependence on ∇u , when we estimate $\chi_{\{\nabla u \neq 0\}}$ by 1, while the left-hand side does involve ∇u . Nonetheless, this estimate is sufficient to play the role of Caccioppoli inequality for some purposes, e.g. in the studies on nonexistence [33, 43]

3.1 Formulation of results

Our first goal is to obtain the following estimate, which is the key tool in our considerations. We call it a ‘local estimate’, because it is stated on a part of the domain where u is not bigger than a given R .

Lemma 3.1 (Local estimate) *Suppose that Assumption A holds except property (d). Assume further that $1 < p < \infty$ and u is a nonnegative solution to PDI*

$$-\Delta_{p,a}u \geq b(x)\Phi(u)\chi_{\{u>0\}} \quad (3.1)$$

in the sense of Definition 2.2.

Then for any nonnegative Lipschitz function ϕ with compact support in Ω such that the integral $\int_{\text{supp } \phi \cap \{\nabla u \neq 0\}} |\nabla \phi|^p \phi^{1-p} a(x) dx$ is finite and for any $R > 0$ the inequality

$$\begin{aligned} & \int_{\{0 < u < R\}} \left(b(x)\Phi(u(x)) + \sigma \frac{a(x)}{g(u(x))} |\nabla u(x)|^p \right) \Psi(u(x))\phi(x) dx \\ & \leq c \int_{S_R} a(x)\Psi(u(x))g^{p-1}(u(x))|\nabla \phi(x)|^p \phi^{1-p}(x) dx + \tilde{C}(R), \end{aligned} \quad (3.2)$$

holds, where $S_R = \{\nabla u(x) \neq 0, 0 < u < R\} \cap \text{supp } \phi$, $c := \frac{1}{p^p} \left(\frac{p-1}{C-\sigma} \right)^{p-1}$,

$$\tilde{C}(R) = \Psi(R) \left[\int_{\Omega \cap \{u \geq \frac{R}{2}\}} a(x)|\nabla u|^{p-1} |\nabla \phi| dx - \int_{\Omega \cap \{u \geq \frac{R}{2}\}} b(x)\Phi(u)\phi dx \right].$$

Moreover, all quantities appearing in (3.2) are finite.

The above local estimate implies the following global estimate of Caccioppoli-type for solutions to (3.1). It may be used to analyze various qualitative properties of them and is the main result of this section.

Theorem 3.1 (Caccioppoli-type estimate) *Suppose that Assumption A holds, $1 < p < \infty$ and $u \in \mathcal{L}_{a,loc}^{1,p}(\Omega)$ is a nonnegative solution to the PDI*

$$-\Delta_{p,a}u \geq b(x)\Phi(u)\chi_{\{u>0\}}$$

in the sense of Definition 2.2. Then for every nonnegative Lipschitz function ϕ with compact support in Ω such that the integral $\int_{\text{supp } \phi} |\nabla \phi|^p \phi^{1-p} a(x) dx$ is finite, we have

$$\begin{aligned} & \int_{\Omega \cap \{u > 0\}} \left(b(x) \Phi(u(x)) + \sigma |\nabla u(x)|^p \frac{a(x)}{g(u(x))} \right) \Psi(u(x)) \phi(x) dx \\ & \leq c \int_{\Omega \cap \{u(x) > 0, \nabla u(x) \neq 0\} \cap \text{supp } \phi} a(x) \Psi(u(x)) g^{p-1}(u(x)) |\nabla \phi(x)|^p \phi(x)^{1-p} dx, \end{aligned} \quad (3.3)$$

with $c = \frac{(p-1)^{p-1}}{p^p (C-\sigma)^{p-1}}$.

Proof We let $R \rightarrow \infty$ in Lemma 3.1 and observe that due to our assumptions we have $\tilde{C}(R) \rightarrow 0$. The appropriate convergence follows from Lebesgue's Monotone Convergence Theorem, because both integrands in (3.2) are nonnegative. $\square$

3.2 Proof of Lemma 3.1

We use the following simple observations (see [55]).

Lemma 3.2 Let $p > 1$, $\tau > 0$ and $s_1, s_2 \geq 0$, then

$$s_1 s_2^{p-1} \leq \frac{1}{p\tau^{p-1}} \cdot s_1^p + \frac{p-1}{p} \tau \cdot s_2^p.$$

We shall start with proving an initial version of (3.2) for truncated and separated from zero functions. For $0 < \delta < R$ let us denote

$$u_{\delta,R}(x) := \min \{u(x) + \delta, R\}, \quad G(x) := \Psi(u_{\delta,R}(x)) \phi(x). \quad (3.4)$$

Lemma 3.3 Let u, ϕ be as in the assumptions of Proposition 3.1 and let $0 < \delta < R$ be arbitrary. Then $u_{\delta,R} \in \mathcal{L}_{a,loc}^{1,p}(\Omega)$, $G \in \mathcal{L}_a^{1,p}(\Omega)$ and G is compactly supported in Ω .

Remark 3.1

- (i) We know that $\mathcal{L}_{a,loc}^{1,p}(\Omega) \subseteq W_{loc}^{1,1}(\Omega)$. This inclusion, together with Nikodym ACL Characterization Theorem [50, Section 1.1.3], implies that we can verify if the function belongs to Sobolev space $\mathcal{L}_{a,loc}^{1,p}(\Omega)$ by checking that it belongs to $W_{loc}^{1,1}(\Omega)$ and that its derivatives computed almost everywhere belong to $L_{a,loc}^p(\Omega)$. The fact that Ψ is locally Lipschitz is used to apply Lemma 3.3 in order to ensure that $\Psi(u_{\delta,R}(x))$ belongs to $W_{loc}^{1,1}(\Omega)$.
- (ii) The nonnegativity of function u allows to deduce that $G \in \mathcal{L}_a^{1,p}(\Omega)$. This fact plays the crucial role in the proof of Lemma 3.1.

We will test the PDI against $u_{\delta,R}$ and pass to the limit with $\delta \searrow 0$. To justify the convergence of some terms we need the following fact.

Lemma 3.4 (e.g. [43], Lemma 3.1) *Let $u \in W_{loc}^{1,1}(\Omega)$ be defined everywhere by (2.2) and let $t \in \mathbf{R}$ be given. Then $\{x \in \Omega : u(x) = t\} \subseteq \{x \in \Omega : \nabla u(x) = 0\} \cup N$, where N is a set of Lebesgue's measure zero.*

Proof of Lemma 3.1

At first we explain the strategy of the proof.

Let the quantities Φ, Ψ, g, a, b, u be as in (3.1) and Assumption A, while ϕ be as in the statement of the lemma.

The proof is performed in four steps:

Step 1. We prove that for every $0 < \delta < R$, the inequality

$$\begin{aligned} & \int_{\Omega \cap \{0 < u \leq R - \delta\}} \left(b(x) \Phi(u) + \sigma \frac{a(x)}{g(u + \delta)} |\nabla u|^p \right) \Psi(u + \delta) \phi \, dx \\ & \leq c \int_{\Omega \cap \text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) \Psi(u + \delta) g^{p-1}(u + \delta) \left(\frac{|\nabla \phi|}{\phi} \right)^p \phi \, dx \\ & \quad + \tilde{C}(\delta, R) \end{aligned} \quad (3.5)$$

holds with σ from Assumption A, (u) and

$$\begin{aligned} \tilde{C}(\delta, R) = & \Psi(R) \left[\int_{\Omega \cap \{u > R - \delta\}} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \right. \\ & \left. - \int_{\Omega \cap \{u > R - \delta\}} b(x) \Phi(u) \phi \, dx \right]. \end{aligned}$$

Step 2. We pass to the limit as $\delta \searrow 0$ and obtain

$$\begin{aligned} & \limsup_{\delta \searrow 0} c \int_{\text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) \Psi(u + \delta) g^{p-1}(u + \delta) \left(\frac{|\nabla \phi|}{\phi} \right)^p \phi \, dx \\ & \quad + \tilde{C}(\delta, R) \\ & \leq c \int_{\text{supp } \phi \cap \{\nabla u(x) \neq 0, 0 < u < R\}} a(x) \Psi(u(x)) g^{p-1}(u(x)) |\nabla \phi(x)|^p \phi^{1-p}(x) \, dx \\ & \quad + \tilde{C}(R). \end{aligned}$$

Step 3. When $\delta \geq 0$, we denote

$$A_\delta(x) := \left(b(x) \Phi(u(x)) + \sigma \frac{a(x)}{g(u(x) + \delta)} |\nabla u(x)|^p \right) \Psi(u(x) + \delta) \chi_{\{u(x) > 0\}}. \quad (3.6)$$

We show that

$$\liminf_{\delta \searrow 0} \int_{\{0 < u \leq R-\delta\}} A_\delta(x) \phi(x) dx \geq \int_{\{0 < u < R\}} A_0(x) \phi(x) dx.$$

Step 4. We complete the proof of the statement.

Proof of Step 1

Let us introduce the following notation for $J_i = J_i(\delta, R)$, $i = 1, \dots, 6$:

$$\begin{aligned} J_1 &= \int_{\Omega \cap \{0 < u \leq R-\delta\}} a(x) |\nabla u|^p \Psi'(u + \delta) \phi dx, \\ J_2 &= \int_{\Omega \cap \{0 < u \leq R-\delta\}} a(x) |\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} \phi dx, \\ J_3 &= \int_{\Omega \cap \{0 < u \leq R-\delta\}} a(x) |\nabla u|^{p-2} \Psi(u + \delta) \nabla u \cdot \nabla \phi dx, \\ J_4 &= \Psi(R) \int_{\Omega \cap \{u > R-\delta\}} b(x) \Phi(u) \phi dx, \\ J_5 &= \Psi(R) \int_{\Omega \cap \{u > R-\delta\}} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx, \\ J_6 &= \int_{\text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R-\delta\}} a(x) \left(\frac{|\nabla \phi|}{\phi} \right)^p \Psi(u + \delta) g^{p-1}(u + \delta) \phi dx. \end{aligned}$$

By our assumptions all the above quantities are finite (for $0 \leq u \leq R - \delta$ we have $\delta \leq u + \delta \leq R$). Let $u_{\delta, R}$, G be given by (3.4). Choose $w := G$ to be a test function in (2.6). Then the right hand side of (2.6) becomes

$$\begin{aligned} -\infty < I &:= \int_{\Omega \cap \{u > 0\}} b(x) \Phi(u) G(x) dx = \int_{\Omega \cap \{u > 0\}} b(x) \Phi(u) \Psi(u_{\delta, R}) \phi dx \\ &= \int_{\Omega \cap \{0 < u \leq R-\delta\}} b(x) \Phi(u) \Psi(u + \delta) \phi dx \\ &\quad + \Psi(R) \int_{\Omega \cap \{u > R-\delta\}} b(x) \Phi(u) \phi dx \\ &= \int_{\Omega \cap \{0 < u \leq R-\delta\}} b(x) \Phi(u) \Psi(u + \delta) \phi dx + J_4, \end{aligned} \quad (3.7)$$

thus I is finite. Using (2.6) we get the following estimate

$$\begin{aligned} I &= \int_{\Omega \cap \{u > 0\}} b(x) \Phi(u(x)) G(x) dx \leq \left\langle -\text{div} \left(a(x) |\nabla u|^{p-2} \nabla u \right), G \right\rangle \\ &= \int_{\Omega \cap \{\nabla u \neq 0\}} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla G dx \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) |\nabla u|^p \Psi'(u + \delta) \phi \, dx \\
&\quad + \int_{\Omega \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) |\nabla u|^{p-2} \Psi(u + \delta) \nabla u \cdot \nabla \phi \, dx \\
&\quad + \Psi(R) \int_{\Omega \cap \{\nabla u \neq 0, u > R - \delta\}} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = J_1 + J_3 + J_5 \\
&\leq -C J_2 + J_3 + J_5.
\end{aligned}$$

The last inequality follows from $J_1 \leq -C J_2$ which holds due to (2.8). Moreover,

$$\begin{aligned}
J_3 &\leq \int_{\Omega \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) |\nabla u|^{p-1} |\nabla \phi| \Psi(u + \delta) \, dx \\
&= \int_{\text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} \left(\frac{|\nabla \phi|}{\phi} g(u + \delta) \right) |\nabla u|^{p-1} a(x) \frac{\Psi(u + \delta)}{g(u + \delta)} \phi \, dx.
\end{aligned}$$

We apply Lemma 3.2 with $s_1 = \frac{|\nabla \phi|}{\phi} g(u + \delta)$, $s_2 = |\nabla u|$ and arbitrary $\tau > 0$, to get

$$\begin{aligned}
J_3 &\leq \frac{p-1}{p} \tau \int_{\text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) |\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} \phi \, dx \\
&\quad + \frac{1}{p\tau^{p-1}} \int_{\text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) \left(\frac{|\nabla \phi|}{\phi} \right)^p \Psi(u + \delta) g^{p-1}(u + \delta) \phi \, dx. \\
&\leq \frac{p-1}{p} \tau J_2 + \frac{1}{p\tau^{p-1}} J_6.
\end{aligned}$$

Combining these estimates we deduce that for $\tau > 0$ such that $C - \frac{p-1}{p} \tau = \sigma$ we have

$$\begin{aligned}
I &\leq -C J_2 + J_3 + J_5 \\
&\leq \left(-C + \frac{p-1}{p} \tau \right) J_2 + \frac{1}{p\tau^{p-1}} J_6 + J_5 = -\sigma J_2 + \frac{1}{p\tau^{p-1}} J_6 + J_5.
\end{aligned}$$

The last inequality and (3.7) imply

$$\int_{\Omega \cap \{0 < u \leq R - \delta\}} b(x) \Phi(u) \Psi(u + \delta) \phi \, dx + \sigma J_2 \leq \frac{1}{p\tau^{p-1}} J_6 + J_5 - J_4,$$

which implies (3.5), because $\tilde{C}(\delta, R) \geq J_5 - J_4$ and $\tau = (C - \sigma) \frac{p}{p-1}$.

Introduction of parameters δ and R is necessary as we need to move the quantities J_2 , J_4 in the estimates to the opposite sides of inequalities. For doing this we have to know that they are finite.

Proof of Step 2

We show that under our assumptions

$$\begin{aligned} & \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u + \delta \leq R\}} a(x) \Psi(u + \delta) g^{p-1}(u + \delta) |\nabla \phi|^p \phi^{1-p} dx \\ & \xrightarrow{\delta \searrow 0} \int_{\text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R\}} a(x) \Psi(u) g^{p-1}(u) |\nabla \phi|^p \phi^{1-p} dx. \end{aligned} \quad (3.8)$$

To verify this we note that for a.e. $x \in \Omega$ we have

$$\begin{aligned} & \Psi(u(x) + \delta) g^{p-1}(u(x) + \delta) \chi_{\{\nabla u(x) \neq 0, u(x) + \delta \leq R\}} \\ & \xrightarrow{\delta \rightarrow 0} \Psi(u(x)) g^{p-1}(u(x)) \chi_{\{\nabla u(x) \neq 0, 0 < u(x) \leq R\}}. \end{aligned}$$

Indeed, when $0 < u(x) < R$ or $u(x) > R$ this follows from the continuity of the involved functions, while according to Lemma 3.4 the set $\{x : u(x) = 0, |\nabla u(x)| \neq 0\} \cup \{x : u(x) = R, |\nabla u(x)| \neq 0\}$ is of measure zero.

For the proof of (3.8) we recall the nonnegative function $\Theta(t) := \Psi(t)g^{p-1}(t)$ given by (2.9), which is nonincreasing or bounded in the neighbourhood of zero.

Let us start with the first case, i.e. there exists $\varepsilon > 0$ such that for $t < \varepsilon$ the function $\Theta(t)$ is nonincreasing. Without loss of generality we may assume $2\delta \leq \varepsilon \leq R$ and

$$E_\varepsilon = \left\{ \nabla u \neq 0, 0 < u < \frac{\varepsilon}{2} \right\} \cap \text{supp } \phi, \quad F_\varepsilon = \left\{ \nabla u \neq 0, \frac{\varepsilon}{2} \leq u \right\} \cap \text{supp } \phi.$$

Then we have

$$\begin{aligned} & \int_{\text{supp } \phi \cap \{\nabla u \neq 0, 0 < u, u + \delta \leq R\}} \Theta(u + \delta) a(x) |\nabla \phi|^p \phi^{1-p} dx \\ & = \int_{E_\varepsilon} \Theta(u + \delta) a(x) |\nabla \phi|^p \phi^{1-p} dx \\ & \quad + \int_{F_\varepsilon} \Theta(u + \delta) \chi_{\{u + \delta \leq R\}} a(x) |\nabla \phi|^p \phi^{1-p} dx. \end{aligned}$$

Let us concentrate on the integral on E_ε . We consider $\delta < \varepsilon/2$, so on E_ε we have $u + \delta < \varepsilon$. Note that mapping $t \mapsto \Theta(t)$ is nonincreasing for $t \in (0, \varepsilon)$. For $\delta \searrow 0$ functions $\Theta_\delta(x) := \Theta(u(x) + \delta)$ converge to $\Theta(u(x))$ for almost every x for which $u(x) > 0$. Therefore, due to Lebesgue's Monotone Convergence Theorem we obtain

$$\lim_{\delta \searrow 0} \int_{E_\varepsilon} \Theta(u + \delta) a(x) |\nabla \phi|^p \phi^{1-p} dx = \int_{E_\varepsilon} \Theta(u) a(x) |\nabla \phi|^p \phi^{1-p} dx.$$

To deal with integrals over F_ε we note that

$$\begin{aligned} & \Theta(u + \delta) \chi_{\{u+\delta \leq R\}} a(\cdot) |\nabla \phi|^p \phi^{1-p} \chi_{F_\varepsilon} \\ & \leq \chi_{\{\varepsilon/2 \leq u+\delta \leq R\} \cap \text{supp } \phi} \Theta(u + \delta) a(\cdot) |\nabla \phi|^p \phi^{1-p} \\ & \leq \sup_{t \in [\varepsilon/2, R]} \Theta(t) \chi_{\text{supp } \phi} a(\cdot) |\nabla \phi|^p \phi^{1-p} \in L^1(\Omega). \end{aligned}$$

Application of Lebesgue's Dominated Convergence Theorem yields

$$\begin{aligned} & \lim_{\delta \searrow 0} \int_{F_\varepsilon \cap \{u+\delta \leq R\}} \Theta(u + \delta) a(x) |\nabla \phi|^p \phi^{1-p} dx \\ & = \int_{F_\varepsilon \cap \{u < R\}} \Theta(u) a(x) |\nabla \phi|^p \phi^{1-p} dx. \end{aligned}$$

This completes the case of Θ decreasing in the neighbourhood of 0. The case of bounded Θ follows from Lebesgue's Dominated Convergence Theorem (cf. as above for integral over F_ε with $\varepsilon = 0$).

To complete the proof of Step 2 it suffices to observe that for $\delta \leq \frac{R}{2}$ we have $\tilde{C}(\delta, R) \leq \tilde{C}(R)$.

Proof of Step 3

We note that, when $A_\delta(x)$ is given by (3.6), we have $A_\delta(x) \rightarrow A_0(x)$ a.e. in Ω as $\delta \searrow 0$, but we do not have information about the sign of A_δ . Therefore we cannot apply for example Lebesgue's Monotone Convergence Theorem directly to justify the convergence of the integrals. Thus we distinguish between two cases: when $\sigma \geq 0$ and when $\sigma < 0$. In both cases we prove the statement under each of the restrictions below on Ψ and Ψ/g . They cover all the cases in Condition (Ψ, g) .

- (3a) Ψ is nonincreasing and Ψ/g is nonincreasing;
- (3b) Ψ is increasing and Ψ/g is nonincreasing;
- (3c) Ψ is nonincreasing and Ψ/g is bounded in some neighbourhood of 0;
- (3d) Ψ is increasing and Ψ/g is bounded in some neighbourhood of 0.

Case $\sigma \geq 0$. In this case Ψ is decreasing because $0 \leq \sigma < C$ by Assumption A, (a). Therefore, we consider restrictions (3a) and (3c) only.

Let us start with restriction (3a). Then $\Psi(u + \delta) \leq \Psi(u)$, $\sigma \frac{\Psi(u+\delta)}{g(u+\delta)} \leq \sigma \frac{\Psi(u)}{g(u)}$ when $u > 0$. For $\delta \geq 0$ we set

$$B_\delta(x) := \left(b^+(x) \Phi(u(x)) + \sigma \frac{a(x)}{g(u(x) + \delta)} |\nabla u|^p \right) \Psi(u(x) + \delta) \chi_{\{u(x) > 0\}}. \quad (3.9)$$

For the ease of the proof in what follows we skip repeating the argument of u . Note that $B_\delta \geq 0$ a.e. in Ω and we have for x such that $u(x) > 0$

$$\begin{aligned} A_\delta(x) &= \left(b^+(x)\Phi(u) + \sigma \frac{a(x)}{g(u+\delta)} |\nabla u|^p \right) \Psi(u+\delta) + b^-(x)\Phi(u)\Psi(u+\delta)\phi \\ &= B_\delta(x) + b^-(x)\Phi(u)\Psi(u+\delta) \geq B_\delta(x) + b^-(x)\Phi(u)\Psi(u). \end{aligned}$$

Lebesgue's Monotone Convergence Theorem yields

$$\begin{aligned} &\lim_{\delta \searrow 0} \int_{\{0 < u \leq R-\delta\}} B_\delta(x)\phi(x)dx \\ &= \int_{\{0 < u < R\}} \left(b^+(x)\Phi(u) + \sigma \frac{a(x)}{g(u)} |\nabla u|^p \right) \Psi(u)\phi(x)dx. \end{aligned}$$

For restriction (3c) we verify the convergence of integrals involving B_δ , given by (3.9), by noticing that

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R-\delta\}} b^+(x)\Phi(u)\Psi(u+\delta)\phi(x)dx = \int_{\{0 < u < R\}} b^+(x)\Phi(u)\Psi(u)\phi(x)dx$$

by Lebesgue's Monotone Convergence Theorem, while the convergence

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R-\delta\}} a(x)|\nabla u|^p \frac{\Psi(u+\delta)}{g(u+\delta)} \phi(x)dx = \int_{\{0 < u < R\}} a(x)|\nabla u|^p \frac{\Psi(u)}{g(u)} \phi(x)dx$$

follows from Lebesgue's Dominated Convergence Theorem, as Ψ/g is bounded near 0. From there we deduce the conclusion for the nonnegative σ .

Case $\sigma < 0$. Let us consider first restriction (3a). Then we have

$$\sigma \frac{\Psi(u+\delta)}{g(u+\delta)} \geq \sigma \frac{\Psi(u(x))}{g(u(x))}, \quad b^-(x)\Psi(u(x)+\delta) \geq b^-(x)\Psi(u(x))$$

when $\delta > 0$ and $u(x) > 0$. Moreover, in that case

$$\begin{aligned} A_\delta(x) &\geq \Phi(u)b^+(x)\Psi(u+\delta) + \sigma \frac{a(x)}{g(u)} |\nabla u|^p \Psi(u) + \Phi(u)b^-(x)\Psi(u) \\ &= \Phi(u)b(x)\Psi(u) + \sigma \frac{a(x)}{g(u)} |\nabla u|^p \Psi(u) + \Phi(u)b^+(x)(\Psi(u+\delta) - \Psi(u)) \\ &= A_0(u) - \Phi(u)b^+(x)(\Psi(u) - \Psi(u+\delta)). \end{aligned}$$

Let us consider the integral over $\Omega \cap \{u > 0\}$ from the last expression above and let δ converge to 0. Note that $b^+(x)\Phi(u)(\Psi(u) - \Psi(u+\delta))$ is nonnegative and decreasing to 0 a.e. in $\Omega \cap \{u > 0\}$ as $\delta \searrow 0$. Moreover, according to Assumption A, (b), we have

$$0 \leq b^+(x)\Phi(u)(\Psi(u) - \Psi(u+\delta))\chi_{0 < u \leq R}\phi(x) \leq b^+(x)\Phi(u)\Psi(u)\chi_{0 < u \leq R}\phi(x),$$

which belongs to $L^1(\Omega)$. Therefore, Lebesgue's Dominated Convergence Theorem gives

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R - \delta\}} b^+(x) \Phi(u) (\Psi(u) - \Psi(u + \delta)) \phi(x) dx = 0.$$

If restriction (3b) applies we have $\sigma \frac{\Psi(u+\delta)}{g(u+\delta)} \geq \sigma \frac{\Psi(u)}{g(u)}$, $b^+(x)\Psi(u+\delta) \geq b^+(x)\Psi(u)$ when $u > 0$, and then

$$A_\delta(x) \geq b^+(x) \Phi(u) \Psi(u) + \sigma \frac{a(x)}{g(u)} |\nabla u|^p \Psi(u) + b^-(x) \Phi(u) \Psi(u + \delta).$$

We note that

$$\begin{aligned} & \lim_{\delta \searrow 0} \int_{\{0 < u \leq R - \delta\}} (-b^-(x)) \Phi(u) \Psi(u + \delta) \phi(x) dx \\ &= \int_{\{0 < u < R\}} (-b^-(x)) \Phi(u) \Psi(u) \phi(x) dx. \end{aligned}$$

Indeed, this is a consequence of Lebesgue's Dominated Convergence Theorem and the fact that $\Psi(u + \delta) \leq \Psi(R)$ on the domain of integration $\{0 < u \leq R - \delta\}$, which can be used due to Assumption A, (u).

In case of restriction (3c) we have $b^-(x)\Psi(u + \delta) \geq b^-(x)\Psi(u)$, therefore

$$A_\delta(x) \geq b^+(x) \Phi(u) \Psi(u + \delta) + \sigma a(x) |\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} + b^-(x) \Phi(u) \Psi(u).$$

The convergence of integrals involving $b^+(x) \Phi(u) \Psi(u + \delta) \chi_{\{u > 0\}}$ follows from Lebesgue's Monotone Convergence Theorem, and the convergence of integrals involving $a(x) |\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)}$ follows from Lebesgue's Dominated Convergence Theorem, because we can estimate $(\Psi/g)(u + \delta) \leq \sup\{(\Psi/g)(\lambda) : \lambda \in (0, R)\}$ on domains of integration.

For restriction (3d) we use the following estimate for $u > 0$:

$$A_\delta(x) \geq b^+(x) \Phi(u) \Psi(u) + \sigma a(x) |\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} + b^-(x) \Phi(u) \Psi(u + \delta).$$

We justify the convergence of integrals from the expression on the right-hand side by Lebesgue's Dominated Convergence Theorem using the fact that $\Psi(u + \delta) \leq \Psi(R)$, $(\Psi/g)(u + \delta) \leq \sup\{(\Psi/g)(\lambda) : \lambda \in (0, R)\}$ on the domain of integration, and taking into account Assumption A, (b).

Proof of Step 4 and thus of Lemma 3.1

Let $x \in \text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}$, $\delta \geq 0$ and

$$C_\delta(x) := ca(x)\Psi(u + \delta)g^{p-1}(u + \delta) \left(\frac{|\nabla \phi|}{\phi} \right)^p.$$

Then we have

$$\begin{aligned} \int_{\Omega \cap \{0 < u < R\}} A_0(x)\phi(x) dx &\stackrel{\text{Step 3}}{\leq} \liminf_{\delta \searrow 0} \int_{\Omega \cap \{0 < u < R\}} A_\delta(x)\phi(x) dx \\ &\stackrel{\text{Step 1}}{\leq} \liminf_{\delta \searrow 0} \int_{\Omega \cap \{0 < u < R\}} C_\delta(x)\phi(x) dx + \tilde{C}(\delta, R) \\ &\leq \limsup_{\delta \searrow 0} \int_{\Omega \cap \{0 < u < R\}} C_\delta(x)\phi(x) dx + \tilde{C}(\delta, R) \\ &\stackrel{\text{Step 2}}{\leq} \int_{\Omega \cap \{0 < u < R\}} C_0(x)\phi(x) dx + \tilde{C}(R). \end{aligned}$$

This ends the proof of the statement. $\square$

4 Hardy-type inequality

As a direct consequence of Caccioppoli-type estimates for solutions to PDI, we obtain Hardy-type inequality for a rather general class of test functions, i.e. Lipschitz and compactly supported functions. In this section we present our second main result, as well as we give comments on special instances holding with the optimal constants.

4.1 Main result

The following theorem is our main result on Hardy-type inequalities.

Theorem 4.1 (Hardy-type inequality) *Suppose $a \in L^1_{loc}(\Omega) \cap B_p(\Omega)$, $b \in L^1_{loc}(\Omega)$. Assume that $1 < p < \infty$ and $u \in \mathcal{L}^{1,p}_{a,loc}(\Omega)$ is a nonnegative solution to the PDI $-\Delta_{p,a}u \geq b(x)\Phi(u)\chi_{\{u>0\}}$ in the sense of Definition 2.2. Moreover, let Assumption A hold.*

Then for every Lipschitz function $\xi \in \mathcal{L}^{1,p}_a(\Omega)$ with compact support in Ω we have

$$\int_{\Omega} |\xi|^p \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_2(dx), \quad (4.1)$$

where

$$\begin{aligned}\mu_1(dx) &= \left(\Phi(u)b(x) + \sigma |\nabla u|^p \frac{a(x)}{g(u)} \chi_{\{u \neq 0\}} \right) \Psi(u) \chi_{\{u > 0\}} dx, \\ \mu_2(dx) &= \left(\frac{p-1}{C-\sigma} \right)^{p-1} a(x) \Psi(u) g^{p-1}(u) \chi_{\{u > 0, \nabla u \neq 0\}} dx.\end{aligned}$$

Proof We apply of Theorem 3.1 with $\phi = \xi^p$, where ξ is nonnegative Lipschitz function with compact support. Then ϕ is Lipschitz and

$$|\nabla \xi|^p = \left(\frac{1}{p} \phi^{\frac{1}{p}-1} |\nabla \phi| \right)^p = \frac{1}{p^p} \left(\frac{|\nabla \phi|}{\phi} \right)^p \phi.$$

Therefore (3.3) becomes (4.1). Note that for every Lipschitz function ξ with compact support in Ω we have $\int_{\Omega} |\nabla \xi|^p a(x) dx < \infty$, equivalently $\int_{\text{supp } \phi} |\nabla \phi|^p \phi^{1-p} a(x) dx < \infty$. As the absolute value of a Lipschitz function is a Lipschitz function as well, we write $|\xi|$ instead of ξ on the left-hand side and do not require its nonnegativeness. $\square$

4.2 Special cases

The above Theorem 4.1 generalizes [55, Theorem 4.1], which implies several examples of inequalities with the best constants. Indeed, in the nonweighted case, i.e. when $a(\cdot) = b(\cdot) \equiv 1$, Theorem 4.1, as well as Theorem 3.1, retrieves the results of [55]. In contrast with [55] our function Ψ need not be increasing here. Hence, broader class of measures μ_1 and μ_2 may appear in (4.1). Therefore our result generalizes that of [55] even in the nonweighted case. Below we mention special cases of [55, Theorem 4.1] that in particular results from Theorem 4.1.

Remark 4.1 The instance of Theorem 4.1 is the classical Hardy inequality on the half-line with power-type weights and optimal constant, see [55, Section 5.1]. Moreover, for a range of parameters the Hardy–Poincaré inequality with weights of a form $(1 + |x|^{\frac{p}{p-1}})^{\alpha}$ obtained in [57] (by application of [55]) is also sharp, while for another range it confirms some constants from [37] and [10]. This is commented below in detail. In addition, the classical (unweighted) Poincaré’s inequality on an arbitrary bounded domain can be concluded from [55] and it is confirmed to hold with best constant in [26, Remark 7.6]. The inequality with weights of the form $x^{\alpha} \exp(\beta x^{\gamma})$ provided in [55, Theorem 5.5] can also be retrieved by the methods from [40] with the same constant, while the inequality with weights of the form $x^{\alpha} \log^{p+\alpha}(e+x)$ from [42, Proposition 5.2] are comparable with [55, Theorem 5.8].

The optimal constants in inequalities of Hardy type are an object of a certain discussion already, which we briefly summarize in Table 2.

Hardy inequalities may be used to construct other inequalities.

Table 2 Results dealing with Hardy–Poincaré-type inequalities and their sharpness

Inequality	Optimality	Comment
Classical Hardy	[45]	Proven in [55]
Hardy–Poincaré	[13,25]	Proven with explicitly computed spectrum of the associated linear operator
	[14,15]	In the case of Caffarelli–Kohn–Nirenberg weights $ x ^\alpha$
	[57]	[57, Theorem 3.1]; improved constants from [10,37]
	[44]	Theorem 4.3 and Remark 4.1
Poincaré	[26]	Concluded from [55]
Exponential-weighted Hardy	Expected	[55, Theorem 5.5] versus [40]
	Expected	[55, Theorem 5.8] versus [42, Proposition 5.2]

Remark 4.2 In [41] it is proven that if the measures $\mu = \exp(-\phi)dx$, $\nu = |\nabla\phi|^p\mu$, where $\phi \in W^{1,\infty}(\Omega)$, are admitted to Hardy inequalities:

$$\|u\|_{L^p(\Omega,\nu)} \leq C \|\nabla u\|_{L^p(\Omega,\mu)}, \quad (4.2)$$

then the following Gagliardo–Nirenberg interpolation inequalities for intermediate derivatives holds for μ :

$$\|\nabla u\|_{L^q(\Omega,\mu)}^2 \leq C \|u\|_{L^r(\Omega,\mu)} \|\nabla^{(2)} u\|_{L^p(\Omega,\mu)}, \quad \text{where } \frac{2}{q} = \frac{1}{r} + \frac{1}{p}.$$

The examples of measures on $\Omega = (0, \infty)$ which admit (4.2), are $\mu(dx) = x^\alpha \exp(-x^\beta) dx$ where $\alpha \geq 0, \beta \geq 0$. Their instances are weights of power growth ($\beta = 0$), exponential one ($\alpha = 0, \beta = 1$), as well as the Gaussian distribution ($\alpha = 0, \beta = 2$). Another example of the admissible measures on domains $\Omega \subseteq \mathbf{R}^n$ are powers of distances from the boundary. See the discussion in [41].

4.3 Hardy inequalities resulted from existence theorems

We are going to derive sharp Hardy type inequality, not knowing u explicitly but only its existence. We assume now that b is nonnegative and that there exists a nonnegative nontrivial locally bounded solution of PDI $-\Delta_{p,a}u \geq b(x)u^{p-1}$ i.e.,

$$\langle -\Delta_{p,a}u, w \rangle \geq \int_{\Omega} b(x)u^{p-1}w \, dx,$$

holds for every nonnegative compactly supported function $w \in \mathcal{L}_a^{1,p}(\Omega)$.

This is the special case of inequality (2.5) for $\Phi(u) = u^{p-1}$. Our result reads as follows.

Theorem 4.2 (Sharp Hardy inequality) *Suppose $1 < p < \infty$, $a, b \in W(\Omega)$, $a \in L_{loc}^1(\Omega) \cap B_p(\Omega)$, and $u \in \mathcal{L}_{a,loc}^{1,p}(\Omega)$, $bu^{p-1} \in L_{loc}^1(\Omega)$, u is a nonnegative nontrivial*

solution to (4.3). Then for every Lipschitz function $\xi \in \mathcal{L}_a^{1,p}(\Omega)$ with compact support in Ω we have

$$\int_{\Omega} |\xi|^p b(x) dx \leq \int_{\Omega} |\nabla \xi|^p a(x) dx. \quad (4.3)$$

Moreover, if there exists nontrivial, nonnegative, $u_0 \in W_{(b,a),0}^{1,p}(\Omega)$ which is the solution to

$$-\Delta_{p,a} u_0 = b(x) u_0^{p-1} \in L_{loc}^1(\Omega) \quad (4.4)$$

then inequality (4.3) is sharp, i.e. for compactly supported $\xi \in \mathcal{L}_a^{1,p}(\Omega)$, the constant $C = 1$ is optimal in the inequality $C \int_{\Omega} |\xi|^p b(x) dx \leq \int_{\Omega} |\nabla \xi|^p a(x) dx$.

Proof We apply Theorem 4.1 with $\Psi(t) = \frac{1}{t^{p-1}}$, $g(t) = t$, $\Phi(t) = t^{p-1}$, $\sigma = 0$ and verify that under our conditions Assumption A is satisfied. This gives (4.3).

Suppose now that there exists u_0 being a solution to (4.4) and satisfying all the requirements of the theorem. Let us consider the sequence $(w_k)_{k \in \mathbb{N}}$ of smooth compactly supported functions, such that $w_k \rightarrow u_0$ in $W_{(b,a)}^{1,p}(\Omega)$. Since each w_k has a compact support and belongs to $\mathcal{L}_a^{1,p}(\Omega)$, according to (4.4), we have the equality

$$\langle -\Delta_{p,a} u_0, w_k \rangle = \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla w_k a(x) dx = \int_{\Omega} b(x) u_0^{p-1} w_k dx.$$

When we let $k \rightarrow \infty$, we get $\int_{\Omega} |\nabla u_0|^p a(x) dx = \int_{\Omega} b(x) u_0^p dx$ which proves sharpness. $\square$

Remark 4.3 Theorem 4.2 is known in the case $a \equiv 1$, $b \equiv 1$, see [1] or Remark 1 on page 163 in [48].

Remark 4.4 We substitute the special value of $\sigma = 0$, in the proof of the above statement. Therefore, we do not expect that the inequality (4.3) holds with the best constant in general.

4.4 Sharp Hardy–Poincaré inequalities

Using the Talenti extremal profile given by (1.5) where $\beta = 0$ in our approach, one obtains the following theorem. Adopting the same nomenclature as in e.g. [10, 13], we call the following inequalities of Hardy–Poincaré type.

Theorem 4.3 (cf. Theorem 2 in [44]) Assume that $1 < p < \infty$, $\gamma > 1 - \frac{n}{p}$, $0 < r < 1 - \frac{p}{n} + \gamma \frac{p}{n}$ and $v_1(x) := \left(1 + r|x|^{\frac{p}{p-1}}\right) \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p}$, $v_2(x) := \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma}$. Then for every $\xi \in W_{v_1, v_2}^{1,p}(\mathbb{R}^n)$ we have

$$\begin{aligned} \bar{C}_{\gamma, n, p, r} \int_{\mathbb{R}^n} |\xi|^p \left(1 + r|x|^{\frac{p}{p-1}}\right) \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p} dx \\ \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma} dx, \end{aligned}$$

where $\bar{C}_{\gamma,n,p,r} = n \left(\frac{p}{p-1} \right)^{p-1} \left(\gamma - 1 + \frac{n}{p}(1-r) \right)^{p-1}$. Moreover, the constant $\bar{C}_{\gamma,n,p,r}$ is optimal when $\gamma > nr + 1 - \frac{n}{p}$ and when $\gamma = 1 + n(1 - \frac{1}{p})$, $r = 1$.

Information about the proof In the construction of the inequality we apply Theorem 4.1 involving the PDE:

$$-\Delta_{p,u_\beta} u_\alpha = C_1(1 + C_2|x|^{p'})\Phi(u_\alpha), \quad (4.5)$$

where $u_\eta(x) := (1 + |x|^{p'})^{-\eta}$, $\Phi(s) = s^\delta$ where $\delta > 0$, $\eta \in \mathbf{R}$, $p' = p/(p-1)$, with the suitable chosen parameters. All the details are provided in [44]. $\square$

Remark 4.5 We would like to point out a typo in the formulation of the above theorem in the original paper, i.e. [44, Theorem 2]. The correct range of parameters is $0 < r < 1 - \frac{p}{n} + \gamma \frac{p}{n}$, but it is wrongly stated therein as $0 < r < \frac{p}{n} + \gamma \frac{p}{n}$. In fact, in the first line on page 171 in [44], it should be written

$$R_{\alpha,\beta}(\mathcal{A}(\alpha, \beta, \gamma)) = [-\frac{\beta p'}{n}, 1 + \frac{p}{n}(\gamma - 1)) \supseteq [0, 1 + \frac{p}{n}(\gamma - 1) =: y(n)) \text{ for } \beta \geq 0, \quad (4.6)$$

instead of $R_{\alpha,\beta}(\mathcal{A}(\alpha, \beta, \gamma)) = [0, \frac{p}{n} + \gamma \frac{p}{n} =: y(n))$.

Remark 4.6 Such inequalities in the case $p = 2$ are very much of interest in the theory of nonlinear diffusions, where one investigates the asymptotic behavior of solutions of the equation $u_t = \Delta u^m$, see [10]. The best constants in the case $p = 2$ have been obtained in [13], where also the whole spectrum of the associated operator has been explicitly calculated, by means of linear spectral methods, different from the one we use here. In the case of Caffarelli–Kohn–Nirenberg weights we refer to [14, 15, 29] for analogous results.

Remark 4.7

- (i) To our best knowledge our inequalities are new if $r \neq 1$ in general. However, as an example dealing with $r \neq 1$ and $p = 2$ we refer to the fourth line on page 434 in [10, Proposition 2], which is precisely our inequality from Theorem 4.3 with $r = \gamma/n$, $p = 2$ (with the same constant)

$$\begin{aligned} & n(n-2+\gamma) \int_{\mathbf{R}^n} |\xi|^2 \left(1 + \frac{\gamma}{n}|x|^2 \right) (1 + |x|^2)^{\gamma-2} dx \\ & \leq \int_{\mathbf{R}^n} |\nabla \xi|^2 (1 + |x|^2)^\gamma dx. \end{aligned}$$

The proof of this inequality in [10] requires knowledge about the best constants in Sobolev inequality, which we do not need.

- (ii) We note that Theorem 4.3 provides a version of [10, Proposition 3] dealing with $p = 2$, for an arbitrary p . Thus, we can interpret it as therein that our method allows to find a sharp, in some cases, first spectral gap (or Poincaré inequality)

with methods different from the already known ones, and that are suitable for generalization to more sophisticated operators.

- (iii) We retrieve [57, Theorem 3.1] as a special case of Theorem 4.2 when one substitutes $r = 1$ and deals with $\gamma > 1$. In Theorem 4.3 we admit some range of negative γ s as well. The need to consider the negative γ s in case of $p = 2$ and to look for the best constants in such inequalities is visible in the studies on asymptotics of fast diffusion equations, see [13, Theorem 1] and its application in the proof of [13, Theorem 2].

We also present the following consequence of Theorem 4.1, which is in a sense complementary to Theorem 4.3, as it holds for with negative r . It essentially requires to deal with the weighted p -Laplacian in the inequality (1.1).

Theorem 4.4 Assume that $1 < p < \infty$, $r < 0$, $\gamma > 1 + (r - 1)\frac{n}{p}$ and

$$v_1(x) := \left(1 + r|x|^{\frac{p}{p-1}}\right) \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p}, \quad v_2(x) := \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma}.$$

Then for every $\xi \in W_{\{v_1, v_2\}, 0}^{1,p}(B)$ where $B = B(0, |r|^{-1/p'})$, we have

$$\begin{aligned} \bar{C}_{\gamma, n, p, r} \int_B |\xi|^p \left(1 + r|x|^{\frac{p}{p-1}}\right) \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p} dx \\ \leq \int_B |\nabla \xi|^p \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma}, \end{aligned}$$

where the constant is $\bar{C}_{\gamma, n, p, r} = n \left(\frac{p}{p-1}\right)^{p-1} \left(\gamma - 1 + \frac{n}{p}(1-r)\right)^{p-1}$.

Proof Let us consider the notation from (4.5). Direct computation or application of [44, Lemma 4] gives for any $\alpha, \beta \in \mathbf{R}$:

$$-\Delta_{p, u_\beta} u_\alpha = n \Phi_p(\alpha p') u_{(\alpha+1)(p-1)+\beta+1} (1 + c_{\alpha, \beta, p, n} |x|^{p'}) \quad \text{in } \mathbf{R}^n, \quad (4.7)$$

where

$$c_{\alpha, \beta, p, n} = 1 - \frac{((\alpha+1)(p-1)+\beta)p'}{n} \quad \text{and} \quad \Phi_p(s) = |s|^{p-2}s.$$

Let

$$\begin{aligned} a(x) &:= u_{-\gamma(p-1)}(x), & b(x) &:= \bar{C}_{\gamma, n, p, r} \left(1 + r|x|^{\frac{p}{p-1}}\right) \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p}, \\ \alpha &:= \gamma - 1 + (1-r)\frac{n}{p} > 0, & \beta &:= -\gamma(p-1). \end{aligned}$$

Substituting such α and β to (4.7), we obtain:

$$-\Delta_{p, a(x)} u_\alpha = b(x) u_\alpha^{p-1} \quad \text{in } B$$

and b is nonnegative in B , because α is positive. Now the result follows directly from Theorem 4.2. $\square$

4.5 Other possible variants of Hardy inequality

It is possible to consider another PDI

$$-\Delta_{p,a}u \geq b(x)\Phi(u), \quad (4.8)$$

that is (2.5) with $\Phi(u)\chi_{\{u>0\}}$ substituted by $\Phi(u)$, where we assume that Φ is continuous up to zero, with obvious modifications of its definition. Clearly, when $\Phi(0) = 0$ then inequalities (3.1) and (4.8) are equivalent, while if $b(x)\chi_{u(x)=0} \geq 0$ a.e. then the formulation (3.1) is weaker, while if $b(x)\chi_{u(x)=0} \leq 0$ then the formulation (4.8) is weaker. In general the formulations might not compare when we deal with dead core solutions and $b(\cdot)$ changes its sign on the dead regim.

To deal with (4.8) we will consider the following assumption.

(u_2) We assume that $u \in \mathcal{L}_{a,loc}^{1,p}(\Omega)$ is nonnegative, (a, b) holds, $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function, such that for every nonnegative compactly supported function $w \in \mathcal{L}_a^{1,p}(\Omega)$ one has $\int_{\Omega} b(x)\Phi(u)w \, dx > -\infty$ and $b(\cdot)\Phi(u) \in L_{loc}^1(\Omega)$.

Moreover, let us consider the set $\mathcal{A}$ of those $\sigma \in \mathbf{R}$ for which

$$b(x)\Phi(u) + \sigma \frac{a(x)}{g(u)} |\nabla u|^p \geq 0 \quad \text{a.e. in } \Omega \cap \{u > 0\}.$$

We suppose that

$$\sigma_0 := \inf \mathcal{A} = \inf \{\sigma \in \mathbf{R} : \sigma \text{ satisfies (2.10)}\} \in \mathbf{R}.$$

We obtain the following statement.

Theorem 4.5 Suppose $a \in L_{loc}^1(\Omega) \cap B_p(\Omega)$, $b \in L_{loc}^1(\Omega)$. Assume that $1 < p < \infty$ and $u \in \mathcal{L}_{a,loc}^{1,p}(\Omega)$ is a nonnegative solution to the PDI (4.8). Moreover, let Assumption A holds where (u) substituted by (u_2) and where additionally $\lim_{\delta \rightarrow 0} \Psi(\delta) = 0$.

Then for every Lipschitz function $\xi \in \mathcal{L}_a^{1,p}(\Omega)$ with compact support in Ω we have

$$\int_{\Omega} |\xi|^p \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_2(dx), \quad (4.9)$$

where

$$\begin{aligned} \mu_1(dx) &= \left(\Phi(u)b(x) + \sigma |\nabla u|^p \frac{a(x)}{g(u)} \chi_{\{u \neq 0\}} \right) \Psi(u) \chi_{\{u>0\}} \, dx, \\ \mu_2(dx) &= \left(\frac{p-1}{C-\sigma} \right)^{p-1} a(x) \Psi(u) g^{p-1}(u) \chi_{\{u>0, \nabla u \neq 0\}} \, dx. \end{aligned}$$

Sketch of the proof At first by minor modifications of the proof of Lemma 3.1 we prove its variant holding under current our assumptions and the same conclusion. Repeating almost the same arguments as in Step 1 we prove that for every $0 < \delta < R$:

$$\begin{aligned} & \int_{\{\Omega \cap \{0=u\}\}} b(x) \Phi(u) \Psi(u + \delta) \phi \, dx \\ & + \int_{\{\Omega \cap \{0 < u \leq R-\delta\}\}} \left(b(x) \Phi(u) + \sigma \frac{a(x)}{g(u + \delta)} |\nabla u|^p \right) \Psi(u + \delta) \phi \, dx \\ & \leq c \int_{\Omega \cap \text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R-\delta\}} a(x) \Psi(u + \delta) g^{p-1}(u + \delta) \left(\frac{|\nabla \phi|}{\phi} \right)^p \phi \, dx \\ & + \tilde{C}(\delta, R). \end{aligned} \quad (4.10)$$

Now arguments of Steps 2 remain the same, because we deal there with integrals over the sets where u is strictly positive. In Step 3 we additionally observe that first term in (4.10) converges to zero. The final conclusion to obtain the local estimates follows from the same arguments as in Step 4.

From there we obtain the Caccioppoli estimates and Hardy inequalities, by precisely the same arguments as in the proofs in Theorems 3.1 and 4.1. $\square$

Remark 4.8 We can weaken the assumptions of Lemma 3.1, and thus in Theorem 3.1, if we have more information about u . For example, if $u \in (0, b)$ a.e. for some $0 < b < \infty$, it suffices to consider continuous $\Phi, \Psi : (0, b) \rightarrow (0, \infty)$, with suitable obvious modifications of the other parts of Assumption A, according to the restriction of the domain of u , that is $(0, b)$. Such a situation appears for example in the study on a simple model of electrostatic micromechanical systems (MEMS) [35], which is reduced to the following problem

$$\begin{cases} -\Delta u = \frac{b(x)}{(1-u)^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ 0 < u < 1 & \text{in } \Omega, \end{cases} \quad (4.11)$$

where $b(\cdot) \geq 0$, $u \in C^1(\overline{\Omega}) \cap W^{2,2}(\Omega)$, Ω is open and bounded, sufficiently regular. Clearly, in that case $\Phi(u) = \frac{1}{(1-u)^2}$ and Φ is not continuous on the whole $(0, \infty)$.

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