Correction to: Ulrich bundles on non-special surfaces with $p_g = 0$ and $q = 1$

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Published online: 20 February 2019
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Correction to: Rev Mat Complut <https://doi.org/10.1007/s13163-017-0248-z>

The proof of Theorem 1.2 in [2] contains a gap which can be filled by simply assuming that the base field k is uncountable. In particular all the results in [2] remain true if we assume such a restriction in Sections 4 and 5. We shortly explain below the due changes in the paper and in the proof. For all the definitions and the notation, we refer to [2].

Recall that S is a smooth, irreducible surface with $p_g(S) = 0$ and $q(S) = 1$ endowed with a very ample line bundle $\mathcal{O}_S(h)$ which is non-special (i.e. $h^1(S, \mathcal{O}_S(h)) = 0$). If K_S is the canonical class of S , the Riemann–Roch theorem implies that $\mathcal{O}_S(h)$ gives an embedding $S \subseteq \mathbb{P}^N$ where

$$N := h^0(S, \mathcal{O}_S(h)) - 1 = \frac{h^2 - hK_S}{2} - 1,$$

because $h^2(S, \mathcal{O}_S(h)) = h^0(S, \mathcal{O}_S(K_S - h)) \leq p_g(S) = 0$.

Each general hyperplane section $C = S \cap H$ is a smooth irreducible curve with genus

$$\pi(\mathcal{O}_S(h)) := \frac{h^2 + hK_S}{2} + 1.$$

The Hilbert scheme $\mathcal{H}_C$ of 0-dimensional subschemes of degree $N + 1$ on C has dimension $N + 1$ and contains an open non-empty subset $\mathcal{R} \subseteq \mathcal{H}_C$ corresponding to reduced subschemes of C consisting of $N + 1$ points in general linear position in H .

The original article can be found online at <https://doi.org/10.1007/s13163-017-0248-z>.

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If $Z \subseteq C \subseteq S$ corresponds to a general point in $\mathcal{R}$ and $\mathcal{O}_S(\eta) \in \text{Pic}^0(S) \setminus \{ \mathcal{O}_S \}$ is such that

$$h^0(S, \mathcal{O}_S(K_S \pm \eta)) = h^1(S, \mathcal{O}_S(h \pm \eta)) = 0.$$

Theorem 1.1 of [2] guarantees the existence of an Ulrich bundle $\mathcal{E}$ of rank 2 fitting into an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_S(h + K_S + \eta) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|S}(2h + \eta) \longrightarrow 0. \quad (1)$$

Assume that k is uncountable: Theorem 1.2 in [2] then holds in the following form.

Theorem 1.2 *Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$.*

If $\pi(\mathcal{O}_S(h)) \geq 2$, then the bundle $\mathcal{E}$ constructed in Theorem 1.1 of [2] from a very general set $Z \subseteq C \subseteq S$ of $h^0(S, \mathcal{O}_S(h))$ points is stable.

Proof We know how to construct a rank 2 Ulrich bundle fitting into Sequence (1) starting from C , Z and $\mathcal{O}_S(\eta)$ as above.

Let $\mathcal{D} \subseteq \text{Pic}(S)$ be the set of Ulrich line bundles. If $\mathcal{O}_S(D) \in \mathcal{D}$, then

$$Dh = \frac{1}{2}(3h^2 + hK_S) \quad (2)$$

(see Proposition 2.1 of [2]). Equality (2) yields $(h + \eta - D)h \leq -1$ because $\pi(\mathcal{O}_S(h)) \geq 2$. Thus

$$h^0(S, \mathcal{I}_{C|S}(2h + \eta - D)) = h^0(S, \mathcal{O}_S(h + \eta - D)) = 0,$$

i.e. there are no divisors $A \in |2h + \eta - D|$ containing C . By definition for each

$$Z \in \mathcal{Z}_D := \{Z \in \mathcal{R} \mid h^0(S, \mathcal{I}_{Z|S}(2h + \eta - D)) \geq 1\}$$

there is $A \in |2h + \eta - D|$ such that $Z \subseteq A$. Thus $Z = C \cap A$, because $(2h + \eta - D)h = N + 1$ and $C \not\subseteq A$. If $i: C \rightarrow S$ denotes the inclusion map, then Z identifies a unique element in $|i^*\mathcal{O}_S(2h + \eta - D)|$. Such a construction gives an injective morphism $\mathcal{Z}_D \rightarrow |i^*\mathcal{O}_S(2h + \eta - D)|$, hence

$$\dim(\mathcal{Z}_D) \leq h^0(C, i^*\mathcal{O}_S(2h + \eta - D)) - 1.$$

If $i^*\mathcal{O}_S(2h + \eta - D)$ is special, then the Clifford theorem and the Equality (2) imply

$$h^0(C, i^*\mathcal{O}_S(2h + \eta - D)) \leq \frac{(2h + \eta - D)h}{2} + 1 = \frac{N + 3}{2} \leq N,$$

because $N \geq 3$. If $i^* \mathcal{O}_S(2h + \eta - D)$ is non-special, the Riemann–Roch theorem on C and the Equality (2) return

$$h^0(C, i^* \mathcal{O}_S(2h + \eta - D)) = N + 2 - \pi(\mathcal{O}_S(h)) \leq N,$$

because $\pi(\mathcal{O}_S(h)) \geq 2$. Thus $\dim(\mathcal{Z}_D) \leq N - 1$ for each Ulrich line bundle $\mathcal{O}_S(D)$.

Let $\nu: \text{Pic}(S) \rightarrow \text{NS}(S)$ be the natural map onto the Néron–Severi group. Thanks to the discussion above the dimension of $\widehat{\mathcal{Z}}_\delta := \bigcup_{D \in \nu^{-1}(\delta)} \mathcal{Z}_D$ is at most N for each $\delta \in \nu(\mathcal{D})$, because the fibres of ν have dimension 1. Since $\text{NS}(S)$ is a finitely generated abelian group, $\dim(\mathcal{R}) = N + 1$ and k is uncountable, it follows the existence of $Z \in \mathcal{R} \setminus \bigcup_{\delta \in \nu(\mathcal{D})} \widehat{\mathcal{Z}}_\delta$ (see Exercise V.4.15 (c) of [3]). We show below that the corresponding bundle $\mathcal{E}$ is stable.

If not, then it is also not μ -stable. Theorem 2.9 of [1] guarantees that $\mathcal{E}$ is μ -semistable; hence there exists a line bundle $\mathcal{O}_S(D) \subseteq \mathcal{E}$ such that $\mu(\mathcal{E}) = \mu(\mathcal{O}_S(D))$. Again the aforementioned theorem also implies that $\mathcal{O}_S(D) \in \mathcal{D}$.

On the one hand, the choice of Z implies that $h^0(S, \mathcal{I}_{Z|S}(2h + \eta - D)) = 0$; hence $\mathcal{O}_S(D)$ must be contained in the kernel $\mathcal{O}_S(h + K_S + \eta)$ of the map $\mathcal{E} \rightarrow \mathcal{I}_{Z|S}(2h + \eta)$ in Sequence (1). On the other hand, Equality (2) implies $(h + K_S + \eta - D)h \leq -2$ because $N \geq 3$, hence $h^0(S, \mathcal{O}_S(h + K_S + \eta - D)) = 0$. We deduce that such an $\mathcal{O}_S(D)$ cannot exist; hence the bundle $\mathcal{E}$ is necessarily stable. $\square$

References

1. Casanellas, M., Hartshorne, R., Geiss, F., Schreyer, F.O.: Stable Ulrich bundles. *Int. J. Math.* **23**, 1250083 (2012)
2. Casnati, G.: Ulrich bundles on non-special surfaces with $p_g = 0$ and $q = 1$. *Rev. Mat. Complut.* (2019)
3. Hartshorne, R.: *Algebraic Geometry*. G.T.M. 52. Springer, Berlin (1977)

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