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ORIGINAL PAPER

Upper estimates for blow-up solutions of a quasi-linear parabolic equation

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Abstract
In this paper, we consider a quasi-linear parabolic equation $u_t = u^p(x_{xx} + u)$. It is known that there exist blow-up solutions and some of them develop Type II singularity. However, only a few results are known about the precise behavior of Type II blow-up solutions for $p > 2$. We investigated the blow-up solutions for the equation with periodic boundary conditions and derived upper estimates of the blow-up rates in the case of $2 < p < 3$ and in the case of $p = 3$, separately. In addition, we assert that if $2 \leq p \leq 3$ then $\lim_{t \nearrow T}(T-t)^{\frac{1}{p}+\varepsilon} \max u(x,t) = 0$ z for any $\varepsilon > 0$ under some assumptions.

Keywords Asymptotic behavior · Quasi-linear parabolic equation · Blow-up phenomena

Mathematics Subject Classification 35B44 · 35B40 · 35K59

1 Introduction

In this paper, we consider classical solutions $u = u(x,t)$ of

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$$u_t = u^p(u_{xx} + u) \quad \text{for } (x,t) \in (-L, L) \times (0, T) \tag{1.1}$$

with the following periodic boundary condition

$$u(-L,t) = u(L,t) \quad \text{and} \quad u_x(-L,t) = u_x(L,t) \quad \text{for } t \in (0,T) \tag{1.2}$$

and positive initial data. If $p > 0$ and $L > \frac{\pi}{2}$, then solutions of (1.1)–(1.2) blow up in finite time, T, which is called the blow-up time. It is well-known that when $0 < p < 2$, they develop Type I singularity, that is,

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{p}} \max_{x \in [-L,L]} u(x,t) < \infty.$$

(For instance, see [9].) On the other hand, if $p \geq 2$ then some of them develop Type II singularity, that is,

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{p}} \max_{x \in [-L,L]} u(x,t) = \infty. \tag{1.3}$$

We call such solutions Type II blow-up solutions. (For instance, see [4, 6, 10].) Since we are interested in Type II blow-up solutions, we consider the case $p \geq 2$.

A background of (1.1)–(1.2) is the motion of the plane curve by the power of its curvature,

$$\frac{d\mathcal{X}}{dt} = -k^\alpha \mathcal{N},$$

where α is a positive parameter and $\mathcal{N}$ and k denote the outer unit normal vector and the curvature of the curve at the point $\mathcal{X}$, respectively. In the case where the curvature is positive everywhere on the closed curve, we can parametrize the curve by the normal angle x and $u(x,t) = \left(\alpha^{-\frac{1}{\alpha+1}} k(x,t)\right)^\alpha$ satisfies (1.1)–(1.2) with $L = m\pi$ for some $m \in \mathbb{N}$ and $p = 1 + 1/\alpha$. Here, $k(x, t)$ is the curvature of the curve at the point with $\mathcal{N} = (\cos x, \sin x)$.

If $m = 1$ and $2 \leq p < 4$, then all solutions of (1.1) blow up of Type I. (See [3, 8].) When $m \geq 2$ and $p \geq 2$, the behavior of solutions is different from the case of $m = 1$. In [4], Angenent proved that there exists a Type II blow-up solution of (1.1)–(1.2) for the case of $p = 2$ and $m \geq 2$. He treated the classical curve-shortening flow of a closed cardioid-like immersed curve with a self-crossing point (Fig. 1) which is corresponding to the case of $p = 2$ and $L = m\pi$ with $m \geq 2$. This is the first result

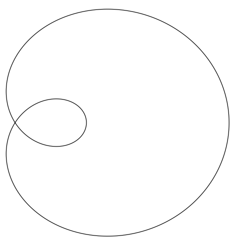

Fig. 1 An example of immersed curves with self-crossing points

for the blow-up rates of $\max_{x\in[-L,L]} u(x,t)$. Furthermore, he also proved that for this blow-up solution, u in the case of $p = 2$,

$$\lim_{t\nearrow T}(T-t)^{\frac{1}{2}+\varepsilon} \max_{x\in[-L,L]} u(x,t) = 0 \quad \text{for any } \varepsilon > 0. \tag{1.4}$$

[1, 5, 7] have investigated the blow-up rates of Type II blow-up solutions under the following conditions for initial data

(I1) $u(x,0) = u(-x,0)$ for any $x \in [-L,L]$,
(I2) $u_x(x,0) < 0$ if $x \in (0,L)$ and $u_x(x,0) > 0$ if $x \in (-L,0)$,
(I3) there exists $\eta_0 > 0$ such that $u(x,0) \geq \eta_0 > 0$ for any $x \in [-L,L]$,
(I4) $u_{xx}(x,0) + u(x,0) \geq 0, \not\equiv 0$ for any $x \in (-L,L)$.

A typical example of the plane curve that satisfies (I1), (I2), (I3), and (I4) is a cardioid-like curve. The precise blow-up rates for $p = 2$ were established by Angenent and Velázquez [5] under the assumptions (I1), (I2), and (I3), that is, solutions of (1.1)–(1.2), u, satisfy

$$\max_{x\in[-L,L]} u(x,t) = \big(1+o(1)\big)\sqrt{\frac{1}{T-t}\log\log\frac{1}{T-t}} \quad \text{as } t \nearrow T. \tag{1.5}$$

Moreover, in [1], the first and the second authors proved the same results as (1.5) for solutions with the Dirichlet boundary condition.

Some results for $p > 2$ and $m \geq 2$ were provided by Poon [7]. Precisely, he showed that solutions of (1.1)–(1.2) satisfy the following.

– Let $p > 2$, $m \geq 2$ and assume (I1), (I2), and (I3). Then there exists $t_* \in (0,T)$ and a constant $C = C(p) > 0$ such that u satisfies

$$\max_{x\in[-L,L]} u(x,t) \geq C(p)\left(\frac{T}{T-t}\right)^{\frac{1}{p}}\left(\frac{1}{p}\log\frac{T}{T-t}\right)^{\frac{p-2}{p}} \quad \text{if } t \in (t_*,T). \tag{1.6}$$

– Let $2 < p < 3$, $m \geq 2$ and assume (I1), (I2), (I3), and (I4). Then there exists $t_* \in (0,T)$ and a constant $C = C(p) > 0$ such that u satisfies

$$\max_{x\in[-L,L]} u(x,t) \leq C(p)\sqrt{\frac{T}{T-t}} \quad \text{if } t \in (t_*,T). \tag{1.7}$$

In addition, [7] showed the same results for solutions with the Dirichlet boundary condition.

Our purpose of this paper is to improve upper estimates of blow-up rates for $2 < p < 3$ and provide one for $p = 3$ for solutions of (1.1) with the periodic boundary condition (1.2). Precisely, our main result is as follows.

Theorem 1 (Main result) *Let u be a solution of* (1.1)–(1.2) *with* (1.3). *Let $L = m\pi$, where $m \geq 2$ is an integer. Assume* (*I*1), (*I*2), (*I*3), *and* (*I*4). *Then the following hold.*

(i) *In the case of* $2 < p < 3$, *there exist* $t_* \in (0, T)$ *and a constant* $C = C(p) > 0$ *such that u satisfies*

$$\max_{x\in[-L,L]} u(x,t) \le C(p)\Big(\frac{1}{T-t}\Big)^{\frac{1}{p}}\Big(\log\frac{1}{T-t}\Big)^{\frac{p-2}{p(3-p)}} \quad \text{if } t \in (t_*, T). \tag{1.8}$$

(ii) *In the case of* $p = 3$, *there exist* $t_* \in (0, T)$ *and a constant* $C > 0$ *such that u satisfies*

$$\max_{x\in[-L,L]} u(x,t) \le \Big(\frac{1}{T-t}\Big)^{\frac{1}{3}} \exp\left(C\sqrt{\log\frac{1}{T-t}}\right) \quad \text{if } t \in (t_*, T). \tag{1.9}$$

We remark that the result of Angenent (1.4) can be extended to the case of $2 < p \le 3$ by Theorem 1 as follows:

Corollary 1 *Let u be a solution of* (1.1)–(1.2) *with* (1.3). *Assume* (I1), (I2), (I3), *and* (I4). *Then if* $2 \le p \le 3$ *then*

$$\lim_{t\nearrow T}(T-t)^{\frac{1}{p}+\varepsilon}\max_{x\in[-L,L]} u(x,t) = 0 \quad \text{for any } \varepsilon > 0.$$

Let us comment on expected blow-up rates for $p = 2$, $2 < p < 3$, and $p = 3$. The precise rate for $p = 2$ is known as (1.5). In the case of $2 < p < 3$, the two inequalities, (1.6) and (1.8), suggest that there exists $\gamma = \gamma(p) > 0$ such that the blow-up rates for $2 < p < 3$ are the form of

$$\max_{x\in[-L,L]} u(x,t) = O\left(\Big(\frac{1}{T-t}\Big)^{\frac{1}{p}}\Big(\log\frac{1}{T-t}\Big)^{\gamma(p)}\right) \quad \text{as } t \nearrow T, \tag{1.10}$$

where $\frac{p-2}{p} \le \gamma(p) \le \frac{p-2}{p(3-p)}$.

The blow-up rate for $2 < p < 3$, (1.10), has a completely different form that for $p = 2$. Since the exponent $\gamma(2) = 0$, the estimate (1.10) fails for $p = 2$. Hence, the more subtle correction term, which has $\log\log$ form, appears for $p = 2$, (1.5). Furthermore, Theorem 1 seems to support that the blow-up rate may drastically change at $p = 3$. Indeed, the divergence of the upper estimate (1.8) as $p \to 3$ is the reason that $\gamma(p)$ might also diverge as $p \to 3$ according to (1.10).

Let us explain our strategy to prove our main theorem briefly. First, we introduce a function,

$$\psi_\lambda(\sigma) := \int_\sigma^\infty \frac{1}{U(0,\tilde{\sigma})^{\frac{2}{p-2}}} e^{-\lambda(\tilde{\sigma}-\sigma)}\, d\tilde{\sigma},$$

where $\lambda > 0$ and $U(x,\sigma) = e^{-\frac{\sigma}{p}} u(x, T - e^{-\sigma})$ ($\sigma = \log\frac{1}{T-t}$), which is sometimes called type I rescaling. The estimates for $\psi_\lambda(\tau)$ play an important role in the proof of our main results. The function ψ_λ is the novel device employed in this paper. Second, the function

$$\begin{aligned} W_p(t) &:= (T-t)u\left(\frac{\pi}{2}, t\right)u(0,t)^{p-1} \\ &= U\left(\frac{\pi}{2}, \sigma\right)U(0,\sigma)^{p-1} \end{aligned} \tag{1.11}$$

can be estimated from above. We note that the upper bound of $W_p(t)$ has different forms in the cases of $2 < p < 3$ and $p = 3$. The former case was obtained in [7], and the latter case is newly proved in this paper. Finally, using the bound of $W_p(t)$, we estimate ψ_λ from below for suitable λ, and we can obtain Theorem 1.

2 Upper bounds for solutions

When (I1), (I2), (I3), and (I4) hold, solutions u of (1.1)–(1.2) also satisfy the following.

- If (I1) holds then $u(x,t) = u(-x,t)$ for any $x \in [-L, L]$ and $t \in (0,T)$.
- If (I2) holds then $u_x(x,t) < 0$ if $x \in (0,L)$ and $t \in (0,T)$ and $u_x(x,t) > 0$ if $x \in (-L, 0)$ and $t \in (0,T)$.
- If (I3) holds then $u(x,t) \geq \eta_0 > 0$ for any $x \in [-L, L]$ and $t \in (0,T)$.
- If (I4) holds then $u_{xx}(x,t) + u(x,t) \geq 0$ for any $x \in (-L, L)$ and $t \in (0,T)$.

Some features for u are already known (for instance, see [7]). For the readers' convenience, we summarize them in the following proposition.

Proposition 1 *Assume* (*I*1) *and* (*I*4). *If* $0 < x < \pi/2$, *then* u *satisfies*

$$u(x,t) \geq u(0,t)\cos x, \tag{2.1}$$

$$u(x,t) \leq u(0,t)\cos x + u\left(\frac{\pi}{2}, t\right)\sin x, \tag{2.2}$$

and

$$-u(0,t)\sin x \leq u_x(x,t) \leq -\frac{u(x,t)\sin x}{\cos x} + \frac{u\left(\frac{\pi}{2}, t\right)}{\cos x}. \tag{2.3}$$

Proof Since (I1) and (I4) imply $u_x(0,t) = 0$ and $u_{xx}(y,t) + u(y,t) \geq 0$ for any $y \in (-L, L)$ and $t \in (0,T)$, we have

$$u(x,t) = u(0,t)\cos x + \int_0^x \Big(u_{xx}(y,t) + u(y,t)\Big)\sin(x-y)\,dy \geq u(0,t)\cos x$$

and

$$
\begin{aligned}
u(x,t) = u(0,t)\cos x + \sin x \int_0^x \Big(u_{xx}(y,t) + u(y,t)\Big)\cos y\, dy \\
- \cos x \int_0^x \Big(u_{xx}(y,t) + u(y,t)\Big)\sin y\, dy \\
\leq u(0,t)\cos x + \sin x \int_0^{\frac{\pi}{2}} \Big(u_{xx}(y,t) + u(y,t)\Big)\cos y\, dy \\
= u(0,t)\cos x + u\left(\frac{\pi}{2}, t\right)\sin x.
\end{aligned}
$$

Next, it holds

$$
\int_0^x \Big(u_{xx}(y,t) + u(y,t)\Big)\sin y\, dy = u_x(x,t)\sin x - u(x,t)\cos x + u(0,t).
$$

This implies that

$$
u_x(x,t) = -u(0,t)\sin x + \int_0^x \Big(u_{xx}(y,t) + u(y,t)\Big)\cos(x-y)\, dy \geq -u(0,t)\sin x
$$

and

$$
\begin{aligned}
u_x(x,t) = -u(0,t)\sin x + \cos x \int_0^x \Big(u_{xx}(y,t) + u(y,t)\Big)\cos y\, dy \\
+ \sin x \int_0^x \Big(u_{xx}(y,t) + u(y,t)\Big)\sin y\, dy \\
\leq \cos x \int_0^{\frac{\pi}{2}} \Big(u_{xx}(y,t) + u(y,t)\Big)\cos y\, dy \\
+ u_x(x,t)\sin^2 x - u(x,t)\sin x\cos x \\
= u\left(\frac{\pi}{2}, t\right)\cos x + u_x(x,t)\sin^2 x - u(x,t)\sin x\cos x.
\end{aligned}
$$

Hence, we have

$$
u_x(x,t) \leq -\frac{u(x,t)\sin x}{\cos x} + \frac{u\left(\frac{\pi}{2}, t\right)}{\cos x}. \qquad \square
$$

Since it holds that $u_t\left(\frac{\pi}{2}, t\right) = u\left(\frac{\pi}{2}, t\right)^p \left(u_{xx}\left(\frac{\pi}{2}, t\right) + u\left(\frac{\pi}{2}, t\right)\right) \geq 0$ due to (I4), in the case of $p \geq 2$, it can be verified

$$
\begin{aligned}
(T-t)u\Big(\frac{\pi}{2},t\Big) &\le \int_t^T u\Big(\frac{\pi}{2},s\Big)\,ds \\
&= -\frac{1}{p-1}\int_t^T \int_0^{\frac{\pi}{2}} \Big(u(y,s)^{-(p-1)}\Big)_s \cos y\,dy\,ds \\
&= \frac{1}{p-1}\int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^{p-1}}\,dy.
\end{aligned}
\tag{2.4}
$$

In addition, by (2.1), $u(y,t) \ge u(0,t)\cos y$ holds under assumptions (I1) and (I4). Therefore, u satisfies

$$
W_p(t) \le \frac{1}{2(p-1)} B\Big(\frac{1}{2}, \frac{3-p}{2}\Big) < \infty \quad \text{for } 2 < p < 3, \tag{2.5}
$$

where $W_p(t)$ is defined by (1.11) and $B(\alpha,\beta) := 2\int_0^{\frac{\pi}{2}} \big(\sin y\big)^{2\alpha-1}\big(\cos y\big)^{2\beta-1}\,dy$ is the beta function. The estimate of (2.5) for $2 < p < 3$ was given in [7]. Furthermore, we prove the following theorem in the case of $p = 3$ in this paper.

Theorem 2 *Let $p = 3$ and u be a solution of (1.1)–(1.2) with (1.3). Assume (I1), (I2), (I3), and (I4). Then u satisfies*

$$
\limsup_{t \nearrow T} \frac{W_3(t)}{\log\Big[(T-t)^{\frac{1}{3}} u(0,t)\Big]} \le \frac{3}{2}, \tag{2.6}
$$

where $W_3(t)$ is given by (1.11) with $p = 3$.

Proof Let t be fixed in $(0,T)$ and

$$
x(t) := \arccos \frac{1}{(T-t)u(0,t)^3}. \tag{2.7}
$$

We note that by the type II singularity of u, (1.3),

$$
x(t) \to \frac{\pi}{2} \quad \text{as } t \nearrow T. \tag{2.8}
$$

It is verified by (2.4) that

$$
\begin{aligned}
(T-t)u\Big(\frac{\pi}{2},t\Big) &\le \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^2}\,dy \\
&= \frac{1}{2}\bigg(\int_0^{x(t)} \frac{\cos y}{u(y,t)^2}\,dy + \int_{x(t)}^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^2}\,dy\bigg).
\end{aligned}
\tag{2.9}
$$

By (2.1), the first term is estimated as follows:

$$\begin{aligned}\int_0^{x(t)} \frac{\cos y}{u(y,t)^2}\,dy &\le \frac{1}{u(0,t)^2}\int_0^{x(t)} \frac{1}{\cos y}\,dy\\ &= \frac{1}{u(0,t)^2}\log\left|\frac{1+\sin x(t)}{\cos x(t)}\right|\\ &< \frac{1}{u(0,t)^2}\log\left[2(T-t)u(0,t)^3\right].\end{aligned} \tag{2.10}$$

Furthermore, by (I2), (2.2), and (2.3), it holds that, for $0<y<\frac{\pi}{2}$, $u_x(y,t)<0$,

$$\sin y \le -\frac{u_x(y,t)\cos y}{u(y,t)} + \frac{u\left(\frac{\pi}{2},t\right)}{u(y,t)} \le -\frac{u_x(y,t)}{u(0,t)} + \frac{u\left(\frac{\pi}{2},t\right)}{u(y,t)}, \tag{2.11}$$

$$\begin{aligned}u(x(t),t) &\le u(0,t)\cos x(t) + u\left(\frac{\pi}{2},t\right)\sin x(t)\\ &< \frac{1}{(T-t)u(0,t)^2} + u\left(\frac{\pi}{2},t\right)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\int_{x(t)}^{\frac{\pi}{2}} \frac{\sin y}{u(y,t)^2}\,dy &= \frac{\cos x(t)}{u(x(t),t)^2} - 2\int_{x(t)}^{\frac{\pi}{2}} \frac{u_x(y,t)\cos y}{u(y,t)^3}\,dy\\ &\le \frac{\cos x(t)}{u(x(t),t)^2} - \frac{2}{u(0,t)}\int_{x(t)}^{\frac{\pi}{2}} \frac{u_x(y,t)}{u(y,t)^2}\,dy\\ &\le \frac{1}{u(0,t)u(x(t),t)} + \frac{2}{u(0,t)}\left(\frac{1}{u\left(\frac{\pi}{2},t\right)} - \frac{1}{u(x(t),t)}\right)\\ &< \frac{2}{u(0,t)u\left(\frac{\pi}{2},t\right)}.\end{aligned} \tag{2.13}$$

Hence, the second term is estimated as follows:

$$
\begin{aligned}
&\int_{x(t)}^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^2}\,dy \\
&\quad \le \frac{1}{u(0,t)} \int_{x(t)}^{\frac{\pi}{2}} \frac{1}{u(y,t)}\,dy \\
&\quad < \frac{1}{u(0,t)\sin x(t)} \int_{x(t)}^{\frac{\pi}{2}} \frac{\sin y}{u(y,t)}\,dy \\
&\quad \le -\frac{1}{u(0,t)^2 \sin x(t)} \int_{x(t)}^{\frac{\pi}{2}} \frac{u_x(y,t)}{u(y,t)}\,dy + \frac{u\left(\frac{\pi}{2},t\right)}{u(0,t)\sin x(t)} \int_{x(t)}^{\frac{\pi}{2}} \frac{1}{u(y,t)^2}\,dy \\
&\quad < \frac{1}{u(0,t)^2 \sin x(t)} \log \left| \frac{u(x(t),t)}{u\left(\frac{\pi}{2},t\right)} \right| + \frac{u\left(\frac{\pi}{2},t\right)}{u(0,t)\sin^2 x(t)} \int_{x(t)}^{\frac{\pi}{2}} \frac{\sin y}{u(y,t)^2}\,dy \\
&\quad < \frac{1}{u(0,t)^2 \sin x(t)} \log \left| \frac{1}{(T-t)u\left(\frac{\pi}{2},t\right)u(0,t)^2} + 1 \right| + \frac{2}{u(0,t)^2 \sin^2 x(t)} \\
&\quad < \frac{1}{u(0,t)^2 \sin x(t)} \cdot \frac{1}{(T-t)u\left(\frac{\pi}{2},t\right)u(0,t)^2} + \frac{2}{u(0,t)^2 \sin^2 x(t)}.
\end{aligned}
\tag{2.14}
$$

Here, we use (2.1) for the first inequality, (2.11) for the third one, and (2.12) and (2.13) for the fifth one. Therefore, by (2.9), (2.10) and (2.14), $W_3(t)$ satisfies

$$
W_3(t) < \frac{1}{2\sin x(t)} \cdot \frac{1}{W_3(t)} + \frac{1}{\sin^2 x(t)} + \frac{1}{2} \log \left[2(T-t)u(0,t)^3 \right].
$$

Since

$$
\begin{aligned}
W_3(t)^2 &< \left[\frac{1}{\sin^2 x(t)} + \frac{1}{2} \log \left[2(T-t)u(0,t)^3 \right] \right] \cdot W_3(t) + \frac{1}{2\sin x(t)} \\
&< \frac{1}{2} \left[\frac{1}{\sin^2 x(t)} + \frac{1}{2} \log \left[2(T-t)u(0,t)^3 \right] \right]^2 + \frac{1}{2} W_3(t)^2 + \frac{1}{2\sin x(t)},
\end{aligned}
$$

it holds that

$$
W_3(t) < \sqrt{ \left[\frac{1}{\sin^2 x(t)} + \frac{1}{2} \log \left[2(T-t)u(0,t)^3 \right] \right]^2 + \frac{1}{\sin x(t)} }.
$$

Noting $\sin x(t) \to 1$ as $t \nearrow T$ by (2.8), this implies that

$$
\limsup_{t \nearrow T} \frac{W_3(t)}{\log \left[(T-t)^{\frac{1}{3}} u(0,t) \right]} \le \frac{3}{2}
$$

which completes this proof. □

In addition, we can give some properties of u for $2 < p < 3$ and $p = 3$.

Corollary 2 *Let $2 < p \leq 3$, u be a solution of (1.1)–(1.2) with (1.3) and assume ($I1$), ($I2$), ($I3$), and ($I4$). Then u satisfies the following.*

(i) $\displaystyle \lim_{t \nearrow T} \frac{u\left(\frac{\pi}{2}, t\right)}{u(0,t)} = 0.$

(ii) *If $\frac{\pi}{2} < x < L$, then there exists $C_x > 0$ such that* $\displaystyle \sup_{t \in (0,T)} u(x,t) \leq C_x < \infty.$

Proof We only prove the case of $L = m\pi$, where $m \geq 2$ is an integer, for the simplicity of the description. The general case can be proved similarly.

It is obtained by (2.5) and Theorem 2 that

$$\frac{u\left(\frac{\pi}{2}, t\right)}{u(0,t)} \leq \frac{1}{2(p-1)} B\left(\frac{1}{2}, \frac{3-p}{2}\right) \cdot \frac{1}{(T-t)u(0,t)^p} \quad \text{if } 2 < p < 3 \tag{2.15}$$

and

$$\limsup_{t \nearrow T} \frac{u\left(\frac{\pi}{2}, t\right)}{u(0,t)} \leq \frac{3}{2} \cdot \limsup_{t \nearrow T} \frac{\log\left[(T-t)^{\frac{1}{3}} u(0,t)\right]}{(T-t)u(0,t)^3} \quad \text{if } p = 3 \tag{2.16}$$

which implies (i) holds because of (1.3). Next, if $\pi/2 < x < \pi$, then

$$\begin{aligned} \int_0^x \frac{\cos y}{u(y,t)^{p-1}}\, dy &= \int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^{p-1}}\, dy + \int_{\frac{\pi}{2}}^x \frac{\cos y}{u(y,t)^{p-1}}\, dy \\ &< \int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^{p-1}}\, dy - \frac{1}{u\left(\frac{\pi}{2}, t\right)^{p-1}}(1 - \sin x). \end{aligned}$$

If $2 < p < 3$, then it is obtained by (2.1) and (2.15) that

$$\begin{aligned} 0 &\leq \lim_{t \nearrow T} u\left(\frac{\pi}{2}, t\right)^{p-1} \int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^{p-1}}\, dy \\ &\leq \lim_{t \nearrow T} \frac{u\left(\frac{\pi}{2}, t\right)^{p-1}}{u(0,t)^{p-1}} \cdot \frac{1}{2} B\left(\frac{1}{2}, \frac{3-p}{2}\right) = 0. \end{aligned}$$

If $p = 3$, then it is verified by (2.10) and (2.14) that

$$
\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^2}\,dy \\
&< \frac{1}{u(0,t)^2}\log\left[2(T-t)u(0,t)^3\right] \\
&\quad + \frac{1}{u(0,t)^2\sin x(t)}\cdot\frac{1}{(T-t)u\left(\frac{\pi}{2},t\right)u(0,t)^2} + \frac{2}{u(0,t)^2\sin^2 x(t)},
\end{aligned}
$$

where $x(t)$ is defined by (2.7). Hence, we have

$$
\begin{aligned}
0 &< u\left(\frac{\pi}{2},t\right)^2\int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^2}\,dy \\
&< \frac{u\left(\frac{\pi}{2},t\right)^2}{u(0,t)^2}\log\left[2(T-t)u(0,t)^3\right] \\
&\quad + \frac{u\left(\frac{\pi}{2},t\right)}{u(0,t)\sin x(t)}\cdot\frac{1}{(T-t)u(0,t)^3} + \frac{2u\left(\frac{\pi}{2},t\right)^2}{u(0,t)^2\sin^2 x(t)}
\end{aligned}
$$

and it is obtained by (2.16) that

$$
\lim_{t\nearrow T} u\left(\frac{\pi}{2},t\right)^2\int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^2}\,dy = 0 \qquad \text{for } p = 3.
$$

They imply that if $2 < p \le 3$ then for $\frac{\pi}{2} < x < \pi$ there exists $t_*(x)$ such that $\displaystyle\int_0^x \frac{\cos y}{u(y,t_*(x))^{p-1}}\,dy < 0$. On the other hand, if $\frac{\pi}{2} < x < \pi$ then $u_x(x,t)\cos x > 0$ and

$$
\begin{aligned}
&\left(\int_0^x \frac{\cos y}{u(y,t)^{p-1}}\,dy\right)_t \\
&= -(p-1)\int_0^x \big(u_{xx}(y,t) + u(y,t)\big)\cos y\,dy \\
&= -(p-1)\big(u_x(x,t)\cos x + u(x,t)\sin x\big) < 0 \qquad \text{for any } t \in (0,T).
\end{aligned}
$$

Hence, there exists $C_*(x) > 0$ such that

$$
\begin{aligned}
-\frac{1-\sin x}{u(x,t)^{p-1}} &< \int_{\frac{\pi}{2}}^x \frac{\cos y}{u(y,t)^{p-1}}\,dy \\
&< \sup_{t_*(x)\le t<T}\int_0^x \frac{\cos y}{u(y,t)^{p-1}}\,dy < -C_*(x) \qquad \text{if } t_*(x) \le t < T.
\end{aligned}
$$

Therefore, if $2 < p \le 3$ and $\dfrac{\pi}{2} < x < \pi$, then

$$u(x,t) \le \max\left\{ \left(\frac{1-\sin x}{C_*(x)}\right)^{\frac{1}{p-1}}, \sup_{0<t<t_*(x)} u(x,t) \right\} < \infty.$$

Furthermore, if $\pi \le x \le L$, then $u(x,t) \le \sup_{0<t<t_*(y)} u(y,t) < \infty$ with $\frac{\pi}{2} < y < \pi$ by the monotonicity of u with respect to $x > 0$ due to (I2), which completes the proof of (ii). □

3 The Proof of Theorem 1

Let u be a solution of (1.1)–(1.2) with (1.3) and consider rescaled function

$$U(x,\sigma) := e^{-\frac{\sigma}{p}} u(x, T - e^{-\sigma}) = (T-t)^{\frac{1}{p}} u(x,t), \tag{3.1}$$

where $\sigma := \log \frac{1}{T-t}$. This rescaling, which is called Type I rescaling, is widely used in the literature, for instance, [4, 5, 7]. Then U is a solution of

$$U_\sigma = U^p(U_{xx} + U) - \frac{1}{p}U \tag{3.2}$$

and satisfies

$$\limsup_{\sigma \nearrow \infty} \max_{x\in[-L,L]} U(x,\sigma) = \infty$$

due to (1.3). In particular, it is shown in [7] under assumptions (I1), (I2), and (I3) that there exists $\tau_* > 0$ such that

$$U_\sigma(0,\sigma) > 0 \quad \text{for } \sigma \ge \tau_* \tag{3.3}$$

and

$$U(0,\sigma) = (T-t)^{\frac{1}{p}} u(0,t) \nearrow \infty \quad \text{as } \sigma \nearrow \infty \text{ or } t \nearrow T.$$

(See Proposition 2.2 in [7].) We also note that [5, 7] provided a special traveling wave solution of (3.2) to prove (1.5) and (1.6). In addition, [2] gave some details for the special traveling wave solution. Precisely, [2] proved that if $\kappa > p^{-\frac{1}{p}}$ then there exist $\varepsilon_\kappa > 0$ and $R = R(\cdot\,;\kappa)$ such that

$$\begin{cases} R''(x;\kappa) + R(x;\kappa) = \dfrac{1}{pR(x;\kappa)^{p-1}} + \dfrac{\varepsilon_\kappa R'(x;\kappa)}{R(x;\kappa)^p} & \text{for } x \in \mathbb{R}, \\ R(0;\kappa) = \kappa \\ R'(0;\kappa) = 0 \end{cases} \tag{3.4}$$

with the following conditions.

- $\varepsilon_\kappa = O\left(\frac{1}{\kappa^{\frac{p}{p-2}}}\right)$ as $\kappa \nearrow \infty$.
- $R'(x;\kappa) < 0$ if $x > 0$.
- $R(x;\kappa) \searrow 0$ as $x \nearrow \infty$.
- $R''(x;\kappa) + R(x;\kappa) > 0$ for any $x \in \mathbb{R}$.

Let $\tau > 0$ and $\kappa = U(0,\tau)$. Then $\mathcal{U}(x,\sigma) := R\big(x + \varepsilon_\kappa(\sigma - \tau);\kappa\big)$ is a solution of (3.2) with $\mathcal{U}(x,\tau) = R(x;\kappa)$.

Remark The traveling wave solution provided in [5] and [7] is the same as $\mathcal{V}(x,\sigma) := R\big(-x - \varepsilon_\kappa(\sigma - \tau);\kappa\big)$. For small $\varepsilon_\kappa > 0$, $\mathcal{U}$ and $\mathcal{V}$ are called "slowly traveling wave".

The following properties for the special traveling wave solution had very important roles in the proof of (1.6) and (1.7). (See Lemma 3.3 and 3.4 in [7].)

(R1) there exist positive constants $E_1(p)$ and $E_2(p)$ such that E_1 and E_2 depend only on p and

$$E_1(p) < \varepsilon_\kappa \kappa^{\frac{p}{p-2}} < E_2(p).$$

(R2) There exists $\tau_R > 0$ such that the solution R of (3.4) with $\kappa = U(0,\tau)$ satisfies

$$U(x,\tau) > R\big(x;U(0,\tau)\big) \qquad \text{for any } x > 0 \text{ and } \tau \geq \tau_R.$$

In addition, in [2], we derived additional information for R as follows. (See Theorem 1 and 2 in [2].)

(R3) If $2 < p < 3$, then $\kappa^{p-1}R\left(\frac{\pi}{2};\kappa\right) = \frac{1}{2p}B\left(\frac{1}{2}, \frac{3-p}{2}\right) + o(1)$ as $\kappa \nearrow \infty$, where $B(\cdot,\cdot)$ is the beta function.

(R4) If $p = 3$, then $\dfrac{\kappa^2 R\left(\frac{\pi}{2};\kappa\right)}{\log \kappa} = 1 + o(1)$ as $\kappa \nearrow \infty$.

(R5) $\dfrac{R'\left(\frac{\pi}{2};\kappa\right)}{\kappa} = -1 + o(1)$ as $\kappa \nearrow \infty$ for $2 < p \leq 3$.

In the following lemma, we list the properties of U and R, which are needed to prove our main result.

Lemma 1 *Assume* $(I1)$, $(I2)$, $(I3)$, *and* $(I4)$. *Let* τ *and* U *be defined by* (3.1). *Then there exist* τ_0, $C_1(p)$, $C_2(p)$ *such that for* $\tau \geq \tau_0$ *and the solution* R *of* (3.4) *with* $\kappa = U(0,\tau)$ *the following hold.*

(i) $U_\tau(0,\tau) > 0$ *and* $U(0,\sigma) > U(0,\tau)$ *if* $\sigma > \tau \geq \tau_0$.

(ii) $\dfrac{C_1(p)}{U(0,\tau)^{\frac{p}{p-2}}} < \varepsilon_{U(0,\tau)} < \dfrac{C_2(p)}{U(0,\tau)^{\frac{p}{p-2}}}$. *Here,* $\varepsilon_{U(0,\tau)}$ *is* ε_κ *which satisfies* (3.4) *for* $\kappa = U(0,\tau)$.

(iii) *If* $2 < p < 3$, *then*

$$\frac{C_1(p)}{U(0,\tau)^{p-1}} < R\Big(\frac{\pi}{2};U(0,\tau)\Big) < U\Big(\frac{\pi}{2},\tau\Big) < \frac{C_2(p)}{U(0,\tau)^{p-1}}.$$

(iv) *If* $p = 3$, *then*

$$\frac{C_1(p)\log U(0,\tau)}{U(0,\tau)^2} < R\Big(\frac{\pi}{2};U(0,\tau)\Big) < U\Big(\frac{\pi}{2},\tau\Big) < \frac{C_2(p)\log U(0,\tau)}{U(0,\tau)^2}.$$

(v) $U(x,\sigma) > R\big(x + \varepsilon_{U(0,\tau)}(\sigma-\tau);U(0,\tau)\big)$ *if* $x > 0$ *and* $\sigma > \tau \geq \tau_0$.

(vi) $-U(0,\tau) < R'\big(x;U(0,\tau)\big) < 0$ *for any* $x > 0$.

Proof (3.3) and (R1) can directly lead to (i) and (ii), respectively. (iii) can be obtained by (R2), (R3), and the upper bound of W_p, (2.5). In addition, (iv) can also be proved by (R2), (R4), and the upper bound of W_3, (2.6).

(v) can be shown by (R2) and the maximum principle because if $\sigma > \tau \geq \tau_0$ then $U(x,\sigma)$ and $\mathcal{U}(x,\sigma) = R\big(x + \varepsilon_{U(0,\tau)}(\sigma-\tau);U(0,\tau)\big)$ are solution of

$$V_\sigma = V^p(V_{xx} + V) - \frac{1}{p}V \quad \text{in } (0,\infty)\times(\tau,\infty)$$

with $U(x,\tau) > R\big(x;U(0,\tau)\big) = \mathcal{U}(x,\tau)$, $U(0,\sigma) > U(0,\tau) = R\big(0;U(0,\tau)\big) \geq R\big(\varepsilon_{U(0,\tau)}(\sigma-\tau);U(0,\tau)\big) = \mathcal{U}(0,\sigma)$, $R\big(x;U(0,\tau)\big) \searrow 0$ as $x \to \infty$, and $U(x,\sigma) \geq e^{-\frac{\sigma}{p}}\eta_0 > 0$, where η_0 is given in (I3). Furthermore, (vi) is obtained by $R'\big(x;U(0,\tau)\big) < 0$ and $R''\big(x;U(0,\tau)\big) + R\big(x;U(0,\tau)\big) > 0$ for any $x > 0$. Indeed, since $R'\big(x;U(0,\tau)\big)R''\big(x;U(0,\tau)\big) + R'\big(x;U(0,\tau)\big)R\big(x;U(0,\tau)\big) < 0$ for any $x > 0$, we have $\big(R'\big(x;U(0,\tau)\big)\big)^2 < \big(R'\big(x;U(0,\tau)\big)\big)^2 + R\big(x;U(0,\tau)\big)^2 < U(0,\tau)^2$ and thus $0 > R'\big(x;U(0,\tau)\big) > -U(0,\tau)$ which complete this proof. □

In addition, we prepare the following lemma. For the solution U of (3.2) and $\lambda > 0$, we define

$$\psi_\lambda(\sigma) := \int_\sigma^\infty \frac{1}{U(0,\tilde{\sigma})^{\frac{2}{p-2}}} e^{-\lambda(\tilde{\sigma}-\sigma)}\, d\tilde{\sigma}. \tag{3.5}$$

Lemma 2 *Assume* (*I*1), (*I*2), (*I*3), *and* (*I*4). *Let* τ_0 *be given in Lemma* 1. *For* $\lambda > 0$ *it holds that*

$$0 < \psi_\lambda(\sigma) < \frac{1}{\lambda U(0,\sigma)^{\frac{2}{p-2}}} < \infty \quad \text{if } \sigma \geq \tau_0. \tag{3.6}$$

Furthermore, ψ_λ *satisfies*

$$\psi_\lambda'(\sigma) < 0 \quad \text{if } \sigma \geq \tau_0 \tag{3.7}$$

Proof Since $U_\tau(0, \tilde{\sigma}) > 0$ if $\tilde{\sigma} \geq \tau_0$, we have

$$\begin{aligned} 0 &< \int_\sigma^{\sigma'} \frac{1}{U(0,\tilde{\sigma})^{\frac{2}{p-2}}} e^{-\lambda(\tilde{\sigma}-\sigma)}\, d\tilde{\sigma} \\ &= -\frac{1}{\lambda U(0,\sigma')^{\frac{2}{p-2}}} e^{-\lambda(\sigma'-\sigma)} \\ &\quad + \frac{1}{\lambda U(0,\sigma)^{\frac{2}{p-2}}} - \frac{2}{(p-2)\lambda} \int_\sigma^{\sigma'} \frac{U_\tau(0,\tilde{\sigma})}{U(0,\tilde{\sigma})^{\frac{p}{p-2}}} e^{-\lambda(\tilde{\sigma}-\sigma)}\, d\tilde{\sigma}. \end{aligned}$$

Hence, it is obtained by letting $\sigma' \nearrow \infty$ that $0 < \psi_\lambda(\sigma) < (\lambda U(0,\sigma)^{\frac{2}{p-2}})^{-1} < \infty$, that is, (3.6) holds. Here, use has been made of

$$\int_\sigma^\infty \frac{U_\tau(0,\tilde{\sigma})}{U(0,\tilde{\sigma})^{\frac{p}{p-2}}} e^{-\lambda(\tilde{\sigma}-\sigma)}\, d\tilde{\sigma} > 0.$$

Furthermore, by (3.6), we have

$$\psi_\lambda'(\sigma) = -\frac{1}{U(0,\sigma)^{\frac{2}{p-2}}} + \lambda\psi_\lambda(\sigma) < 0$$

which implies (3.7) holds. ◻

Next, we give lower estimates for ψ_λ as follows.

Lemma 3 *Assume* (*I*1), (*I*2), (*I*3), *and* (*I*4). *Let* τ_0 *be given in Lemma* 1 *and* $\mu \in (\tau_0, \infty)$ *be fixed arbitrarily. Then* ψ_λ *defined in* (3.5) *satisfies the following*.

– If $2 < p < 3$ and $\lambda = U(0,\mu)^{-\frac{p(3-p)}{p-2}}$, then there exists a positive constant $C_* = C_*(p)$ such that C_* is independent of μ and

$$\psi_\lambda(\tau) \geq \frac{C_*}{U(0,\tau)^{p-1}} \quad \text{if } \mu > \tau \geq \tau_0. \tag{3.8}$$

– If $p = 3$ and $\lambda = \left(\log U(0,\mu)\right)^{-1}$, then there exists a positive constant $C_* = C_*(p)$ such that C_* is independent of μ and

$$\psi_\lambda(\tau) \geq \frac{C_* \log U(0,\tau)}{U(0,\tau)^2} \quad \text{if } \mu > \tau \geq \tau_0. \tag{3.9}$$

Proof The following notations are used in the proofs below:

$$M_1(p, U(0,\sigma)) := \begin{cases} C_2(p) & (2 < p < 3), \\ C_2(p) \log U(0,\sigma) & (p = 3) \end{cases}$$

and

$$M_2(p, U(0,\sigma)) := \begin{cases} C_3(p) & (2 < p < 3), \\ C_3(p)(\log U(0,\sigma))^{-\frac{3}{2}} & (p = 3), \end{cases}$$

where $C_2(p)$ is defined in Lemma 1 and

$$C_3(p) := \frac{(p-1)(p-2)}{C_2(p)^{\frac{p}{(p-1)(p-2)}}(p^2 - 2p + 2)}.$$

In contrast to the case of $p = 3$, M_1 and M_2 depend on p only in the case of $2 < p < 3$
.

Let $\tau \in [\tau_0, \mu)$ be fixed and R be a solution of (3.2) with $\kappa = U(0, \tau)$. By Lemma 1 (i), (iii), (v) and (vi), if $\sigma > \tau \geq \tau_0$, then U and R satisfy

$$R\left(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma - \tau); U(0,\tau)\right) < U\left(\frac{\pi}{2}, \sigma\right) < \frac{M_1(p, U(0,\sigma))}{U(0,\sigma)^{p-1}} \tag{3.10}$$

and

$$-U(0,\sigma) < -U(0,\tau) < R'\left(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma - \tau); U(0,\tau)\right) < 0. \tag{3.11}$$

This implies that if $\sigma > \tau \geq \tau_0$ then

$$\begin{aligned} \psi_\lambda'(\sigma) - \lambda\psi_\lambda(\sigma) &= -\frac{1}{U(0,\sigma)^{\frac{2}{p-2}}} \\ &< \frac{1}{U(0,\sigma)^{\frac{p}{p-2}}} R'\left(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma - \tau); U(0,\tau)\right) \\ &< \frac{1}{M_1(p, U(0,\sigma))^{\frac{p}{(p-1)(p-2)}}} R\left(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma - \tau); U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}} \\ &\qquad \times R'\left(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma - \tau); U(0,\tau)\right) \\ &= \frac{M_2(p, U(0,\sigma))}{\varepsilon_{U(0,\tau)}} \left(R\left(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma - \tau); U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}+1} \right)_\sigma. \end{aligned}$$

Hence, if $\sigma > \tau \geq \tau_0$, then

$$
\begin{aligned}
&\left(e^{-\lambda(\sigma-\tau)}\psi_\lambda(\sigma)\right)_\sigma \\
&\quad < \frac{M_2(p,U(0,\sigma))}{\varepsilon_{U(0,\tau)}} e^{-\lambda(\sigma-\tau)} \left(R\left(\frac{\pi}{2}+\varepsilon_{U(0,\tau)}(\sigma-\tau);U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}+1} \right)_\sigma
\end{aligned}
$$

and it is obtained by integrating from τ to ∞ with respect to σ that

$$
\begin{aligned}
&-\psi_\lambda(\tau) \\
&= \int_\tau^\infty \left(e^{-\lambda(\sigma-\tau)}\psi_\lambda(\sigma)\right)_\sigma d\sigma \\
&< \frac{M_2(p,U(0,\sigma))}{\varepsilon_{U(0,\tau)}} \int_\tau^\infty e^{-\lambda(\sigma-\tau)} \left(R\left(\frac{\pi}{2}+\varepsilon_{U(0,\tau)}(\sigma-\tau);U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}+1} \right)_\sigma d\sigma.
\end{aligned}
\tag{3.12}
$$

When $2 < p < 3$, we have

$$
\begin{aligned}
-\psi_\lambda(\tau) &< -\frac{C_3(p)}{\varepsilon_{U(0,\tau)}} R\left(\frac{\pi}{2};U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}+1} \\
&\quad + \frac{C_3(p)\lambda}{\varepsilon_{U(0,\tau)}} \int_\tau^\infty e^{-\lambda(\sigma-\tau)} R\left(\frac{\pi}{2}+\varepsilon_{U(0,\tau)}(\sigma-\tau);U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}+1} d\sigma.
\end{aligned}
$$

Since it is verified by (3.10) that

$$
\begin{aligned}
&\int_\tau^\infty e^{-\lambda(\sigma-\tau)} R\left(\frac{\pi}{2}+\varepsilon_{U(0,\tau)}(\sigma-\tau);U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}+1} d\sigma \\
&\quad < C_2(p)^{\frac{p}{(p-1)(p-2)}+1} \int_\tau^\infty \frac{1}{U(0,\sigma)^{\frac{2}{p-2}+p}} e^{-\lambda(\sigma-\tau)} d\sigma \\
&\quad < \frac{C_2(p)^{\frac{p}{(p-1)(p-2)}+1}}{U(0,\tau)^p} \psi_\lambda(\tau),
\end{aligned}
$$

if $\lambda = U(0,\mu)^{-\frac{p(3-p)}{p-2}}$, then $\lambda U(0,\tau)^{\frac{p(3-p)}{p-2}} < 1$ for $\mu > \tau \geq \tau_0$ and thus it is obtained by Lemma 1 (ii) that

$$
\begin{aligned}
&\frac{C_3(p)}{\varepsilon_{U(0,\tau)}} R\left(\frac{\pi}{2};U(0,\tau)\right)^{\frac{p}{(p-1)(p-2)}+1} \\
&\quad < \left(1+\frac{C_3(p)C_2(p)^{\frac{p}{(p-1)(p-2)}+1}\lambda}{\varepsilon_{U(0,\tau)}U(0,\tau)^p}\right)\psi_\lambda(\tau) \\
&\quad < \left(1+\frac{C_3(p)C_2(p)^{\frac{p}{(p-1)(p-2)}+1}\lambda U(0,\tau)^{\frac{p(3-p)}{p-2}}}{C_1(p)}\right)\psi_\lambda(\tau) \\
&\quad < \left(1+\frac{C_3(p)C_2(p)^{\frac{p}{(p-1)(p-2)}+1}}{C_1(p)}\right)\psi_\lambda(\tau) \quad \text{if } \mu > \tau \geq \tau_0.
\end{aligned}
$$

Furthermore, Lemma 1 (ii) and (iii) imply that if $2 < p < 3$ then

$$\frac{1}{\varepsilon_{U(0,\tau)}} R\Big(\frac{\pi}{2}; U(0,\tau)\Big)^{\frac{p}{(p-1)(p-2)}+1} > \frac{C_1(p)^{\frac{p}{(p-1)(p-2)}+1}}{C_2(p)} \cdot \frac{1}{U(0,\tau)^{p-1}}.$$

Therefore, it is obtained that there exists $C_* = C_*(p)$ such that C_* is independent of μ and

$$\psi_\lambda(\tau) \geq \frac{C_*}{U(0,\tau)^{p-1}} \quad \text{if } 2 < p < 3, \mu > \tau \geq \tau_0 \text{ and } \lambda = \frac{1}{U(0,\mu)^{\frac{p(3-p)}{p-2}}}.$$

Next, when $p = 3$, by (3.12), we have

$$\begin{aligned}
&-\psi_\lambda(\tau) \\
&= \int_\tau^\infty \Big(e^{-\lambda(\sigma-\tau)}\psi_\lambda(\sigma)\Big)_\sigma d\sigma \\
&\quad < \int_\tau^\infty \frac{M_2(3, U(0,\sigma))e^{-\lambda(\sigma-\tau)}}{\varepsilon_{U(0,\tau)}} \Big(R\Big(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma-\tau); U(0,\tau)\Big)^{\frac{5}{2}}\Big)_\sigma d\sigma \\
&\quad < -\frac{C_3(3)}{\varepsilon_{U(0,\tau)}\big(\log U(0,\sigma)\big)^{\frac{3}{2}}} R\Big(\frac{\pi}{2}; U(0,\tau)\Big)^{\frac{5}{2}} \\
&\qquad + \frac{C_3(3)\lambda}{\varepsilon_{U(0,\tau)}} \int_\tau^\infty \frac{e^{-\lambda(\sigma-\tau)}}{\big(\log U(0,\sigma)\big)^{\frac{3}{2}}} R\Big(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma-\tau); U(0,\tau)\Big)^{\frac{5}{2}} d\sigma \\
&\qquad + \frac{3C_3(3)}{2\varepsilon_{U(0,\tau)}} \int_\tau^\infty \frac{U_\tau(0,\sigma)e^{-\lambda(\sigma-\tau)}}{U(0,\sigma)\big(\log U(0,\sigma)\big)^{\frac{5}{2}}} R\Big(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma-\tau); U(0,\tau)\Big)^{\frac{5}{2}} d\sigma.
\end{aligned} \tag{3.13}$$

Now, we can assume that $U(0,\tau_0) \geq e^{\frac{1}{3}}$ and then it holds that

$$\frac{\log U(0,\sigma)}{U(0,\sigma)^3} < \frac{\log U(0,\tau)}{U(0,\tau)^3} \quad (\sigma > \tau \geq \tau_0)$$

because $f(s) = \frac{\log s}{s^3}$ is decreasing for $s > e^{\frac{1}{3}}$. Hence, it is verified by (3.10) that

$$\begin{aligned}
&\int_\tau^\infty \frac{1}{\big(\log U(0,\sigma)\big)^{\frac{3}{2}}} e^{-\lambda(\sigma-\tau)} R\Big(\frac{\pi}{2} + \varepsilon_{U(0,\tau)}(\sigma-\tau); U(0,\tau)\Big)^{\frac{5}{2}} d\sigma \\
&< C_2(3)^{\frac{5}{2}} \int_\tau^\infty \frac{\log U(0,\sigma)}{U(0,\sigma)^5} e^{-\lambda(\sigma-\tau)} d\sigma \\
&< \frac{C_2(3)^{\frac{5}{2}} \log U(0,\tau)}{U(0,\tau)^3} \psi_\lambda(\tau)
\end{aligned} \tag{3.14}$$

and

$$
\begin{aligned}
&\int_\tau^\infty \frac{U_\tau(0,\sigma)}{U(0,\sigma)\big(\log U(0,\sigma)\big)^{\frac{5}{2}}} e^{-\lambda(\sigma-\tau)} R\Big(\frac{\pi}{2}+\varepsilon_{U(0,\tau)}(\sigma-\tau); U(0,\tau)\Big)^{\frac{5}{2}}\,d\sigma \\
&< C_2(3)^{\frac{5}{2}} \int_\tau^\infty \frac{U_\tau(0,\sigma)}{U(0,\sigma)^6} e^{-\lambda(\sigma-\tau)}\,d\sigma \\
&< C_2(3)^{\frac{5}{2}} \left(\frac{1}{5U(0,\tau)^5} - \frac{\lambda}{5}\int_\tau^\infty \frac{1}{U(0,\sigma)^5} e^{-\lambda(\sigma-\tau)}\,d\sigma \right) \\
&< \frac{C_2(3)^{\frac{5}{2}}}{5U(0,\tau)^5}.
\end{aligned}
\tag{3.15}
$$

If $\lambda = \big(\log U(0,\mu)\big)^{-1}$, then $\lambda \log U(0,\tau) < 1$ for $\mu > \tau \geq \tau_0$ and thus, by (3.13), (3.14) and (3.15), we have

$$
\begin{aligned}
&\frac{C_3(3)}{\varepsilon_{U(0,\tau)}\big(\log U(0,\sigma)\big)^{\frac{3}{2}}} R\Big(\frac{\pi}{2}; U(0,\tau)\Big)^{\frac{5}{2}} - \frac{3C_3(3)C_2(3)^{\frac{5}{2}}}{10\varepsilon_{U(0,\tau)}U(0,\tau)^5} \\
&< \left(1 + \frac{C_3(3)C_2(3)^{\frac{5}{2}}\lambda \log U(0,\tau)}{\varepsilon_{U(0,\tau)}U(0,\tau)^3}\right)\psi_\lambda(\tau) \\
&< \left(1 + \frac{C_3(3)C_2(3)^{\frac{5}{2}}}{C_1(3)}\right)\psi_\lambda(\tau) \quad \text{if } \mu > \tau \geq \tau_0.
\end{aligned}
$$

Furthermore, Lemma 1 (ii) and (iv) imply that if $p = 3$ then

$$
\frac{1}{\varepsilon_{U(0,\tau)}\big(\log U(0,\tau)\big)^{\frac{3}{2}}} R\Big(\frac{\pi}{2}; U(0,\tau)\Big)^{\frac{5}{2}} > \frac{C_1(3)^{\frac{5}{2}} \log U(0,\tau)}{C_2(3)U(0,\tau)^2}
$$

and

$$
\frac{1}{\varepsilon_{U(0,\tau)}U(0,\tau)^5} < \frac{1}{C_1(3)U(0,\tau)^2}.
$$

Therefore, it is obtained that there exists $C_* = C_*(p)$ such that C_* is independent of μ and

$$
\psi_\lambda(\tau) \geq \frac{C_* \log U(0,\tau)}{U(0,\tau)^2} \quad \text{if } p = 3, \mu > \tau \geq \tau_0 \text{ and } \lambda = \frac{1}{\log U(0,\mu)}
$$

which completes this proof. □

We make use of ψ_λ defined by (3.5) with Lemmas 2 and 3 and prove the following theorem.

Theorem 3 *Assume* $(I1)$, $(I2)$, $(I3)$, *and* $(I4)$. *There exists a positive constant* $C = C(p)$ *such that the following hold.*

- If $2 < p < 3$ then $\limsup_{\mu \nearrow \infty} \frac{U(0,\mu)}{\mu^{\frac{p-2}{p(3-p)}}} \le C$.
- If $p = 3$ then $\limsup_{\mu \nearrow \infty} \frac{U(0,\mu)}{\exp\left(C\sqrt{\mu}\right)} \le 1$.

Proof First, we consider the case of $2 < p < 3$. By (3.7) and (3.8), if $2 < p < 3$, $\mu > \tau \ge \tau_0$ and $\lambda = U(0,\mu)^{-\frac{p(3-p)}{p-2}}$, then it holds that

$$
\begin{aligned}
0 > \psi_\lambda'(\tau) &= -\frac{1}{U(0,\tau)^{\frac{2}{p-2}}} + \lambda\psi_\lambda(\tau) \\
&> -\left(\frac{1}{U(0,\tau)^{p-1}}\right)^{\frac{2}{(p-1)(p-2)}} \\
&> -\frac{1}{C_*^{\frac{2}{(p-1)(p-2)}}}\psi_\lambda(\tau)^{\frac{2}{(p-1)(p-2)}}.
\end{aligned}
$$

Hence, we have

$$
\left(\psi_\lambda(\tau)^{-\frac{p(3-p)}{(p-1)(p-2)}}\right)_\tau < C_4(p) \quad \text{if } \tau \ge \tau_0
$$

and thus

$$
\psi_\lambda(\mu)^{-\frac{p(3-p)}{(p-1)(p-2)}} - \psi_\lambda(\tau_0)^{-\frac{p(3-p)}{(p-1)(p-2)}} < C_4(p)(\mu - \tau_0) \quad \text{if } \mu > \tau_0,
$$

where

$$
C_4(p) = \frac{p(3-p)}{(p-1)(p-2)C_*^{\frac{2}{(p-1)(p-2)}}}.
$$

In addition, (3.6) and (3.8) imply that if $\lambda = U(0,\mu)^{-\frac{p(3-p)}{p-2}}$ then

$$
\psi_\lambda(\mu) < \frac{1}{\lambda U(0,\mu)^{\frac{2}{p-2}}} = \frac{1}{U(0,\mu)^{p-1}} \quad \text{and} \quad \psi_\lambda(\tau_0) \ge \frac{C_*(p)}{U(0,\tau_0)^{p-1}}.
$$

Therefore, it is obtained that

$$
\begin{aligned}
&U(0,\mu)^{\frac{p(3-p)}{p-2}} \\
&< \psi_\lambda(\mu)^{-\frac{p(3-p)}{(p-1)(p-2)}} \\
&< C_4(p)(\mu-\tau_0) + C_*(p)^{-\frac{p(3-p)}{(p-1)(p-2)}} U(0,\tau_0)^{\frac{p(3-p)}{p-2}}.
\end{aligned}
\tag{3.16}
$$

Since (3.16) holds for any $\mu \in (\tau_0, \infty)$ and τ_0, C_4 and C_* are independent of μ, it can be shown that there exists $C = C(p)$ such that

$$
\limsup_{\mu \nearrow \infty} \frac{U(0,\mu)}{\mu^{\frac{p-2}{p(3-p)}}} \le C.
$$

Next, we consider the case of $p = 3$. (3.7) and (3.9) imply that if $\mu > \tau \ge \tau_0$ and $\lambda = \big(\log U(0,\mu)\big)^{-1}$ then

$$
\begin{aligned}
0 > \psi_\lambda'(\tau) &= -\frac{1}{U(0,\tau)^2} + \lambda \psi_\lambda(\tau) \\
&> -\frac{1}{\log U(0,\tau)} \cdot \frac{\log U(0,\tau)}{U(0,\tau)^2} \\
&> -\frac{\psi_\lambda(\tau)}{C_* \log U(0,\tau)}.
\end{aligned}
$$

Hence, there exists C_5 such that

$$
\begin{aligned}
\left[\left(\log \frac{1}{\psi_\lambda(\tau)}\right)^2\right]' &= -\frac{2\psi_\lambda'(\tau)}{\psi_\lambda(\tau)} \log \frac{1}{\psi_\lambda(\tau)} \\
&< \frac{2}{C_* \log U(0,\tau)} \left(\log \frac{U(0,\tau)^2}{C_* \log U(0,\tau)}\right) \\
&= \frac{2\big(2\log U(0,\tau) - \log(C_* \log U(0,\tau))\big)}{C_* \log U(0,\tau)} \\
&< C_5 \quad \text{if } \mu > \tau \ge \tau_0.
\end{aligned}
$$

We note that C_5 is independent of μ. Since it is verified by (3.6) and (3.9) that

$$
\psi_\lambda(\mu) < \frac{1}{\lambda U(0,\mu)^2} = \frac{\log U(0,\mu)}{U(0,\mu)^2}
$$

and

$$
\psi_\lambda(\tau_0) \ge \frac{C_* \log U(0,\tau_0)}{U(0,\tau_0)^2} \quad \text{if } \lambda = \frac{1}{\log U(0,\mu)},
$$

we have

$$
\begin{aligned}
\Big(\log U(0,\mu)\Big)^2 &< \left(\log \frac{U(0,\mu)^2}{\log U(0,\mu)}\right)^2 \\
&< \left(\log \frac{1}{\psi_\lambda(\mu)}\right)^2 \\
&< C_5(\mu-\tau_0) + \left(\log \frac{1}{\psi_\lambda(\tau_0)}\right)^2 \\
&\le C_5(\mu-\tau_0) + \left(\log \frac{U(0,\tau_0)^2}{C_* \log U(0,\tau_0)}\right)^2
\end{aligned}
$$

Here, we note that $\log U(0,\mu) < U(0,\mu)$ for any μ. Hence, it holds that

$$
\begin{aligned}
&\frac{U(0,\mu)}{\exp(\sqrt{C_5\mu})} \\
&< \exp\left(\sqrt{C_5(\mu-\tau_0) + \left(\log \frac{U(0,\tau_0)^2}{C_* \log U(0,\tau_0)}\right)^2} - \sqrt{C_5\mu}\right).
\end{aligned}
\tag{3.17}
$$

Since (3.17) holds for any $\mu \in (\tau_0, \infty)$ and τ_0, C_5 and C_* are independent of μ, it can be obtained that there exists $C = C(p)$ such that

$$
\limsup_{\mu \nearrow \infty} \frac{U(0,\mu)}{\exp\left(C\sqrt{\mu}\right)} \le 1
$$

which completes this proof. □

Theorem 3 directly leads to the results of Theorem 1, that is, there exists $t_* \in (0,T)$ and $C = C(p) > 0$ such that if $t_* < t < T$ then

$$
\max_{x\in[-L,L]} u(x,t) = u(0,t) \le C(p)\Big(\frac{1}{T-t}\Big)^{\frac{1}{p}}\Big(\log \frac{1}{T-t}\Big)^{\frac{p-2}{p(3-p)}} \quad \text{for } 2<p<3
$$

and

$$
\max_{x\in[-L,L]} u(x,t) = u(0,t) \le \Big(\frac{1}{T-t}\Big)^{\frac{1}{3}} \exp\left(C\sqrt{\log \frac{1}{T-t}}\right) \quad \text{for } p=3
$$

which completes the proof of Theorem 1.

Remark It has been shown in [4] (the case of $p=2$) or [7] (the case of $2 \le p < 3$) that

$$
\lim_{t\nearrow T} u\Big(\frac{\pi}{2}, t\Big) = \infty. \tag{3.18}
$$

We can mention that (3.18) also holds in the case of $p = 3$ because (2.4) and Theorem 1 (ii) imply u satisfies

$$\begin{aligned}
&\frac{1}{T-t}\int_t^T u\left(\frac{\pi}{2}, s\right) ds \\
&= \frac{1}{2(T-t)}\int_0^{\frac{\pi}{2}} \frac{\cos y}{u(y,t)^2}\, dy \\
&\geq \frac{1}{2(T-t)^{\frac{1}{3}}} \exp\left(-2C\sqrt{\log \frac{1}{T-t}}\right) \\
&= \frac{1}{2}\exp\left(\frac{1}{3}\log\frac{1}{T-t} - 2C\sqrt{\log\frac{1}{T-t}}\right) \qquad \text{for any } t \in (t_*, T)
\end{aligned}$$

and thus, if $u\left(\frac{\pi}{2}, t\right)$ would be bounded as $t \nearrow T$, then we have a contradiction. □

In addition, we can obtain Corollary 1.

Proof of Corollary 1 [4] has shown that if $p = 2$ then

$$\lim_{t \nearrow T}(T-t)^{\frac{1}{2}+\varepsilon} \max_{x\in[-L,L]} u(x,t) = 0 \qquad \text{for any } \varepsilon > 0.$$

Furthermore, Theorem 1 implies that the same features hold in the case of $2 < p \leq 3$ under assumptions (I1), (I2), (I3), and (I4) because of

$$\lim_{t \nearrow T}(T-t)^{\varepsilon}\left(\log\frac{1}{T-t}\right)^{\frac{p-2}{p(3-p)}} = 0 \qquad \text{in the case of } 2 < p < 3$$

and

$$\lim_{t \nearrow T}(T-t)^{\varepsilon} \exp\left(C\sqrt{\log\frac{1}{T-t}}\right) = 0 \qquad \text{in the case of } p = 3$$

for any $\varepsilon > 0$. □

4 Conclusion

In this paper, we provided the upper estimation of the blow-up rates for solutions of (1.1) with the periodic boundary condition (1.2) in the case of $2 < p < 3$ and $p = 3$ in Theorem 1. No results on the upper estimate of blow-up rate are previously known for $p \geq 3$. Our upper estimate for $p = 3$ is the first result for this issue.

Clarifying the relationship between the value of p and the blow-up rate of the solution is an interesting problem. As a known result, it was shown in [5] that solution with the rate of Type II appears at $p = 2$. Another known result was given in [7], where the blow-up rate changes between $p = 2$ and $2 < p < 3$. In addition to these, our results in this paper suggest the need for a discussion on the possible change in the rate between $2 < p < 3$ and $p = 3$. The reason why the upper estimates for $p = 3$ of Theorem 1 differ from $2 < p < 3$ is due to the drastic change in the behavior at $\frac{\pi}{2}$ of the slowly traveling waves R at $2 < p < 3$ and $p = 3$ as proved in [2]. On the other hand, it is still unclear whether the blow-up rate of the solution, in fact, changes for $2 < p < 3$ and $p = 3$, which is one of our future issues.

Although the precise form of the blow-up rate for $2 < p \leq 3$ and the reason that generates the difference in blow-up rate between $p = 2$ and $p > 2$ is unclear, combining our results in this paper with [7], we are closer to the conclusion that the blow-up rate for $2 < p < 3$ would have the form (1.10), which is different from $p = 2$.

In our proof, we use the slowly traveling wave R to evaluate the blow-up solutions. This method is valid under periodic boundary conditions. However, this is not directly applicable to the case of the Dirichlet boundary condition since the comparison between the solution and R fails due to the boundary condition. And, in the case of $p > 3$, the estimation of $R(\frac{\pi}{2}; \kappa)$ is more involved than the case of $p \leq 3$. Hence, our strategy for upper estimates of the blow-up rate does not work well so far and there are no results on upper estimates of the blow-up rates for $p > 3$. These are also our future works.

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