



Recent progress on integrally convex functions

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Abstract

Integrally convex functions constitute a fundamental function class in discrete convex analysis, including M-convex functions, L-convex functions, and many others. This paper aims at a rather comprehensive survey of recent results on integrally convex functions with some new technical results. Topics covered in this paper include characterizations of integral convex sets and functions, operations on integral convex sets and functions, optimality criteria for minimization with a proximity-scaling algorithm, integral biconjugacy, and the discrete Fenchel duality. While the theory of M-convex and L-convex functions has been built upon fundamental results on matroids and submodular functions, developing the theory of integrally convex functions requires more general and basic tools such as the Fourier–Motzkin elimination.

Keywords Discrete convex analysis · Integrally convex function · Box-total dual integrality · Fenchel duality · Integral subgradient · Minimization

Mathematics Subject Classification 52A41 · 90C27 · 90C25

1 Introduction

Discrete convex analysis [39, 40] has found applications and connections to a wide variety of disciplines. Early applications include those to combinatorial optimization, matrix theory, and economics, as described in the book [40] and a survey paper [43].

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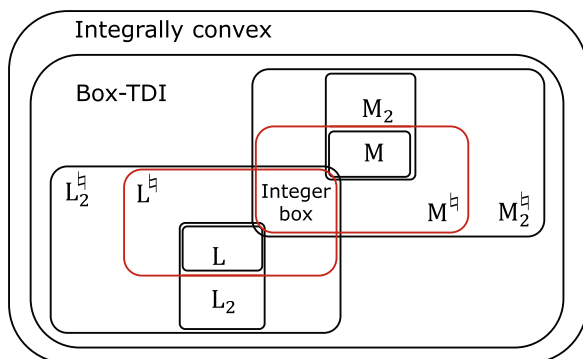
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More recently, there have been active interactions with operations research (inventory theory) [4, 5, 60], economics and game theory [44, 53, 59, 63], and algebra (Lorentzian polynomials, in particular) [3]. The reader is referred to [40] for basic concepts and terminology in discrete convex analysis, and to [17, 22, 31–33, 35, 45, 46, 58] for recent theoretical and algorithmic developments.

Integrally convex functions, due to Favati–Tardella [13], constitute a fundamental function class in discrete convex analysis. A function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *integrally convex* if its local convex extension $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is (globally) convex in the ordinary sense, where \tilde{f} is defined as the collection of convex extensions of f in each unit hypercube $\{x \in \mathbb{R}^n \mid a_i \leq x_i \leq a_i + 1 \ (i = 1, \dots, n)\}$ with $a \in \mathbb{Z}^n$; see Sect. 3.2 for more precise statements. A subset of \mathbb{Z}^n is called *integrally convex* if its indicator function $\delta_S : \mathbb{Z}^n \rightarrow \{0, +\infty\}$ ($\delta_S(x) = 0$ for $x \in S$ and $= +\infty$ for $x \notin S$) is an integrally convex function. The concept of integral convexity is used in formulating discrete fixed point theorems [25, 26, 28, 68], and designing solution algorithms for discrete systems of nonlinear equations [30, 67]. In game theory the integral concavity of payoff functions guarantees the existence of a pure strategy equilibrium in finite symmetric games [27].

Integrally convex functions serve as a common framework for discrete convex functions. Indeed, separable convex, L-convex, L^{\natural} -convex, M-convex, M^{\natural} -convex, L_2^{\natural} -convex, and M_2^{\natural} -convex functions are known to be integrally convex [40]. Multimodular functions [23] are also integrally convex, as pointed out in [42]. Moreover, BS-convex and UJ-convex functions [18] are integrally convex. Discrete midpoint convex functions [38] and directed discrete midpoint convex functions [64] are also integrally convex. The relations among those discrete convexity concepts are investigated in [34, 47]. Figure 1 is an overview of the inclusion relations among the most fundamental classes of discrete convex sets. It is noted that the class of integrally convex sets contains the other classes of discrete convex sets.

In the last several years, significant progress has been made in the theory of integrally convex functions. A proximity theorem for integrally convex functions is established



$$(L_2^{\natural}\text{-convex} \cap M_2^{\natural}\text{-convex} = L^{\natural}\text{-convex} \cap M^{\natural}\text{-convex} = \text{Integer box})$$

Fig. 1 Classes of discrete convex sets

in [37] together with a proximity-scaling algorithm for minimization. Fundamental operations for integrally convex functions such as projection and convolution are investigated in [33, 45, 46]. It is revealed that integer-valued integrally convex functions enjoy integral biconjugacy [50], and a discrete Fenchel-type min-max formula is established for a pair of integer-valued integrally convex and separable convex functions [51]. The present paper aims at a rather comprehensive survey of those recent results on integrally convex functions with some new technical results. While the theory of M-convex and L-convex functions has been built upon fundamental results on matroids and submodular functions, developing the theory of integrally convex functions requires more general and basic tools such as the Fourier–Motzkin elimination.

This paper is organized as follows. In Sect. 2 we review integrally convex sets, with new observations on their polyhedral properties. In Sect. 3 the concept of integrally convex functions is reviewed with emphasis on their characterizations. Section 4 deals with properties related to minimization and minimizers, including a proximity-scaling algorithm. Section 5 is concerned with integral subgradients and biconjugacy, and Sect. 6 with the discrete Fenchel duality.

2 Integrally convex sets

2.1 Hole-free property

Let n be a positive integer and $N = \{1, 2, \dots, n\}$. For a subset I of N , we denote by $\mathbf{1}^I$ the characteristic vector of I ; the i th component of $\mathbf{1}^I$ is equal to 1 or 0 according to whether $i \in I$ or not. We use a short-hand notation $\mathbf{1}^i$ for $\mathbf{1}^{\{i\}}$, which is the i th unit vector. The vector with all components equal to 1 is denoted by $\mathbf{1}$, that is, $\mathbf{1} = (1, 1, \dots, 1) = \mathbf{1}^N$.

For two vectors $\alpha \in (\mathbb{R} \cup \{-\infty\})^n$ and $\beta \in (\mathbb{R} \cup \{+\infty\})^n$ with $\alpha \leq \beta$, we define notation $[\alpha, \beta]_{\mathbb{R}} = \{x \in \mathbb{R}^n \mid \alpha \leq x \leq \beta\}$, which represents the set of real vectors between α and β . An *integral box* will mean a set B of real vectors represented as $B = [\alpha, \beta]_{\mathbb{R}}$ for integer vectors $\alpha \in (\mathbb{Z} \cup \{-\infty\})^n$ and $\beta \in (\mathbb{Z} \cup \{+\infty\})^n$ with $\alpha \leq \beta$. The set of integer vectors contained in an integral box will be called a *box of integers* or an *interval of integers*. We use notation $[\alpha, \beta]_{\mathbb{Z}} := [\alpha, \beta]_{\mathbb{R}} \cap \mathbb{Z}^n = \{x \in \mathbb{Z}^n \mid \alpha \leq x \leq \beta\}$ for $\alpha \in (\mathbb{Z} \cup \{-\infty\})^n$ and $\beta \in (\mathbb{Z} \cup \{+\infty\})^n$ with $\alpha \leq \beta$.

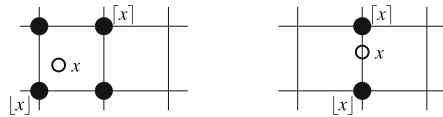
For a subset S of \mathbb{R}^n , we denote its *convex hull* by \bar{S} , which is, by definition, the smallest convex set containing S . As is well known, \bar{S} coincides with the set of all convex combinations of (finitely many) elements of S . We say that a set $S \subseteq \mathbb{Z}^n$ is *hole-free* if

$$S = \bar{S} \cap \mathbb{Z}^n. \quad (2.1)$$

Since the inclusion $S \subseteq \bar{S} \cap \mathbb{Z}^n$ is trivially true for any S , the content of this condition lies in

$$S \supseteq \bar{S} \cap \mathbb{Z}^n, \quad (2.2)$$

Fig. 2 Integral neighborhood $N(x)$ of x (●: point of $N(x)$)



stating that the integer points contained in the convex hull of S all belong to S itself. A finite set of integer points is hole-free if and only if it is the set of integer points in some integral polytope.

For a set $S \subseteq \mathbb{R}^n$ we define its *indicator function* $\delta_S : \mathbb{R}^n \rightarrow \{0, +\infty\}$ by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases} \tag{2.3}$$

Remark 2.1 In a standard textbook [24, Section A.1.3], the convex hull of a subset S of the n -dimensional Euclidean space \mathbb{R}^n is denoted by $\text{co } S$ and the *closed convex hull* by $\overline{\text{co } S}$, where the closed convex hull of S is defined to be the intersection of all closed convex set containing S . It is known that $\overline{\text{co } S}$ coincides with the (topological) closure of $\text{co } S$, which is expressed as $\overline{\text{co } S} = \text{cl}(\text{co } S)$ with the use of notation cl for closure operation. Using our notation \overline{S} , we have $\text{co } S = \overline{S}$ and $\overline{\text{co } S} = \text{cl}(\overline{S})$. For a finite set S , we have $\text{co } S = \overline{\text{co } S}$. To see the difference of $\text{co } S$ and $\overline{\text{co } S}$ for an infinite set S , consider $S = \{(0, 1)\} \cup \{(k, 0) \mid k \in \mathbb{Z}\}$. The convex hull is $\text{co } S = \overline{S} = \{(0, 1)\} \cup \{(x_1, x_2) \mid 0 \leq x_2 < 1\}$ and the closed convex hull is $\overline{\text{co } S} = \text{cl}(\overline{S}) = \{(x_1, x_2) \mid 0 \leq x_2 \leq 1\}$. We have $\overline{S} \cap \mathbb{Z}^2 = S$, which shows that this set S is hole-free, while $\text{cl}(\overline{S}) \cap \mathbb{Z}^2 \neq S$. ■

2.2 Definition of integrally convex sets

For $x \in \mathbb{R}^n$ the *integral neighborhood* of x is defined by

$$N(x) = \{z \in \mathbb{Z}^n \mid |x_i - z_i| < 1 \ (i = 1, 2, \dots, n)\}. \tag{2.4}$$

It is noted that strict inequality “ $<$ ” is used in this definition and $N(x)$ admits an alternative expression

$$N(x) = \{z \in \mathbb{Z}^n \mid \lfloor x_i \rfloor \leq z_i \leq \lceil x_i \rceil \ (i = 1, 2, \dots, n)\}, \tag{2.5}$$

where, for $t \in \mathbb{R}$ in general, $\lfloor t \rfloor$ denotes the largest integer not larger than t (rounding-down to the nearest integer) and $\lceil t \rceil$ is the smallest integer not smaller than t (rounding-up to the nearest integer). That is, $N(x)$ consists of all integer vectors z between $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \dots, \lfloor x_n \rfloor)$ and $\lceil x \rceil = (\lceil x_1 \rceil, \lceil x_2 \rceil, \dots, \lceil x_n \rceil)$. See Fig. 2 for $N(x)$ when $n = 2$.

For a set $S \subseteq \mathbb{Z}^n$ and $x \in \mathbb{R}^n$ we call the convex hull of $S \cap N(x)$ the *local convex hull* of S around x . A nonempty set $S \subseteq \mathbb{Z}^n$ is said to be *integrally convex* if the union of the local convex hulls $\overline{S \cap N(x)}$ over $x \in \mathbb{R}^n$ is convex [40]. In other words, a set

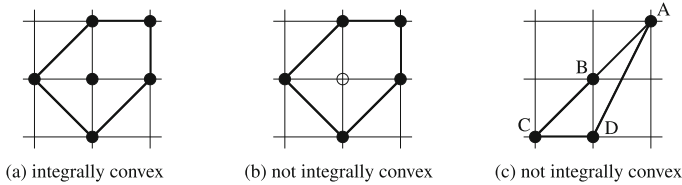


Fig. 3 Concept of integrally convex sets

$S \subseteq \mathbb{Z}^n$ is called integrally convex if

$$\bar{S} = \bigcup_{x \in \mathbb{R}^n} \overline{S \cap N(x)}. \tag{2.6}$$

Since the inclusion $\bar{S} \supseteq \bigcup_x \overline{S \cap N(x)}$ is trivially true for any S , the content of the condition (2.6) lies in

$$\bar{S} \subseteq \bigcup_{x \in \mathbb{R}^n} \overline{S \cap N(x)}. \tag{2.7}$$

Example 2.1 The concept of integrally convex sets is illustrated by simple examples. The six-point set in Fig. 3(a) is integrally convex. The removal of the middle point (Fig. 3(b)) breaks integral convexity (cf., Proposition 2.2). The four-point set $S = \{A, B, C, D\}$ in Fig. 3(c) is not integrally convex, since its convex hull \bar{S} , which is the triangle ACD , does not coincide with the union of the local convex hulls $\overline{S \cap N(x)}$, which is the union of the line segment AB and the triangle BCD . ■

Example 2.2 Obviously, every subset of $\{0, 1\}^n$ is integrally convex, and every interval of integers ($\subseteq \mathbb{Z}^n$) is integrally convex. ■

Integral convexity can be defined by seemingly different conditions. Here we mention the following two.

- Every point x in the convex hull of S is contained in the convex hull of $S \cap N(x)$, i.e.,

$$x \in \bar{S} \implies x \in \overline{S \cap N(x)}. \tag{2.8}$$

- For each $x \in \mathbb{R}^n$, the intersection of the convex hulls of S and $N(x)$ is equal to the convex hull of the intersection of S and $N(x)$, i.e.,

$$\bar{S} \cap \overline{N(x)} = \overline{S \cap N(x)}. \tag{2.9}$$

Since the inclusion $\bar{S} \cap \overline{N(x)} \supseteq \overline{S \cap N(x)}$ is trivially true for any S and x , the content of this condition lies in

$$\bar{S} \cap \overline{N(x)} \subseteq \overline{S \cap N(x)}. \tag{2.10}$$

The following proposition states the equivalence of the five conditions (2.6) to (2.10). Thus, any one of these conditions characterizes integral convexity of a set of integer points.

Proposition 2.1 *For $S \subseteq \mathbb{Z}^n$, the five conditions (2.6) to (2.10) are all equivalent.*

Proof The proof is easy and straightforward, but we include it for completeness. We already mentioned the equivalences [(2.6) \Leftrightarrow (2.7)] and [(2.9) \Leftrightarrow (2.10)]. [(2.7) \Rightarrow (2.8)]: Take any $x \in \bar{S}$. By (2.7) we obtain $x \in \overline{S \cap N(y)}$ for some y . This means that there exist integer points $x^{(1)}, x^{(2)}, \dots, x^{(m)} \in S \cap N(y)$ and coefficients $\lambda_1, \lambda_2, \dots, \lambda_m > 0$ such that $x = \sum_{k=1}^m \lambda_k x^{(k)}$ and $\sum_{k=1}^m \lambda_k = 1$. Since all the points $x, x^{(1)}, x^{(2)}, \dots, x^{(m)}$ lie between $\lfloor y \rfloor$ and $\lceil y \rceil$, we must have $x^{(k)} \in N(x)$ for all k . Therefore, $x \in \overline{S \cap N(x)}$.

[(2.8) \Rightarrow (2.7)]: Take any $y \in \bar{S}$. By (2.8) we obtain $y \in \overline{S \cap N(y)}$, whereas $S \cap N(y) \subseteq \bigcup_{x \in \mathbb{R}^n} \overline{S \cap N(x)}$. Thus (2.7) is shown.

[(2.8) \Rightarrow (2.10)]: Take any $y \in \bar{S} \cap \overline{N(x)}$. By $y \in \bar{S}$ and (2.8) we obtain $y \in S \cap N(y)$, whereas $y \in N(x)$ implies $N(y) \subseteq N(x)$. Hence $y \in \overline{S \cap N(x)}$.

[(2.10) \Rightarrow (2.8)]: Take any $x \in \bar{S}$. Since $x \in \overline{N(x)}$, we have $x \in \overline{S \cap N(x)}$. Then $x \in \overline{S \cap N(x)}$ by (2.10). □

As an application of Proposition 2.1 we give a formal proof to the statement that an integrally convex set is hole-free.

Proposition 2.2 *For an integrally convex set S , we have $S = \bar{S} \cap \mathbb{Z}^n$.*

Proof It suffices to show $\bar{S} \cap \mathbb{Z}^n \subseteq S$ in (2.2). Take any $x \in \bar{S} \cap \mathbb{Z}^n$. Since $N(x) = \{x\}$, (2.8) shows $x \in \overline{S \cap \{x\}}$, which implies $x \in S$. □

The following (new) characterization of an integrally convex set is often useful, which is used indeed in the proof of Theorem 3.2 in Sect. 3.4. While the condition (2.8) refers to all real vectors x , the condition (2.11) below restricts x to the midpoints of two vectors in S .

Theorem 2.1 *A nonempty set $S \subseteq \mathbb{Z}^n$ is integrally convex if and only if*

$$\frac{y + y'}{2} \in \overline{S \cap N\left(\frac{y + y'}{2}\right)} \tag{2.11}$$

for every $y, y' \in S$ with $\|y - y'\|_\infty \geq 2$.

Proof The only-if-part is obvious from (2.8), since $(y + y')/2 \in \bar{S}$. To prove the if-part, let $x \in \bar{S}$. This implies the existence of $y^{(1)}, y^{(2)}, \dots, y^{(m)} \in S$ such that

$$x = \sum_{i=1}^m \lambda_i y^{(i)}, \tag{2.12}$$

where $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i > 0$ ($i = 1, 2, \dots, m$). In the following we modify the generating points $y^{(1)}, y^{(2)}, \dots, y^{(m)}$ repeatedly and eventually arrive at an expression

of the form (2.12) with the additional condition that $y^{(i)} \in N(x)$ for all i , showing $x \in S \cap N(x)$. Then the integral convexity of S is established by (2.8).

For each $j = n, n - 1, \dots, 1$, we look at the j -th component of $y^{(1)}, y^{(2)}, \dots, y^{(m)}$. Let $j = n$ and define

$$\alpha_n := \min_i y_n^{(i)}, \beta_n := \max_i y_n^{(i)}, I_{\min} := \{i \mid y_n^{(i)} = \alpha_n\}, I_{\max} := \{i \mid y_n^{(i)} = \beta_n\}. \tag{2.13}$$

If $\beta_n - \alpha_n \leq 1$, we are done with $j = n$. Suppose that $\beta_n - \alpha_n \geq 2$. By translation and reversal of the n -th coordinate, we may assume $0 \leq x_n \leq 1, \alpha_n \leq 0$, and $\beta_n \geq 2$. By renumbering the generators we may assume $1 \in I_{\min}$ and $2 \in I_{\max}$, i.e., $y_n^{(1)} = \alpha_n$ and $y_n^{(2)} = \beta_n$. We have $\|y^{(1)} - y^{(2)}\|_\infty \geq 2$.

By (2.11) for $(y^{(1)}, y^{(2)})$ we have

$$\frac{y^{(1)} + y^{(2)}}{2} = \sum_{k=1}^l \mu_k z^{(k)}, \quad z^{(k)} \in S \cap N\left(\frac{y^{(1)} + y^{(2)}}{2}\right) \quad (k = 1, 2, \dots, l) \tag{2.14}$$

with $\mu_k > 0$ ($k = 1, 2, \dots, l$) and $\sum_{k=1}^l \mu_k = 1$. With notation $\lambda = \min(\lambda_1, \lambda_2)$, it follows from (2.12) and (2.14) that

$$x = (\lambda_1 - \lambda)y^{(1)} + (\lambda_2 - \lambda)y^{(2)} + 2\lambda \sum_{k=1}^l \mu_k z^{(k)} + \sum_{i=3}^m \lambda_i y^{(i)},$$

which is another representation of the form (2.12).

With reference to this new representation we define $\hat{\alpha}_n, \hat{\beta}_n, \hat{I}_{\min}$, and \hat{I}_{\max} , as in (2.13). Since $\beta_n - \alpha_n \geq 2$, we have

$$\alpha_n + 1 \leq (y_n^{(1)} + y_n^{(2)})/2 \leq \beta_n - 1,$$

which implies $\alpha_n + 1 \leq z_n^{(k)} \leq \beta_n - 1$ for all k . Hence, $\alpha_n \leq \hat{\alpha}_n$ and $\hat{\beta}_n \leq \beta_n$. Moreover, if $(\hat{\alpha}_n, \hat{\beta}_n) = (\alpha_n, \beta_n)$, then $|\hat{I}_{\min}| + |\hat{I}_{\max}| \leq |I_{\min}| + |I_{\max}| - 1$. Therefore, by repeating the above process with $j = n$, we eventually arrive at a representation of the form of (2.12) with $\beta_n - \alpha_n \leq 1$.

We next apply the above procedure for the $(n - 1)$ -st component. What is crucial here is that the condition $\beta_n - \alpha_n \leq 1$ is maintained in the modification of the generators via (2.14) for the $(n - 1)$ -st component. Indeed, for each k , the inequality $\alpha_n \leq z_n^{(k)} \leq \beta_n$ follows from $\alpha_n \leq (y_n^{(1)} + y_n^{(2)})/2 \leq \beta_n$ and $z^{(k)} \in N((y^{(1)} + y^{(2)})/2)$. Therefore, we can obtain a representation of the form of (2.12) with $\beta_n - \alpha_n \leq 1$ and $\beta_{n-1} - \alpha_{n-1} \leq 1$, where $\alpha_{n-1} = \min_i y_{n-1}^{(i)}$ and $\beta_{n-1} = \max_i y_{n-1}^{(i)}$.

Then we continue the above process for $j = n - 2, n - 3, \dots, 1$, to finally obtain a representation of the form of (2.12) with $|y_j^{(i)} - y_j^{(i')}| \leq 1$ for all i, i' and $j = 1, 2, \dots, n$. This means, in particular, that $y^{(i)} \in S \cap N(x)$ for all i . □

2.3 Polyhedral aspects

A subset of \mathbb{R}^n is called a *polyhedron* if it is described by a finite number of linear inequalities. A polyhedron is said to be *rational* if it is described by a finite number of linear inequalities with rational coefficients. A polyhedron is called an *integer polyhedron* if $P = \overline{P \cap \mathbb{Z}^n}$, i.e., if it coincides with the convex hull of the integer points contained in it, or equivalently, if P is rational and each face of P contains an integer vector. See [56, 57] for terminology about polyhedra. For two vectors $a, x \in \mathbb{R}^n$, we use notation $\langle a, x \rangle = a^\top x = \sum_{i=1}^n a_i x_i$.

The convex hull of an integrally convex set is an integer polyhedron (see [50, Sec. 4.1] for a rigorous proof). However, not much is known about the inequality system $Ax \leq b$ to describe an integrally convex set. This is not surprising because every subset of $\{0, 1\}^n$ is integrally convex (as noted in Example 2.2), and most of the NP-hard combinatorial optimization problems can be formulated on $\{0, 1\}^n$ polytopes.

When $n = 2$, the following fact is easy to see.

Proposition 2.3 ([37]) *A set $S \subseteq \mathbb{Z}^2$ is integrally convex if and only if it can be represented as $S = \{(x_1, x_2) \in \mathbb{Z}^2 \mid a^i x_1 + b^i x_2 \leq c^i \ (i = 1, 2, \dots, m)\}$ for some $a^i, b^i \in \{-1, 0, 1\}$ and $c^i \in \mathbb{Z} \ (i = 1, 2, \dots, m)$.*

Polyhedral descriptions are known for major subclasses of integrally convex sets. The present knowledge for various kinds of discrete convex sets is summarized in Table 1, which shows the possible forms of the vector a for an inequality $a^\top x \leq b$ to describe the convex hull \overline{S} of a discrete convex set S . It should be clear that each vector a corresponds (essentially) to the normal vector of a face of \overline{S} . Since an M_2 -convex (resp., M_2^{\natural} -convex) set is, by definition, the intersection of two M -convex (resp., M^{\natural} -convex) sets [40], the polyhedral description of an M_2 -convex (resp., M_2^{\natural} -convex) set is obtained immediately as the union of the inequality systems for the constituent M -convex (resp., M^{\natural} -convex) sets.

In all cases listed in Table 1, we have $a \in \{-1, 0, +1\}^n$, that is, every component of a belongs to $\{-1, 0, +1\}$. However, this is not the case with a general integrally convex set.

Example 2.3 Let $S = \{(1, 1, 0, 0), (0, 1, 1, 0), (1, 0, 1, 0), (0, 0, 0, 1)\}$, which is obviously an integrally convex set since $S \subseteq \{0, 1\}^4$. Because all these points lie on the hyperplane $x_1 + x_2 + x_3 + 2x_4 = 2$, we need a vector $a = \pm(1, 1, 1, 2)$ to describe the convex hull \overline{S} . ■

The following example illustrates a use of the results in Table 1.

Example 2.4 Consider $S = \{x \in \mathbb{Z}^4 \mid x_1 + x_2 = x_3 + x_4\}$. This set is described by two inequalities of the form of $a^\top x \leq 0$ with $a = (1, 1, -1, -1), (-1, -1, 1, 1)$. This shows that S is not L^{\natural} -convex, because we must have $a = \mathbf{1}^i - \mathbf{1}^j$ or $\pm \mathbf{1}^i$ for an L^{\natural} -convex set (see Table 1). M^{\natural} -convexity of S is also denied because $a = \pm \mathbf{1}^I$ for an M^{\natural} -convex set. The set S is, in fact, an L_2 -convex set with $a = \mathbf{1}^J - \mathbf{1}^I$ for $(I, J) = (\{1, 2\}, \{3, 4\})$ and $(I, J) = (\{3, 4\}, \{1, 2\})$. ■

Each face of the convex hull \overline{S} of an integrally convex set S induces an integrally convex set.

Table 1 Polyhedral descriptions of discrete convex sets [35]

	Vector a for $a^\top x \leq b$	Refs.
Box (interval)	$\pm \mathbf{1}^i$	obvious
L-convex set	$\mathbf{1}^j - \mathbf{1}^i$	[40, Sec.5.3]
L^\sharp -convex set	$\mathbf{1}^j - \mathbf{1}^i, \pm \mathbf{1}^i$	[40, Sec.5.5]
L_2 -convex set	$\mathbf{1}^J - \mathbf{1}^I$ ($ I = J $)	[35]
L_2^\sharp -convex set	$\mathbf{1}^J - \mathbf{1}^I$ ($ I - J \in \{-1, 0, 1\}$)	[35]
M-convex set	$\mathbf{1}^I, -\mathbf{1}^N (= -\mathbf{1})$	[40, Sec.4.4]
M^\sharp -convex set	$\pm \mathbf{1}^I$	[40, Sec.4.7]
M_2 -convex set	$\mathbf{1}^I, -\mathbf{1}^N (= -\mathbf{1})$	by M-convex
M_2^\sharp -convex set	$\pm \mathbf{1}^I$	by M^\sharp -convex
Multimodular set	$\pm \mathbf{1}^I$ (I : consecutive)	[34]

$\mathbf{1}^I$: characteristic vector of $I \subseteq N$; $\mathbf{1}^i$: i -th unit vector ($= \mathbf{1}^{\{i\}}$)

Proposition 2.4 *Let $S \subseteq \mathbb{Z}^n$ be an integrally convex set. For any face F of \overline{S} , $F \cap \mathbb{Z}^n$ is integrally convex.*

Proof Consider inequality descriptions of \overline{S} and F , say, $\overline{S} = \{x \in \mathbb{R}^n \mid \langle a^{(i)}, x \rangle \leq b^{(i)} \ (i \in I)\}$ and $F = \{x \in \mathbb{R}^n \mid \langle a^{(j)}, x \rangle = b^{(j)} \ (j \in J), \langle a^{(i)}, x \rangle \leq b^{(i)} \ (i \in I \setminus J)\}$ with some index sets $I \supseteq J$. Since \overline{S} is an integer polyhedron, we may assume $\mathbf{0} \in F$, which implies $b^{(j)} = 0$ for $j \in J$. Take any $x \in F$. Since $x \in \overline{S}$ and S is integrally convex, there exist $y^{(1)}, y^{(2)}, \dots, y^{(m)} \in S \cap N(x)$ and coefficients $\lambda_1, \lambda_2, \dots, \lambda_m > 0$ such that $x = \sum_{k=1}^m \lambda_k y^{(k)}$ and $\sum_{k=1}^m \lambda_k = 1$. For each $j \in J$, we have $0 = \langle a^{(j)}, x \rangle = \langle a^{(j)}, \sum_{k=1}^m \lambda_k y^{(k)} \rangle = \sum_{k=1}^m \lambda_k \langle a^{(j)}, y^{(k)} \rangle$, where $\langle a^{(j)}, y^{(k)} \rangle \leq 0$ since $y^{(k)} \in S$. Therefore $\langle a^{(j)}, y^{(k)} \rangle = 0$ for all k , which implies $y^{(k)} \in F$, and hence $y^{(k)} \in (F \cap \mathbb{Z}^n) \cap N(x)$ for all k . \square

The edges of \overline{S} have a remarkable property that the direction of an edge is given by a $\{-1, 0, +1\}$ -vector. This property follows from a basic fact that every edge (line) of \overline{S} contains a pair of lattice points in a translated unit hypercube, whose difference is a $\{-1, 0, +1\}$ -vector. In this connection, the following fact is known.

Proposition 2.5 ([50, Proposition 1]) *Let $S \subseteq \mathbb{Z}^n$ be an integrally convex set. For any face F of \overline{S} , the smallest affine subspace containing F is given as $\{x + \sum_{k=1}^h c_k d^{(k)} \mid c_1, c_2, \dots, c_h \in \mathbb{R}\}$ for any point x in F and some direction vectors $d^{(k)} \in \{-1, 0, +1\}^n$ ($k = 1, 2, \dots, h$).*

Remark 2.2 ([50]) The property mentioned in Proposition 2.5 does not characterize integral convexity of a set. For example, let $S = \{(0, 0, 0), (1, 0, 1), (1, 1, -1), (2, 1, 0)\}$. The convex hull \overline{S} is a parallelogram with edge directions $(1, 0, 1)$ and $(1, 1, -1)$, and hence is an integer polyhedron with the property that the smallest affine subspace containing each face is spanned by $\{-1, 0, +1\}$ -vectors. However, S is not integrally convex, since (2.8) is violated by $x = [(1, 0, 1) + (1, 1, -1)]/2 = (1, 1/2, 0) \in \overline{S}$, for which $N(x) = \{(1, 0, 0), (1, 1, 0)\}$ and $S \cap N(x) = \emptyset$. \blacksquare

The following result is concerned with a direction of infinity in the discrete setting, to be used in the proof of Proposition 3.3 in Sect. 3.7.

Proposition 2.6 *Let $S \subseteq \mathbb{Z}^n$ be an integrally convex set, $y \in S$, and $d \in \mathbb{Z}^n$. If $y + kd \in S$ for all integers $k \geq 1$, then for any $z \in S$, we have $z + kd \in S$ for all integers $k \geq 1$.*

Proof It suffices to prove that, for any $x \in S$, we have $x + d \in S$. For an integer $k \geq 1$ (to be specified later), consider $u = \frac{k-1}{k}x + \frac{1}{k}(y + kd) = x + d + \frac{1}{k}(y - x)$, which is a convex combination of x and $y + kd$. By taking k large enough, we can assume $\|u - (x + d)\|_2 = \|y - x\|_2/k < 1/\sqrt{n}$, which implies that $x + d \in N(u)$ and $u \notin N(u) \setminus \{x + d\}$. On the other hand, we have $u \in \bar{S} \cap N(u)$, since $u \in \bar{S}$ and S is integrally convex. Therefore, we must have $x + d \in S$. \square

Box-integer and box-TDI polyhedra

A polyhedron $P \subseteq \mathbb{R}^n$ is called *box-integer* if $P \cap [l, u]_{\mathbb{R}} (= P \cap \{x \in \mathbb{R}^n \mid l \leq x \leq u\})$ is an integer polyhedron for each choice of integer vectors $l, u \in \mathbb{Z}^n$ with $l \leq u$ ([57, Section 5.15]). This concept is closely related (or essentially equivalent) to that of integrally convex sets, as follows.

Proposition 2.7 ([46]) *If a set $S \subseteq \mathbb{Z}^n$ is integrally convex, then its convex hull \bar{S} is a box-integer polyhedron. Conversely, if P is a box-integer polyhedron, then $S = P \cap \mathbb{Z}^n$ is an integrally convex set.*

It follows from Proposition 2.7 that a set S of integer points is integrally convex if and only if it is hole-free and its convex hull \bar{S} is a box-integer polyhedron.

The concept of (box-)total dual integrality has long played a major role in combinatorial optimization [7, 8, 11, 12, 56, 57]. A linear inequality system $Ax \leq b$ is said to be *totally dual integral (TDI)* if the entries of A and b are rational numbers and the minimum in the linear programming duality equation

$$\max\{w^T x \mid Ax \leq b\} = \min\{y^T b \mid y^T A = w^T, y \geq 0\}$$

has an integral optimal solution y for every integral vector w such that the minimum is finite. A linear inequality system $Ax \leq b$ is said to be *box-totally dual integral (box-TDI)* if the system $[Ax \leq b, d \leq x \leq c]$ is TDI for each choice of rational (finite-valued) vectors c and d . It is known [57, Theorem 5.35] that a system $Ax \leq b$ is box-TDI if the matrix A is totally unimodular.

A polyhedron is called a *box-TDI polyhedron* if it can be described by a box-TDI system. It was pointed out in [8] that every TDI system describing a box-TDI polyhedron is a box-TDI system, which fact indicates that box-TDI is a property of a polyhedron. In this connection it is worth mentioning that every rational polyhedron can be described by a TDI system, showing that TDI is a property of a system of inequalities and not of a polyhedron.

An integral box-TDI polyhedron is box-integer [57, (5.82), p. 83]. Although the converse is not true (see Example 2.5 below), it is possible to characterize a box-TDI

polyhedron in terms of box-integrality of its dilations. For a positive integer α , the α -dilation of a polyhedron $P = \{x \mid Ax \leq b\}$ means the polyhedron $\alpha P := \{x \mid Ax \leq \alpha b\} = \{x \mid \frac{1}{\alpha}x \in P\}$.

Proposition 2.8 ([6, Theorem 2 & Prop. 2]) *An integer polyhedron P is box-TDI if and only if the α -dilation αP is box-integer for any positive integer α .*

Example 2.5 Here is an example of a $\{0, 1\}$ -polyhedron that is not box-TDI. Let $P (\subseteq \mathbb{R}^4)$ be the convex hull of $S = \{(1, 1, 0, 0), (0, 1, 1, 0), (1, 0, 1, 0), (0, 0, 0, 1)\}$ (considered in Example 2.3). Since P is a $\{0, 1\}$ -polyhedron, it is obviously box-integer. However, the 2-dilation $2P$ is not box-integer. To see this we note that

$$(1, 1, 1, 1/2) = \frac{1}{4}((2, 2, 0, 0) + (0, 2, 2, 0) + (2, 0, 2, 0) + (0, 0, 0, 2)) \in 2P,$$

whereas $(1, 1, 1, 1/2) \in [l, u]_{\mathbb{R}}$ for $l = (1, 1, 1, 0)$ and $u = (1, 1, 1, 1)$, and $l \notin 2P$ and $u \notin 2P$. We can easily show that $(2P) \cap [l, u]_{\mathbb{R}}$ consists of $(1, 1, 1, 1/2)$ only. Thus $2P$ is not box-integer, which implies, by Proposition 2.8, that P is not box-TDI. ■

Following [15] we call the set of integral elements of an integral box-TDI polyhedron a *discrete box-TDI set*, or just a *box-TDI set*. A box-TDI set is an integrally convex set, but the converse is not true (Example 2.5). That is, box-TDI sets form a proper subclass of integrally convex sets. On the other hand, the major classes of discrete convex sets considered in discrete convex analysis are known to be box-TDI as follows.

Proposition 2.9 ([35]) *An L_2^{\natural} -convex set is a box-TDI set.*

Proposition 2.10 *An M_2^{\natural} -convex set is a box-TDI set.*

Proposition 2.9 for L_2^{\natural} -convex sets is established recently in [35] and Proposition 2.10 for M_2^{\natural} -convex sets is a reformulation of the fundamental fact about polymatroid intersection [57] in the language of discrete convex analysis. These propositions imply, in particular, that L_2 -, L^{\natural} -, L -, M_2 -, M^{\natural} -, M -convex sets are all box-TDI.

The following are examples of a box-TDI set S that is neither L_2^{\natural} -convex nor M_2^{\natural} -convex. The former consists of $\{0, 1\}$ -vectors and the latter arises from a cone.

Example 2.6 Consider $S = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$. This set is described by four inequalities

$$x_1 + x_2 + x_3 \leq 2, \quad x_1 - x_2 - x_3 \leq 0, \quad -x_1 + x_2 - x_3 \leq 0, \quad -x_1 - x_2 + x_3 \leq 0.$$

The first inequality, of the form of $a^{\top}x \leq b$ with $a = (1, 1, 1)$, denies L_2^{\natural} -convexity of S , because we must have $a = \mathbf{1}^J - \mathbf{1}^I$ with $|I| - |J| \in \{-1, 0, 1\}$ for an L_2^{\natural} -convex set (see Table 1). In the second inequality we have $a = (1, -1, -1)$, which denies M_2^{\natural} -convexity of S , because $a = \pm \mathbf{1}^J$ for an M_2^{\natural} -convex set. The set S is box-TDI, that is, its convex hull \bar{S} is a box-TDI polyhedron, which we can verify on the basis of Proposition 2.8. ■

Example 2.7 The set $S = \{x \in \mathbb{Z}^2 \mid x_1 + x_2 \leq 0, x_1 - x_2 \leq 0\}$ is neither L_2^{\natural} -convex nor M_2^{\natural} -convex, whereas it is box-TDI since the convex hull \bar{S} is a box-TDI polyhedron by Proposition 2.8. ■

We can summarize the above argument as

$$\{L_2^{\natural}\text{-convex sets}\} \cup \{M_2^{\natural}\text{-convex sets}\} \subsetneq \{\text{box-TDI sets}\} \subsetneq \{\text{integrally convex sets}\},$$

where Examples 2.5, 2.6, and 2.7 demonstrate the strict inclusions (\subsetneq). See Fig. 1.

2.4 Basic operations

In this section we show how integral convexity of a set behaves under basic operations. Let S be a subset of \mathbb{Z}^n , i.e., $S \subseteq \mathbb{Z}^n$.

Origin shift:

For an integer vector $b \in \mathbb{Z}^n$, the *origin shift* of S by b means a set T defined by $T = \{x - b \mid x \in S\}$. The origin shift of an integrally convex set is an integrally convex set.

Inversion of coordinates:

The *independent coordinate inversion* of S means a set T defined by

$$T = \{(\tau_1x_1, \tau_2x_2, \dots, \tau_nx_n) \mid (x_1, x_2, \dots, x_n) \in S\}$$

with an arbitrary choice of $\tau_i \in \{+1, -1\}$ ($i = 1, 2, \dots, n$). The independent coordinate inversion of an integrally convex set is an integrally convex set. This is a nice property of integral convexity, not shared by L^{\natural} -, L_2^{\natural} -, M^{\natural} , or M_2^{\natural} -convexity.

Permutation of coordinates:

For a permutation σ of $(1, 2, \dots, n)$, the *permutation* of S by σ means a set T defined by

$$T = \{(y_1, y_2, \dots, y_n) \mid (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)}) \in S\}.$$

The permutation of an integrally convex set is an integrally convex set.

Remark 2.3 Integral convexity is not preserved under a transformation by a (totally) unimodular matrix. For example, $S = \{(0, 0), (1, 0), (1, 1)\}$ is integrally convex and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is totally unimodular. However, $\{Ax \mid x \in S\} = \{(0, 0), (1, 0), (2, 1)\}$ is not integrally convex. ■

Scaling: For a positive integer α , the *scaling* of S by α means a set T defined by

$$T = \{(y_1, y_2, \dots, y_n) \in \mathbb{Z}^n \mid (\alpha y_1, \alpha y_2, \dots, \alpha y_n) \in S\}. \tag{2.15}$$

Note that the same scaling factor α is used for all coordinates. If $\alpha = 2$, for example, this operation amounts to considering the set of even points contained in S . The scaling of an integrally convex set is not necessarily integrally convex (Example 2.8 below).

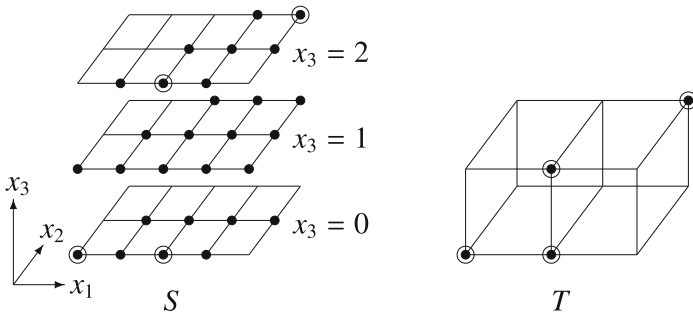


Fig. 4 An integrally convex set S and its scaled set T (Example 2.8) [37]

However, when $n = 2$, integral convexity admits the scaling operation. That is, if $S \subseteq \mathbb{Z}^2$ is integrally convex, then $T = \{y \in \mathbb{Z}^2 \mid \alpha y \in S\}$ is integrally convex ([37, Proposition 3.1]).

Example 2.8 ([37, Example 3.1]) This example shows that integral convexity is not preserved under scaling. Let S be a subset of \mathbb{Z}^3 defined by

$$\begin{aligned}
 S = & \{(x_1, x_2, 0) \mid 0 \leq x_2 \leq 1, 0 \leq x_1 - x_2 \leq 3\} \\
 & \cup \{(x_1, x_2, 1) \mid 0 \leq x_2 \leq 2, x_2 \leq x_1 \leq 4\} \\
 & \cup \{(x_1, x_2, 2) \mid 0 \leq x_2 \leq 2, 1 \leq x_1 - x_2 \leq 3, x_1 \leq 4\},
 \end{aligned}$$

which is an integrally convex set (Fig. 4, left). With the scaling factor $\alpha = 2$, however, the scaled set $T = \{y \in \mathbb{Z}^3 \mid 2y \in S\} = \{(0, 0, 0), (1, 0, 0), (1, 0, 1), (2, 1, 1)\}$ is not integrally convex (Fig. 4, right). ■

Dilation: The dilation operation for a polyhedron (described in Sect. 2.3) is another kind of scaling operation. An adaptation of this operation to a hole-free discrete set $S \subseteq \mathbb{Z}^n$, we may call the set $T' = (\alpha S) \cap \mathbb{Z}^n$ the α -dilation of S , where α is a positive integer. Note that the scaling in (2.15) can be expressed as $T = (\frac{1}{\alpha} S) \cap \mathbb{Z}^n$ when S is hole-free.

The dilation operation does not always preserve integral convexity. Indeed, Example 2.5 shows that the 2-dilation of an integrally convex set is not necessarily integrally convex.

Remark 2.4 Failure of dilation operation is rather exceptional for discrete convex sets. Indeed, all kinds of discrete convexity (box, L-, L^{\natural} -, L_2 -, L_2^{\natural} -, M-, M^{\natural} -, M_2 -, M_2^{\natural} -convexity, and multimodularity) listed in Table 1 are preserved under the dilation operation. In contrast, the scaling operation in (2.15) preserves L-convexity and its relatives (box, L-, L^{\natural} -, L_2 -, L_2^{\natural} -convexity, and multimodularity), and not M-convexity and its relatives (M-, M^{\natural} -, M_2 -, M_2^{\natural} -convexity). ■

Restriction:

For a set $S \subseteq \mathbb{Z}^N$ and a subset U of the index set $N = \{1, 2, \dots, n\}$, the *restriction* of S to U is a subset T of \mathbb{Z}^U defined by

$$T = \{y \in \mathbb{Z}^U \mid (y, \mathbf{0}_{N \setminus U}) \in S\},$$

where $\mathbf{0}_{N \setminus U}$ denotes the zero vector in $\mathbb{Z}^{N \setminus U}$. The notation $(y, \mathbf{0}_{N \setminus U})$ means the vector in \mathbb{Z}^N whose i th component is equal to y_i for $i \in U$ and to 0 for $i \in N \setminus U$. The restriction of an integrally convex set is integrally convex (if the resulting set is nonempty).

Projection:

For a set $S \subseteq \mathbb{Z}^N$ and a subset U of the index set $N = \{1, 2, \dots, n\}$, the *projection* of S to U is a subset T of \mathbb{Z}^U defined by

$$T = \{y \in \mathbb{Z}^U \mid (y, z) \in S \text{ for some } z \in \mathbb{Z}^{N \setminus U}\}, \tag{2.16}$$

where the notation (y, z) means the vector in \mathbb{Z}^N whose i th component is equal to y_i for $i \in U$ and to z_i for $i \in N \setminus U$. The projection of an integrally convex set is integrally convex ([33, Theorem 3.1]).

Splitting:

Suppose that we are given a family $\{U_1, U_2, \dots, U_n\}$ of disjoint nonempty sets indexed by $N = \{1, 2, \dots, n\}$. Let $m_i = |U_i|$ for $i = 1, 2, \dots, n$ and define $m = \sum_{i=1}^n m_i$, where $m \geq n$. For each $i \in N$ we define an m_i -dimensional vector $y_{[i]} = (y_j \mid j \in U_i)$ and express $y \in \mathbb{Z}^m$ as $y = (y_{[1]}, y_{[2]}, \dots, y_{[n]})$. For a set $S \subseteq \mathbb{Z}^m$, the subset of \mathbb{Z}^m defined by

$$T = \{(y_{[1]}, y_{[2]}, \dots, y_{[n]}) \in \mathbb{Z}^m \mid y_{[i]} \in \mathbb{Z}^{m_i}, y_{[i]}(U_i) = x_i \ (i \in N), x \in S\}$$

is called the *splitting* of S by $\{U_1, U_2, \dots, U_n\}$, where $y_{[i]}(U_i) = \sum \{y_j \mid j \in U_i\}$. For example, $T = \{(y_1, y_2, y_3) \in \mathbb{Z}^3 \mid (y_1, y_2 + y_3) \in S\}$ is a splitting of $S \subseteq \mathbb{Z}^2$ for $U_1 = \{1\}$ and $U_2 = \{2, 3\}$, where $n = 2$ and $m = 3$. The splitting of an integrally convex set is integrally convex ([46, Proposition 3.4]).

Aggregation:

Let $\mathcal{P} = \{N_1, N_2, \dots, N_m\}$ be a partition of $N = \{1, 2, \dots, n\}$ into disjoint nonempty subsets: $N = N_1 \cup N_2 \cup \dots \cup N_m$ and $N_i \cap N_j = \emptyset$ for $i \neq j$. For a set $S \subseteq \mathbb{Z}^N$ the subset of \mathbb{Z}^m , where $m \leq n$, defined by

$$T = \{(y_1, y_2, \dots, y_m) \in \mathbb{Z}^m \mid y_j = x(N_j) \ (j = 1, 2, \dots, m), x \in S\}$$

is called the *aggregation* of S by \mathcal{P} . For example, $T = \{(y_1, y_2) \in \mathbb{Z}^2 \mid y_1 = x_1, y_2 = x_2 + x_3 \text{ for some } (x_1, x_2, x_3) \in S\}$ is an aggregation of $S \subseteq \mathbb{Z}^3$ for $N_1 = \{1\}$ and $N_2 = \{2, 3\}$, where $n = 3$ and $m = 2$. The aggregation of an integrally convex set is not necessarily integrally convex.

Example 2.9 ([46, Example 3.4]) Set $S = \{(0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 0), (1, 1, 0, 1)\}$ is an integrally convex set. For the partition of $N = \{1, 2, 3, 4\}$ into

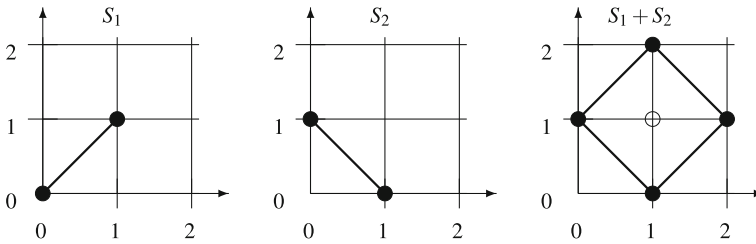


Fig. 5 Failure of convexity in Minkowski sum (2.18)

$N_1 = \{1, 3\}$ and $N_2 = \{2, 4\}$, the aggregation of S by $\{N_1, N_2\}$ is given by $T = \{(1, 0), (0, 1), (2, 1), (1, 2)\}$, which is not integrally convex. ■

Intersection:

The intersection $S_1 \cap S_2$ of integrally convex sets $S_1, S_2 \subseteq \mathbb{Z}^n$ is not necessarily integrally convex (Example 2.10 below). However, it is obviously true (almost from definition) that the intersection of an integrally convex set with a box of integers is integrally convex.

Example 2.10 ([47, Example 4.4]) The intersection of two integrally convex sets is not necessarily integrally convex. Let $S_1 = \{(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 2, 1)\}$ and $S_2 = \{(0, 0, 0), (0, 1, 0), (1, 1, 1), (1, 2, 1)\}$, for which $S_1 \cap S_2 = \{(0, 0, 0), (1, 2, 1)\}$. The sets S_1 and S_2 are integrally convex, whereas $S_1 \cap S_2$ is not. ■

Minkowski sum:

The *Minkowski sum* of two sets $S_1, S_2 \subseteq \mathbb{Z}^n$ means the subset of \mathbb{Z}^n defined by

$$S_1 + S_2 = \{x + y \mid x \in S_1, y \in S_2\}. \tag{2.17}$$

The Minkowski sum of integrally convex sets is not necessarily integrally convex (Example 2.11 below). However, the Minkowski sum of an integrally convex set with a box of integers is integrally convex ([33, Theorem 4.1]).

Example 2.11 ([40, Example 3.15]) The Minkowski sum of $S_1 = \{(0, 0), (1, 1)\}$ and $S_2 = \{(1, 0), (0, 1)\}$ is equal to $S_1 + S_2 = \{(1, 0), (0, 1), (2, 1), (1, 2)\}$, which has a “hole” at $(1, 1)$, i.e., $(1, 1) \in \overline{S_1 + S_2}$ and $(1, 1) \notin S_1 + S_2$. ■

Remark 2.5 The Minkowski sum is often a source of difficulty in a discrete setting, because

$$S_1 + S_2 = \overline{(S_1 + S_2)} \cap \mathbb{Z}^n \tag{2.18}$$

is not always true (Example 2.11). In other words, the equality (2.18), if true, captures a certain essence of discrete convexity. The property (2.18) is called “convexity in Minkowski sum” in [40, Section 3.3]. We sometimes call (2.17) the *discrete (or integral) Minkowski sum* of S_1 and S_2 to emphasize discreteness. ■

Remark 2.6 The Minkowski sum plays a central role in discrete convex analysis. The Minkowski sum of two (or more) M^{\natural} -convex sets is M^{\natural} -convex. The Minkowski sum of two L^{\natural} -convex sets is not necessarily L^{\natural} -convex, but it is integrally convex. The Minkowski sum of three L^{\natural} -convex sets is no longer integrally convex. For example ([47, Example 4.12]), $S_1 = \{(0, 0, 0), (1, 1, 0)\}$, $S_2 = \{(0, 0, 0), (0, 1, 1)\}$, and $S_3 = \{(0, 0, 0), (1, 0, 1)\}$ are L^{\natural} -convex sets, and their Minkowski sum $S = S_1 + S_2 + S_3$ is given as

$$S = \{(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 1), (2, 1, 1), (1, 1, 2), (1, 2, 1), (2, 2, 2)\},$$

which is not integrally convex, since $(1, 1, 1) \in \bar{S}$ and $(1, 1, 1) \notin S$. ■

The following theorem is a discrete analogue of a well-known decomposition of a polyhedron into a bounded part and a conic part (recession cone or characteristic cone) [56, Theorem 8.5]. An integrally convex set is called *conic* if its convex hull is a cone.

Theorem 2.2 ([52]) *Every integrally convex set S can be represented as a (discrete) Minkowski sum of a bounded integrally convex set Q and a conic integrally convex set C , that is, $S = Q + C$.*

3 Integrally convex functions

3.1 Convex extension

For a function $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ in general, $\text{dom } g := \{x \in \mathbb{R}^n \mid -\infty < g(x) < +\infty\}$ is called the *effective domain* of g . In this section we always assume that $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\text{dom } f \neq \emptyset$, that is, f is a function defined on \mathbb{Z}^n taking values in $\mathbb{R} \cup \{+\infty\}$ and $\text{dom } f = \{x \in \mathbb{Z}^n \mid f(x) < +\infty\}$ is nonempty.

We say that f is *convex-extendible* if there exists a convex function $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying $g(x) = f(x)$ for all $x \in \mathbb{Z}^n$. When $n = 1$, $f : \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex-extendible if and only if $\text{dom } f$ is an interval of integers and $f(k - 1) + f(k + 1) \geq 2f(k)$ for all $k \in \mathbb{Z}$. In this case, a convex extension of f is given by the piecewise-linear function whose graph consists of line segments connecting $(k, f(k))$ and $(k + 1, f(k + 1))$ for all $k \in \mathbb{Z}$.

We say that a function $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ *minorizes* f if $g(x) \leq f(x)$ for all $x \in \mathbb{Z}^n$. In this section we always assume that f is minorized by some affine function $g(x) = \langle p, x \rangle + \alpha$, where $p \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, and $\langle p, x \rangle := \sum_{i=1}^n p_i x_i$ denotes the inner product (or duality pairing, to be more precise) of $p = (p_1, p_2, \dots, p_n)$ and $x = (x_1, x_2, \dots, x_n)$. Note that every convex-extendible function is minorized by an affine function.

The *convexification* of f , to be denoted by \check{f} , is defined as

$$\check{f}(x) := \inf_{\lambda} \left\{ \sum_y \lambda_y f(y) \mid \sum_y \lambda_y y = x, (\lambda_y) \in \Lambda \right\} \quad (x \in \mathbb{R}^n), \tag{3.1}$$

where Λ denotes the set of coefficients for convex combinations indexed by $y \in \mathbb{Z}^n$:

$$\Lambda = \{(\lambda_y \mid y \in \mathbb{Z}^n) \mid \sum_y \lambda_y = 1, \lambda_y \geq 0 \text{ for all } y, \lambda_y > 0 \text{ for finitely many } y\}.$$

It is known [24, Section B.2.5] that \check{f} is a convex function and that \check{f} coincides with the pointwise supremum of all convex functions minorizing f , that is,

$$\check{f}(x) = \sup\{g(x) \mid g \text{ is convex, } g(y) \leq f(y) \text{ for all } y \in \mathbb{Z}^n\} \quad (x \in \mathbb{R}^n).$$

Therefore, f is convex-extensible if and only if $\check{f}(x) = f(x)$ for all $x \in \mathbb{Z}^n$.

The *convex envelope* of f , to be denoted by \bar{f} , is defined as the pointwise supremum of all affine functions minorizing f , that is,

$$\bar{f}(x) := \sup_{p,\alpha} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \leq f(y) \ (\forall y \in \mathbb{Z}^n)\} \quad (x \in \mathbb{R}^n). \quad (3.2)$$

This function \bar{f} is a closed convex function and coincides with the pointwise supremum of all closed convex functions minorizing f , that is,

$$\bar{f}(x) = \sup\{g(x) \mid g \text{ is closed convex, } g(y) \leq f(y) \text{ for all } y \in \mathbb{Z}^n\} \quad (x \in \mathbb{R}^n).$$

In this paper we often refer to the condition

$$f(x) = \bar{f}(x) \quad (x \in \mathbb{Z}^n) \quad (\text{i.e., } f = \bar{f}|_{\mathbb{Z}^n}) \quad (3.3)$$

as the convex-extensibility of f , although this condition is slightly stronger than the condition $f = \check{f}|_{\mathbb{Z}^n}$ mentioned above. Accordingly, we often refer to \bar{f} as the *convex extension* of f if (3.3) is the case.

Example 3.1 The quadratic function $f(x) = x^2$ defined for $x \in \mathbb{Z}$ is convex-extensible, where $g(x) = x^2$ ($x \in \mathbb{R}$) is an obvious convex extension of f . The convex envelope \bar{f} in (3.2) is a piecewise-linear function given by

$$\bar{f}(x) = (2k + 1)|x| - k(k + 1) \quad \text{with } k = \lfloor |x| \rfloor \quad (x \in \mathbb{R}).$$

It is noted that $\bar{f}(x) = x^2$ for integers x and $\bar{f}(x) > x^2$ for non-integral x ; for example, $\bar{f}(1/2) = 1/2 > 1/4$. The convexification \check{f} in (3.1) coincides with \bar{f} . ■

Remark 3.1 In a standard textbook [24, Section B.2.5], (3.1) is called the *convex hull* of f and denoted by $\text{co} f$, whereas (3.2) is called the *closed convex hull* of f and denoted by $\overline{\text{co}} f$ or $\text{cl}(\text{co} f)$. Using our notation we have $\text{co} f = \check{f}$ and $\overline{\text{co}} f = \bar{f}$. ■

Remark 3.2 We have $\bar{f}(x) \leq \check{f}(x)$ for all $x \in \mathbb{R}^n$, and the equality may fail in general (Example 3.2 below). However, when $\text{dom } f$ is bounded, we have $\bar{f}(x) = \check{f}(x)$ for

all $x \in \mathbb{R}^n$. The proofs are as follows. In (3.2) we have $\langle p, y \rangle + \alpha \leq f(y)$ for each y . By using $\lambda \in \Lambda$ satisfying $\sum_y \lambda_y y = x$, we obtain

$$\langle p, x \rangle + \alpha = \sum_y \lambda_y (\langle p, y \rangle + \alpha) \leq \sum_y \lambda_y f(y),$$

from which $\bar{f}(x) = \sup_{(p,\alpha)} \{\langle p, x \rangle + \alpha\} \leq \inf_{\lambda} \{\sum_y \lambda_y f(y)\} = \check{f}(x)$. When $\text{dom } f$ is bounded, $\text{dom } f$ is a finite set. For each $x \in \mathbb{R}^n$, consider a pair of (mutually dual) linear programs:

$$\begin{aligned} (P) \quad & \text{Maximize } \langle p, x \rangle + \alpha \\ & \text{subject to } \langle p, y \rangle + \alpha \leq f(y) \quad (y \in \text{dom } f), \\ (D) \quad & \text{Minimize } \sum_{y \in \text{dom } f} \lambda_y f(y) \\ & \text{subject to } \sum_{y \in \text{dom } f} \lambda_y y = x, \quad \sum_{y \in \text{dom } f} \lambda_y = 1, \lambda_y \geq 0 \quad (y \in \text{dom } f), \end{aligned}$$

where $(p, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ and $(\lambda_y \mid y \in \text{dom } f)$ are the variables of (P) and (D), respectively. The optimal values of (P) and (D) are equal to $\bar{f}(x)$ and $\check{f}(x)$, respectively. Problem (P) is feasible (e.g., take $p = 0$ and a sufficiently small α). By LP duality, (P) and (D) have the same (finite or infinite) optimal values, that is, $\bar{f}(x) = \check{f}(x)$. Note that (D) is feasible if and only if $x \in \overline{\text{dom } f}$, in which case the optimal values are finite. ■

Example 3.2 Let $f : \mathbb{Z}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ be the indicator function δ_S of the set S considered in Remark 2.1. For $x = (x_1, 1)$ with $x_1 \neq 0$, we have $x \in \text{cl}(\bar{S}) \setminus \bar{S}$. Hence $0 = \bar{f}(x) < \check{f}(x) = +\infty$. ■

Remark 3.3 For a set $S \subseteq \mathbb{Z}^n$, the convexification of the indicator function δ_S coincides with the indicator function of its convex hull \bar{S} , that is, $\check{\delta}_S = \delta_{\bar{S}}$. A set $S \subseteq \mathbb{Z}^n$ is hole-free if and only if the indicator function δ_S is convex-extensible. ■

3.2 Definition of integrally convex functions

Recall the notation $N(x)$ for the integral neighborhood of $x \in \mathbb{R}^n$ (cf., (2.4), Fig. 2). For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the *local convex extension* $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ of f is defined as the union of all convex extensions (convexifications) of f on $N(x)$. That is,

$$\tilde{f}(x) = \min \left\{ \sum_{y \in N(x)} \lambda_y f(y) \mid \sum_{y \in N(x)} \lambda_y y = x, (\lambda_y) \in \Lambda(x) \right\} \quad (x \in \mathbb{R}^n), \quad (3.4)$$

where $\Lambda(x)$ denotes the set of coefficients for convex combinations indexed by $N(x)$:

$$\Lambda(x) = \{(\lambda_y \mid y \in N(x)) \mid \sum_{y \in N(x)} \lambda_y = 1, \lambda_y \geq 0 (y \in N(x))\}. \quad (3.5)$$

It follows from this definition that, for each $x \in \mathbb{R}^n$, the function \tilde{f} restricted to $\overline{N(x)}$ is a convex function. In general, we have $\tilde{f}(x) \geq \check{f}(x) \geq \bar{f}(x)$ for all $x \in \mathbb{R}^n$, where \check{f} and \bar{f} are defined by (3.1) and (3.2), respectively.

We say that a function f is *integrally convex* if its local convex extension \tilde{f} is (globally) convex on the entire space \mathbb{R}^n . In this case, \tilde{f} is a convex function satisfying $\tilde{f}(x) = f(x)$ for all $x \in \mathbb{Z}^n$, which means that f is convex-extensible. Moreover, \tilde{f} coincides with \check{f} and \bar{f} , that is,

$$\tilde{f}(x) = \check{f}(x) = \bar{f}(x) \quad (x \in \mathbb{R}^n). \tag{3.6}$$

In particular, we have $\text{dom } \tilde{f} = \overline{\text{dom } f}$. Since $\tilde{f}(x) = f(x)$ for $x \in \mathbb{Z}^n$, (3.6) implies

$$\tilde{f}(x) = \check{f}(x) = \bar{f}(x) = f(x) \quad (x \in \mathbb{Z}^n). \tag{3.7}$$

Proposition 3.1 (1) *The effective domain of an integrally convex function is integrally convex.*

(2) *A set $S \subseteq \mathbb{Z}^n$ is integrally convex if and only if its indicator function δ_S is integrally convex.*

The following is an example of a convex-extensible function that is not integrally convex.

Example 3.3 Let $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$ be defined by $f(x_1, x_2) = |2x_1 - x_2|$ for all $(x_1, x_2) \in \mathbb{Z}^2$. Obviously, this function is convex-extensible and the convex envelope is given by $\bar{f}(x_1, x_2) = |2x_1 - x_2|$ for all $(x_1, x_2) \in \mathbb{R}^2$. For $y = (1/2, 1)$ we have $N(y) = \{(0, 1), (1, 1)\}$ and the local convex extension \tilde{f} of f around y is given by

$$\tilde{f}(1/2, 1) = (f(0, 1) + f(1, 1))/2 = (1 + 1)/2 = 1.$$

On the other hand, $y = (1/2, 1)$ is the midpoint of $u = (0, 0)$ and $v = (1, 2)$ with $\tilde{f}(u) = f(0, 0) = 0$ and $\tilde{f}(v) = f(1, 2) = 0$. This shows that the function \tilde{f} is not convex, and f is not integrally convex. Also note that $0 = \bar{f}(1/2, 1) \neq \tilde{f}(1/2, 1) = 1$. ■

Integrally convex functions in two variables ($n = 2$) can be defined by simple inequality conditions without referring to the local convex extension \tilde{f} (see Theorem 3.3 in Sect. 3.4).

Remark 3.4 The concept of integrally convex functions is introduced in [13] for functions defined on a box of integers. The extension to functions with general integrally convex effective domains is straightforward, which is found in [40]. ■

3.3 Examples

Three classes of integrally convex functions are given below.

Example 3.4 A function $\Phi : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ in $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ is called *separable convex* if it can be represented as

$$\Phi(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n) \quad (3.8)$$

with univariate discrete convex functions $\varphi_i : \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$, which means, by definition, that $\text{dom } \varphi_i$ is an interval of integers and

$$\varphi_i(k-1) + \varphi_i(k+1) \geq 2\varphi_i(k) \quad (k \in \mathbb{Z}). \quad (3.9)$$

A separable convex function is integrally convex (actually, both L^{\natural} - and M^{\natural} -convex). ■

Example 3.5 A symmetric matrix $Q = (q_{ij})$ that satisfies the condition

$$q_{ii} \geq \sum_{j \neq i} |q_{ij}| \quad (i = 1, 2, \dots, n) \quad (3.10)$$

is called a diagonally dominant matrix (with nonnegative diagonals). If Q is diagonally dominant in the sense of (3.10), then $f(x) = x^{\top} Qx$ is integrally convex [13, Proposition 4.5]. The converse is also true if $n \leq 2$ [13, Remark 4.3]. Recently it has been shown in [64, Theorem 9] that the diagonally dominance (3.10) of Q is equivalent to the directed discrete midpoint convexity of $f(x) = x^{\top} Qx$; see [64] for details. ■

Example 3.6 A function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *2-separable convex* if it can be expressed as the sum of univariate convex, diff-convex, and sum-convex functions, i.e., if

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \varphi_i(x_i) + \sum_{i \neq j} \varphi_{ij}(x_i - x_j) + \sum_{i \neq j} \psi_{ij}(x_i + x_j),$$

where $\varphi_i, \varphi_{ij}, \psi_{ij} : \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ ($i, j = 1, 2, \dots, n; i \neq j$) are univariate convex functions. A 2-separable convex function is known to be integrally convex [64, Theorem 4], whereas it is L^{\natural} -convex if $\psi_{ij} \equiv 0$ for all (i, j) with $i \neq j$. A quadratic function $f(x) = x^{\top} Qx$ with Q satisfying (3.10) is an example of a 2-separable convex function. ■

In addition to the above, almost all kinds of discrete convex functions treated in discrete convex analysis are integrally convex. It is known that separable convex, L -convex, L^{\natural} -convex, M -convex, M^{\natural} -convex, L_2^{\natural} -convex, and M_2^{\natural} -convex functions are integrally convex [40]. Multimodular functions [23] are also integrally convex [42]. Moreover, BS-convex and UJ-convex functions [18] are integrally convex.

3.4 Characterizations

In this section we give two characterizations of integrally convex functions in terms of an inequality of the form

$$\tilde{f}\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)), \quad (3.11)$$

where \tilde{f} denotes the local convex extension of f defined by (3.4). By the definition of \tilde{f} , the inequality (3.11) above is true for (x, y) with $\|x - y\|_\infty \leq 1$ for any function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$. If f is integrally convex, the inequality (3.11) holds for any (x, y) , as follows.

Proposition 3.2 *If f is integrally convex, then (3.11) holds for every $x, y \in \text{dom } f$.*

Proof The function \tilde{f} is convex by integral convexity of f , and hence

$$\tilde{f}\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(\tilde{f}(x) + \tilde{f}(y)) = \frac{1}{2}(f(x) + f(y)),$$

where the equalities $\tilde{f}(x) = f(x)$ and $\tilde{f}(y) = f(y)$ by (3.7) are used. \square

Integral convexity of a function can be characterized by a local condition under the assumption that the effective domain is an integrally convex set.

Theorem 3.1 ([13, 37]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function with an integrally convex effective domain. Then the following properties are equivalent.*

- (a) f is integrally convex.
- (b) Inequality (3.11) holds for every $x, y \in \text{dom } f$ with $\|x - y\|_\infty = 2$.

Proof [(a) \Rightarrow (b)]: This is shown in Proposition 3.2.

[(b) \Rightarrow (a)]: (The proof given in [37, Appendix A] is sketched here.) For an integer vector $a \in \mathbb{Z}^n$, define a box $B \subseteq \mathbb{R}^n$ of size two by

$$B = [a, a + 2\mathbf{1}]_{\mathbb{R}} = \{x \in \mathbb{R}^n \mid a_i \leq x_i \leq a_i + 2 \ (i = 1, 2, \dots, n)\}. \quad (3.12)$$

It can be shown ([37, Lemma A.1]) that, if $\text{dom } f$ is integrally convex and the condition (b) is satisfied, then \tilde{f} is convex on $B \cap \overline{\text{dom } f}$.

Fix arbitrary $x, y \in \overline{\text{dom } f}$, and denote by L the (closed) line segment connecting x and y . We show that \tilde{f} is convex on L . Consider the boxes B of the form of (3.12) that intersect L . There exists a finite number of such boxes, say, B_1, B_2, \dots, B_m , and L is covered by the line segments $L_j = L \cap B_j$ ($j = 1, 2, \dots, m$). That is, $L = \bigcup_{j=1}^m L_j$. For each point $z \in L \setminus \{x, y\}$, there exists some L_j that contains z in its interior, and \tilde{f} is convex on L_j by the above-mentioned fact. Hence \tilde{f} is convex on L (cf. [66, Lemma 2]). This implies the convexity of \tilde{f} , that is, the integral convexity of f . \square

The second characterization of integral convexity of a function is free from the assumption on the effective domain, but is not a local condition as it refers to all pairs (x, y) with $\|x - y\|_\infty \geq 2$.

Theorem 3.2 ([38, Theorem A.1]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function with $\text{dom } f \neq \emptyset$. Then the following properties are equivalent.*

- (a) *f is integrally convex.*
- (b) *Inequality (3.11) holds for every $x, y \in \text{dom } f$ with $\|x - y\|_\infty \geq 2$.*

Proof [(a) \Rightarrow (b)]: This is shown in Proposition 3.2.

[(b) \Rightarrow (a)]: By Theorem 3.1, it suffices to show that $\text{dom } f$ is an integrally convex set, which follows from Theorem 2.1 applied to $S = \text{dom } f$. Note that the condition (2.11) in Theorem 2.1 holds by the assumption (b). □

Remark 3.5 Theorem 3.1 originates in [13, Proposition 3.3], which shows the equivalence of (a) and (b) when the effective domain is a box of integers, while their equivalence for a general integrally convex effective domain is proved in [37, Appendix A]. Theorem 3.2 is given in [38, Theorem A.1] with a direct proof without using Theorem 3.1, while here we have given an alternative proof that relies on Theorem 3.1 via Theorem 2.1. ■

Integrally convex functions in two variables ($n = 2$) can be characterized by simple inequality conditions as follows. We use notation $f_z(x) := f(z + x)$.

Theorem 3.3 *A function $f : \mathbb{Z}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ is integrally convex if and only if its effective domain is an integrally convex set and the following five inequalities*

$$\begin{aligned}
 g(0, 0) + g(2, 1) &\geq g(1, 1) + g(1, 0), \\
 g(0, 0) + g(2, -1) &\geq g(1, -1) + g(1, 0), \\
 g(0, 0) + g(2, 0) &\geq 2g(1, 0), \\
 g(0, 0) + g(2, 2) &\geq 2g(1, 1), \\
 g(0, 0) + g(2, -2) &\geq 2g(1, -1)
 \end{aligned}$$

are satisfied by both $g(x_1, x_2) = f_z(x_1, x_2)$ and $g(x_1, x_2) = f_z(x_2, x_1)$ for any $z \in \text{dom } f$.

Proof This follows immediately from Theorem 3.1, since when $n = 2$ and $\|x - y\|_\infty = 2$, (3.11) is equivalent to

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right). \tag{3.13}$$

For example, if $x = z$ and $y = z + (2, 1)$, then $\lceil(x + y)/2\rceil = z + (1, 1)$ and $\lfloor(x + y)/2\rfloor = z + (1, 0)$, and (3.13) gives the first inequality for $g(x_1, x_2) = f_z(x_1, x_2)$. □

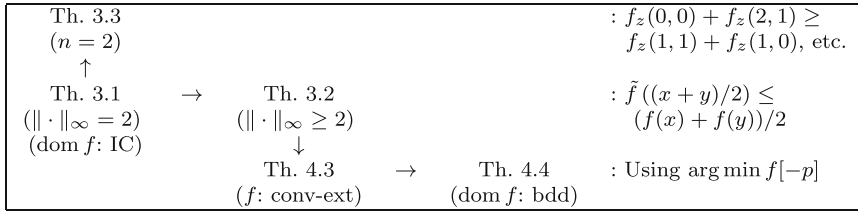


Fig. 6 Characterizations of integrally convex functions

Remark 3.6 For a function $g : \mathbb{Z}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ (in general), an inequality of the form

$$g(0, 0) + g(a + b, a) \geq g(a, a) + g(b, 0) \quad (a, b \geq 0; a, b \in \mathbb{Z}) \quad (3.14)$$

is called the (basic) *parallelogram inequality* in [37]. It is shown in [37, Proposition 3.3] that for any integrally convex function f in two variables and a point $z \in \text{dom } f$, the function $g(x) = f_z(x)$ satisfies the inequality (3.14). Note that (3.14) with $(a, b) = (1, 1)$ coincides with the first inequality in Theorem 3.3. Furthermore, the inequality (3.14) holds also for $g(x_1, x_2) = f_z(x_2, x_1)$, $f_z(x_1, -x_2)$, and $f_z(-x_2, x_1)$, as integral convexity is preserved under such coordinate inversions (cf., (3.17), (3.18)). ■

In this section we have given three theorems (Theorems 3.1, 3.2, and 3.3) to characterize integrally convex functions. In Sect. 4.2 we give two additional theorems (Theorems 4.3 and 4.4). Their logical dependence (in our presentation) is illustrated in Fig. 6.

3.5 Simplicial divisions

As is well known ([17, Section 16.3], [40, Section 7.7]), the convex extension of an L^1 -convex function can be constructed in a systematic manner using a regular simplicial division (the Freudenthal simplicial division) of unit hypercubes. This is a generalization of the Lovász extension for a submodular set function. In addition, the concepts of BS-convex and UJ-convex functions are introduced on the basis of other regular simplicial divisions in [18].

By definition, an integrally convex function f is convex-extensible, and its convex envelope \bar{f} can be constructed locally within each unit hypercube, since \bar{f} coincides with the local convex extension \tilde{f} . However, general integrally convex functions are not associated with a regular simplicial division. Indeed, the following construction shows that, when $n = 2$, an arbitrary triangulation can arise from an integrally convex function.

Consider the rectangular domain $[0, a]_{\mathbb{R}}$, where $a = (a_1, a_2)$ with positive integers a_1 and a_2 , and assume that we are given an arbitrary triangulation of each unit square in the domain $[0, a]_{\mathbb{R}}$ such as the one in Fig. 7(a). We can construct an integrally convex function f such that the convex envelope \bar{f} corresponds to the given triangulation.

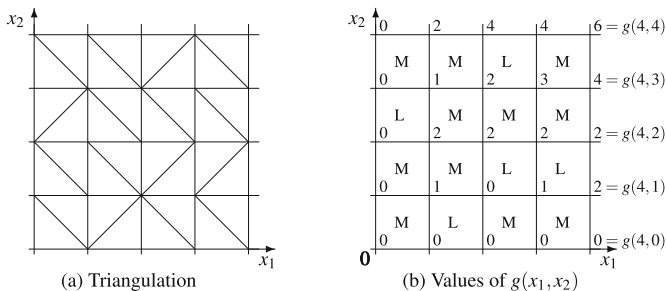


Fig. 7 A given triangulation and the corresponding function g

According to the given triangulation, we classify the unit squares into two types, type M and type L. We say that a unit square is of type M (resp., type L) if it has a diagonal line segment on $x_1 + x_2 = c$ (resp., $x_1 - x_2 = c$) for some c ; see Fig. 7(b). For each $x = (x_1, x_2) \in [0, a]_{\mathbb{Z}}$, we denote the number of unit squares of type M (resp., type L) contained in the domain $[0, x]_{\mathbb{R}}$ by $g(x; M)$ (resp., $g(x; L)$), and define $g(x) := g(x; M) - g(x; L)$. For $x = (2, 2)$ in Fig. 7(b), for example, we have $g(x; M) = 3$ and $g(x; L) = 1$, and therefore $g(2, 2) = g(x; M) - g(x; L) = 3 - 1 = 2$. Finally, we define a function f on $[0, a]_{\mathbb{Z}}$ by

$$f(x_1, x_2) = A(x_1^2 + x_2^2) + g(x_1, x_2) \quad (x \in [0, a]_{\mathbb{Z}}) \tag{3.15}$$

with a positive constant A . If $A \geq a_1 + a_2$, this function f is integrally convex and the associated triangulation of each unit square coincides with the given one (proved in Remark 3.7). It is noted that, while f is integrally convex, g itself may not be integrally convex. For example, in Fig. 7(b), we have

$$\begin{aligned} [f(0, 0) + f(2, 1)] - [f(1, 0) + f(1, 1)] &= 2A - 1 > 0, \\ [g(0, 0) + g(2, 1)] - [g(1, 0) + g(1, 1)] &= -1 < 0 \end{aligned}$$

(cf., Theorem 3.3).

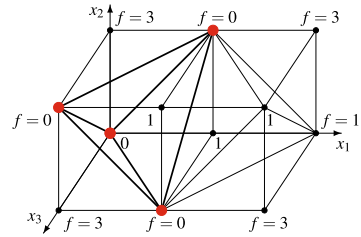
Remark 3.7 First, we prove the integral convexity of f in (3.15) by showing that

$$f(x) + f(y) - f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) - f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) \geq 0 \tag{3.16}$$

holds for every $x, y \in [0, a]_{\mathbb{Z}}$ with $\|x - y\|_{\infty} = 2$. By symmetry between x and y and that between coordinate axes, we have five cases to consider (cf., Theorem 3.3): (i) $y = x + (2, 1)$, (ii) $y = x + (2, -1)$, (iii) $y = x + (2, 0)$, (iv) $y = x + (2, 2)$, and (v) $y = x + (2, -2)$. Let $h(x) := x_1^2 + x_2^2$, for which we have

$$h(x) + h(y) - h\left(\left\lceil \frac{x+y}{2} \right\rceil\right) - h\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) = \begin{cases} 2 & \text{(case (i), (ii), (iii))}, \\ 4 & \text{(case (iv), (v))}. \end{cases}$$

Fig. 8 Simplicial division associated with an integrally convex function



On the other hand, we have

$$\left| g(x) + g(y) - g\left(\left\lceil \frac{x+y}{2} \right\rceil\right) - g\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) \right| \leq 2(a_1 + a_2)$$

in either case. Therefore, if $A \geq a_1 + a_2$, the inequality in (3.16) holds.

Next, we observe that the function $f = Ah + g$ induces a triangulation of the specified type within each unit square. Consider a square $[x, y]_{\mathbb{R}}$ with $y = x + \mathbf{1}$ and $x, y \in [0, a]_{\mathbb{Z}}$. Let $u := (x_1 + 1, x_2)$ and $v := (x_1, x_2 + 1)$. Then $h(x) + h(y) - h(u) - h(v) = 0$, while $g(x) + g(y) - g(u) - g(v)$ is equal to $+1$ and -1 according to whether the square $[x, y]_{\mathbb{R}}$ is of type M or L. This shows that the triangulation of $[x, y]_{\mathbb{R}}$ induced by f coincides with the given one. ■

Next we give an example of a simplicial division associated with an integrally convex function in three variables.

Example 3.7 Consider $S = [(0, 0, 0), (2, 1, 1)]_{\mathbb{Z}} = \{x \in \mathbb{Z}^3 \mid 0 \leq x_1 \leq 2, 0 \leq x_i \leq 1 \ (i = 2, 3)\}$ (see Fig. 8), and define $f : \mathbb{Z}^3 \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\text{dom } f = S$ by

$$(f(x_1, 0, 0))_{x_1=0,1,2} = (0, 1, 1), \quad (f(x_1, 1, 0))_{x_1=0,1,2} = (3, 0, 3),$$

and $f(x_1, 0, 0) = f(x_1, 1, 1), f(x_1, 1, 0) = f(x_1, 0, 1)$ for $x_1 = 0, 1, 2$. By Theorem 3.1 we can verify that f is integrally convex. For example, for $x = (0, 0, 0)$ and $y = (2, 1, 1)$, we have

$$\tilde{f}\left(\frac{x+y}{2}\right) = \tilde{f}\left(1, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(f(1, 1, 0) + f(1, 0, 1)) = 0 \leq \frac{1}{2}(f(x) + f(y)) = \frac{1}{2}$$

in (3.11). The simplicial division of $\bar{S} = [(0, 0, 0), (2, 1, 1)]_{\mathbb{R}}$ for the convex extension of f is symmetric with respect to the plane $x_1 = 1$. The left cube $[(0, 0, 0), (1, 1, 1)]_{\mathbb{R}}$ is decomposed into five simplices; one of them has vertices at $(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)$ and has volume $1/3$ (drawn in bold line), whereas the other four simplices are congruent to the standard simplex, having volume $1/6$. The right cube $[(1, 0, 0), (2, 1, 1)]_{\mathbb{R}}$ is decomposed similarly into five simplices; one of them has vertices at $(2, 0, 0), (1, 1, 0), (1, 0, 1), (2, 1, 1)$, and has volume $1/3$, whereas the other four simplices are congruent to the standard simplex, having volume $1/6$. Thus the simplices are not uniform in volume, whereas they have the same volume ($= 1/6$) for an L^{\natural}_2 -convex function. It is added that this function f is neither L^{\natural}_2 -convex nor

M_2^{\natural} -convex, since $\arg \min f = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is neither L_2^{\natural} -convex nor M_2^{\natural} -convex (as discussed in Example 2.6). \blacksquare

3.6 Basic operations

In this section we show how integral convexity of a function behaves under basic operations. Let f be a function on \mathbb{Z}^n , i.e., $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$.

Origin shift:

For an integer vector $b \in \mathbb{Z}^n$, the *origin shift* of f by b means a function g on \mathbb{Z}^n defined by $g(y) = f(y - b)$. The origin shift of an integrally convex function is an integrally convex function.

Inversion of coordinates:

The *independent coordinate inversion* of f means a function g on \mathbb{Z}^n defined by

$$g(y_1, y_2, \dots, y_n) = f(\tau_1 y_1, \tau_2 y_2, \dots, \tau_n y_n) \quad (3.17)$$

with an arbitrary choice of $\tau_i \in \{+1, -1\}$ ($i = 1, 2, \dots, n$). The independent coordinate inversion of an integrally convex function is an integrally convex function. This is a nice property of integral convexity, not shared by L^{\natural} -, L_2^{\natural} -, M^{\natural} -, or M_2^{\natural} -convexity.

Permutation of coordinates:

For a permutation σ of $(1, 2, \dots, n)$, the *permutation* of f by σ means a function g on \mathbb{Z}^n defined by

$$g(y_1, y_2, \dots, y_n) = f(y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)}). \quad (3.18)$$

The permutation of an integrally convex function is integrally convex.

Variable-scaling:

For a positive integer α , the *variable-scaling* (or *scaling* for short) of f by α means a function g on \mathbb{Z}^n defined by

$$g(y_1, y_2, \dots, y_n) = f(\alpha y_1, \alpha y_2, \dots, \alpha y_n). \quad (3.19)$$

Note that the same scaling factor α is used for all coordinates. If $\alpha = 2$, for example, this operation amounts to considering the function values at even points. The scaling operation is used effectively in minimization algorithms (see Sect. 4). The scaling of an integrally convex function is not necessarily integrally convex. The indicator function $f = \delta_S$ of the integrally convex set $S \subseteq \mathbb{Z}^3$ in Example 2.8 is such an example. Another example of a function on the integer box $[(0, 0, 0), (4, 2, 2)]_{\mathbb{Z}}$ can be found in [37, Example 3.1]. In the case of $n = 2$, integral convexity admits the scaling operation. That is, if $f : \mathbb{Z}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ is integrally convex, then $g : \mathbb{Z}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ is integrally convex ([37, Theorem 3.2]).

As another kind of scaling, the dilation operation can be defined for a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ if it is convex-extensible. For any positive integer α , the

α -dilation of f is defined as a function g on \mathbb{Z}^n given by

$$g(y_1, y_2, \dots, y_n) = \overline{f}(y_1/\alpha, y_2/\alpha, \dots, y_n/\alpha), \quad (3.20)$$

where \overline{f} denotes the convex envelope of f . The dilation of an integrally convex function is not necessarily integrally convex. For example, the indicator function $f = \delta_S$ of the integrally convex set $S \subseteq \mathbb{Z}^4$ in Example 2.5 is an integrally convex function for which the 2-dilation is not integrally convex.

Remark 3.8 Although integral convexity is not compatible with the dilation operation, other kinds of discrete convexity such as L-, L^{\natural} -, L_2 -, L_2^{\natural} -, M-, M^{\natural} -, M_2 -, M_2^{\natural} -convexity, and multimodularity are preserved under dilation. In contrast, the scaling operation in (3.19) preserves L-convexity and its relatives (box, L-, L^{\natural} -, L_2 -, L_2^{\natural} -convexity, and multimodularity), and not M-convexity and its relatives (M -, M^{\natural} -, M_2 -, M_2^{\natural} -convexity). ■

Value-scaling:

For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a nonnegative factor $a \geq 0$, the *value-scaling* of f by a means a function $g : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by $g(y) = af(y)$ for $y \in \mathbb{Z}^n$. We may also introduce an additive constant $b \in \mathbb{R}$ and a linear function $\langle c, y \rangle = \sum_{i=1}^n c_i y_i$, where $c \in \mathbb{R}^n$, to obtain

$$g(y) = af(y) + b + \langle c, y \rangle \quad (y \in \mathbb{Z}^n). \quad (3.21)$$

The operation (3.21) preserves integral convexity of a function.

Restriction:

For a function $f : \mathbb{Z}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ and a subset U of the index set $N = \{1, 2, \dots, n\}$, the *restriction* of f to U is a function $g : \mathbb{Z}^U \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$g(y) = f(y, \mathbf{0}_{N \setminus U}) \quad (y \in \mathbb{Z}^U),$$

where $\mathbf{0}_{N \setminus U}$ denotes the zero vector in $\mathbb{Z}^{N \setminus U}$. The notation $(y, \mathbf{0}_{N \setminus U})$ means the vector whose i th component is equal to y_i for $i \in U$ and to 0 for $i \in N \setminus U$. The restriction of an integrally convex function is integrally convex (if the effective domain of the resulting function is nonempty).

Projection:

For a function $f : \mathbb{Z}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ and a subset U of the index set $N = \{1, 2, \dots, n\}$, the *projection* of f to U is a function $g : \mathbb{Z}^U \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g(y) = \inf\{f(y, z) \mid z \in \mathbb{Z}^{N \setminus U}\} \quad (y \in \mathbb{Z}^U), \quad (3.22)$$

where the notation (y, z) means the vector whose i th component is equal to y_i for $i \in U$ and to z_i for $i \in N \setminus U$. The projection is also called *partial minimization*. The resulting function g is referred to as the *marginal function* of f in [24]. The projection

of an integrally convex function is integrally convex ([33, Theorem 3.1]) if $g > -\infty$, or else we have $g \equiv -\infty$ (see Proposition 3.4 in Sect. 3.7).

Splitting:

Suppose that we are given a family $\{U_1, U_2, \dots, U_n\}$ of disjoint nonempty sets indexed by $N = \{1, 2, \dots, n\}$. Let $m_i = |U_i|$ for $i = 1, 2, \dots, n$ and define $m = \sum_{i=1}^n m_i$, where $m \geq n$. For each $i \in N$ we define an m_i -dimensional vector $y_{[i]} = (y_j \mid j \in U_i)$ and express $y \in \mathbb{Z}^m$ as $y = (y_{[1]}, y_{[2]}, \dots, y_{[n]})$. For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the *splitting* of f by $\{U_1, U_2, \dots, U_n\}$ is defined as a function $g : \mathbb{Z}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$g(y_{[1]}, y_{[2]}, \dots, y_{[n]}) = f(y_{[1]}(U_1), y_{[2]}(U_2), \dots, y_{[n]}(U_n)),$$

where $y_{[i]}(U_i) = \sum\{y_j \mid j \in U_i\}$. For example, $g(y_1, y_2, y_3) = f(y_1, y_2 + y_3)$ is a splitting of $f : \mathbb{Z}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ for $U_1 = \{1\}$ and $U_2 = \{2, 3\}$, where $n = 2$ and $m = 3$. The splitting of an integrally convex function is integrally convex ([46, Proposition 4.4]).

Aggregation:

Let $\mathcal{P} = \{N_1, N_2, \dots, N_m\}$ be a partition of $N = \{1, 2, \dots, n\}$ into disjoint nonempty subsets: $N = N_1 \cup N_2 \cup \dots \cup N_m$ and $N_i \cap N_j = \emptyset$ for $i \neq j$. We have $m \leq n$. For a function $f : \mathbb{Z}^N \rightarrow \mathbb{R} \cup \{+\infty\}$, the *aggregation* of f with respect to \mathcal{P} is defined as a function $g : \mathbb{Z}^m \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ given by

$$g(y_1, y_2, \dots, y_m) = \inf\{f(x) \mid x(N_j) = y_j \ (j = 1, 2, \dots, m)\}.$$

For example, $g(y_1, y_2) = \inf\{f(x_1, x_2, x_3) \mid x_1 = y_1, x_2 + x_3 = y_2\}$ is an aggregation of $f : \mathbb{Z}^3 \rightarrow \mathbb{R} \cup \{+\infty\}$ for $N_1 = \{1\}$ and $N_2 = \{2, 3\}$, where $n = 3$ and $m = 2$. The aggregation of an integrally convex function is not necessarily integrally convex (Example 2.9).

Direct sum:

The *direct sum* of two functions $f_1 : \mathbb{Z}^{n_1} \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{Z}^{n_2} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a function $f_1 \oplus f_2 : \mathbb{Z}^{n_1+n_2} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$(f_1 \oplus f_2)(x, y) = f_1(x) + f_2(y) \quad (x \in \mathbb{Z}^{n_1}, y \in \mathbb{Z}^{n_2}).$$

The direct sum of two integrally convex functions is integrally convex.

Addition:

The *sum* of two functions $f_1, f_2 : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad (x \in \mathbb{Z}^n). \tag{3.23}$$

For two sets $S_1, S_2 \subseteq \mathbb{Z}^n$, the sum of their indicator functions δ_{S_1} and δ_{S_2} coincides with the indicator function of their intersection $S_1 \cap S_2$, that is, $\delta_{S_1} + \delta_{S_2} = \delta_{S_1 \cap S_2}$. The sum of integrally convex functions is not necessarily integrally convex (Example 2.10).

However, the sum of an integrally convex function with a separable convex function

$$g(x) = f(x) + \sum_{i=1}^n \varphi_i(x_i) \quad (x \in \mathbb{Z}^n) \tag{3.24}$$

is integrally convex.

Convolution:

The (infimal) *convolution* of two functions $f_1, f_2 : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$(f_1 \square f_2)(x) = \inf\{f_1(y) + f_2(z) \mid x = y + z, y, z \in \mathbb{Z}^n\} \quad (x \in \mathbb{Z}^n), \tag{3.25}$$

where it is assumed that the infimum is bounded from below (i.e., $(f_1 \square f_2)(x) > -\infty$ for every $x \in \mathbb{Z}^n$). For two sets $S_1, S_2 \subseteq \mathbb{Z}^n$, the convolution of their indicator functions δ_{S_1} and δ_{S_2} coincides with the indicator function of their Minkowski sum $S_1 + S_2 = \{y + z \mid y \in S_1, z \in S_2\}$, that is, $\delta_{S_1} \square \delta_{S_2} = \delta_{S_1 + S_2}$. The convolution of integrally convex functions is not necessarily integrally convex (Example 2.11). The convolution of an integrally convex function and a separable convex function is integrally convex [33, Theorem 4.2] (also [45, Proposition 4.17]).

Remark 3.9 The convolution operation plays a central role in discrete convex analysis. The convolution of two (or more) M^\natural -convex functions is M^\natural -convex. The convolution of two L^\natural -convex functions is not necessarily L^\natural -convex, but it is integrally convex. The convolution of three L^\natural -convex functions is no longer integrally convex (Remark 2.6). ■

3.7 Technical supplement

This section is a technical supplement concerning the projection operation defined in (3.22). We first consider a direction d in which the function value diverges to $-\infty$.

Proposition 3.3 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function, $y \in \text{dom } f$, and $d \in \mathbb{Z}^n$. If $\lim_{k \rightarrow \infty} f(y + kd) = -\infty$, then for any $z \in \text{dom } f$, we have*

$$\lim_{k \rightarrow \infty} f(z + kd) = -\infty.$$

Proof Let $S = \text{dom } f$. For each $x \in S$, $g_x(k) = f(x + kd)$ is a convex function in $k \in \mathbb{Z}$, which follows from the convex-extensibility of f . Let T denote the set of $x \in S$ for which $\lim_{k \rightarrow \infty} g_x(k) = -\infty$. We want to show that $T = \emptyset$ or $T = S$. To prove this by contradiction, assume that both T and $S \setminus T$ are nonempty. Consider $y \in T$ and $z \in S \setminus T$ that minimize $C\|y - z\|_\infty + \|y - z\|_1$, where $C \gg 1$. From integral convexity of S , we can easily show that $\|y - z\|_\infty = 1$ and $S \cap N\left(\frac{y+z}{2}\right) = \{y, z\}$. Moreover, for $y^{(k)} = y + kd$ and $z^{(k)} = z + kd$, we have

$$S \cap N\left(\frac{y^{(k)} + z^{(k)}}{2}\right) = \{y^{(k)}, z^{(k)}\} \quad (k = 0, 1, 2, \dots). \tag{3.26}$$

(Proof of (3.26): Since g_y is convex, $y \in S$, and $\lim_{k \rightarrow \infty} g_y(k) = -\infty$, we have $y^{(k)} = y + kd \in S$ for all k . This implies, by Proposition 2.6, that $z^{(k)} = z + kd \in S$ for all k . We also have $C\|y^{(k)} - z^{(k)}\|_\infty + \|y^{(k)} - z^{(k)}\|_1 = C\|y - z\|_\infty + \|y - z\|_1$.) Since f is integrally convex, we have

$$\tilde{f}\left(\frac{y^{(k+1)} + z^{(k-1)}}{2}\right) \leq \frac{1}{2}(f(y^{(k+1)}) + f(z^{(k-1)}))$$

by Proposition 3.2, where the left-hand side can be expressed as

$$\tilde{f}\left(\frac{y^{(k+1)} + z^{(k-1)}}{2}\right) = \tilde{f}\left(\frac{y^{(k)} + z^{(k)}}{2}\right) = \frac{1}{2}(f(y^{(k)}) + f(z^{(k)}))$$

by $y^{(k+1)} + z^{(k-1)} = y^{(k)} + z^{(k)}$ and (3.26). Therefore, we have

$$f(y^{(k)}) + f(z^{(k)}) \leq f(y^{(k+1)}) + f(z^{(k-1)}).$$

By adding these inequalities for $k = 1, 2, \dots, \hat{k}$, we obtain

$$f(z^{(\hat{k})}) \leq f(y^{(\hat{k}+1)}) + f(z) - f(y^{(1)}).$$

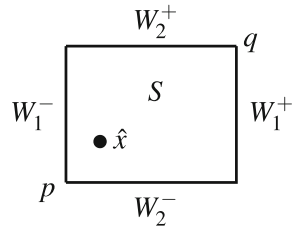
By letting $\hat{k} \rightarrow \infty$ we obtain a contradiction, since the right-hand side tends to $-\infty$ while the left-hand side does not. □

Using the above proposition we can show that the projection $g(y) = \inf\{f(y, z) \mid z \in \mathbb{Z}^{N \setminus U}\}$, defined in (3.22), is away from the value of $-\infty$ unless it is identically equal to $-\infty$.

Proposition 3.4 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function, and g be the projection of f to U . If $g(y^0) = -\infty$ for some $y^0 \in \mathbb{Z}^U$, then $g(y) = -\infty$ for all $y \in \mathbb{Z}^U$.*

Proof First suppose $|N \setminus U| = 1$ with $N \setminus U = \{v\}$, and assume $g(y^0) = -\infty$ for some $y^0 \in \mathbb{Z}^U$. Then $x^0 := (y^0, z^0) \in \text{dom } f$ for some $z^0 \in \mathbb{Z}$, and $\lim_{k \rightarrow \infty} f(x^0 + kd) = -\infty$ for $d = \mathbf{1}^v$ or $d = -\mathbf{1}^v$. This implies, by Proposition 3.3, that $\lim_{k \rightarrow \infty} f(x + kd) = -\infty$ for all $x \in \text{dom } f$, which shows $g \equiv -\infty$. Next we consider the case where $|N \setminus U| \geq 2$. Let $N \setminus U = \{v_1, v_2, \dots, v_r\}$ with $2 \leq r < n$, and $g^{(k)}$ denote the projection of f to $U \cup \{v_{k+1}, v_{k+2}, \dots, v_r\}$ for $k = 0, 1, \dots, r$. Then $g^{(0)} = f$, $g^{(r)} = g$, and $g^{(k)}$ is the projection of $g^{(k-1)}$ for $k = 1, 2, \dots, r$. By the argument for the case with $|N \setminus U| = 1$, we have $g^{(1)} > -\infty$ or $g^{(1)} \equiv -\infty$. In the latter case we obtain $g^{(k)} \equiv -\infty$ for $k = 1, 2, \dots, r$. In the former case, $g^{(1)}$ is an integrally convex function by [33, Theorem 3.1], as already mentioned in Sect. 3.6. This allows us to apply the same argument to $g^{(1)}$ to obtain that $g^{(2)} > -\infty$ or $g^{(2)} \equiv -\infty$. Continuing this way we arrive at the conclusion that $g > -\infty$ or $g \equiv -\infty$. □

Fig. 9 Box barrier property



4 Minimization and minimizers

4.1 Optimality conditions

The global minimum of an integrally convex function can be characterized by a local condition.

Theorem 4.1 ([13, Proposition 3.1]; see also [40, Theorem 3.21]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function and $x^* \in \text{dom } f$. Then x^* is a minimizer of f if and only if $f(x^*) \leq f(x^* + d)$ for all $d \in \{-1, 0, +1\}^n$.*

A more general form of this local optimality criterion is known as “box-barrier property” in Theorem 4.2 below (see Fig. 9). A special case of Theorem 4.2 with $\hat{x} = x^*$, $p = x^* - \mathbf{1}$, and $q = x^* + \mathbf{1}$ coincides with Theorem 4.1 above.

Theorem 4.2 (Box-barrier property [37, Theorem 2.6]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function, and $p \in (\mathbb{Z} \cup \{-\infty\})^n$ and $q \in (\mathbb{Z} \cup \{+\infty\})^n$, where $p \leq q$. Define*

$$\begin{aligned}
 S &= \{x \in \mathbb{Z}^n \mid p_i < x_i < q_i \ (i = 1, 2, \dots, n)\}, \\
 W_i^+ &= \{x \in \mathbb{Z}^n \mid x_i = q_i, \ p_j \leq x_j \leq q_j \ (j \neq i)\} \ (i = 1, 2, \dots, n), \\
 W_i^- &= \{x \in \mathbb{Z}^n \mid x_i = p_i, \ p_j \leq x_j \leq q_j \ (j \neq i)\} \ (i = 1, 2, \dots, n),
 \end{aligned}$$

and $W = \bigcup_{i=1}^n (W_i^+ \cup W_i^-)$. Let $\hat{x} \in S \cap \text{dom } f$. If $f(\hat{x}) \leq f(y)$ for all $y \in W$, then $f(\hat{x}) \leq f(z)$ for all $z \in \mathbb{Z}^n \setminus S$.

Proof (The proof of [37] is described here.) Let U_i^+ and U_i^- denote the convex hulls of W_i^+ and W_i^- , respectively, and define $U = \bigcup_{i=1}^n (U_i^+ \cup U_i^-)$. Then $W = U \cap \mathbb{Z}^n$. For a point $z \in \mathbb{Z}^n \setminus S$, the line segment connecting \hat{x} and z intersects U at a point, say, $u \in \mathbb{R}^n$. Then its integral neighborhood $N(u)$ is contained in W . Since the local convex extension $\tilde{f}(u)$ is a convex combination of the $f(y)$ ’s with $y \in N(u)$, and $f(y) \geq f(\hat{x})$ for every $y \in W$, we have $\tilde{f}(u) \geq f(\hat{x})$. On the other hand, it follows from the convexity of f that $\tilde{f}(u) \leq (1 - \lambda)f(\hat{x}) + \lambda f(z)$ for some λ with $0 < \lambda \leq 1$. Hence $f(\hat{x}) \leq \tilde{f}(u) \leq (1 - \lambda)f(\hat{x}) + \lambda f(z)$, and therefore, $f(\hat{x}) \leq f(z)$. \square

Remark 4.1 The optimality criterion in Theorem 4.1 is certainly local, but not satisfactory from the computational complexity viewpoint. We need $O(3^n)$ function evaluations to verify the local optimality condition. \blacksquare

4.2 Minimizer sets

It is (almost) always the case that if a function is equipped with some kind of discrete convexity, then the set of its minimizers is equipped with the discrete convexity of the same kind. This is indeed the case with an integrally convex function.

Proposition 4.1 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function. If it attains a (finite) minimum, the set of its minimizers is integrally convex.*

Proof Let α denote the minimum value of f , and $S := \arg \min f$. Take any $x \in \overline{S}$. We have $\alpha = \overline{f}(x) = \tilde{f}(x)$. By the definition of \tilde{f} in (3.4), this implies $x \in \overline{S \cap N(x)}$. Thus (2.8) holds, showing the integral convexity of S . □

In the following we discuss how integral convexity of a function can be characterized in terms of the integral convexity of the minimizer sets. For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a vector $p \in \mathbb{R}^n$, $f[-p]$ will denote the function defined by

$$f[-p](x) = f(x) - \langle p, x \rangle = f(x) - \sum_{i=1}^n p_i x_i \quad (x \in \mathbb{Z}^n). \tag{4.1}$$

We use notation

$$\arg \min f[-p] = \{x \in \mathbb{Z}^n \mid f[-p](x) \leq f[-p](y) \text{ for all } y \in \mathbb{Z}^n\} \tag{4.2}$$

for the set of the minimizers of $f[-p]$.

Theorem 4.3 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and assume that the convex envelope \overline{f} is a polyhedral convex function and*

$$\overline{f}(x) = f(x) \quad (x \in \mathbb{Z}^n). \tag{4.3}$$

Then f is integrally convex if and only if $\arg \min f[-p]$ is an integrally convex set for each $p \in \mathbb{R}^n$ for which $f[-p]$ attains a (finite) minimum.

Proof The only-if-part is immediate from Proposition 4.1, since if f is integrally convex, so is $f[-p]$ for any p . We prove the if-part by using Theorem 3.2. Take any $x, y \in \text{dom } f$ with $\|x - y\|_\infty \geq 2$, and let $u := (x + y)/2$. Our goal is to show

$$\tilde{f}(u) \leq \frac{1}{2}(f(x) + f(y)) \tag{4.4}$$

in (3.11), where \tilde{f} is the local convex extension of f . The midpoint u belongs to the convex hull $\overline{\text{dom } f}$ of $\text{dom } f$, which implies that $\overline{f}(u)$ is finite and $u \in \arg \min \overline{f}[-p]$ for some p . Let $S_p := \arg \min f[-p]$ for this p , where $S_p \subseteq \mathbb{Z}^n$. Since $\arg \min \overline{f}[-p] = \overline{\arg \min f[-p]} = \overline{S_p}$, we have $u \in \overline{S_p}$. By the assumed integral convexity of S_p , this implies $u \in \overline{S_p \cap N(u)}$ (see (2.8)). Therefore, there

exist $z^{(1)}, z^{(2)}, \dots, z^{(m)} \in S_p \cap N(u)$ as well as positive numbers $\lambda_1, \lambda_2, \dots, \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$ such that

$$u = \sum_{i=1}^m \lambda_i z^{(i)}, \quad \tilde{f}(u) = \sum_{i=1}^m \lambda_i f(z^{(i)}). \quad (4.5)$$

Since each $z^{(i)} (\in S_p)$ is a minimizer of $f[-p]$, we have

$$\sum_{i=1}^m \lambda_i f[-p](z^{(i)}) \leq \frac{1}{2}(f[-p](x) + f[-p](y)),$$

that is,

$$\sum_{i=1}^m \lambda_i f(z^{(i)}) - \sum_{i=1}^m \lambda_i \langle p, z^{(i)} \rangle \leq \frac{1}{2}(f(x) + f(y)) - \frac{1}{2}(\langle p, x \rangle + \langle p, y \rangle). \quad (4.6)$$

For the linear parts in this expression we have

$$\begin{aligned} \sum_{i=1}^m \lambda_i \langle p, z^{(i)} \rangle &= \langle p, \sum_{i=1}^m \lambda_i z^{(i)} \rangle = \langle p, u \rangle, \\ \frac{1}{2}(\langle p, x \rangle + \langle p, y \rangle) &= \langle p, \frac{1}{2}(x + y) \rangle = \langle p, u \rangle, \end{aligned}$$

and therefore, (4.6) is equivalent to

$$\sum_{i=1}^m \lambda_i f(z^{(i)}) \leq \frac{1}{2}(f(x) + f(y)).$$

Combining this with the expression of $\tilde{f}(u)$ in (4.5), we obtain (4.4). This completes the proof of Theorem 4.3. \square

The next theorem gives a similar characterization of integrally convex functions under a different assumption.

Theorem 4.4 ([40, Theorem 3.29]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function with a bounded nonempty effective domain. Then f is integrally convex if and only if $\arg \min f[-p]$ is an integrally convex set for each $p \in \mathbb{R}^n$.*

Proof The only-if-part is immediate from Proposition 4.1, since if f is integrally convex, so is $f[-p]$ for any p . To prove the if-part by Theorem 4.3, define $S_p := \arg \min f[-p]$, which is integrally convex by assumption. Note that S_p is nonempty for every p by the assumed boundedness of $\text{dom } f$. The boundedness of $\text{dom } f$ also implies that the convex envelope \bar{f} is a polyhedral convex function. The integral convexity of S_p implies that S_p is hole-free ($S_p = \overline{S_p} \cap \mathbb{Z}^n$), from which we obtain

$\bar{f}(x) = f(x)$ ($x \in \mathbb{Z}^n$) in (4.3). Then the integral convexity of f follows from Theorem 4.3. \square

Remark 4.2 The characterization of integral convexity of f by $\arg \min f[-p]$ originates in [40, Theorem 3.29], which is stated as Theorem 4.4 above. In this paper, we have given an alternative proof to this theorem by first establishing Theorem 4.3 that employs the assumption of convex-extensibility. See also Fig. 6 in Sect. 3.4. \blacksquare

Remark 4.3 Theorems 4.3 and 4.4 impose the assumption of convex-extensibility of f or boundedness of $\text{dom } f$. Such an assumption seems inevitable. Consider a function $f : \mathbb{Z} \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & (x = 0), \\ 1 & (x \neq 0). \end{cases}$ Then $\arg \min f[-p]$ is equal to $\{0\}$ or the empty set for each $p \in \mathbb{R}$. However, this function is not integrally convex. Note that f is not convex-extensible nor $\text{dom } f$ is bounded. \blacksquare

4.3 Proximity theorems

The proximity-scaling approach is a fundamental technique in designing efficient algorithms for discrete or combinatorial optimization. For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ in integer variables and a positive integer $\alpha \in \mathbb{Z}_{++}$, called a scaling unit, the α -scaling of f means the function f^α defined by $f^\alpha(x) = f(\alpha x)$ ($x \in \mathbb{Z}^n$) (cf., (3.19)). A proximity theorem is a result guaranteeing that a (local) minimum of the scaled function f^α is (geometrically) close to a minimizer of the original function f . More precisely, we say that $x^\alpha \in \text{dom } f$ is an α -local minimizer of f (or α -local minimal for f) if $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{-1, 0, +1\}^n$, and a proximity theorem gives a bound $B(n, \alpha)$ such that for any α -local minimizer x^α of f , there exists a minimizer x^* of f satisfying $\|x^\alpha - x^*\|_\infty \leq B(n, \alpha)$. The scaled function f^α is expected to be simpler and hence easier to minimize, whereas the quality of the obtained minimizer of f^α as an approximation to the minimizer of f is guaranteed by a proximity theorem. The proximity-scaling approach consists in applying this idea for a decreasing sequence of α , often by halving the scale unit α .

In discrete convex analysis the following proximity theorems are known for L^\natural -convex and M^\natural -convex functions.

Theorem 4.5 ([29]; [40, Theorem 7.18]) *Suppose that f is an L^\natural -convex function, $\alpha \in \mathbb{Z}_{++}$, and $x^\alpha \in \text{dom } f$. If $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{0, 1\}^n \cup \{0, -1\}^n$, then there exists a minimizer x^* of f satisfying $\|x^\alpha - x^*\|_\infty \leq n(\alpha - 1)$.*

Theorem 4.6 ([36]; [40, Theorem 6.37]) *Suppose that f is an M^\natural -convex function, $\alpha \in \mathbb{Z}_{++}$, and $x^\alpha \in \text{dom } f$. If $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{\mathbf{1}^i, -\mathbf{1}^i \mid 1 \leq i \leq n\} \cup \{\mathbf{1}^i - \mathbf{1}^j \mid i \neq j\}$, then there exists a minimizer x^* of f satisfying $\|x^\alpha - x^*\|_\infty \leq n(\alpha - 1)$.*

The proximity bounds in Theorems 4.5 and 4.6 are known to be tight [37, Examples 4.2 and 4.3]. It is noteworthy that we have the same proximity bound $B(n, \alpha) = n(\alpha - 1)$ for L^\natural -convex and M^\natural -convex functions, and that $B(n, \alpha)$ is linear in n . For integrally convex functions with $n \geq 3$, this bound $n(\alpha - 1)$ is no longer valid, which

is demonstrated by Examples 4.4 and 4.5 in [37]. More specifically, the latter example shows a quadratic lower bound $B(n, \alpha) \geq (n - 2)^2(\alpha - 1)/4$ for an integrally convex function arising from bipartite graphs.

The following is a proximity theorem for integrally convex functions.

Theorem 4.7 ([37, Theorem 5.1]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function, $\alpha \in \mathbb{Z}_{++}$, and $x^\alpha \in \text{dom } f$. If $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{-1, 0, +1\}^n$, then $\arg \min f \neq \emptyset$ and there exists $x^* \in \arg \min f$ with*

$$\|x^\alpha - x^*\|_\infty \leq \beta_n(\alpha - 1), \tag{4.7}$$

where β_n is defined by

$$\beta_1 = 1, \quad \beta_2 = 2; \quad \beta_n = \frac{n + 1}{2} \beta_{n-1} + 1 \quad (n = 3, 4, \dots). \tag{4.8}$$

The proximity bound β_n satisfies

$$\beta_n \leq \frac{(n + 1)!}{2^{n-1}} \quad (n = 3, 4, \dots). \tag{4.9}$$

The bound $\beta_n(\alpha - 1)$ in (4.7) is superexponential in n , as seen from (4.9). The numerical values of β_n are as follows:

Dimension n	2	3	4	5	6	7	
Value (4.8) of β_n	2	5	13.5	41.5	146.25	586	(4.10)
Bound (4.9) on β_n	—	6	15	45	157.5	630	

While the proof of Theorem 4.7 for general n is quite long [37], its special case for $n = 2$ admits an alternative method based on the box-barrier property (Theorem 4.2) and the parallelogram inequality (Remark 3.6). See the proof of [37, Theorem 4.1] for the detail.

Finally we mention that proximity theorems are also available for L_2^{\natural} -convex and M_2^{\natural} -convex functions [49]. The proximity bound for L_2^{\natural} -convex functions is linear in n [49, Theorem 6] and that for M_2^{\natural} -convex functions is quadratic in n [49, Theorem 10].

4.4 Scaling algorithm

In spite of the fact that integral convexity is not preserved under variable-scaling, it is possible to design a scaling algorithm for minimizing an integrally convex function with a bounded effective domain.

In the following we briefly describe the algorithm of [37]. The algorithm is justified by the proximity bound in Theorem 4.7 and the optimality criterion in Theorem 4.1. Let K_∞ denote the ℓ_∞ -size of the effective domain of f , i.e.,

$$K_\infty := \max\{\|x - y\|_\infty \mid x, y \in \text{dom } f\}.$$

Initially, the scaling unit α is set to $2^{\lceil \log_2 K_\infty \rceil} \approx K_\infty$. In Step S1 of the algorithm, the function $\hat{f}(y)$ is the restriction of $f(x + \alpha y)$ to the set

$$Y := \{y \in \mathbb{Z}^n \mid \|\alpha y\|_\infty \leq \beta_n(2\alpha - 1)\},$$

which is a box of integers. Then a local minimizer y^* of $\hat{f}(y)$ is found to update x to $x + \alpha y^*$. A local minimizer of $\hat{f}(y)$ can be found, e.g., by any descent method (the steepest descent method, in particular), although $\hat{f}(y)$ is not necessarily integrally convex.

Scaling algorithm for integrally convex functions

S0: Find an initial vector x with $f(x) < +\infty$, and set $\alpha := 2^{\lceil \log_2 K_\infty \rceil}$.

S1: Find an integer vector y^* that locally minimizes

$$\hat{f}(y) = \begin{cases} f(x + \alpha y) & (\|\alpha y\|_\infty \leq \beta_n(2\alpha - 1)), \\ +\infty & (\text{otherwise}), \end{cases}$$

in the sense of $\hat{f}(y^*) \leq \hat{f}(y^* + d)$ ($\forall d \in \{-1, 0, +1\}^n$)

(e.g., by the steepest descent method), and set $x := x + \alpha y^*$.

S2: If $\alpha = 1$, then stop (x is a minimizer of f).

S3: Set $\alpha := \alpha/2$, and go to S1.

The steepest descent method to locally minimize $\hat{f}(y)$

D0: Set $z := \mathbf{0}$.

D1: Find $d \in \{-1, 0, +1\}^n$ that minimizes $\hat{f}(z + d)$.

D2: If $\hat{f}(z) \leq \hat{f}(z + d)$, then set $y^* := z$ and stop

(y^* is a local minimizer of \hat{f}).

D3: Set $z := z + d$, and go to D1.

In the final phase with $\alpha = 1$, \hat{f} is an integrally convex function, and hence, by Theorem 4.1, the local minimizer in Step S1 is a global minimizer of \hat{f} . Furthermore, it can be shown, with the use of Theorem 4.7, that this point is a global minimizer of f .

The complexity of the algorithm is as follows. The number of iterations in the descent method is bounded by $|Y| \leq (4\beta_n)^n$. For each z , the 3^n neighboring points are examined to find a descent direction or verify its local minimality. Thus Step S1 can be done with at most $(12\beta_n)^n$ function evaluations. The number of scaling phases is $\log_2 K_\infty$. Therefore, the total number of function evaluations in the algorithm is bounded by $(12\beta_n)^n \log_2 K_\infty$. For a fixed n , this gives a polynomial bound $O(\log_2 K_\infty)$ in the problem size. It is emphasized in [37, Remark 6.2] that the linear dependence of $B(n, \alpha) = \beta_n(\alpha - 1)$ on α is critical for the complexity $O(\log_2 K_\infty)$.

Finally, we mention that no algorithm can minimize every integrally convex function in time polynomial in n , since any function on the unit cube $\{0, 1\}^n$ is integrally convex.

5 Subgradient and biconjugacy

5.1 Subgradient

In convex analysis [2, 24, 55], the *subdifferential* of a convex function $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ at $x \in \text{dom } g$ is defined by

$$\partial g(x) = \{p \in \mathbb{R}^n \mid g(y) - g(x) \geq \langle p, y - x \rangle \text{ for all } y \in \mathbb{R}^n\}, \quad (5.1)$$

and an element p of $\partial g(x)$ is called a *subgradient* of g at x . Analogously, for a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the subdifferential of f at $x \in \text{dom } f$ is defined as

$$\partial f(x) = \{p \in \mathbb{R}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \text{ for all } y \in \mathbb{Z}^n\}, \quad (5.2)$$

and an element p of $\partial f(x)$ is called a subgradient of f at x . An alternative expression

$$\partial f(x) = \{p \in \mathbb{R}^n \mid x \in \arg \min f[-p]\} \quad (5.3)$$

is often convenient, where $f[-p](x) = f(x) - \langle p, x \rangle$ defined in (4.1). If f is convex-extendible in the sense of $f = \overline{f}|_{\mathbb{Z}^n}$ in (3.3), where \overline{f} is the convex envelope of f defined in (3.2), then $\partial f(x) = \partial \overline{f}(x)$ for each $x \in \mathbb{Z}^n$.

When $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is integrally convex, $\partial f(x)$ is nonempty for $x \in \text{dom } f$, since $f = \overline{f}|_{\mathbb{Z}^n}$ by (3.7). Furthermore, we can rewrite (5.2) by making use of Theorem 4.1 for the minimality of an integrally convex function. Namely, by Theorem 4.1 applied to $f[-p]$, we may restrict y in (5.2) to the form of $y = x + d$ with $d \in \{-1, 0, +1\}^n$, to obtain

$$\partial f(x) = \{p \in \mathbb{R}^n \mid \sum_{j=1}^n d_j p_j \leq f(x + d) - f(x) \text{ for all } d \in \{-1, 0, +1\}^n\}. \quad (5.4)$$

This expression shows, in particular, that $\partial f(x)$ is a polyhedron described by inequalities with coefficients taken from $\{-1, 0, +1\}$.

To discuss an integrality property of $\partial f(x)$ in Sect. 5.2, it is useful to investigate the projections of $\partial f(x)$ along coordinate axes. Let $P := \partial f(x)$ for notational simplicity, and for each $l = 1, 2, \dots, n$, let $[P]_l$ denote the projection of P to the space of $(p_l, p_{l+1}, \dots, p_n)$. Inequality systems to describe the projections $[P]_l$ for $l = 1, 2, \dots, n$ can be obtained by applying the Fourier–Motzkin elimination procedure [48, 56] to the system of inequalities in (5.4), where the variable p_1 is eliminated first, and then p_2, p_3, \dots , to finally obtain an inequality in p_n only. By virtue of the integral convexity of f , a drastic simplification occurs in this elimination process. The inequalities that are generated in the elimination process are actually redundant and need not be added to the current system of inequalities, which is a crucial observation made in [50]. Thus we obtain the following theorem.

Theorem 5.1 ([50]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function and $x \in \text{dom } f$. Then $\partial f(x)$ is a nonempty polyhedron, and for each $l = 1, 2, \dots, n$, the projection $[\partial f(x)]_l$ of $\partial f(x)$ to the space of $(p_l, p_{l+1}, \dots, p_n)$ is given by*

$$[\partial f(x)]_l = \{(p_l, p_{l+1}, \dots, p_n) \mid \sum_{j=l}^n d_j p_j \leq f(x + d) - f(x) \text{ for all } d \in \{-1, 0, +1\}^n \text{ with } d_j = 0 (1 \leq j \leq l - 1)\}. \tag{5.5}$$

While the Fourier–Motzkin elimination for the proof of Theorem 5.1 depends on the linear ordering of $N = \{1, 2, \dots, n\}$, it is possible to formulate the obtained identity (5.5) without referring to the ordering of N . This is stated below as a corollary.

Corollary 5.1 *Let J be any nonempty subset of $N = \{1, 2, \dots, n\}$. Under the same assumption as in Theorem 5.1, the projection of $\partial f(x)$ to the space of $(p_j \mid j \in J)$ is given by $\{(p_j \mid j \in J) \mid \sum_{j \in J} d_j p_j \leq f(x + d) - f(x)$ for all $d \in \{-1, 0, +1\}^n$ with $d_j = 0 (j \in N \setminus J)\}$.*

5.2 Integral subgradient

For an integer-valued function $f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$, we are naturally interested in integral vectors in $\partial f(x)$. An integer vector p belonging to $\partial f(x)$ is called an *integral subgradient* and the condition

$$\partial f(x) \cap \mathbb{Z}^n \neq \emptyset \tag{5.6}$$

is sometimes referred to as the *integral subdifferentiability* of f at x .

Integral subdifferentiability of integer-valued integrally convex functions is established recently by the present authors [50].

Theorem 5.2 ([50, Theorem 3]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ be an integer-valued integrally convex function. For every $x \in \text{dom } f$, we have $\partial f(x) \cap \mathbb{Z}^n \neq \emptyset$.*

Proof The proof is based on Theorem 5.1 concerning projections of $\partial f(x)$. While the reader is referred to [50] for the formal proof, we indicate the basic idea here. By (5.5) for $l = n$, we have

$$\{p_n \mid p \in \partial f(x)\} = \{p_n \mid p_n \leq f(x + \mathbf{1}^n) - f(x), -p_n \leq f(x - \mathbf{1}^n) - f(x)\},$$

which is an interval $[\alpha_n, \beta_n]_{\mathbb{R}}$ with $\alpha_n = f(x) - f(x - \mathbf{1}^n)$ and $\beta_n = f(x + \mathbf{1}^n) - f(x)$. We have $\alpha_n \leq \beta_n$ since $\partial f(x) \neq \emptyset$, while $\alpha_n, \beta_n \in \mathbb{Z}$ since f is integer-valued. Therefore, the interval $[\alpha_n, \beta_n]_{\mathbb{R}}$ contains an integer, say, p_n^* . Next, by (5.5) for $l = n - 1$, the set $\{p_{n-1} \mid p \in \partial f(x), p_n = p_n^*\}$ is described by six inequalities

$$\sigma p_{n-1} + \tau p_n^* \leq f(x + \sigma \mathbf{1}^{n-1} + \tau \mathbf{1}^n) - f(x) \quad (\sigma \in \{+1, -1\}, \tau \in \{+1, -1, 0\}).$$

This implies that $\{p_{n-1} \mid p \in \partial f(x), p_n = p_n^*\}$ is a nonempty interval, say, $[\alpha_{n-1}, \beta_{n-1}]_{\mathbb{R}}$ with $\alpha_{n-1}, \beta_{n-1} \in \mathbb{Z}$, which contains an integer, say, p_{n-1}^* . This means that there exists $p \in \partial f(x)$ such that $p_n = p_n^* \in \mathbb{Z}$ and $p_{n-1} = p_{n-1}^* \in \mathbb{Z}$. Continuing in this way (with $l = n - 2, n - 3, \dots, 2, 1$), we can construct $p^* \in \partial f(x) \cap \mathbb{Z}^n$. \square

Some supplementary facts concerning Theorem 5.2 are shown below.

- Theorem 5.2 states that $\partial f(x) \cap \mathbb{Z}^n \neq \emptyset$, but it does not claim a stronger statement that $\partial f(x)$ is an integer polyhedron. Indeed, $\partial f(x)$ is not necessarily an integer polyhedron. For example, let $f : \mathbb{Z}^3 \rightarrow \mathbb{Z} \cup \{+\infty\}$ be defined by $f(0, 0, 0) = 0$ and $f(1, 1, 0) = f(0, 1, 1) = f(1, 0, 1) = 1$ with $\text{dom } f = \{(0, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 1)\}$. This f is integrally convex and $\partial f(\mathbf{0}) = \{p \in \mathbb{R}^3 \mid p_1 + p_2 \leq 1, p_2 + p_3 \leq 1, p_1 + p_3 \leq 1\}$ is not an integer polyhedron, having a non-integral vertex at $p = (1/2, 1/2, 1/2)$. See [50, Remark 4] for details. In the special cases where f is L^{\natural} -convex, M^{\natural} -convex, L_2^{\natural} -convex, or M_2^{\natural} -convex, the subdifferential $\partial f(x)$ is known [40] to be an integer polyhedron.
- If $\partial f(x)$ is bounded, $\partial f(x)$ has an integral vertex, although not every vertex of $\partial f(x)$ is integral. For example, let $D = \{x \in \{-1, 0, +1\}^3 \mid |x_1| + |x_2| + |x_3| \leq 2\}$ and define f with $\text{dom } f = D$ by $f(\mathbf{0}) = 0$ and $f(x) = 1$ ($x \in D \setminus \{\mathbf{0}\}$). This f is an integer-valued integrally convex function and $\partial f(\mathbf{0})$ is a bounded polyhedron that has eight non-integral vertices $(\pm 1/2, \pm 1/2, \pm 1/2)$ (with arbitrary combinations of double-signs) and six integral vertices $(\pm 1, 0, 0), (0, \pm 1, 0),$ and $(0, 0, \pm 1)$. See [50, Remark 7] for details.
- Integral subdifferentiability is not guaranteed without the assumption of integral convexity. Consider $D = \{(0, 0, 0), \pm(1, 1, 0), \pm(0, 1, 1), \pm(1, 0, 1)\}$ and define $f : \mathbb{Z}^3 \rightarrow \mathbb{Z} \cup \{+\infty\}$ with $\text{dom } f = D$ by $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)/2$. This function is not integrally convex, $\partial f(\mathbf{0}) = \{(1/2, 1/2, 1/2)\}$, and $\partial f(\mathbf{0}) \cap \mathbb{Z}^3 = \emptyset$. See [39, Example 1.1] or [50, Example 1] for details.

The integral subdifferentiability formulated in Theorem 5.2 can be strengthened with an additional box condition. This stronger form plays the key role in the proof of the Fenchel-type min-max duality theorem (Theorem 6.1) discussed in Sect. 6.

Recall that an integral box means a set B of real vectors represented as $B = \{p \in \mathbb{R}^n \mid \alpha \leq p \leq \beta\}$ for integer vectors $\alpha \in (\mathbb{Z} \cup \{-\infty\})^n$ and $\beta \in (\mathbb{Z} \cup \{+\infty\})^n$ satisfying $\alpha \leq \beta$. The following theorem states that

$$\partial f(x) \cap B \neq \emptyset \implies \partial f(x) \cap B \cap \mathbb{Z}^n \neq \emptyset, \tag{5.7}$$

which may be referred to as *box-integral subdifferentiability*.

Theorem 5.3 ([51, Theorem 1.2]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ be an integer-valued integrally convex function, $x \in \text{dom } f$, and B be an integral box. If $\partial f(x) \cap B$ is nonempty, then $\partial f(x) \cap B$ is a polyhedron containing an integer vector. If, in addition, $\partial f(x) \cap B$ is bounded, then $\partial f(x) \cap B$ has an integral vertex.*

We briefly describe how Theorem 5.3 has been proved in [51]. Recall that Theorem 5.2 for integral subdifferentiability (without a box) is proved from a hierarchical system of inequalities (Theorem 5.1) to describe the projection $[\partial f(x)]_l$ of $\partial f(x)$

to the space of $(p_l, p_{l+1}, \dots, p_n)$ for $l = 1, 2, \dots, n$, where Theorem 5.1 itself is proved by means of the Fourier–Motzkin elimination. This approach is extended in [51] to prove Theorem 5.3. Namely, Theorem 4.3 of [51] gives a hierarchical system of inequalities to describe the projection $[\partial f(x) \cap B]_l$ of $\partial f(x) \cap B$ to the space of $(p_l, p_{l+1}, \dots, p_n)$ for $l = 1, 2, \dots, n$. The proof of this theorem is based on the Fourier–Motzkin elimination. Once inequalities for the projections $[\partial f(x) \cap B]_l$ are obtained, the derivation of box-integral subdifferentiability (Theorem 5.3) is almost the same as that of integral subdifferentiability (Theorem 5.2) from Theorem 5.1. Finally we mention that alternative proofs of Theorems 5.2 and 5.3 can be found in [19] and [20], respectively.

5.3 Biconjugacy

For an integer-valued function $f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$, we define $f^\bullet : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ by

$$f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbb{Z}^n\} \quad (p \in \mathbb{Z}^n), \quad (5.8)$$

called the *integral conjugate* of f . For any $x, p \in \mathbb{Z}^n$ we have

$$f(x) + f^\bullet(p) \geq \langle p, x \rangle, \quad (5.9)$$

which is a discrete analogue of Fenchel’s inequality [24, (1.1.3), p. 211] or the Fenchel–Young inequality [2, Proposition 3.3.4]. When $x \in \text{dom } f$, we have

$$f(x) + f^\bullet(p) = \langle p, x \rangle \iff p \in \partial f(x) \cap \mathbb{Z}^n. \quad (5.10)$$

Note the asymmetric roles of f and f^\bullet in (5.10).

The integral conjugate f^\bullet of an integer-valued function f is also an integer-valued function defined on \mathbb{Z}^n . So we can apply the transformation (5.8) to f^\bullet to obtain $f^{\bullet\bullet} = (f^\bullet)^\bullet$, which is called the *integral biconjugate* of f . It follows from (5.9) and (5.10) that, for each $x \in \text{dom } f$ we have

$$f^{\bullet\bullet}(x) = f(x) \iff \partial f(x) \cap \mathbb{Z}^n \neq \emptyset. \quad (5.11)$$

See [39, Lemma 4.1] or [50, Lemma 1] for the proof. We say that f enjoys *integral biconjugacy* if

$$f^{\bullet\bullet}(x) = f(x) \quad \text{for all } x \in \mathbb{Z}^n. \quad (5.12)$$

Example 5.1 Let $D = \{(0, 0, 0), \pm(1, 1, 0), \pm(0, 1, 1), \pm(1, 0, 1)\}$ and consider the function $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)/2$ on $\text{dom } f = D$. (This is the function used in Sect. 5.2 as an example of an integer-valued function lacking in integral subdifferentiability.) According to the definition (5.8), the integral conjugate f^\bullet is given

by

$$f^\bullet(p) = \max\{|p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_1 + p_3 - 1|\} \quad (p \in \mathbb{Z}^3). \tag{5.13}$$

For $x = \mathbf{0}$ we have $f(x) = 0$, while (5.13) shows $f^\bullet(p) \geq 1$ for every integer vector $p \in \mathbb{Z}^3$. Therefore we have strict inequality $f(x) + f^\bullet(p) > \langle p, x \rangle$ for $x = \mathbf{0}$ and every $p \in \mathbb{Z}^3$. This is consistent with (5.10) since $\partial f(\mathbf{0}) = \{(1/2, 1/2, 1/2)\}$ and hence $\partial f(\mathbf{0}) \cap \mathbb{Z}^3 = \emptyset$. For the integral biconjugate $f^{\bullet\bullet}(x) = \sup\{\langle p, x \rangle - f^\bullet(p) \mid p \in \mathbb{Z}^3\}$ we have

$$f^{\bullet\bullet}(\mathbf{0}) = - \inf_{p \in \mathbb{Z}^3} \max\{|p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_3 + p_1 - 1|\} = -1.$$

Therefore we have $f^{\bullet\bullet}(\mathbf{0}) \neq f(\mathbf{0})$. This is consistent with (5.11) since $\partial f(\mathbf{0}) \cap \mathbb{Z}^3 = \emptyset$. ■

We now assume that $f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ is an integer-valued integrally convex function. The integral conjugate f^\bullet is not necessarily integrally convex ([47, Example 4.15], [50, Remark 2.3]). Nevertheless, the integral biconjugate $f^{\bullet\bullet}$ coincides with f itself, that is, integral biconjugacy holds for an integer-valued integrally convex function. This theorem is established by the present authors [50] based on the integral subdifferentiability $\partial f(x) \cap \mathbb{Z}^n \neq \emptyset$ given in Theorem 5.2; see Remark 5.1 below for some technical aspects.

Theorem 5.4 ([50, Theorem 4]) *For any integer-valued integrally convex function $f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$, we have $f^{\bullet\bullet}(x) = f(x)$ for all $x \in \mathbb{Z}^n$.*

As special cases of Theorem 5.4 we obtain integral biconjugacy for L-convex, L^{\natural} -convex, M-convex, M^{\natural} -convex, L_2^{\natural} -convex, and M_2^{\natural} -convex functions given in [40, Theorems 8.12, 8.36, 8.46], and that for BS-convex and UJ-convex functions given in [50, Corollary 2].

Remark 5.1 There is a subtle gap between integral subdifferentiability in (5.11) and integral biconjugacy in (5.12) for a general integer-valued function $f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ (which is not necessarily integrally convex). While the latter imposes the condition $f^{\bullet\bullet}(x) = f(x)$ on all $x \in \mathbb{Z}^n$, the former refers to x in $\text{dom } f$ only. This means that integral subdifferentiability may possibly be weaker than integral biconjugacy, and this is indeed the case in general (see [39, Remark 4.1] or [50, Remark 6]). However, it is known [39, Lemma 4.2] that integral subdifferentiability does imply integral biconjugacy under the technical conditions that $\text{dom } f = \text{cl}(\text{dom } f) \cap \mathbb{Z}^n$ and $\text{cl}(\text{dom } f)$ is rationally-polyhedral, where $\text{cl}(\text{dom } f)$ denotes the closure of the convex hull (or closed convex hull) of $\text{dom } f$; see Remark 2.1 for this notation. ■

Remark 5.2 In convex analysis [2, 24, 55], the *conjugate function* of $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\text{dom } g \neq \emptyset$ is defined to be a function $g^{\bullet\mathbb{R}} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$g^{\bullet\mathbb{R}}(p) := \sup\{\langle p, x \rangle - g(x) \mid x \in \mathbb{R}^n\} \quad (p \in \mathbb{R}^n), \tag{5.14}$$

where a (non-standard) notation $g^{\bullet\mathbb{R}}$ is introduced for discussion here. The *biconjugate* of g is defined as $(g^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}$ by using the transformation (5.14) twice. We have biconjugacy $(g^{\bullet\mathbb{R}})^{\bullet\mathbb{R}} = g$ for closed convex functions g . For a real-valued function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ in discrete variables, we may also define

$$f^{\bullet\mathbb{R}}(p) := \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbb{Z}^n\} \quad (p \in \mathbb{R}^n). \tag{5.15}$$

Then the convex envelope \bar{f} coincides with $(f^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}$, which denotes the function obtained by applying (5.15) to f and then (5.14) to $g = f^{\bullet\mathbb{R}}$. Therefore, if f is convex-extensible in the sense of $f = \bar{f}|_{\mathbb{Z}^n}$ in (3.3), we have $(f^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}|_{\mathbb{Z}^n} = f$, which is a kind of biconjugacy. If f is integer-valued, we can naturally consider $(f^\bullet)^\bullet$ using (5.8) twice as well as $(f^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}$ using (5.15) and then (5.14). It is most important to recognize that for any f we have $(f^\bullet)^\bullet(x) \leq (f^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}(x)$ for $x \in \mathbb{Z}^n$ and that the equality may fail even when $f = \bar{f}|_{\mathbb{Z}^n}$. As an example, consider the function $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)/2$ in Example 5.1. The convex envelope \bar{f} is given by $\bar{f}(x_1, x_2, x_3) = (x_1 + x_2 + x_3)/2$ on the convex hull of D , and therefore $f = \bar{f}|_{\mathbb{Z}^n}$ holds. Similarly to (5.13) we have

$$f^{\bullet\mathbb{R}}(p) = \max\{|p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_1 + p_3 - 1|\} \quad (p \in \mathbb{R}^3)$$

and hence

$$(f^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}(\mathbf{0}) = - \inf_{p \in \mathbb{R}^3} \max\{|p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_3 + p_1 - 1|\} = 0,$$

where the infimum over $p \in \mathbb{R}^3$ is attained by $p = (1/2, 1/2, 1/2)$. Therefore $(f^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}(\mathbf{0}) = f(\mathbf{0})$, whereas $(f^\bullet)^\bullet(\mathbf{0}) < f(\mathbf{0})$ as we have seen in Example 5.1. Thus, integral biconjugacy $f = (f^\bullet)^\bullet$ in (5.12) is much more intricate than the equality $f = (f^{\bullet\mathbb{R}})^{\bullet\mathbb{R}}|_{\mathbb{Z}^n}$. ■

5.4 Discrete DC programming

A discrete analogue of the theory of DC functions (difference of two convex functions), or discrete DC programming, has been proposed in [31] using L^{\natural} -convex and M^{\natural} -convex functions. As already noted in [31, Remark 4.7], such theory of discrete DC functions can be developed for functions that satisfy integral biconjugacy and integral subdifferentiability. It is pointed out in [50] that Theorems 5.2 and 5.4 for integrally convex functions enable us to extend the theory of discrete DC functions to integrally convex functions. In particular, an analogue of the Toland–Singer duality [61, 65] can be established for integrally convex functions as follows.

Theorem 5.5 ([50, Theorem 5]) *Let $g, h : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ be integer-valued integrally convex functions. Then*

$$\inf\{g(x) - h(x) \mid x \in \mathbb{Z}^n\} = \inf\{h^\bullet(p) - g^\bullet(p) \mid p \in \mathbb{Z}^n\}. \tag{5.16}$$

As mentioned already in [50], the assumption of integral convexity of g is not needed for (5.16) to be true. That is, (5.16) holds for any $g : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ as long as $h : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ is integrally convex.

6 Discrete Fenchel duality

6.1 General framework of Fenchel duality

The Fenchel duality is one of the expressions of the duality principle in the form of a min-max relation between a pair of convex and concave functions (f, g) and their conjugate functions. As is well known, the existence of such min-max formula guarantees the existence of a certificate of optimality for the problem of minimizing $f - g$ over \mathbb{R}^n or \mathbb{Z}^n .

First we recall the framework for functions in continuous variables. For $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$, the function $f^\bullet : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f^\bullet(p) := \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbb{R}^n\} \quad (p \in \mathbb{R}^n) \quad (6.1)$$

is called the *conjugate* (or *convex conjugate*) of f . For $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ with $\text{dom } g \neq \emptyset$, the function $g^\circ : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by

$$g^\circ(p) := \inf\{\langle p, x \rangle - g(x) \mid x \in \mathbb{R}^n\} \quad (p \in \mathbb{R}^n) \quad (6.2)$$

is called the *concave conjugate* of g . We have $g^\circ(p) = -f^\bullet(-p)$ if $g(x) = -f(x)$. It follows from the definitions that

$$f(x) - g(x) \geq g^\circ(p) - f^\bullet(p) \quad (6.3)$$

for any $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$. The relation (6.3) is called *weak duality*.

The Fenchel duality theorem says that a min-max formula

$$\inf\{f(x) - g(x) \mid x \in \mathbb{R}^n\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbb{R}^n\} \quad (6.4)$$

holds for convex and concave functions $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfying certain regularity conditions. The relation (6.4) is called *strong duality* in contrast to weak duality. See Bauschke–Combettes [1, Section 15.2], Borwein–Lewis [2, Theorem 3.3.5], Hiriart-Urruty–Lemaréchal [24, (2.3.2), p. 228], Rockafellar [55, Theorem 31.1], Stoer–Witzgall [62, Corollary 5.1.4] for precise statements.

We now turn to functions in discrete variables. For any functions $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ we define

$$f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbb{Z}^n\} \quad (p \in \mathbb{R}^n), \quad (6.5)$$

$$g^\circ(p) = \inf\{\langle p, x \rangle - g(x) \mid x \in \mathbb{Z}^n\} \quad (p \in \mathbb{R}^n), \quad (6.6)$$

where $\text{dom } f \neq \emptyset$ and $\text{dom } g \neq \emptyset$ are assumed. In this case, the generic form of the Fenchel duality reads:

$$\inf\{f(x) - g(x) \mid x \in \mathbb{Z}^n\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbb{R}^n\}, \quad (6.7)$$

which is expected to be true when f and g are equipped with certain discrete convexity and concavity, respectively. Moreover, when f and g are integer-valued ($f : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ and $g : \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{-\infty\}$), we are particularly interested in the dual problem with an integer vector, that is,

$$\inf\{f(x) - g(x) \mid x \in \mathbb{Z}^n\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbb{Z}^n\}. \quad (6.8)$$

To relate the discrete case to the continuous case, it is convenient to consider the convex envelope \bar{f} of f and the concave envelope \bar{g} of g , where $\bar{g} := -(-g)$, that is, \bar{g} is defined to be the negative of the convex envelope of $-g$. By the definitions of convex and concave envelopes and conjugate functions we have

$$\begin{aligned} \bar{f}(x) &\leq f(x), & \bar{g}(x) &\geq g(x) & (x \in \mathbb{Z}^n), \\ \bar{f}^\bullet(p) &= f^\bullet(p), & \bar{g}^\circ(p) &= g^\circ(p) & (p \in \mathbb{R}^n) \end{aligned}$$

as well as weak dualities

$$f(x) - g(x) \geq g^\circ(p) - f^\bullet(p) \quad (x \in \mathbb{Z}^n, p \in \mathbb{R}^n), \quad (6.9)$$

$$\bar{f}(x) - \bar{g}(x) \geq \bar{g}^\circ(p) - \bar{f}^\bullet(p) \quad (x \in \mathbb{R}^n, p \in \mathbb{R}^n). \quad (6.10)$$

Thus we have the following chain of inequalities:

$$P(\mathbb{Z}) \geq \bar{P}(\mathbb{R}) \geq \bar{D}(\mathbb{R}) = D(\mathbb{R}) \geq D(\mathbb{Z}) \quad (6.11)$$

with notations

$$\begin{aligned} P(\mathbb{Z}) &:= \inf\{f(x) - g(x) \mid x \in \mathbb{Z}^n\}, \\ \bar{P}(\mathbb{R}) &:= \inf\{\bar{f}(x) - \bar{g}(x) \mid x \in \mathbb{R}^n\}, \\ \bar{D}(\mathbb{R}) &:= \sup\{\bar{g}^\circ(p) - \bar{f}^\bullet(p) \mid p \in \mathbb{R}^n\}, \\ D(\mathbb{R}) &:= \sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbb{R}^n\}, \\ D(\mathbb{Z}) &:= \sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbb{Z}^n\} \end{aligned}$$

for the optimal values of the problems, where $P(\cdot)$ stands for ‘‘Primal problem’’ and $D(\cdot)$ for ‘‘Dual problem’’.

The desired min-max relations (6.7) and (6.8) can be written as $P(\mathbb{Z}) = D(\mathbb{R})$ and $P(\mathbb{Z}) = D(\mathbb{Z})$, respectively. The inequality between $\bar{P}(\mathbb{R})$ and $\bar{D}(\mathbb{R})$ in the middle of (6.11) becomes an equality if the Fenchel duality (6.4) holds for (\bar{f}, \bar{g}) . The first and the last inequality in (6.11) express possible discrepancy between discrete and

continuous cases, and we are naturally concerned with when these inequalities turn into equalities. The concept of subdifferential plays the essential role here.

For the subdifferential $\partial f(x)$ defined in (5.2) we observe that

$$p \in \partial f(x) \iff f(x) + f^\bullet(p) = \langle p, x \rangle \quad (6.12)$$

holds for any $p \in \mathbb{R}^n$ and $x \in \text{dom } f$. Similarly, we have

$$p \in \partial' g(x) \iff g(x) + g^\circ(p) = \langle p, x \rangle \quad (6.13)$$

for any $p \in \mathbb{R}^n$ and $x \in \text{dom } g$, where $\partial' g(x)$ means the concave version of the subdifferential defined by

$$\partial' g(x) = -(\partial(-g))(x) = \{p \in \mathbb{R}^n \mid g(y) - g(x) \leq \langle p, y - x \rangle \text{ for all } y \in \mathbb{Z}^n\}. \quad (6.14)$$

Suppose that the infimum $P(\mathbb{Z})$ is attained by some $x^* \in \mathbb{Z}^n$. It follows from (6.9), (6.12), and (6.13) that $P(\mathbb{Z}) = D(\mathbb{R})$ and the supremum $D(\mathbb{R})$ is attained by $p^* \in \mathbb{R}^n$ if and only if $p^* \in \partial f(x^*) \cap \partial' g(x^*)$. Therefore, $P(\mathbb{Z}) = D(\mathbb{R})$ if

$$\partial f(x^*) \cap \partial' g(x^*) \neq \emptyset. \quad (6.15)$$

Furthermore, if

$$\partial f(x^*) \cap \partial' g(x^*) \cap \mathbb{Z}^n \neq \emptyset, \quad (6.16)$$

then we have $P(\mathbb{Z}) = D(\mathbb{Z})$. Thus (6.15) and (6.16), respectively, imply the Fenchel-type min-max formulas (6.7) for real-valued functions and (6.8) for integer-valued functions. It is noted that this implication does not presuppose the Fenchel duality $\overline{P}(\mathbb{R}) = \overline{D}(\mathbb{R})$ for $(\overline{f}, \overline{g})$.

When $\overline{P}(\mathbb{R}) = \overline{D}(\mathbb{R})$ is known to hold, the min-max relation $P(\mathbb{Z}) = D(\mathbb{R})$ for real-valued functions follows, by (6.11), from $P(\mathbb{Z}) = \overline{P}(\mathbb{R})$, where the latter condition $P(\mathbb{Z}) = \overline{P}(\mathbb{R})$ holds if

$$\overline{f - g} = \overline{f} - \overline{g}. \quad (6.17)$$

It is emphasized that (6.17) does not follow from the individual convex or concave-extendibility of f and g . See Example 6.1 in Sect. 6.2.

6.2 Fenchel duality for integrally convex functions

The following two examples show that the min-max formula (6.7) or (6.8) is not necessarily true when f and g are integrally convex and concave functions. (Naturally, function g is called *integrally concave* if $-g$ is integrally convex.)

Example 6.1 ([43, Example 5.6]) Let $f, g : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ be defined as

$$f(x_1, x_2) = |x_1 + x_2 - 1|, \quad g(x_1, x_2) = 1 - |x_1 - x_2|.$$

The function f is integrally convex (actually M^{\natural} -convex) and g is integrally concave (actually L^{\natural} -concave). We have

$$\min_{\mathbb{Z}}\{f - g\} > \min_{\mathbb{R}}\{\bar{f} - \bar{g}\} = \max_{\mathbb{R}}\{\bar{g}^{\circ} - \bar{f}^{\bullet}\} = \max_{\mathbb{R}}\{g^{\circ} - f^{\bullet}\} = \max_{\mathbb{Z}}\{g^{\circ} - f^{\bullet}\}.$$

(0) (-1) (-1) (-1) (-1)

Thus the min-max identity (6.8) as well as (6.7) fails because of the primal integrality gap $P(\mathbb{Z}) > \bar{P}(\mathbb{R})$. Indeed, the condition $\bar{f} - g = \bar{f} - \bar{g}$ in (6.17) fails as follows. Let $h := f - g$. Since $h(0, 0) = h(1, 0) = h(0, 1) = h(1, 1) = 0$, we have $\bar{h}(1/2, 1/2) = 0$, whereas $\bar{f}(1/2, 1/2) - \bar{g}(1/2, 1/2) = 0 - 1 = -1$. Thus $\bar{f} - g \neq \bar{f} - \bar{g}$. ■

Example 6.2 ([43, Example 5.7]) Let $f, g : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ be defined as

$$f(x_1, x_2) = \max(0, x_1 + x_2), \quad g(x_1, x_2) = \min(x_1, x_2).$$

The function f is integrally convex (actually M^{\natural} -convex) and g is integrally concave (actually L^{\natural} -concave). We have

$$\min_{\mathbb{Z}}\{f - g\} = \min_{\mathbb{R}}\{\bar{f} - \bar{g}\} = \max_{\mathbb{R}}\{\bar{g}^{\circ} - \bar{f}^{\bullet}\} = \max_{\mathbb{R}}\{g^{\circ} - f^{\bullet}\} > \max_{\mathbb{Z}}\{g^{\circ} - f^{\bullet}\}.$$

(0) (0) (0) (0) (-∞)

Although the min-max identity (6.7) with real-valued p holds, the formula (6.8) with integer-valued p fails because of the dual integrality gap $D(\mathbb{R}) > D(\mathbb{Z})$. The optimal value $\min_{\mathbb{Z}}\{f - g\} = 0$ is attained by $x^* = \mathbf{0}$, at which the subdifferentials are given by $\partial f(\mathbf{0}) = \{(p_1, p_2) \mid 0 \leq p_1 = p_2 \leq 1\}$ and $\partial' g(\mathbf{0}) = \{(p_1, p_2) \mid p_1 + p_2 = 1, 0 \leq p_1 \leq 1\}$. We have $\partial f(\mathbf{0}) \cap \partial' g(\mathbf{0}) = \{(1/2, 1/2)\}$ and $\partial f(\mathbf{0}) \cap \partial' g(\mathbf{0}) \cap \mathbb{Z}^2 = \emptyset$, which shows the failure of (6.16). ■

As the min-max formula (6.7) is not true (in general) when f and $-g$ are integrally convex, we are motivated to restrict g to a subclass of integrally concave functions, while allowing f to be a general integrally convex function. However, the possibility of g being M^{\natural} -concave or L^{\natural} -concave is denied by the above examples. That is, we cannot hope for the combination of (integrally convex, M^{\natural} -convex) nor (integrally convex, L^{\natural} -convex) for $(f, -g)$. Furthermore, since a function in two variables is M^{\natural} -convex if and only if it is multimodular [32, Remark 2.2], the possibility of the combination of (integrally convex, multimodular) for $(f, -g)$ is also denied. Thus we are motivated to consider the combination of (integrally convex, separable concave).

In the following we address the Fenchel-type min-max formula for a pair of an integrally convex function and a separable concave function. A function $\Psi : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ in $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ is called *separable concave* if it can be represented as

$$\Psi(x) = \psi_1(x_1) + \psi_2(x_2) + \dots + \psi_n(x_n) \tag{6.18}$$

with univariate discrete concave functions $\psi_i : \mathbb{Z} \rightarrow \mathbb{R} \cup \{-\infty\}$, which means, by definition, that $\text{dom } \psi_i$ is an interval of integers and

$$\psi_i(k-1) + \psi_i(k+1) \leq 2\psi_i(k) \quad (k \in \mathbb{Z}). \quad (6.19)$$

The concave conjugate of Ψ is denoted by Ψ° , that is,

$$\Psi^\circ(p) = \inf\{\langle p, x \rangle - \Psi(x) \mid x \in \mathbb{Z}^n\} \quad (p \in \mathbb{R}^n). \quad (6.20)$$

This is a separable concave function represented as

$$\Psi^\circ(p) = \psi_1^\circ(p_1) + \psi_2^\circ(p_2) + \cdots + \psi_n^\circ(p_n), \quad (6.21)$$

where

$$\psi_i^\circ(l) = \inf\{kl - \psi_i(k) \mid k \in \mathbb{Z}\} \quad (l \in \mathbb{R}). \quad (6.22)$$

It is often possible to obtain an explicit form of the (integral) conjugate function of an (integer-valued) separable convex (or concave) function; see [14, 15].

The following is the Fenchel-type min-max formula for a pair of an integrally convex function and a separable concave function. The case of integer-valued functions, which is more interesting, is given in [51, Theorem 1.1], while we include here the case of real-valued functions for completeness.

Theorem 6.1 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function and $\Psi : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ a separable concave function. Assume that $\text{dom } f \cap \text{dom } \Psi \neq \emptyset$ and $\inf\{f(x) - \Psi(x) \mid x \in \mathbb{Z}^n\}$ is attained by some x^* . Then*

$$\inf\{f(x) - \Psi(x) \mid x \in \mathbb{Z}^n\} = \sup\{\Psi^\circ(p) - f^\bullet(p) \mid p \in \mathbb{R}^n\} \quad (6.23)$$

and the supremum in (6.23) is attained by some $p^* \in \mathbb{R}^n$. If, in addition, f and Ψ are integer-valued, then

$$\inf\{f(x) - \Psi(x) \mid x \in \mathbb{Z}^n\} = \sup\{\Psi^\circ(p) - f^\bullet(p) \mid p \in \mathbb{Z}^n\} \quad (6.24)$$

and the supremum in (6.24) is attained by some $p^* \in \mathbb{Z}^n$.

Proof The proof of the real-valued case (6.23) consists in showing (6.15) for $(f, g) = (f, \Psi)$, which is proved in Sect. 6.3. The integral case (6.24) has been proved in [51, Theorem 1.1] by showing (6.16) for $(f, g) = (f, \Psi)$ as a consequence of box-integral subdifferentiability described in Theorem 5.3. \square

Remark 6.1 In the integer-valued case, the existence of x^* attaining the infimum in (6.24) is guaranteed if (and only if) the set $\{f(x) - \Psi(x) \mid x \in \mathbb{Z}^n\}$ of function values is bounded from below. It is known [51, Lemma 3.2] that if the supremum on the right-hand side of (6.24) is finite, then the infimum on the left-hand side is also finite. \blacksquare

Theorem 6.1 implies a min-max theorem for separable convex minimization on a box-integer polyhedron. The case of integer-valued functions, which is more interesting, is stated in [51, Theorem 3.1]. We define notation $\mu_P(p) = \inf\{\langle p, x \mid x \in P\}$ for a polyhedron P .

Theorem 6.2 *Let $P (\subseteq \mathbb{R}^n)$ be a nonempty box-integer polyhedron, and $\Phi : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ a separable convex function. Assume that $\inf\{\Phi(x) \mid x \in P \cap \mathbb{Z}^n\}$ is (finite and) attained by some x^* . Then*

$$\inf\{\Phi(x) \mid x \in P \cap \mathbb{Z}^n\} = \sup\{\mu_P(p) - \Phi^\bullet(p) \mid p \in \mathbb{R}^n\} \tag{6.25}$$

and the supremum is attained by some $p^* \in \mathbb{R}^n$. If, in addition, Φ is integer-valued, then

$$\inf\{\Phi(x) \mid x \in P \cap \mathbb{Z}^n\} = \sup\{\mu_P(p) - \Phi^\bullet(p) \mid p \in \mathbb{Z}^n\} \tag{6.26}$$

and the supremum is attained by some $p^* \in \mathbb{Z}^n$.

Proof Denote the indicator function of $P \cap \mathbb{Z}^n$ by δ , which is an integer-valued integrally convex function because P is a box-integer polyhedron. Then the statements follow from Theorem 6.1 for $f = \delta$ and $\Psi = -\Phi$. □

This theorem generalizes a recent result of Frank–Murota [15, Theorem 3.4], which asserts the min-max formula (6.26) for integer-valued Φ when P is an integral box-TDI polyhedron.

6.3 Proof of (6.23) for real-valued functions (Theorem 6.1)

In this section we prove the min-max formula (6.23) for real-valued functions in Theorem 6.1. Let x^* denote an element of $\text{dom } f \cap \text{dom } \Psi (\subseteq \mathbb{Z}^n)$ that attains the infimum in (6.23). According to the general framework described in Sect. 6.1, it suffices to show $\partial f(x^*) \cap \partial' \Psi(x^*) \neq \emptyset$.

Since $\overline{f - \Psi} = \overline{f} - \overline{\Psi}$ by Proposition 2.1 of [51], we have

$$\inf_{x \in \mathbb{Z}^n} \{f(x) - \Psi(x)\} = \inf_{x \in \mathbb{R}^n} \{(\overline{f - \Psi})(x)\} = \inf_{x \in \mathbb{R}^n} \{\overline{f}(x) - \overline{\Psi}(x)\},$$

which implies that x^* is also a minimizer of $\overline{f} - \overline{\Psi}$ over \mathbb{R}^n . Define $\alpha := \inf\{f(x) - \Psi(x) \mid x \in \mathbb{Z}^n\} = f(x^*) - \Psi(x^*)$ and $f_1(x) := f(x) - \alpha$ for $x \in \mathbb{Z}^n$. Then f_1 is an integrally convex function satisfying $\overline{f_1} \geq \overline{\Psi}$ and $f_1(x^*) = \Psi(x^*)$. The desired nonemptiness $\partial f(x^*) \cap \partial' \Psi(x^*) \neq \emptyset$ follows from Proposition 6.1 below applied to $(f, g) = (f_1, \Psi)$.

Proposition 6.1 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ be integrally convex and concave functions, respectively, and $x^* \in \text{dom } f \cap \text{dom } g$. If $\overline{f} \geq \overline{g}$ on \mathbb{R}^n and $f(x^*) = g(x^*)$, then $\partial f(x^*) \cap \partial' g(x^*) \neq \emptyset$.*

Proof We may assume that $x^* = \mathbf{0}$ and $f(\mathbf{0}) = g(\mathbf{0}) = 0$. By (5.4) we have

$$\partial f(\mathbf{0}) = \{p \in \mathbb{R}^n \mid \langle d^{(i)}, p \rangle \leq f(d^{(i)}) \text{ for all } i \in I\}, \tag{6.27}$$

$$\partial' g(\mathbf{0}) = \{p \in \mathbb{R}^n \mid \langle -\hat{d}^{(j)}, p \rangle \leq -g(\hat{d}^{(j)}) \text{ for all } j \in J\} \tag{6.28}$$

where $\{-1, 0, +1\}^n \cap \text{dom } f$ is represented as $\{d^{(i)} \mid i \in I\}$ and $\{-1, 0, +1\}^n \cap \text{dom } g$ as $\{\hat{d}^{(j)} \mid j \in J\}$. By the Farkas lemma (or linear programming duality) [56], there exists $p \in \partial f(\mathbf{0}) \cap \partial' g(\mathbf{0})$ if and only if

$$\sum_{i \in I} u_i f(d^{(i)}) - \sum_{j \in J} v_j g(\hat{d}^{(j)}) \geq 0 \tag{6.29}$$

for any $u_i \geq 0$ and $v_j \geq 0$ satisfying $\sum_{i \in I} u_i d^{(i)} - \sum_{j \in J} v_j \hat{d}^{(j)} = \mathbf{0}$. Let $\hat{x} := \sum_{i \in I} u_i d^{(i)} = \sum_{j \in J} v_j \hat{d}^{(j)}$, $U := \sum_{i \in I} u_i$, and $V := \sum_{j \in J} v_j$. By homogeneity we may assume $U + V \leq 1$. Then $\hat{x} \in \text{dom } \bar{f} \cap \text{dom } \bar{g}$. If $U > 0$, it follows from the convexity of \bar{f} as well as $\bar{f}(\mathbf{0}) = 0$ and $U \leq 1$ that

$$\sum_{i \in I} u_i f(d^{(i)}) = U \sum_{i \in I} \frac{u_i}{U} \bar{f}(d^{(i)}) \geq U \bar{f}\left(\sum_{i \in I} \frac{u_i}{U} d^{(i)}\right) = U \bar{f}\left(\frac{1}{U} \hat{x}\right) \geq \bar{f}(\hat{x}).$$

The resulting inequality $\sum_{i \in I} u_i f(d^{(i)}) \geq \bar{f}(\hat{x})$ is also true when $U = 0$. Similarly, we obtain $\sum_{j \in J} v_j g(\hat{d}^{(j)}) \leq \bar{g}(\hat{x})$, whereas $\bar{f}(\hat{x}) - \bar{g}(\hat{x}) \geq 0$ by the assumption $\bar{f} \geq \bar{g}$. Therefore, (6.29) holds. □

6.4 Connection to min-max theorems on bisubmodular functions

Let $N = \{1, 2, \dots, n\}$ and denote by 3^N the set of all pairs (X, Y) of disjoint subsets X, Y of N , that is, $3^N = \{(X, Y) \mid X, Y \subseteq N, X \cap Y = \emptyset\}$. A function $f : 3^N \rightarrow \mathbb{R}$ is called *bisubmodular* if

$$\begin{aligned} & f(X_1, Y_1) + f(X_2, Y_2) \\ & \geq f(X_1 \cap X_2, Y_1 \cap Y_2) + f((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) \end{aligned}$$

holds for all $(X_1, Y_1), (X_2, Y_2) \in 3^N$. In the following we assume $f(\emptyset, \emptyset) = 0$. The associated *bisubmodular polyhedron* is defined by

$$P(f) = \{z \in \mathbb{R}^n \mid z(X) - z(Y) \leq f(X, Y) \text{ for all } (X, Y) \in 3^N\},$$

which, in turn, determines f by

$$f(X, Y) = \max\{z(X) - z(Y) \mid z \in P(f)\} \quad ((X, Y) \in 3^N). \tag{6.30}$$

If f is integer-valued, $P(f)$ is an integral polyhedron. The reader is referred to [17, Section 3.5(b)] and [18] for bisubmodular functions and polyhedra.

In a study of b -matching degree-sequence polyhedra, Cunningham–Green–Krótki [9] obtained a min-max formula for the maximum component sum $z(N) = \sum_{i \in N} z_i$ of $z \in P(f)$ upper-bounded by a given vector w .

Theorem 6.3 ([9, Theorem 4.6]) *Let $f : 3^N \rightarrow \mathbb{R}$ be a bisubmodular function with $f(\emptyset, \emptyset) = 0$, and $w \in \mathbb{R}^n$. If there exists $z \in P(f)$ with $z \leq w$, then*

$$\begin{aligned} & \max\{z(N) \mid z \in P(f), z \leq w\} \\ & = \min\{f(X, Y) + w(N \setminus X) + w(Y) \mid (X, Y) \in 3^N\}. \end{aligned} \tag{6.31}$$

Moreover, if f and w are integer-valued, then there exists an integral vector z that attains the maximum on the left-hand side of (6.31).

The min-max formula (6.31) can be extended to a box constraint (with both upper and lower bounds on z). This extension is given in (6.32) below. Although this formula is not explicit in Fujishige–Patkar [21], it can be derived without difficulty from the results of [21]; see Remark 6.2.

Theorem 6.4 ([21]) *Let $f : 3^N \rightarrow \mathbb{R}$ be a bisubmodular function with $f(\emptyset, \emptyset) = 0$, and α and β be real vectors with $\alpha \leq \beta$. If there exists $z \in P(f)$ with $\alpha \leq z \leq \beta$, then, for each $(A, B) \in 3^N$, we have*

$$\begin{aligned} & \max\{z(A) - z(B) \mid z \in P(f), \alpha \leq z \leq \beta\} \\ & = \min\{f(X, Y) + \beta(A \setminus X) + \beta(Y \setminus B) - \alpha(B \setminus Y) - \alpha(X \setminus A) \mid (X, Y) \in 3^N\}. \end{aligned} \tag{6.32}$$

Moreover, if f , α , and β are integer-valued, then there exists an integral vector z that attains the maximum on the left-hand side of (6.32).

Theorem 6.4 can be derived from Theorem 6.1 as follows. Let \hat{f} denote the convex extension of the given bisubmodular function $f : 3^N \rightarrow \mathbb{R}$, as defined by Qi [54, Eqn (5)] as a generalization of the Lovász extension of a submodular function. This function $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ is a positively homogeneous convex function and it is an extension of f in the sense that $\hat{f}(\mathbf{1}^X - \mathbf{1}^Y) = f(X, Y)$ for all $(X, Y) \in 3^N$. It follows from the positive homogeneity of \hat{f} and Lemma 11 of [54] that

$$\frac{1}{2} \left(\hat{f}(x) + \hat{f}(y) \right) \geq \hat{f} \left(\frac{x+y}{2} \right) \quad (x, y \in \mathbb{Z}^n).$$

This implies, by Theorem 3.1, that the function \hat{f} restricted to \mathbb{Z}^n is an integrally convex function. In the following we denote the restriction of \hat{f} to \mathbb{Z}^n also by \hat{f} .

Fix $(A, B) \in 3^N$ and let $C = N \setminus (A \cup B)$. We define a separable concave function $\Psi(x) = \sum_{i \in N} \psi_i(x_i)$ with $\psi_i : \mathbb{Z} \rightarrow \mathbb{R}$ given as follows: For $i \in A$,

$$\psi_i(k) = \begin{cases} \alpha_i(k - 1) & (k \geq 1), \\ \beta_i(k - 1) & (k \leq 1); \end{cases} \tag{6.33}$$

For $i \in B$,

$$\psi_i(k) = \begin{cases} \alpha_i(k + 1) & (k \geq -1), \\ \beta_i(k + 1) & (k \leq -1); \end{cases} \tag{6.34}$$

For $i \in C$,

$$\psi_i(k) = \begin{cases} \alpha_i k & (k \geq 0), \\ \beta_i k & (k \leq 0). \end{cases} \tag{6.35}$$

We apply Theorem 6.1 to the integrally convex function \hat{f} and the separable concave function Ψ . For these functions the min-max formula (6.23) reads

$$\min\{\hat{f}(x) - \Psi(x) \mid x \in \mathbb{Z}^n\} = \max\{\Psi^\circ(p) - \hat{f}^\bullet(p) \mid p \in \mathbb{R}^n\}, \tag{6.36}$$

where (6.24) gives integrality of p in the integer-valued case. In the following we show

$$\min \text{ in (6.36)} = \min \text{ in (6.32)}, \tag{6.37}$$

$$\max \text{ in (6.36)} = \max \text{ in (6.32)} \tag{6.38}$$

to obtain “max = min” in (6.32).

The proof of (6.37) consists of showing two equations

$$\min\{\hat{f}(x) - \Psi(x) \mid x \in \mathbb{Z}^n\} = \min\{\hat{f}(x) - \Psi(x) \mid x \in \{-1, 0, +1\}^n\}, \tag{6.39}$$

$$\min\{\hat{f}(x) - \Psi(x) \mid x \in \{-1, 0, +1\}^n\} = \min \text{ in (6.32)}. \tag{6.40}$$

We first show (6.40) while postponing the proof of (6.39). On identifying a vector $x \in \{-1, 0, +1\}^n$ with $(X, Y) \in 3^N$ by $x = \mathbf{1}^X - \mathbf{1}^Y$, we have $\hat{f}(x) = \hat{f}(\mathbf{1}^X - \mathbf{1}^Y) = f(X, Y)$ and

$$-\Psi(x) = \beta(A \setminus X) + \beta(Y \setminus B) - \alpha(B \setminus Y) - \alpha(X \setminus A),$$

which is easily verified from (6.33)–(6.35). Hence follows (6.40).

Next we turn to (6.38) for the maximum in (6.36). The conjugate function \hat{f}^\bullet is equal to the indicator function of $P(f)$. That is, $\hat{f}^\bullet(p)$ is equal to 0 if $p \in P(f)$, and $+\infty$ otherwise. The concave conjugate Ψ° is given by

$$\Psi^\circ(p) = \begin{cases} p(A) - p(B) & (\alpha \leq p \leq \beta), \\ -\infty & (\text{otherwise}) \end{cases}$$

for $p \in \mathbb{R}^n$. Indeed, $\text{dom } \psi_i^\circ = [\alpha_i, \beta_i]_{\mathbb{R}}$ for all $i \in N$ and, for $l \in [\alpha_i, \beta_i]_{\mathbb{R}}$, we have

$$\psi_i^\circ(l) = \begin{cases} l & (i \in A), \\ -l & (i \in B), \\ 0 & (i \in C) \end{cases}$$

from (6.33), (6.34), and (6.35). Therefore, we have

$$\begin{aligned} & \max\{\Psi^\circ(p) - \hat{f}^\bullet(p) \mid p \in \mathbb{R}^n\} \\ &= \max\{p(A) - p(B) \mid p \in P(f), \alpha \leq p \leq \beta\} = \max \text{ in (6.32)}, \end{aligned} \tag{6.41}$$

where the variable p corresponds to z in (6.32).

It remains to show (6.39). The function $\hat{f} - \bar{\Psi}$ is a polyhedral convex function and is bounded from below since the value of (6.41) is finite. Therefore, $\hat{f} - \Psi$ has a minimizer. Let $\hat{x} \in \mathbb{Z}^n$ be a minimizer of $\hat{f} - \Psi$ with $\|\hat{x}\|_\infty$ minimum. To prove by contradiction, assume $\|\hat{x}\|_\infty \geq 2$. Define $U = \{i \in N \mid \hat{x}_i = \|\hat{x}\|_\infty\}$ and $V = \{i \in N \mid \hat{x}_i = -\|\hat{x}\|_\infty\}$, and let $d = \mathbf{1}^U - \mathbf{1}^V$. By (6.33)–(6.35), each ψ_i is a linear (affine) function on each of the intervals $(-\infty, -1]$ and $[+1, +\infty)$. Combining this with the fundamental property of the extension \hat{f} , we see that there exists a vector $q \in \mathbb{R}^n$ for which

$$(\hat{f} - \Psi)(\hat{x} \pm d) = (\hat{f} - \Psi)(\hat{x}) \pm (f(U, V) - \langle q, d \rangle)$$

holds, where the double-sign corresponds. Since \hat{x} is a minimizer of $\hat{f} - \Psi$, we must have $f(U, V) - \langle q, d \rangle = 0$. This implies, however, that $\hat{x} - d$ is also a minimizer of $\hat{f} - \Psi$, whereas we have $\|\hat{x} - d\|_\infty < \|\hat{x}\|_\infty$, a contradiction. We have thus completed the derivation of Theorem 6.4 from Theorem 6.1.

Remark 6.2 The min-max formula (6.32) can be derived from the results of [21] as follows. Given $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \leq \beta$, we can consider a bisubmodular function $w_{\alpha\beta}$ defined by $w_{\alpha\beta}(X, Y) = \beta(X) - \alpha(Y)$ for disjoint subsets X and Y . The convolution of f with $w = w_{\alpha\beta}$ is defined (and denoted) as

$$\begin{aligned} & (f \circ w)(A, B) \\ &= \min\{f(X, Y) + w(A \setminus X, B \setminus Y) + w(Y \setminus B, X \setminus A) \mid (X, Y) \in 3^N\} \\ &= \min\{f(X, Y) + \beta(A \setminus X) + \beta(Y \setminus B) - \alpha(B \setminus Y) - \alpha(X \setminus A) \mid (X, Y) \in 3^N\}. \end{aligned} \tag{6.42}$$

This function is bisubmodular [21, Theorem 3.2]. By (6.30) applied to $f \circ w$, we obtain

$$(f \circ w)(A, B) = \max\{z(A) - z(B) \mid z \in P(f \circ w)\} \quad ((A, B) \in 3^N). \tag{6.43}$$

On the other hand, Theorem 3.3 of [21] shows

$$P(f \circ w) = P(f) \cap P(w) = \{z \mid z \in P(f), \alpha \leq z \leq \beta\}. \quad (6.44)$$

By substituting this expression into $P(f \circ w)$ on the right-hand side of (6.43) we obtain

$$\begin{aligned} (f \circ w)(A, B) &= \max\{z(A) - z(B) \mid z \in P(f \circ w)\} \\ &= \max\{z(A) - z(B) \mid z \in P(f), \alpha \leq z \leq \beta\}. \end{aligned} \quad (6.45)$$

The combination of (6.42) and (6.45) gives the desired equality (6.32).

Finally we mention that the paper [21] considers a more general setting where $\alpha \in (\mathbb{R} \cup \{-\infty\})^n$, $\beta \in (\mathbb{R} \cup \{+\infty\})^n$, and f is a bisubmodular function defined on a subset \mathcal{F} of 3^N such that

$$\begin{aligned} (X_1, Y_1), (X_2, Y_2) \in \mathcal{F} &\implies (X_1 \cap X_2, Y_1 \cap Y_2) \in \mathcal{F}, \\ (X_1, Y_1), (X_2, Y_2) \in \mathcal{F} &\implies ((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) \in \mathcal{F}. \end{aligned}$$

The min-max formula (6.32) remains true in this general case. ■

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