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ORIGINAL PAPER

Equivalence between substitutability and M$^\natural$-concavity for set functions under discrete transfers

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Abstract

Fujishige and Yang (2003) prove the equivalence between two fundamental conditions of a valuation function in a market model with indivisibilities, i.e., the gross substitutes (GS) condition and M$^\natural$-concavity. We introduce a weaker variant of the GS condition that concerns discrete price changes rather than continuous price changes. We show that this weaker variant is equivalent to M$^\natural$-concavity if the valuation function takes integer values and has an M$^\natural$-convex effective domain containing the empty set. Our result indicates that assuming the weaker GS condition is sufficient for M$^\natural$-concavity in existing auction models.

Keywords M-concave function · Exhange property · Gross substitutes condition · Indivisibility

Mathematics Subject Classification 90C27 · 91B42 · 52B40

1 Introduction

In recent years, discrete convex analysis [1] has been applied to the analysis of market models with indivisibilities (see [2] for s survey). One of the major advantages of discrete convex analysis is to enable the analysis of computational issues; minimization algorithms of a discrete convex function are executed in polynomial time. This property has been utilized to construct a computationally efficient algorithm for finding equilibrium/stable outcomes; see [3–6]. To apply discrete convex analysis, it is essential to reveal a connection between the assumptions in market models and discrete convexity/concavity. Fujishige and Yang [7] identify one such connection by proving

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that an agent's valuation function satisfies the gross substitutes (GS) condition [8] if and only if her valuation function is M$^\natural$-concave.[1]

The purpose of this note is to show that, under certain conditions, assuming a weaker variant of the GS condition is sufficient for M$^\natural$-concavity. The GS condition requires a form of consistency of demand behavior for any continuous price change. We weaken this condition by requiring consistency of demand behavior only for discrete (integer) price changes. Our main theorem states that this weaker variant is equivalent to M$^\natural$-concavity if the valuation function takes integer values and has an M$^\natural$-convex effective domain containing the empty set. We point out that the assumptions in the if-clause of the above assertion are satisfied in existing auction models.

2 Preliminaries

Let K be a nonempty and finite set of **commodities**. A subset $A \subseteq K$ is called a **bundle**. Consider an agent who has a **valuation function** $U : 2^K \to \mathbb{Z} \cup \{-\infty\}$ over the bundles; note that $U(\cdot)$ takes only integer values (except $-\infty$). The **effective domain** of $U(\cdot)$ is given by

$$\mathrm{dom}\, U \equiv \{A \subseteq K : U(A) > -\infty\}.$$

Throughout the analysis, we assume that dom $U \neq \emptyset$.

A **price vector** is given by $p \in \mathbb{Z}^K$. For $k \in K$, let $\mathbb{1}^k$ denote the k-**th unit vector**. For $p \in \mathbb{Z}^K$ and $A \in 2^K$, we define $U[-p](A) \equiv U(A) - \sum_{k \in A} p_k$, representing the agent's **utility** from consuming A under p. We define the **demand correspondence** $D : \mathbb{Z}^K \rightrightarrows 2^K$ by

$$D(p) = \{A \in 2^K : U[-p](A) \geq U[-p](A') \text{ for all } A' \in 2^K\} \text{ for all } p \in \mathbb{Z}^K.$$

This set collects the bundles that bring the highest utility.

For $A \subseteq K$, $k \in A$ and $\ell, \ell', \ell'' \in K \setminus A$, we use the following short-hand notation:

$$\begin{aligned} &A - k \equiv A \setminus \{k\},\ A + \ell \equiv A \cup \{\ell\},\\ &A + \ell + \ell' \equiv (A + \ell) + \ell',\ A + \ell - k \equiv (A + \ell) - k,\ A - k + \ell \equiv (A - k) + \ell,\\ &A + \ell + \ell' + \ell'' \equiv (A + \ell + \ell') + \ell''. \end{aligned}$$

For an auxiliary symbol θ, we use notation $A + \theta \equiv A$ and $A - \theta \equiv A$.

We say that $\mathcal{B} \subseteq 2^K$ with $\mathcal{B} \neq \emptyset$ is an **M$^\natural$-convex family** if, for any $A, B \in \mathcal{B}$ and $k \in A \setminus B$, there exists $\ell \in (B \setminus A) \cup \{\theta\}$ such that

$$A - k + \ell \in \mathcal{B} \text{ and } B + k - \ell \in \mathcal{B}.$$

We say that $U(\cdot)$ is an **M$^\natural$-concave function** [1] if it satisfies the following property:

[1] Connections between substitutes conditions and discrete concavity have been studied extensively in the literature; see [9] or [10].

($\mathbf{B}^\natural$**-EXC**) For any $A, B \subseteq K$ and $k \in A \backslash B$, there exists $\ell \in (B \backslash A) \cup \{\theta\}$ such that

$$U(A - k + \ell) + U(B + k - \ell) \geq U(A) + U(B).$$

This property, known as the *exchange property*, has the following local versions[2]:

($\mathbf{B}^\natural$-$\mathbf{EXC}^1_{\text{loc}}$) For any $A \subseteq K$ with $|K \backslash A| \geq 2$ and any $k, k' \in K \backslash A$ with $k \neq k'$, it holds that

$$U(A + k) + U(A + k') \geq U(A + k + k') + U(A).$$

($\mathbf{B}^\natural$-$\mathbf{EXC}^2_{\text{loc}}$) For any $A \subseteq K$ with $|K \backslash A| \geq 3$ and any distinct elements $k, k', \ell \in K \backslash A$, it holds that

$$\max\left\{U(A + k + \ell) + U(A + k'), U(A + k' + \ell) + U(A + k)\right\} \\ \geq U(A + k + k') + U(A + \ell). \tag{1}$$

Next, we introduce the gross substitutes condition introduced by Kelso and Crawford [8].

($\mathbf{GS}_\mathrm{R}$) For any $p, q \in \mathbb{R}^K$ with $p \leq q$ and any $A \in D(p)$, there exists $B \in D(q)$ such that

$$\left[k \in A \text{ and } p_k = q_k\right] \Longrightarrow k \in B.$$

This condition states that, if some commodities become less appealing due to price increase, then the agent can shift her demand toward other commodities whose prices are kept unchanged. This shift indicates that the commodities are substitutable. We consider a weaker variant that replaces the continuous domain of price vectors ($\mathbb{R}^K$) with a discrete domain ($\mathbb{Z}^K$).

($\mathbf{GS}_\mathbf{Z}$) For any $p, q \in \mathbb{Z}^K$ with $p \leq q$ and any $A \in D(p)$, there exists $B \in D(q)$ such that

$$\left[k \in A \text{ and } p_k = q_k\right] \Longrightarrow k \in B.$$

3 Main theorem

Fujishige and Yang [7] prove the following theorem:

Theorem FY *The following are equivalent:*

$U(\cdot)$ satisfies (GS_R).

[2] Prior work has investigated the relationship between local exchange properties and discrete convexity; see, for example, Section 3.1 of [2].

$U(\cdot)$ satisfies (B$^\natural$-EXC).

We prove that, under certain assumptions, (GS$_\mathrm{R}$) of the above theorem can be replaced with (GS$_\mathrm{Z}$).

Theorem 1 *Let $U : 2^K \to \mathbb{Z} \cup \{-\infty\}$ be a valuation function such that the effective domain dom U is an M$^\natural$-convex family with $\emptyset \in$ dom U. Then, the following are equivalent:*

(i) $U(\cdot)$ *satisfies* (GS$_\mathrm{Z}$).
(ii) $U(\cdot)$ satisfies (B$^\natural$-EXC).

Before proving Theorem 1, we refer to a related result and explain the difference in proof ideas. Corollary 2.3 of Murota [11] shows that $U : 2^K \to \mathbb{Z} \cup \{-\infty\}$ satisfies (B$^\natural$-EXC) if and only if $D(p)$ is an M$^\natural$-convex family for all $p \in \mathbb{Z}^K$. Murota's proof of the "if" part and our proof of (i) $\Longrightarrow$ (ii) of Theorem 1 are similar in that both proofs make use of local versions of (B$^\natural$-EXC) (see Lemma 2.1 of Murota's proof). The critical step of Murota's proof is to apply the duality theorem: rephrasing the maximum value in the inequality of M$^\natural$-concavity (see the left-hand side of (1) of this paper) as the maximum weight of a matching, the dual variable $p \in \mathbb{Z}^K$ allows us to apply M$^\natural$-convexity of $D(p)$ (see Murota's proof of Lemma 2.5). If U takes only integer values, the dual varial p can be taken from $\mathbb{Z}^K$, and hence, considering only integer price vectors suffices. We cannot directly apply this proof technique for proving (i) $\Longrightarrow$ (ii) because the GS condition is defined for a change in p, rather than for a fixed p. We need to change prices in a tractable manner, which is addressed in the proof below.

Proof of Theorem 1 The implication (ii) $\Longrightarrow$ (i) follows from Theorem FY. We prove the converse implication (i) $\Longrightarrow$ (ii). Under the assumption that dom$U$ is an M$^\natural$-convex family with $\emptyset \in$ dom U, (B$^\natural$-EXC) is known to be equivalent to the combination of (B$^\natural$-EXC$^1_\mathrm{loc}$) and (B$^\natural$-EXC$^2_\mathrm{loc}$).[3] We prove that $U(\cdot)$ satisfies these two properties in Lemmas 1 and 2, respectively.

Lemma 1 *If $U(\cdot)$ satisfies* (GS$_\mathrm{Z}$), *then it satisfies* (B$^\natural$-EXC$^1_\mathrm{loc}$).

Proof Let $A \subseteq K$ with $|K \backslash A| \geq 2$ and $k, k' \in K \backslash A$ with $k \neq k'$. Our goal is to prove that

$$U(A + k) + U(A + k') \geq U(A + k + k') + U(A).$$

If $A \notin$ dom U or $A + k + k' \notin$ dom U, then the right-hand side is equal to $-\infty$ and hence the desired inequality holds. In what follows we assume $A \in$ dom U and $A + k + k' \in$ dom U.

By choosing sufficiently low prices for the commodities in $A+k+k'$ and sufficiently high prices for the commodities outside $A+k+k'$, we construct a price vector $p \in \mathbb{Z}^K$ such that

$$D(p) = \{A + k + k'\}. \tag{2}$$

[3] See, for example, Theorem 3.3 of [2].

We define $\chi \in \mathbb{Z}^K$ by

$$\chi_m = \begin{cases} 1 & \text{if } m \notin A + k', \\ 0 & \text{if } m \in A + k'. \end{cases}$$

For any $\alpha \in \mathbb{Z}_+$, the following hold[4]:

$$\begin{aligned} & U[-p-(\alpha+1)\chi](A') = U[-p-\alpha\chi](A') \text{ if } A' \subseteq A+k', \\ & U[-p-(\alpha+1)\chi](A') \leq U[-p-\alpha\chi](A') - 1 \text{ if } A' \nsubseteq A+k'; \\ & \text{in particular, } U[-p-(\alpha+1)\chi](A+k+k') = U[-p-\alpha\chi](A+k+k') - 1. \end{aligned} \tag{3}$$

Hence, together with (2) and the facts that $U(\cdot)$ takes only integer values (except $-\infty$) and $A \in \text{dom}\, U$, the following integer $\alpha' \in \mathbb{Z}_+$ given by

$$\alpha' = \min\{\alpha \in \mathbb{Z}_+ : D(p + \alpha\chi) \backslash D(p) \neq \emptyset\}$$

is well-defined. Let $p' \equiv p + \alpha'\chi$. By (2), (3) and the construction of α',

$$A + k + k' \in D(p'), \tag{4}$$

$$A' \in D(p') \backslash D(p) \Longrightarrow A' \subseteq A + k'. \tag{5}$$

By (2), (4) and (5),

$$A' \in D(p') \Longrightarrow [A' = A + k + k'] \text{ or } [A' \subseteq A + k']. \tag{6}$$

The following condition holds:

$$A + k' \in D(p'). \tag{7}$$

Proof of (7)
Suppose to the contrary that

$$A + k' \notin D(p').$$

Together with $D(p') \backslash D(p) \neq \emptyset$ and (5), we have $D(p' + \mathbb{1}^k) \cap \{A+k', A+k+k'\} = \emptyset$. Combining this condition with (6),

$$\nexists A' \in D(p' + \mathbb{1}^k) \text{ s.t. } A' \supseteq A + k',$$

a contradiction to (GS_Z) applied to p', $p' + \mathbb{1}^k$ and $A + k + k' \in D(p')$.

[4] $\mathbb{Z}_+$ denotes the set of non-negative integers including 0.

end of proof of (7)

Furthermore, the following condition holds:

$$k' \in A' \text{ for all } A' \in D(p')\backslash\{A + k', A + k + k'\}. \tag{8}$$

Proof of (8)

Suppose to the contrary that

$$\exists A' \in D(p')\backslash\{A + k', A + k + k'\} \text{ s.t. } k' \notin A'.$$

This implies $D(p' + \mathbb{1}^{k'}) \cap \{A + k', A + k + k'\} = \emptyset$. In particular, $A + k + k' \notin D(p' + \mathbb{1}^{k'})$. Together with (6),

$$\nexists A' \in D(p' + \mathbb{1}^{k'}) \text{ s.t. } A' \supseteq A + k,$$

a contradiction to ($\text{GS}_\mathbb{Z}$) applied to p', $p' + \mathbb{1}^{k'}$ and $A + k + k' \in D(p')$.

end of proof of (8)

We define $\chi' \in \mathbb{Z}^K$ by

$$\chi'_m = \begin{cases} 1 & \text{if } m \notin A + k, \\ 0 & \text{if } m \in A + k. \end{cases}$$

For any $\alpha \in \mathbb{Z}_+$, the following hold:

$$\begin{aligned} & U[-p' - (\alpha + 1)\chi'](A') = U[-p' - \alpha\chi'](A') \text{ if } A' \subseteq A + k, \\ & U[-p' - (\alpha + 1)\chi'](A') \leq U[-p' - \alpha\chi'](A') - 1 \text{ if } A' \nsubseteq A + k; \\ & \text{in particular, } U[-p' - (\alpha + 1)\chi'](A + k') = U[-p' - \alpha\chi'](A + k') - 1, \text{ and} \\ & \qquad U[-p' - (\alpha + 1)\chi'](A + k + k') = U[-p' - \alpha\chi'](A + k + k') - 1. \end{aligned} \tag{9}$$

Together with (8) and the facts that $U(\cdot)$ takes only integer values (except $-\infty$) and $A \in \text{dom}\, U$, the following integer $\alpha'' \in \mathbb{Z}_+$ given by

$$\alpha'' = \min\{\alpha \in \mathbb{Z}_+ : D(p' + \alpha\chi')\backslash D(p') \neq \emptyset\}$$

is well-defined. Let $p'' \equiv p' + \alpha''\chi'$. By (4), (7), (9), and the construction of α'',

$$A + k', A + k + k' \in D(p''), \tag{10}$$

$$A' \in D(p'')\backslash D(p') \Longrightarrow A' \subseteq A + k. \tag{11}$$

By (6) and (11),

$$A' \in D(p'') \Longrightarrow A' \subseteq A + k + k'. \tag{12}$$

The following condition holds:

$$A + k \in D(p''). \tag{13}$$

Proof of (13)
Suppose to the contrary that

$$A + k \notin D(p'').$$

Together with $D(p'')\backslash D(p') \neq \emptyset$ and (11), we have $D(p''+\mathbb{1}^{k'}) \cap \{A+k,\ A+k+k'\} = \emptyset$. Combining this condition with (12),

$$\nexists A' \in D(p'' + \mathbb{1}^{k'}) \text{ s.t. } A' \supseteq A + k,$$

a contradiction to ($\mathrm{GS}_{\mathbb{Z}}$) applied to p'', $p'' + \mathbb{1}^{k'}$ and $A + k + k' \in D(p')$.

end of proof of (13)
By (10) and (13),

$$\begin{aligned} & A + k \in D(p'') \text{ and } A + k' \in D(p'') \\ \Longrightarrow\ & U[-p''](A + k) \geq U[-p''](A + k + k') \text{ and } U[-p''](A + k') \geq U[-p''](A) \\ \Longrightarrow\ & U[-p''](A + k) + U[-p''](A + k') \geq U[-p''](A + k + k') + U[-p''](A) \\ \Longrightarrow\ & U(A + k) + U(A + k') \geq U(A + k + k') + U(A), \end{aligned} \tag{14}$$

as desired. □

Lemma 2 *If* $U(\cdot)$ *satisfies* ($\mathrm{GS}_{\mathbb{Z}}$), *then it satisfies* ($\mathrm{B}^{\natural}\text{-}\mathrm{EXC}^{2}_{\mathrm{loc}}$).

Proof Let $A \subseteq K$ with $|K\backslash A| \geq 3$ and let $k, k', \ell \in K\backslash A$ be distinct elements. Our goal is to prove that

$$\begin{aligned} & \max\{U(A + k + \ell) + U(A + k'), U(A + k' + \ell) + U(A + k)\} \\ & \geq U(A + k + k') + U(A + \ell). \end{aligned} \tag{15}$$

If $A + \ell \notin \mathrm{dom}\, U$ or $A + k + k' \notin \mathrm{dom}\, U$, then the right-hand side is equal to $-\infty$ and hence the desired inequality holds. In what follows we assume $A + \ell \in \mathrm{dom}\, U$ and $A + k + k' \in \mathrm{dom}\, U$.

By choosing sufficiently low prices for the commodities in $A+k+k'$ and sufficiently high prices for the commodities outside $A+k+k'$, we construct a price vector $p \in \mathbb{Z}^K$ such that

$$D(p) = \{A + k + k'\}. \tag{16}$$

As we did in the proof of Lemma 1, we iterative increase the prices of commodities from p by adding vectors χ and χ', with the only difference that the price of ℓ is kept unchanged, and derive a price vector p'' with the properties stated below.

Claim 1 *There exists a price vector p'' such that*

$$A' \in D(p'') \Longrightarrow A' \subsetneq A + k + k' + \ell, \tag{17}$$

and one of the following conditions holds:

$$\{A + k + k', A + k', A + k + \ell\} \subseteq D(p''), \tag{18}$$
$$\{A + k + k', A + k' + \ell, A + k\} \subseteq D(p''), \tag{19}$$
$$\{A + k + k', A + k', A + k\} \subseteq D(p''), \tag{20}$$
$$\{A + k + k', A + k' + \ell, A + k + \ell\} \subseteq D(p''). \tag{21}$$

Proof See Appendix A. □

If (18) or (19) holds, then

$$[A + k' \in D(p'') \text{ and } A + k + \ell \in D(p'')] \text{ or}$$
$$[A + k \in D(p'') \text{ and } A + k' + \ell \in D(p'')].$$

Hence, by following the same argument as in (14), the desired inequality follows. Below we deal with the remaining two possibilities, (20) and (21). The proof strategy is the same for both cases, except for the following: in the case of (20), we adjust p'' so that $A + k + \ell$ or $A + k' + \ell$ becomes a utility-maximizing bundle, whereas in the case of (21), we adjust p'' so that $A + k$ or $A + k'$ becomes a utility-maximizing bundle. Below we only deal with the case of (20) and relegate the proof for the case of (21) to Appendix B.

Suppose that (20) holds. We define $\chi'' \in \mathbb{Z}^K$ by

$$\chi''_m = \begin{cases} -1 & \text{if } m \in A + \ell, \\ 0 & \text{if } m \in \{k, k'\}, \\ 1 & \text{if } m \notin A + k + k' + \ell. \end{cases}$$

For any $\alpha \in \mathbb{Z}_+$, the following hold:

$$\begin{aligned} &U[-p' - (\alpha + 1)\chi''](A') = U[-p' - \alpha\chi''](A') + |A| + 1 \\ &\qquad\qquad \text{if } A' \supseteq A + \ell \text{ and } A' \subseteq A + k + k' + \ell, \\ &U[-p' - (\alpha + 1)\chi''](A') \leq U[-p' - \alpha\chi''](A') + |A| \\ &\qquad\qquad \text{if } A' \not\supseteq A + \ell \text{ or } A' \not\subseteq A + k + k' + \ell; \\ &\text{in particular, } U[-p' - (\alpha + 1)\chi''](A') = U[-p' - \alpha\chi''](A') + |A| \\ &\qquad\qquad \text{if } A' \in \{A + k + k', A + k', A + k\}. \end{aligned}$$

Together with (20) and the facts that $A + \ell \in \mathrm{dom}\, U$ and $U(\cdot)$ takes only integer values (except $-\infty$), there exists $\alpha''' \in \mathbb{Z}_+$ such that[5]:

$$\{A + k + k', A + k', A + k\} \subseteq D(p'' + \alpha'''\chi''), \tag{22}$$
$$\exists A' \in D(p'' + \alpha'''\chi'') \text{ s.t. } A' \supseteq A + \ell \text{ and } A' \subseteq A + k + k' + \ell. \tag{23}$$

Let $p''' \equiv p'' + \alpha'''\chi''$. Note that A' in (23) takes only four possible forms: $A + \ell$, $A + k + \ell$, $A + k' + \ell$, or $A + k + k' + \ell$. We divide the remaining part into three cases.

Case 1: Suppose $A + k + \ell \in D(p''')$ or $A + k' + \ell \in D(p''')$. Together with (22),

$$[A + k' \in D(p'') \text{ and } A + k + \ell \in D(p'')] \text{ or}$$
$$[A + k \in D(p'') \text{ and } A + k' + \ell \in D(p'')].$$

Hence, by following the same argument as in (14), the desired inequality follows.

Case 2: Suppose $A + k + k' + \ell \in D(p''')$. By Lemma 1,

$$U[-p'''](A + k + \ell) + U[-p'''](A + k + k')$$
$$\geq U[-p'''](A + k + k' + \ell) + U[-p'''](A + k).$$

Together with $A+k+k'$, $A+k \in D(p''')$ (which follows from (22)) and $A+k+k'+\ell \in D(p''')$, we have $A + k + \ell \in D(p''')$. Namely, this case is subsumed in Case 1 and hence the desired inequality holds.

Case 3: The remaining possibility is that $A+k+\ell \notin D(p''')$ and $A+k'+\ell \notin D(p''')$ and $A + k + k' + \ell \notin D(p''')$. Together with (22) and (23),

$$A + k + k' \in D(p'''),\ \ A + \ell \in D(p'''). \tag{24}$$

By (17) and the definition of χ'',

$$A' \in D(p''') \Longrightarrow A' \subseteq A + k + k' + \ell. \tag{25}$$

We define $\chi''' \in \mathbb{Z}^K$ by

$$\chi'''_m = \begin{cases} -1 & \text{if } m \in A + k + \ell, \\ 0 & \text{if } m \notin A + k + \ell. \end{cases}$$

Note that $U[-p''' - \chi'''](A') \leq U[-p'''](A') + |A| + 2$ for all $A' \subseteq K$, with equality holding only if $A' \supseteq A + k + \ell$. By $A + k + \ell \notin D(p''')$, $A + k + k' + \ell \notin D(p''')$ and (25),

$$U[-p''' - \chi'''](A') = U[-p'''](A') + |A| + 2 \Longrightarrow A' \notin D(p'''). \tag{26}$$

[5] We allow the possibility of $\alpha''' = 0$.

Furthermore, we have

$$U[-p''' - \chi'''](A') = U[-p'''](A') + |A| + 1 \text{ for all } A' \in \{A + k + k', A + \ell\}. \tag{27}$$

By (24), (26), (27), and the fact that $U(\cdot)$ takes only integer values (except $-\infty$), we obtain

$$A + k + k' \in D(p''' + \chi'''),\ \ A + \ell \in D(p''' + \chi'''). \tag{28}$$

Moreover, the following condition holds:

$$A' \in D(p''' + \chi''') \text{ and } A' \supseteq A + k' \Longrightarrow k \in A'. \tag{29}$$

Proof of (29)

Let $A' \in D(p''' + \chi''')$ with $A' \supseteq A + k'$. Suppose that $A' \not\subseteq A + k + k' + \ell$. By the contrapositive of (25), $A' \notin D(p''')$. Given that (24) and (27) hold, in order for A' to be a utility-maximizing bundle at $p''' + \chi'''$, we must have

$$U[-p''' - \chi'''](A') = U[-p'''](A') + |A| + 2,$$

which implies $A' \supseteq A + k + \ell$, in particular $k \in A'$. Thus, (29) holds.

The remaining possibility is that $A' \subseteq A + k + k' + \ell$. To obtain the desired claim, it suffices to prove that

$$A + k' \notin D(p''' + \chi''') \text{ and } A + k' + \ell \notin D(p''' + \chi'''). \tag{30}$$

For these two bundles, the following conditions hold:

$$\begin{aligned}
&U[-p''' - \chi'''](A + k') = U[-p'''](A + k') + |A|,\\
&A + k' + \ell \notin D(p'''),\ \text{ and}\\
&U[-p''' - \chi'''](A + k' + \ell) = U[-p'''](A + k' + \ell) + |A| + 1,
\end{aligned}$$

where $A + k' + \ell \notin D(p''')$ in the second line follows from the supposition of Case 3. Together with (24) and (27), we obtain (30).

end of proof of (29)

Now, consider $p''' + \chi''' + 1\!\!1^k$. By (28),

$$A + \ell \in D(p''' + \chi''' + 1\!\!1^k) \text{ and } A + k + k' \notin D(p''' + \chi''' + 1\!\!1^k).$$

By (29), any bundle $A' \subseteq K$ with $A' \supseteq A + k'$ satisfies $A' \notin D(p''' + \chi''' + 1\!\!1^k)$. We obtain a contradiction to (GS_Z) applied to $p''' + \chi'''$, $p''' + \chi''' + 1\!\!1^k$ and $A + k + k' \in D(p''' + \chi''')$ (which follows from (28)). It follows that Case 3 never occurs. □

We resume the proof of Theorem 1. By Lemmas 1 and 2, $U(\cdot)$ satisfies (B$^\natural$-EXC$^1_{\text{loc}}$) and (B$^\natural$-EXC$^2_{\text{loc}}$). As noted in the beginning of the proof of Theorem 1, the combination of these two properties is equivalent to M$^\natural$-concavity. □

In Theorem 1, we assume that the effective domain dom U is an M$^\natural$-convex family containing the empty set. In an auction setting, it is often assumed that every agent has an integer-valued valuation function with dom $U = \{A \subseteq K : A \subseteq \bar{K}\}$ for some $\bar{K} \subseteq K$ (see, e.g., [12] or [13]), in which case our theorem is applicable. It remains an open question whether the assumption can be weakened.

Appendix A: proof of the Claim in the proof of Lemma 2

We define $\chi \in \mathbb{Z}^K$ by

$$\chi_m = \begin{cases} 1 & \text{if } m \notin A + k' + \ell, \\ 0 & \text{if } m \in A + k' + \ell. \end{cases}$$

For any $\alpha \in \mathbb{Z}_+$, the following hold:

$$\begin{aligned} &U[-p - (\alpha+1)\chi](A') = U[-p - \alpha\chi](A') \text{ if } A' \subseteq A + k' + \ell, \\ &U[-p - (\alpha+1)\chi](A') \leq U[-p - \alpha\chi](A') - 1 \text{ if } A' \not\subseteq A + k' + \ell; \qquad (31) \\ &\text{in particular, } U[-p - (\alpha+1)\chi](A + k + k') = U[-p - \alpha\chi](A + k + k') - 1. \end{aligned}$$

Hence, together with (16) and the facts that $U(\cdot)$ takes only integer values (except $-\infty$) and $A + \ell \in \text{dom}\, U$, the following integer $\alpha' \in \mathbb{Z}_+$ given by

$$\alpha' = \min\{\alpha \in \mathbb{Z}_+ : D(p + \alpha\chi) \setminus D(p) \neq \emptyset\}$$

is well-defined. Let $p' \equiv p + \alpha'\chi$. By (16), (31), and the construction of α',

$$A + k + k' \in D(p'), \qquad (32)$$

$$A' \in D(p') \setminus D(p) \Longrightarrow A' \subseteq A + k' + \ell. \qquad (33)$$

By (16), (32) and (33),

$$A' \in D(p') \Longrightarrow [A' = A + k + k'] \text{ or } [A' \subseteq A + k' + \ell]. \qquad (34)$$

The following condition holds:

$$\{A + k', A + k' + \ell\} \cap \big(D(p') \setminus D(p)\big) \neq \emptyset. \qquad (35)$$

Proof of (35)
Suppose to the contrary that

$$\{A + k', A + k' + \ell\} \cap \big(D(p') \setminus D(p)\big) = \emptyset.$$

Together with $D(p')\backslash D(p) \neq \emptyset$ and (33), we have

$$D(p' + \mathbb{1}^k) \cap \{A + k', A + k' + \ell, A + k + k'\} = \emptyset.$$

Combining this condition with (34),

$$\nexists A' \in D(p' + \mathbb{1}^k) \text{ s.t. } A' \supseteq A + k',$$

a contradiction to (GS$_Z$) applied to p', $p' + \mathbb{1}^k$ and $A + k + k' \in D(p')$.

end of proof of (35)

Furthermore, the following condition holds:

$$k' \in A' \text{ for all } A' \in D(p')\backslash\{A + k', A + k' + \ell, A + k + k'\}. \tag{36}$$

Proof of (36)

Suppose to the contrary that

$$\exists A' \in D(p')\backslash\{A + k', A + k' + \ell, A + k + k'\} \text{ s.t. } k' \notin A'.$$

This implies $D(p' + \mathbb{1}^{k'}) \cap \{A + k', A + k' + \ell, A + k + k'\} = \emptyset$. In particular, $A + k + k' \notin D(p' + \mathbb{1}^{k'})$. Together with (33),

$$\nexists A' \in D(p' + \mathbb{1}^{k'}) \text{ s.t. } A' \supseteq A + k,$$

a contradiction to (GS$_Z$) applied to p', $p' + \mathbb{1}^{k'}$ and $A + k + k' \in D(p')$.

end of proof of (36)

We define $\chi' \in \mathbb{Z}^K$ by

$$\chi' = \begin{cases} 1 & \text{if } m \notin A + k + \ell, \\ 0 & \text{if } m \in A + k + \ell. \end{cases}$$

For any $\alpha \in \mathbb{Z}_+$, the following hold:

$$\begin{aligned} &U[-p' - (\alpha + 1)\chi'](A') = U[-p' - \alpha\chi'](A') \text{ if } A' \subseteq A + k + \ell, \\ &U[-p' - (\alpha + 1)\chi'](A') \leq U[-p' - \alpha\chi'](A') - 1 \text{ if } A' \nsubseteq A + k + \ell; \end{aligned} \tag{37}$$

in particular, $U[-p' - (\alpha + 1)\chi'](A + k') = U[-p' - \alpha\chi'](A + k') - 1$,

$$U[-p' - (\alpha + 1)\chi'](A + k' + \ell) = U[-p' - \alpha\chi'](A + k' + \ell) - 1,$$
$$U[-p' - (\alpha + 1)\chi'](A + k + k') = U[-p' - \alpha\chi'](A + k + k') - 1.$$

Hence, together with (36) and the facts that $U(\cdot)$ takes only integer values (except $-\infty$) and $A + \ell \in \text{dom}\, U$, the following integer $\alpha'' \in \mathbb{Z}_+$ given by

$$\alpha'' = \min\{\alpha \in \mathbb{Z}_+ : D(p' + \alpha\chi')\backslash D(p') \neq \emptyset\}$$

is well-defined. Let $p'' \equiv p' + \alpha''\chi'$. By (32), (35), (37), and the construction of α'',

$$A + k + k' \in D(p''), \tag{38}$$
$$\{A + k', A + k' + \ell\} \cap D(p'') \neq \emptyset, \tag{39}$$
$$A' \in D(p'')\backslash D(p') \Longrightarrow A' \subseteq A + k + \ell. \tag{40}$$

By (34) and (40),

$$A' \in D(p'') \Longrightarrow A' \subsetneq A + k + k' + \ell,$$

which establishes (17) in the statement of the claim.

The following condition holds:

$$\{A + k, A + k + \ell\} \cap D(p'') \neq \emptyset. \tag{41}$$

Proof of (41)

Suppose to the contrary that

$$\{A + k, A + k + \ell\} \cap D(p'') = \emptyset.$$

Together with $D(p'')\backslash D(p') \neq \emptyset$ and (40), we have

$$D(p'' + \mathbb{1}^{k'}) \cap \{A + k, A + k + \ell, A + k + k'\} = \emptyset.$$

Combining this condition with (17),

$$\nexists A' \in D(p'' + \mathbb{1}^{k'}) \text{ s.t. } A' \supseteq A + k,$$

a contradiction to (GS_Z) applied to p'', $p'' + \mathbb{1}^{k'}$ and $A + k + k' \in D(p'')$.

end of proof of (41)

Now, given that (38) holds, (39) and (41) give rise to four possible combinations of utility-maximizing bundles:

$$\{A + k + k', A + k', A + k + \ell\} \subseteq D(p''),$$
$$\{A + k + k', A + k' + \ell, A + k\} \subseteq D(p''),$$
$$\{A + k + k', A + k', A + k\} \subseteq D(p''),$$
$$\{A + k + k', A + k' + \ell, A + k + \ell\} \subseteq D(p''),$$

which establishes (18)-(21) in the statement of the claim. □

Appendix B: proof of Lemma 2 when (21) holds

Suppose that (21) holds. Recall that p'' is chosen from the Claim. We define $\chi'' \in \mathbb{Z}^K$ by

$$\chi''_m = \begin{cases} 1 & \text{if } m \in \{k, k', \ell\} \\ 2 & \text{if } m \notin A + k + k' + \ell, \\ 0 & \text{if } m \in A. \end{cases}$$

Then, for any $\alpha \in \mathbb{Z}_+$, the following hold:

$$\begin{aligned} &U[-p'' - (\alpha+1)\chi''](A') = U[-p'' - \alpha\chi''](A') \text{ if } A' \subseteq A, \\ &U[-p'' - (\alpha+1)\chi''](A') = U[-p'' - \alpha\chi''](A') - 1 \\ &\qquad\qquad\qquad\qquad \text{if } A' \subseteq A + k + k' + \ell \text{ and } |A' \cap \{k, k', \ell\}| = 1, \\ &U[-p'' - (\alpha+1)\chi''](A') \leq U[-p'' - \alpha\chi''](A') - 2 \\ &\qquad\qquad\qquad\qquad \text{if } A' \nsubseteq A + k + k' + \ell \text{ or } |A' \cap \{k, k', \ell\}| \geq 2; \end{aligned} \tag{42}$$

in particular,

$$\begin{aligned} U[-p'' - (\alpha+1)\chi''](A + k + k') &= U[-p'' - \alpha\chi''](A + k + k') - 2, \\ U[-p'' - (\alpha+1)\chi''](A + k' + \ell) &= U[-p'' - \alpha\chi''](A + k' + \ell) - 2, \\ U[-p'' - (\alpha+1)\chi''](A + k + \ell) &= U[-p'' - \alpha\chi''](A + k + \ell) - 2. \end{aligned}$$

Let $\alpha''' \in \mathbb{Z}_+$ be the solution to the following problem:

$$\min\left\{\alpha \in \mathbb{Z}_+ : \exists A' \in D(p'' + \alpha\chi'') \text{ s.t. } |A' \cap \{k, k', \ell\}| \leq 1\right\}.$$

By (42) and the facts that $A + \ell \in \operatorname{dom} U$ and $U(\cdot)$ takes only integer values (except $-\infty$), there exists a solution to the above problem. Let $p''' \equiv p'' + \alpha'''\chi''$. By (17) and (42),

$$A' \in D(p''') \Longrightarrow A' \subsetneq A + k + k' + \ell. \tag{43}$$

We consider two cases.

Case 1: Suppose that $\{A + k + k', A + k' + \ell, A + k + \ell\} \cap D(p''') \neq \emptyset$. By (21) and (42),

$$\{A + k + k', A + k' + \ell, A + k + \ell\} \subseteq D(p'''). \tag{44}$$

By (42) and the definition of α''', there exists $A' \in D(p''')$ that satisfies

$$\left[A' \subseteq A\right] \text{ or } \left[A' \subseteq A + k + k' + \ell \text{ and } |A' \cap \{k, k', \ell\}| = 1\right]. \tag{45}$$

We consider two subcases.

Subcase 1–1: Suppose that there exists $A' \in D(p''')$ with $A' \subseteq A$. If $A \notin D(p''')$, then

$$\nexists A'' \in D(p''' + \chi'') \text{ with } A'' \supseteq A,$$

because, by changing p''' to $p''' + \chi''$, the utility from A' is invariant, while the utility from any bundle A'' with $A'' \supsetneq A$ decreases by at least 1. We obtain a contradiction to ($\mathrm{GS}_\mathbb{Z}$) applied to p''', $p''' + \chi''$, and $A + k + k' \in D(p''')$ (which follows from (44)). Hence, $A \in D(p''')$. By Lemma 1,

$$U[-p'''](A + k) + U[-p'''](A + k') \geq U[-p'''](A + k + k') + U[-p'''](A).$$

By $A \in D(p''')$ and $A + k + k' \in D(p''')$ (which follows from (44)), we obtain $A + k \in D(p''')$ and $A + k' \in D(p''')$. Together with (44),

$$[A + k \in D(p''') \text{ and } A + k' + \ell \in D(p''')] \text{ and}$$
$$[A + k' \in D(p''') \text{ and } A + k + \ell \in D(p''')].$$

By following the same argument as in (14), the desired inequality follows.

Subcase 1–2: Suppose that

$$\nexists A' \in D(p''') \text{ with } A' \subseteq A. \tag{46}$$

Then, by (45), there exists $A' \in D(p''')$ such that

$$A' \subseteq A + k + k' + \ell \text{ and } |A' \cap \{k, k', \ell\}| = 1. \tag{47}$$

The following claim holds:

$$\exists A' \in D(p''') \text{ satisfying } A' \supseteq A \text{ and (47)}. \tag{48}$$

Proof of (48)
Suppose for a contradiction that

$$\nexists A'' \in D(p''') \text{ satisfying } A'' \supseteq A \text{ and (47)}. \tag{49}$$

For an arbitrarily chosen $\tilde{A} \in D(p''')$ satisfying (47), let $\tilde{\chi} \in \mathbb{Z}^K$ be such that

$$\tilde{\chi}_m = \begin{cases} 0 & \text{if } m \in \tilde{A} \cap \{k, k', \ell\} \text{ or } m \in A, \\ 1 & \text{otherwise.} \end{cases}$$

Then, by changing p''' to $p''' + \tilde{\chi}$, the utility from $\tilde{A}$ is invariant, while the utility from any bundle A'' with $|A'' \cap \{k, k', \ell\}| \geq 2$ decreases by at least 1. Together with (43),

(46) and (49), we have

$$\nexists A'' \in D(p''' + \tilde{\chi}) \text{ with } A'' \supseteq A,$$

a contradiction to ($\mathrm{GS_Z}$) applied to p''', $p''' + \tilde{\chi}$ and $A + k + k' \in D(p''')$ (which follows from (44)).

end of proof of (48)

Note that A' in (48) takes only three possible forms: $A + k$, $A + k'$ or $A + \ell$. If $A + k \in D(p''')$ or $A + k' \in D(p''')$, then, together with (44), we obtain

$$\begin{aligned}&[A + k \in D(p''') \text{ and } A + k' + \ell \in D(p''')] \text{ or}\\&[A + k' \in D(p''') \text{ and } A + k + \ell \in D(p''')].\end{aligned}$$

By following the same argument as in (14), the desired inequality follows. Hence, in the remaining part, suppose that

$$A + k \notin D(p''') \text{ and } A + k' \notin D(p''') \text{ and } A + \ell \in D(p'''). \tag{50}$$

We prove by way of contradiction that (50) never occurs. We define $\chi''' \in \mathbb{Z}^K$ by

$$\chi'''_m = \begin{cases} 1 & \text{if } m \in \{k, k'\},\\ 2 & \text{if } m = \ell,\\ -3 & \text{if } m \in A,\\ 3 & \text{otherwise.}\end{cases}$$

At $p''' + \chi'''$, the following hold:

$$\begin{aligned}&U[-p''' - \chi'''](A') = U[-p'''](A') + 3|A| \text{ if } A' = A,\\&U[-p''' - \chi'''](A') = U[-p'''](A') + 3|A| - 1 \text{ if } A' \in \{A + k, A + k'\},\\&U[-p''' - \chi'''](A') = U[-p'''](A') + 3|A| - 2 \text{ if } A' \in \{A + k + k', A + \ell\}, \quad (51)\\&U[-p''' - \chi'''](A') \leq U[-p'''](A') + 3|A| - 3\\&\qquad\qquad \text{if } A' \notin \{A, A + k, A + k', A + k + k', A + \ell\}.\end{aligned}$$

We consider two subcases.

Subcase 1–2-1: Suppose that $\{A + k + k', A + \ell\} \cap D(p''' + \chi''') = \emptyset$. This means that $A + k + k'$ and $A + \ell$ are not included in $D(p''' + \chi''')$, despite the fact that both of these bundles are included in $D(p''')$ (which follows from (44) and (50)). Note here that, by changing p''' to $p''' + \chi'''$, the utilities from $A + k + k'$ and $A + \ell$ change by $3|A| - 2$. Together with (51) and the fact that $U(\cdot)$ takes only integer values (except $-\infty$), (i) or (ii) below is true:

(i) There exists $A' \in D(p''')$ such that the utility from A' changes by $3|A| - 1$ as a result of changing p''' to $p''' + \chi'''$.

(ii) It holds that $A \in D(p''' + \chi''')$.

However, (i) is never true; the only candidate for A' is $A + k$ or $A + k'$, neither of which is included in $D(p''')$ because of (50). Thus, (ii) holds. Together with (46) (which implies $A \notin D(p''')$) and (51),[6]

$$U[-p'''](A) = U[-p'''](A + k + k') - 1. \tag{52}$$

By Lemma 1,

$$U[-p'''](A + k) + U[-p'''](A + k') \geq U[-p'''](A + k + k') + U[-p'''](A).$$

By (44), (52), and the fact that $U(\cdot)$ takes only integer values (except $-\infty$), we have $A+k \in D(p''')$ or $A+k' \in D(p''')$. We obtain a contradiction to (50). Thus, Subcase 1–2-1 never occurs.

Subcase 1–2-2: Suppose that $\{A + k + k', A + \ell\} \cap D(p''' + \chi''') \neq \emptyset$. Since $A + k + k', A + \ell \in D(p''')$ (which follows from (44) and (50)), together with (51), we have

$$\{A + k + k', A + \ell\} \subseteq D(p''' + \chi''').$$

Note that, by changing p''' to $p''' + \chi'''$, the utilities from $A + k + k'$ and $A + \ell$ change by $3|A| - 2$. Again by (51), we obtain

$$\{A + k + k', A + \ell\} \subseteq D(p''' + \chi''') \subseteq \{A, A + k, A + k', A + k + k', A + \ell\}. \tag{53}$$

If $A + k \notin D(p''' + \chi''')$, then

$$\nexists A'' \in D(p''' + \chi''' + \mathbb{1}^{k'}) \text{ s.t. } A'' \supseteq A + k,$$

a contradiction to (GS_Z) applied to $p''' + \chi'''$, $p''' + \chi''' + \mathbb{1}^{k'}$, and $A + k + k' \in D(p''' + \chi''')$. Similarly, if $A + k' \notin D(p''' + \chi''')$, we obtain a contradiction to (GS_Z). It follows that $A + k, A + k' \in D(p''' + \chi''')$. In this case, by applying the same technique used in Case 3 in the proof of Lemma 2 in the main text, we obtain a contradiction to (GS_Z).[7] Hence, Subcase 1–2-2 never occurs.

Case 2: Suppose that $\{A+k+k', A+k'+\ell, A+k+\ell\} \cap D(p''') = \emptyset$. By (21), (42), the definition of α''', and the fact that $U(\cdot)$ takes only integer values (except $-\infty$), this

[6] Intuitively, the utility from A can "surpass" the utility from $A + k + k'$ at $p''' + \chi'''$ only if the utility from A is smaller than the utility from $A + k + k'$ by 1 at p'''.

[7] At $p''' + \chi'''$, we have $A + k \in D(p''' + \chi''')$, $A + k' \in D(p''' + \chi''')$, $A + k + k' \in D(p''' + \chi''')$, $A+\ell \in D(p'''+\chi''')$, $A+k+\ell \notin D(p'''+\chi''')$, $A+k'+\ell \notin D(p'''+\chi''')$, $A+k+k'+\ell \notin D(p'''+\chi''')$, where the latter five conditions follow from (53). By letting $p''' + \chi'''$ play the role of p''' in Case 3, the desired contradiction follows.

is true only if $\alpha''' \geq 1$ and the following implication holds[8]:

$$\begin{aligned} & A' \in D(p''') \\ \Longrightarrow & \Big[A' \subseteq A\Big] \text{ and } \Big[U[-p'''](A') = U[-p'''](A'') + 1 \\ & \text{for all } A'' \in \{A + k + k', A + k + \ell, A + k' + \ell\}\Big]. \end{aligned} \tag{54}$$

If $A \notin D(p''')$, then there does not exist $A' \in D(p''')$ with $A' \supseteq A$. We obtain a contradiction to (GS_Z) applied to

$p''' - \chi''$ (which is equal to $p'' + (\alpha''' - 1)\chi''$),
p''', and
$A + k + k' \in D(p''' - \chi'')$ (which follows from the minimality of α''').

Thus, $A \in D(p''')$. Moreover, we have

$$\{A + k + k', A + k + \ell, A + k' + \ell\} \subseteq D(p''' - \chi''), \tag{55}$$

$$U[-p''' + \chi''](A) = U[-p''' + \chi''](A') - 1 \quad \text{for all } A' \in \{A + k + k', A + k + \ell, A + k' + \ell\}, \tag{56}$$

where the former set-inclusion follows from the minimality of α''' and the latter equality follows from the definition of χ'' and (54).

By Lemma 1,

$$\begin{aligned} & U[-p''' + \chi''](A + k) + U[-p''' + \chi''](A + k') \\ \geq & U[-p''' + \chi''](A + k + k') + U[-p''' + \chi''](A). \end{aligned}$$

By (55), (56), and the fact that $U(\cdot)$ takes only integer values (except $-\infty$), we have $A + k \in D(p''' - \chi'')$ or $A + k' \in D(p''' - \chi'')$. Together with (55), we obtain

$$\begin{aligned} & [A + k \in D(p''' - \chi'') \text{ and } A + k' + \ell \in D(p''' - \chi'')] \text{ or} \\ & [A + k' \in D(p''' - \chi'') \text{ and } A + k + \ell \in D(p''' - \chi'')]. \end{aligned}$$

By following the same argument as in (14), the desired inequality follows. □

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[8] Here is an intuitive explanation of this implication. Notice that $A + k + k'$, $A + k' + \ell$, and $A + k + \ell$ are all utility-maximizing bundles at p'' but they are not utility-maiximizing bundles at $p''' \equiv p'' + \alpha''' \chi''$, despite that fact that α''' is chosen as the minimum value where a new utility-maximizing bundle appears. This happens only if the utilities from $A + k + k'$, $A + k' + \ell$, and $A + k + \ell$ "cross over" the utility from $A' \subseteq A$.

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