A note on blow-up solutions for a scalar differential equation with a discrete delay

Tetsuya Ishiwata[1] · Yukihiko Nakata[2]

Abstract

We study blow-up solutions for a general scalar differential equation with a discrete delay. It is shown that the existence of blow-up solutions for a discrete delay differential equation (DDE) is proven by finding blow-up solutions to an associated autonomous ordinary differential equation (ODE). We give an example that the existence of blow-up solutions to an associated autonomous ODE does not necessarily imply the existence of blow-up solutions to DDEs. Nevertheless, for a class of discrete DDEs, we prove that the existence of blow-up solutions implies the existence of blow-up solutions to associated autonomous ODEs. For special cases, we study the asymptotic behavior and the blow-up rate of the blow-up solutions.

Keywords Delay differential equations · Blow-up solutions · Delay-induced blow-up

Mathematics Subject Classification 34K99 · 34K13

✉ Yukihiko Nakata
ynakata@math.aoyama.ac.jp

Tetsuya Ishiwata
tisiwata@shibaura-it.ac.jp

1   Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama City 337-8570, Japan

2   Department of Mathematical Sciences, Aoyama Gakuin University, 5-10-1 Fuchinobe, Chuo-ku, Sagamihara City 252-5258, Japan

1 Introduction

Finite-time blow-up of the solution, forming a finite-time singularity of the solution, has been actively investigated in the study of partial differential equations, in particular, for elliptic and parabolic partial differential equations, see e.g. [8, 16] and references therein. The blow-up phenomenon is also studied in Volterra type integral equations in many papers, see [5, 17] and references therein. Many papers have been devoted to the study of blow-up of ordinary differential equations (ODEs). For ODEs, the blow-up solutions and the asymptotic behavior of the solutions are investigated, see [2, 3, 9, 10, 15, 18] and references therein.

Continuation of solutions is a crucial property to investigate global existence solutions. For ODEs (and perturbed equations), continuation of solutions is studied in e.g. [4, 12]. For delay differential equations (DDEs), results concerning continuation of solutions is given in Chapter 2.3 of [13]. For a DDE, behavior of a non-continuable solution in which the maximal existence interval is finite is shown in Chapter 2.3 of [13], see also [14]. However, as far as the authors' best knowledge, much elaboration has not been made to determine if a given DDE has a noncontinuable solution.

The paper [7] provides, for a class of DDEs, conditions for the existence of blow-up solutions, of which the solution diverges and its existence interval is finite. The study is motivated by the estimation of the energy function for a delay reaction-diffusion equation. In the paper [7] the authors pay attention mainly to DDEs of the following forms:

$$x'(t) = f(x(t)) + g(x(t-1))$$
(1.1)

and

$$x'(t) = f(x(t))g(x(t-1)).$$
(1.2)

Assuming that ODE $x'(t) = f(x(t))$ blows up in a finite-time, the authors obtain an estimation for the initial functions so that the solutions of (1.1) and (1.2) blow up in finite time.

The blow-up phenomenon has been studied in some DDEs. The paper [1] studies blow-up and asymptotic behavior of solutions for a differential equation with distributed delay (Volterra integro-differential equation). In the paper [11], the authors study a logistic equation with a discrete delay and show the existence of blow-up solutions. In the paper [6] the authors show that the system of DDEs:

$$\begin{aligned}
x'(t) &= x(t) - y(t) - x(t-\tau)(x^2(t) + y^2(t)), \\
y'(t) &= x(t) + y(t) - y(t-\tau)(x^2(t) + y^2(t)),
\end{aligned}$$
(1.3)

where $\tau(> 0)$ is a time delay, has blow-up solutions, while the system (1.3) with $\tau = 0$ has no blow-up solutions, that is, the delay induces blow-up of solutions. In these papers [11] and [6], we exploit the nature of differential equations with a discrete delay (discrete DDEs), which can be solved, for a given initial function, by the method of steps. We thus analyze discrete DDEs for $t \le \tau$, where τ is the delay

of the equations and, in this interval, the equations can be seen as nonautonomous ODEs, of which we find solutions that blow up in finite time $t \leq \tau$.

In this paper, we apply the idea used in [6, 11] for proving the existence of blow-up solutions to a general scalar discrete DDEs. Our aim is to, in a general setting, formulate a sufficient condition for the scalar discrete DDE in which blow-up solutions exist, associating the problem to the existence of blow-up solutions to an ODE with a parameter, where the ODE generates a solution of DDE through an initial function having a constant value for an interval. We also show an example that such an associated ODE does not necessarily yield blow-up solutions of a discrete DDE. If we restrict the class of DDEs, including (1.1) and (1.2), the existence of blow-up solutions to DDEs indeed implies the existence of blow-up solutions of the corresponding ODEs. For those equations, we study the blow-up rate of the blow-up solutions. As it is expected, the asymptotic behavior is determined by the "ODE part" in a sense.

The paper is organized as follows. In Sect. 2, as a preliminary, we collect results concerning the blow-up of autonomous and nonautonomous ODEs. In Sect. 3, we study blow-up of a general scalar DDE, relating the problem to blow-up of an ODE, given by the DDE with an initial function that is constant for an interval. We also propose a condition that blow-up of DDE implies blow-up of a corresponding ODE for some parameter. In Sect. 4, we revisit equations (1.1) and (1.2) and draw a connection to ODEs. In Sect. 5, we introduce a case study using a simple example. In Sect. 6, we discuss our results obtained in this paper.

2 Blow-up results for ODEs

We here first revisit the following ODE:

$$x'(t) = f(x(t)), \tag{2.1}$$

where f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$. By the Peano's existence theorem, Eq. (2.1) has a local solution satisfying the initial condition

$$x(0) = x_0 \in \mathbb{R}. \tag{2.2}$$

Definition 2.1 We say that a solution x to Eq. (2.1) blows up in a finite time if the maximal existence time is finite, i.e., there exists $T \in (0, \infty)$ such that

$$\lim_{t \to T-} |x(t)| = \infty,$$

where T is called blow-up time.

For the sake of simplicity, if it is necessary, we consider Eq. (2.1) satisfying $f(x) > 0$ for large x and solutions that blows up to $+\infty$. Similarly, one can consider

the case that the solution blows up to $-\infty$. Let us introduce the following elementary result.

Proposition 2.1 *There exists $\delta \in \mathbb{R}$ such that the solution of Eq. (2.1) with the initial condition (2.2) satisfying $x_0 > \delta$ monotonically increases and blows up in a finite time, if and only if, there exists $\delta \in \mathbb{R}$ such that*

$$f(x) > 0, \text{ and } \int_x^\infty \frac{d\xi}{f(\xi)} < \infty \quad \text{for } \forall x > \delta. \tag{2.3}$$

Remark 2.1 We here present Definition 2.1 and Proposition 2.1 for Eq. (2.1) to discuss a blow-up phenomenon in DDEs below in Sects. 3 and 4, see also Definition 3.1 and results in Sect. 3.

Next, we state necessary conditions concerning the nonlinear function $f(x)$ for the existence of blow-up solutions to (2.1).

Lemma 2.1 *Let us assume that Eq. (2.1) has blow-up solutions. Then*

$$0 \leq \liminf_{x \to \infty} f(x) \leq \limsup_{x \to \infty} f(x) = \infty,$$

and there exists $\delta \in \mathbb{R}$ such that

$$f(x) > 0 \text{ for } \forall x > \delta. \tag{2.4}$$

Proof Let us suppose that $\limsup_{x \to \infty} f(x) < \infty$. Let $\limsup_{x \to \infty} f(x) = L$. Then for any $\varepsilon > 0$ there exists M such that $x > M$ implies $f(x) < L + \varepsilon$. Then for $x > M$, one has $x' < L + \varepsilon$. This leads to a contradiction that the solution blows up in a finite time, thus $\limsup_{x \to \infty} f(x) = \infty$ holds. We easily obtain other conditions from Proposition 2.1. □

Consider the following perturbed ODEs

$$x'(t) = f(x(t)) + c, \tag{2.5}$$

where $c \in \mathbb{R}$ is a constant.

Lemma 2.2 *Let us suppose that, there exists $c \in \mathbb{R}$ such that Eq. (2.5) has blow-up solutions. If $\liminf_{x \to \infty} f(x) > 0$, then Eq. (2.1) also has blow-up solutions.*

Proof Applying Proposition 2.1 to (2.5), there exists δ such that

$$f(x) + c > 0, \text{ and } \int_x^\infty \frac{d\xi}{f(\xi) + c} < \infty \quad \text{for } \forall x > \delta.$$

We then show that (2.1) also has blow-up solutions. Let us suppose that $c \leq 0$. From the comparison principle, it is easy to see that the solutions of (2.1) satisfying the initial condition $x(0) = x_0 > \delta$ also blow up in finite time. Let us suppose that

$c > 0$. Let $d = \frac{1}{2} \liminf_{x\to\infty} f(x) > 0$. Then there exists M' such that $x > M'$ implies $f(x) > d > 0$. Thus

$$x > M' \implies \frac{d}{d+c} < \frac{f(x)}{f(x)+c} < 1.$$

We then have the following estimates:

$$\frac{d}{d+c} \int_{M'}^{\infty} \frac{dx}{f(x)} < \int_{M'}^{\infty} \frac{f(x)}{f(x)+c} \frac{1}{f(x)} dx < \infty.$$

Hence, by Proposition 2.1, the solutions of Eq. (2.1) through the initial condition $x(0) = x_0 > M'$ blow-up in finite time. □

3 Blow-up solutions for a DDE

Let F be a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$. Let us consider the following DDE:

$$x'(t) = F(x(t), x(t - \tau)), \tag{3.1}$$

where $\tau \geq 0$. For $\tau > 0$, without loss of generality, redefining the time scale and the function F, for our purpose, it is sufficient to consider Eq. (3.1) with $\tau = 1$, i.e.,

$$x'(t) = F(x(t), x(t - 1)). \tag{3.2}$$

We consider Eq. (3.2) with the following initial condition:

$$x(\theta) = \varphi(\theta), \ \theta \in [-1, 0], \tag{3.3}$$

where $\varphi : [-1, 0] \to \mathbb{R}$ is a continuous function.

Definition 3.1 We say that a solution x to DDE (3.2) blows up in finite time if the maximal existence time is finite, i.e., there exists $T \in (0, \infty)$ such that

$$\limsup_{t \to T-} |x(t)| = \infty.$$

Here T is called blow-up time.

Remark 3.1 The author in [14] shows an example of DDE, in which a blow-up solution satisfy $\limsup_{t\to T-} |x(t)| = \infty$ for some T but $\lim_{t\to T-} |x(t)| \neq \infty$.

We remark that if F does not depend on the first argument, then Eq. (3.2) does not admit any blow-up solutions. We thus assume that F does depend on both arguments.

For $t \in [0, 1]$ the solution of Eq. (3.2) with the initial condition (3.3) satisfies the following nonautonomous ODE:

$$x'(t) = F(x(t), \varphi(t-1)) \tag{3.4}$$

with the initial condition $x(0) = \varphi(0)$. By the method of steps, one can successively solve Eq. (3.2), as far as the solution exists.

From the time-translation invariance, we have the following result.

Proposition 3.1 *Equation* (3.2) *has a blow-up solution if and only if Eq.* (3.2) *has a blow-up solution that blows up in finite time* $0 < t \leq 1$.

Proof If there is a noncontinuable solution $x : [-1, T) \to \mathbb{R}$ to Eq. (3.2), where $T > 1$ is the maximal existence time, Eq. (3.2) has a blow-up solution $\tilde{x}$ with blow-up time $\tilde{T} \in (0, 1]$, where $\tilde{x}(t) = x(t - \tilde{T} + T)$. The converse is trivial. Thus we obtain the conclusion. $\square$

Therefore, from Proposition 3.1, to prove the existence of blow-up solutions for DDE (3.2), we would like to find an initial function φ such that the solution with the initial function φ blows up in $t \in (0, 1]$. We mean that by saying that a DDE has blow-up solutions, there exist initial functions such that the solutions with the initial functions blow up in finite time.

As a candidate, we consider an initial function which takes a constant value for an interval. This leads Eq. (3.4) to an autonomous ODE (temporarily), in which we investigate if the solution of DDE (3.2) blows up in a finite time. Consequently, it is shown that the existence of blow-up solutions for Eq. (3.2) can be studied by the following ODE:

$$x'(t) = F(x(t), a) \tag{3.5}$$

with a parameter $a \in \mathbb{R}$.

A similar result is Proposition 9 in Ezzinbi and Jazar [7]. Here we do not pay much attention to the estimation of the initial function φ that generates blow-up solutions to (3.2). To characterize the existence of blow-up solutions for (3.2), we do not explicitly impose conditions (e.g. the monotonicity condition and the positivity condition) for the function $F(x, y)$ as in Proposition 9 in [7].

Theorem 3.1 *If there exists* $a \in \mathbb{R}$ *such that ODE* (3.5) *has a blow-up solution*, *then DDE* (3.2) *has blow-up solutions*.

Proof Suppose that a solution of ODE (3.5) for some a with the initial condition $x(0) = b \in \mathbb{R}$ blows up in a finite time. Without loss of generality, we also assume that the solution blows up to $+\infty$, i.e., $\lim_{t \to T_-} x(t) = \infty$, where T is the blow-up time given as

$$T(b) := \int_b^\infty \frac{1}{F(x, a)} dx < \infty.$$

Since $\lim_{b \to \infty} T(b) = 0$, for any $\varepsilon > 0$, there exists δ such that $T(b) < \varepsilon$ for $b > \delta$. We now fix $0 < \bar{\varepsilon} < 1$. For $\bar{\varepsilon}$, there exists $\bar{\delta}$ such that $T(b) < \bar{\varepsilon}$ for $b > \bar{\delta}$. Then, for Eq. (3.2), we consider an initial function φ satisfying (3.6a) and (3.6b)

$$\varphi(\theta) = a, \ -1 \leq \theta \leq -1 + \bar{\varepsilon}, \tag{3.6a}$$

$$\varphi(0) = b > \bar{\delta}. \tag{3.6b}$$

The solution of Eq. (3.2) with the initial condition (3.3) satisfying (3.6) solves Eq. (3.5) with the initial condition $x(0) = b$ and thus blows up at $t = \bar{\varepsilon} < 1$. It is also easy to see that the solutions of DDE (3.2) with the initial condition (3.3) satisfying (3.6a) and $\varphi(0) > b$ blow up in finite time $t < \bar{\varepsilon} < 1$. Therefore there exists a family of blow-up solutions. Hence we obtain the conclusion. $\qquad \square$

In [6], to study the blow-up of a system of DDEs, we consider a system of ODEs that is obtained replacing the delay term with a constant parameter, due to the initial function which takes constant for an interval. This idea is behind in Theorem 3.1. Theorem 3.1 offers a sufficient condition for the existence of blow-up solutions of DDE (3.2): by studying ODE (3.5), one may find blow-up solutions of DDE (3.2).

There is, however, a DDE in which this technique can not yield the blow-up solutions.

Example 3.1 Consider DDE (3.2) and ODE (3.5) with

$$F(x, y) = x^2 \sin(xy). \tag{3.7}$$

One can see that ODE (3.5) does not admit any blow-up solutions for every a. However, DDE (3.2) is shown to have blow-up solutions. For DDE (3.2), consider the initial functions φ satisfying

$$\varphi(0) = x_0 > 0,$$
$$\varphi(\theta) = \frac{\pi}{2} \left(x_0^{-1} - 1 - \theta \right), \ \theta \in \left[-1, x_0^{-1} - 1 \right].$$

From an elementary calculation, one can see that, for $x_0 > 1$, the solution of DDE (3.2) is $x(t) = 1/(x_0^{-1} - t)$ and hence blows up as $t \to x_0^{-1}-$. Therefore, the converse of Theorem 3.1 is not necessarily true.

We also note that for (3.1) with (3.7), every solution is bounded for $\tau = 0$, and the blow-up solutions appear for $\tau > 0$. Thus this is also an example of "delay-induced blow-up" phenomenon.

If we restrict the class of functions $F(x, y)$, the converse of Theorem 3.1 is true: for a class of equations, the existence of blow-up solutions of DDE (3.2) is determined by ODE (3.5). We now propose to assume the following condition for the function F:

(H) For any closed bounded interval $I \subset \mathbb{R}$ there exist $a_i \in \mathbb{R}$ ($i \in \{1,2\}$) such that

$$(x, y) \in \mathbb{R} \times I \Longrightarrow F(x, a_1) \leq F(x, y) \leq F(x, a_2).$$

The condition (H) is used to estimate the solution of DDE (3.2) using the solution of ODE (3.5).

Theorem 3.2 *Suppose that* (*H*) *holds. If DDE* (3.2) *has a blow-up solution, then there exists a* $\in \mathbb{R}$ *such that ODE* (3.5) *has blow-up solutions.*

Proof Suppose that for every $a \in \mathbb{R}$, any solutions of ODE (3.5) do not blow up in finite time. The blow-up solution of DDE (3.2) satisfies Eq. (3.4), as far as the solution exists. Since φ is a continuous function defined in the closed interval $[-1, 0]$, from the condition (H), there exist constants a_1, a_2 (which can be chosen uniformly with respect to t) such that

$$F(x(t), a_1) \leq F(x(t), \varphi(t-1)) \leq F(x(t), a_2)$$

for any t (up to the maximal existence time). Therefore, one obtains the following estimates:

$$F(x(t), a_1) \leq x'(t) \leq F(x(t), a_2).$$

From the comparison principle, it is easy to obtain the contradiction. Hence we obtain the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Special classes of DDEs

Let f and g be continuous functions $\mathbb{R} \to \mathbb{R}$. Since continuous function g on a closed bounded interval has a maximum and a minimum, if $F(x, y)$ is given as $F(x, y) = f(x) + g(y)$ or $F(x, y) = f(x)g(y)$, then the condition (H) is automatically satisfied. Thus, from Theorems 3.1 and 3.2, we have the following direct consequences for Eqs. (1.1) and (1.2).

Corollary 4.1 *The following statements are true.*

1. *DDE* (1.1) *has blow-up solutions if and only if there exists a* $\in \mathbb{R}$ *such that the following ODE*

$$x'(t) = f(x(t)) + g(a) \tag{4.1}$$

   has blow-up solutions.
2. *DDE* (1.2) *has blow-up solutions if and only if there exists a* $\in \mathbb{R}$ *such that the following ODE*

$$x'(t) = g(a)f(x(t)) \tag{4.2}$$

has blow-up solutions.

In the following, we impose further assumptions on the function $f(x)$ to obtain the blow-up rate of the solution.

Theorem 4.1 *Let us suppose that Eq. (1.1) has blow-up solutions. Let us assume that* $\liminf_{x \to \infty} f(x) > 0$ *holds. Then the following statements are true.*

1. *Equation (2.1) also has blow-up solutions.*
2. *If* $\lim_{x \to \infty} f(x) = \infty$ *then the blow-up solutions of (1.1) satisfies* $\lim_{t \to T-} x(t) = \infty$ *and*

$$\lim_{t \to T-} \frac{1}{T-t} \int_{x(t)}^{\infty} \frac{dx}{f(x)} = 1,$$

where T is the blow-up time.

Proof From Corollary 4.1, there exists $a \in \mathbb{R}$ such that Eq. (4.1) has blow-up solutions. Applying Lemma 2.2, one obtains the conclusion 1.

Since $g(x(t-1))$, $t < T$ is bounded, from the assumption that $\lim_{x \to \infty} f(x) = \infty$, it can be shown that $x(t)$ monotonically increases, when t is sufficiently close to T. This implies $\lim_{t \to T-} x(t) = \infty$. Then there exists $T_1 < T$ such that

$$f(x(t)) > 0, \ f(x(t)) + g(x(t-1)) > 0, \ t \in (T_1, T).$$

For $t \in (T_1, T)$, from (1.1), one has

$$x'(t) = f(x(t))\left(1 + \frac{g(x(t-1))}{f(x(t))}\right).$$

Integrating both sides of the above equation from $t \in (T_1, T)$ to T, we obtain

$$\int_{x(t)}^{\infty} \frac{dx}{f(x)} = \int_t^T 1 + \frac{g(x(s-1))}{f(x(s))} ds.$$

Since $\lim_{x \to \infty} f(x) = \infty$, by the mean value theorem, one obtains

$$\lim_{t \to T-} \frac{1}{T-t} \int_t^T 1 + \frac{g(x(s-1))}{f(x(s))} ds = 1.$$

Therefore, we obtain the conclusion 2. $\qquad\square$

For Eq. (1.2), we have the following result. We implicitly characterize the initial functions which yield the solutions blow up in finite time.

Theorem 4.2 *Let us suppose that Eq.* (1.2) *has blow-up solutions. Then there exists $a \in \mathbb{R}$ such that* (4.2) *has blow-up solutions. If $g(a) > 0$, then the following statements are true.*

1. *Equation* (2.1) *also has blow-up solutions.*
2. *If the blow-up solution of* (1.2) *satisfies $\lim_{t \to T_-} x(t) = \infty$ then*

$$\lim_{t \to T_-} \frac{1}{T - t} \int_{x(t)}^{\infty} \frac{dx}{f(x)} = g(x(T - 1)), \tag{4.3}$$

   where T is the blow-up time.

Proof Applying Proposition 2.1 to Eq. (4.2), we easily obtain the conclusion 1. From Corollary 4.1 and Proposition 2.1, without loss of generality, one can assume that there exists δ such that (2.3) holds, i.e., $f(x) > 0$ for sufficiently large x.

From Eq. (1.2) we have

$$\int_{x(t)}^{\infty} \frac{dx}{f(x)} = \int_{t}^{T} g(x(s - 1))ds.$$

Dividing both sides of equation by $T - t$ and taking the limit of both sides, we obtain (4.3). □

Remark 4.1 In Theorem 4.2 we study the case that the solution of (1.2) blows up to $+\infty$ for the sake of a simple exposition. For Eq. (1.2), we can see that for the initial function if there is $T \in (0, 1]$ such that

$$\int_{\varphi(0)}^{\infty} \frac{d\xi}{f(\xi)} = \int_{0}^{T} g(\varphi(s - 1))ds$$

then the solution blows up in finite time $0 < t \le 1$, see also Sect. 5. In the case that the solution blows up to $-\infty$, a similar formula to (4.3) concerning the blow-up rate holds.

5 Example

As a simple example, we consider the case $F(x, y) = -x^2 y$. For this case Eq. (3.2) is the following DDE:

$$x'(t) = -x(t - 1)x^2(t) \tag{5.1}$$

and that the initial condition is given as in (3.3).

It is easy to see that the following ODE does not admit any blow-up solutions.

$$x'(t) = -x^3(t). \tag{5.2}$$

One can see that $\lim_{t\to\infty} x(t) = 0$ for every solution of (5.2).

Applying Theorem 3.1, we see that Eq. (5.1) has blow-up solutions in finite time. The result shows "delay-induced blow-up" phenomenon ([6]). Furthermore, Theorem 4.2 characterizes the initial conditions for the blow-up solutions.

Theorem 5.1 *For the initial function in* (3.3), *there exists* $0 < T \le 1$ *such that*

$$1 + \varphi(0) \int_0^T \varphi(s-1)ds = 0, \tag{5.3}$$

if and only if the solution of Eq. (5.1) *with the initial condition* (3.3) *blows up in a finite time* $0 < T \le 1$.

Proof By Theorem 4.2, for the initial function in (3.3), if there exists $T \le 1$ such that

$$\int_{\varphi(0)}^{\infty} \frac{dx}{x^2} = \int_0^T -\varphi(s-1)ds, \ \varphi(0) > 0, \tag{5.4}$$

then the solution blows up, satisfying $\lim_{t\to T_-} x(t) = \infty$. Similarly, if there exists $T \le 1$ such that

$$\int_{\varphi(0)}^{-\infty} \frac{dx}{x^2} = \int_0^T -\varphi(s-1)ds, \ \varphi(0) < 0, \tag{5.5}$$

then the solution blows up, satisfying $\lim_{t\to T_-} x(t) = -\infty$. Since conditions (5.4) and (5.5) are reduced to (5.3), we obtain the conclusion. $\qquad\square$

The solution can be explicitly computed, as long as the solution exists, as

$$x(t) = \frac{\varphi(0)}{1 + \varphi(0) \int_0^t \varphi(s-1)ds}.$$

Let $\varphi(0) \ne 0$. If the solution exists for $t \in [0, 1]$ then one has either $x(t) > 0$ or $x(t) < 0$ for $t \in [0, 1]$, i.e., the solution does not oscillate about 0. Theorem 5.1 implies that if the initial function does not change the sign then the solution does not blow up. Therefore, if the solution exists for $t \in [0, 1]$, the solution exists for $t \ge 0$ and the solution does not blow up in a finite time. Consequently, the blow up time T should satisfy $T \le 1$. One can also see that if the solution does not blow up in a finite time, then the solution tends to 0 as $t \to \infty$.

6 Discussion

In this note, we analyze the blow-up of a general scalar differential equation with a discrete delay (discrete DDE). Except for some case studies, no systematic studies seem to be available for the blow-up phenomena in DDEs in the literature. Our attention is to understand if the delay induces the blow-up of the solutions and, if so, to clarify the mechanisms behind the phenomena.

Our aim in this paper is to formulate the conditions for the existence of blow-up solutions of DDE (3.2). The study is motivated by the paper [7]. The authors in [7] are interested in the relation of blow-up of DDEs and of corresponding ODEs. Under weaker conditions than in [7], we establish a link between DDE (3.2) and ODE with a parameter (3.5) with respect to the existence of blow-up solutions. Theorem 3.1 offers a sufficient condition for the existence of blow-up solutions to DDE (3.2). From Theorem 3.1, one can consider a nonlinear function $F(x, y)$ such that delay induces the blow-up solutions to DDE (3.2), finding nonlinear functions $F(x, y)$ such that ODE $x'(t) = F(x, x)$ does not have any blow-up solutions, but ODE (3.5) has blow-up solutions for some a.

The example presented below Theorem 3.1 shows that ODE (3.5), which "freezes" the delay term, does not capture the blow-up solutions of a DDE. It is suggested, in general, finding initial functions for DDE (3.2) that blow up solutions may not be a trivial problem, but see [7] for several criteria. We also clarify, in Theorem 3.2, for a class of DDEs, where the nonlinear function F satisfies the condition (H), that the existence of blow-up solutions of DDE (3.2) implies the existence of blow-up solutions of ODE (3.5) for some a. Consequently, Theorem 3.2 shows that special cases (1.1) and (1.2) have clear connections to the corresponding ODEs (4.1) and (4.2), respectively. This aspect is analyzed in detail in Sect. 4.

In our analysis, discrete delay is crucial. The delay part can be considered as a nonautonomous perturbation. Thus the results are closely related to the studies of continuation of solutions to perturbed ODEs [4, 12]. The situation seems to change if we consider distributed delay differential equations.

It is conjectured that the behavior of the blow-up solutions of DDE (3.2) could be complicated, compared to the scalar autonomous ODE such as (2.1). However, this aspect is not analyzed in detail.

References

1. Appleby, J.A.D., Patterson, D.D.: Blow-up and superexponential growth in superlinear volterra equations. DCDS-A **38**(8), 3993–4017 (2018)
2. Balabane, M., Jazar, M., Souplet, P.: Oscillatory blow-up in nonlinear second order ODE's: The critical case. Disc. Cont. Dyn. Syst. **9**(3), 577–584 (2003)
3. Baris, J., Wawiórko, E.: Blow-up solutions of cubic differential systems. J. Math. Anal. Appl. **341**(2), 1155–1162 (2008)
4. Bernfeld, S.R.: The extendability of solutions of perturbed scalar differential equations. Pacific J. Math. **42**(2), 277–288 (1972)
5. Brunner, H., Yang, Z.W.: Blow-up behavior of Hammerstein-type Volterra integral equations. J. Int. Equ. Appl. **24**(4), 487–512 (2012)
6. Eremin, A., Ishiwata, E., Ishiwata, T., Nakata, Y.: Delay-induced blow-up in a planar oscillation model. Jpn. J. Indust. Appl. Math. **38**, 1037–1061 (2021)
7. Ezzinbi, K., Jazar, M.: Blow-up results for some nonlinear delay differential equations. Positivity **10**(2), 329–341 (2006)
8. Filla, M., Ninomiya, H.: Reaction versus diffusion: blow-up induced and inhibited by diffusivity. Russ. Math. Surv. **60**(6), 1217–1235 (2005)
9. Gazzola, F., Karageorgis, P.: Refined blow-up results for nonlinear fourth order differential equations. Comm. Pure Appl. Anal. **14**(2), 677–693 (2015)
10. Goriely, A., Hyde, C.: Necessary and sufficient conditions for finite time blow-up in ordinary differential equations. J. Diff. Eq. **161**(422–448), 1–28 (1997)
11. Győri, I., Nakata, Y., Röst, G.: Unbounded and blow-up solutions for a delay logistic equation with positive feedback. Comm. Pure Appl. Anal. **17**(6), 2845–2854 (2018)
12. Hara, T., Yoneyama, T., Sugie, J.: Continuability of solutions of perturbed differential equations. Nonlinear Anal.:TMA **8**(8), 963–975 (1984)
13. Hale, J.K., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations. Springer (1993)
14. Herdman, T.L.: A note on noncontinuable solutions of a delay differential equation, pp. 187–192. Academic Press, Differential Equations (1980)
15. Matsue, K.: On blow-up solutions of differential equations with Poincaré-type compactifications. SIAM J. Appl. Dyn. Syst. **17**(3), 2249–2288 (2018)
16. Quittner, P., Souplet, P.: Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States. Birkhäuser (2007)
17. Roberts, C.A.: Recent results on blow-up and quenching for nonlinear Volterra equations. J. Comp. Appl. Math. **205**(2), 736–743 (2007)
18. Souplet, P.: Critical exponents, special large-time behavior and oscillatory blow-up in nonlinear ODE's. Diff. Int. Eq. **11**(1), 147–16 (1998)