Directed discrete midpoint convexity

Akihisa Tamura¹ · Kazuya Tsurumi¹

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Abstract

For continuous functions, midpoint convexity characterizes convex functions. By considering discrete versions of midpoint convexity, several types of discrete convexities of functions, including integral convexity, $L^{\natural}$ -convexity and global/local discrete midpoint convexity, have been studied. We propose a new type of discrete midpoint convexity that lies between $L^{\natural}$ -convexity and integral convexity and is independent of global/local discrete midpoint convexity. The new convexity, named DDM-convexity, has nice properties satisfied by $L^{\natural}$ -convexity and global/local discrete midpoint convexity. DDM-convex functions are stable under scaling, satisfy the so-called parallelogram inequality and a proximity theorem with the same small proximity bound as that for $L^{\natural}$ -convex functions. Several characterizations of DDM-convexity are given and algorithms for DDM-convex function minimization are developed. We also propose DDM-convexity in continuous variables and give proximity theorems on these functions.

Keywords Midpoint convexity · Discrete midpoint convexity · Integral convexity · $L^{\natural}$ -convexity · Proximity theorem · Scaling algorithm

Mathematics Subject Classification 52A41 · 90C27 · 90C25

1 Introduction

For a continuous function f defined on a convex set $S \subseteq \mathbb{R}^n$, it was proved by Jensen [12] that *midpoint convexity* defined by

$$f(x) + f(y) \geq 2f\left(\frac{x+y}{2}\right) \quad (\forall x, y \in S)$$

✉ Akihisa Tamura
aki-tamura@math.keio.ac.jp

Kazuya Tsurumi
goro56koroppa0@keio.jp

¹ Department of Mathematics, Keio University, Yokohama 223-8522, Japan

is equivalent to the inequality defining convex functions

$$\alpha f(x) + (1 - \alpha)f(y) \geq f(\alpha x + (1 - \alpha)y) \quad (\forall x, y \in S; \forall \alpha \in [0, 1]).$$

By capturing the concept of midpoint convexity, several types of ‘discrete’ midpoint convexities for functions defined on the integer lattice $\mathbb{Z}^n$ have been proposed.

A weak version of ‘discrete’ midpoint convexity is obtained by replacing $f((x + y)/2)$ by the smallest value of a linear extension of f among the integer points neighboring $(x + y)/2$. More precisely, for any point $x \in \mathbb{R}^n$, we consider its integer neighborhood

$$N(x) = \{z \in \mathbb{Z}^n \mid |z_i - x_i| < 1 \ (i = 1, \dots, n)\}$$

and the set $\Lambda(x)$ of all coefficients $(\lambda_z \mid z \in N(x))$ for convex combinations indexed by $N(x)$. For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, we define the local convex envelope $\tilde{f}$ of f by

$$\tilde{f}(x) = \min \left\{ \sum_{z \in N(x)} \lambda_z f(z) \mid \sum_{z \in N(x)} \lambda_z z = x, (\lambda_z) \in \Lambda(x) \right\} \quad (x \in \mathbb{R}^n).$$

We say that f satisfies *weak discrete midpoint convexity* if the following inequality holds

$$f(x) + f(y) \geq 2\tilde{f}\left(\frac{x+y}{2}\right) \quad (1.1)$$

for all $x, y \in \mathbb{Z}^n$. On the other hand, f is said to be *integrally convex* [3] if $\tilde{f}$ is convex on $\mathbb{R}^n$. Characterizations of integral convexity by using weak discrete midpoint convexity have been discussed in [3, 17, 18]. The simplest characterization, Theorem A.1 in [18], says that f is integrally convex if and only if f satisfies (1.1) for all $x, y \in \text{dom } f$ with¹ $\|x - y\|_\infty \geq 2$, where the effective domain $\text{dom } f$ of f is defined by

$$\text{dom } f = \{x \in \mathbb{Z}^n \mid f(x) < +\infty\}.$$

The class of integrally convex functions establishes a general framework of discrete convex functions, including separable convex, $L^\natural$ -convex, $M^\natural$ -convex, $L_2^\natural$ -convex, $M_2^\natural$ -convex functions [21], BS-convex and UJ-convex functions [4], and globally/locally discrete midpoint convex functions [18]. The concept of integral convexity is used in formulating discrete fixed point theorems [8, 9, 29], designing algorithms for discrete systems of nonlinear equations [14, 28], and guaranteeing the existence of a pure strategy equilibrium in finite symmetric games [10].

A strong version of ‘discrete’ midpoint convexity is obtained by replacing $f((x + y)/2)$ by the average of the values of f at two integer points obtained by rounding-up and rounding-down of all components of $(x + y)/2$. More precisely, for a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, we say that f satisfies *discrete midpoint convexity* if it has

¹ If $\|x - y\|_\infty \leq 1$, then (1.1) obviously holds.

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) \quad (1.2)$$

for all $x, y \in \mathbb{Z}^n$, where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the integer vectors obtained by rounding up and rounding down all components of a given real vector, respectively. It is known that discrete midpoint convexity characterizes the class of $L^{\natural}$ -convex functions [5, 20] which play important roles in both theoretical and practical aspects. $L^{\natural}$ -convex functions are applied to several fields, including auction theory [15, 25], image processing [13], inventory theory [2, 27, 30] and scheduling [1]. Since discrete midpoint convexity (1.2) obviously implies weak discrete midpoint convexity (1.1), $L^{\natural}$ -convex functions forms a subclass of integrally convex functions.

Moriguchi et al. [18] classified discrete convex functions between $L^{\natural}$ -convex and integrally convex functions in terms of discrete midpoint convexity with ℓ_{∞} -distance requirements, and proposed two new classes of discrete convex functions, namely, globally/locally discrete midpoint convex functions. A function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *globally discrete midpoint convex* if (1.2) holds for any pair $(x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $\|x - y\|_{\infty} \geq 2$. A set $S \subseteq \mathbb{Z}^n$ is called a *discrete midpoint convex set* if its indicator function δ_S defined by

$$\delta_S(x) = \begin{cases} 0 & (x \in S) \\ +\infty & (x \notin S) \end{cases} \quad (x \in \mathbb{Z}^n)$$

is globally discrete midpoint convex, that is, if

$$x, y \in S \text{ with } \|x - y\|_{\infty} \geq 2 \Rightarrow \left\lceil \frac{x+y}{2} \right\rceil, \left\lfloor \frac{x+y}{2} \right\rfloor \in S.$$

A function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *locally discrete midpoint convex* if $\text{dom} f$ is a discrete midpoint convex set and (1.2) holds for any pair $(x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $\|x - y\|_{\infty} = 2$. It is shown in [18] that the following inclusion relations among function classes hold:

$$\begin{aligned} \{L^{\natural}\text{-convex}\} &\subsetneq \{\text{globally discrete midpoint convex}\} \\ &\subsetneq \{\text{locally discrete midpoint convex}\} \subsetneq \{\text{integrally convex}\}, \end{aligned}$$

and globally/locally discrete midpoint convex functions inherit nice features from $L^{\natural}$ -convex functions, that is, for a globally/locally discrete midpoint convex f and a positive integer α ,

- the scaled function f^{α} defined by $f^{\alpha}(x) = f(\alpha x)$ ($x \in \mathbb{Z}^n$) belongs to the same class, that is, global/local discrete midpoint convexity is closed with respect to scaling operations,
- a proximity theorem with the same proximity distance with $L^{\natural}$ -convexity holds, that is, given an x^{α} with $f(x^{\alpha}) \leq f(x^{\alpha} + \alpha d)$ for all $d \in \{-1, 0, +1\}^n$, there exists a minimizer x^* of f with $\|x^{\alpha} - x^*\|_{\infty} \leq n(\alpha - 1)$,
- when f has a minimizer, a steepest descent algorithm for the minimization of f is developed such that the number of local minimizations in the neighborhood

of ℓ_∞ -distance 2 (the 2-neighborhood minimizations) is bounded by the shortest ℓ_∞ -distance from a given initial feasible point to a minimizer of f , and

- when $\text{dom} f$ is bounded and K_∞ denotes the ℓ_∞ -size of $\text{dom} f$, a scaling algorithm minimizing f with $O(n \log_2 K_\infty)$ calls of the 2-neighborhood minimization is developed.

This paper, strongly motivated by [18], proposes a new type of discrete midpoint convexity between L^h -convexity and integral convexity, but it is independent of global/local discrete midpoint convexity with respect to inclusion relation. We name the new convexity *directed discrete midpoint convexity* (DDM-convexity) which forms the following classification

$$\{L^h\text{-convex}\} \subsetneq \{\text{DDM-convex}\} \subsetneq \{\text{integrally convex}\}.$$

The same features as mentioned above are satisfied by DDM-convexity. The merits of DDM-convexity relative to global/local discrete midpoint convexity are the following properties:

- DDM-convexity is closed with respect to individual sign inversion of variables, that is, for a DDM-convex function f and $\tau_i \in \{+1, -1\}$ ($i = 1, \dots, n$), $f(\tau_1 x_1, \dots, \tau_n x_n)$ is also DDM-convex (see Proposition 4 (3)). Neither L^h -convexity nor global nor local discrete midpoint convexity has this property, while integral convexity is closed with respect to individual sign inversion of variables.
- For a quadratic function $f(x) = x^T Q x$ with a symmetric matrix $Q = [q_{ij}]$, DDM-convexity is characterized by the diagonal dominance with nonnegative diagonals of Q :

$$q_{ii} \geq \sum_{j:j \neq i} |q_{ij}| \quad (\forall i = 1, \dots, n)$$

(see Theorem 9). While L^h -convexity is characterized by the combination of diagonal dominance with nonnegative diagonals and nonpositivity of all off-diagonal components of Q , global/local discrete midpoint convexity is independent of the diagonal dominance with nonnegative diagonals.

- A function $g: \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be discrete convex if $g(t-1) + g(t+1) \geq 2g(t)$ for all $t \in \mathbb{Z}$. For univariate discrete convex functions $\xi_i, \varphi_{ij}, \psi_{ij}: \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ ($i = 1, \dots, n; j \in \{1, \dots, n\} \setminus \{i\}$), a 2-separable convex function [7] defined as a function represented as

$$f(x) = \sum_{i=1}^n \xi_i(x_i) + \sum_{i,j:j \neq i} \varphi_{ij}(x_i - x_j) + \sum_{i,j:j \neq i} \psi_{ij}(x_i + x_j) \quad (x \in \mathbb{Z}^n).$$

The class of DDM-convex functions includes all 2-separable convex functions (see Theorem 4). It is known that if all ψ_{ij} are identically zero, then f is L^h -convex, whereas there exists a 2-separable convex function not contained in the class of globally/locally discrete midpoint convex functions.

- A steepest descent algorithm for the minimization of DDM-convex functions requires only the 1-neighborhood minimization in contrast to the 2-neighborhood minimization (see Sect. 8.1).

In the next section, we give the definition of DDM-convexity and basic properties of DDM-convex functions. In Sect. 3, we discuss a relationship between DDM-convexity and known discrete convexities. For globally/locally discrete midpoint convex functions, Moriguchi et al. [18] revealed a useful property, which is expressed by the so-called parallelogram inequality. We show that a similar parallelogram inequality holds for DDM-convex functions in Sect. 4. Sections 5 and 6 are devoted to characterizations and operations for DDM-convexity. We prove a proximity theorem for DDM-convex functions in Sect. 7, while in Sect. 8 we propose a steepest descent algorithm and a scaling algorithm for DDM-convex function minimization. In Sect. 9, we define DDM-convex functions in continuous variables and give proximity theorems for such functions.

2 Directed discrete midpoint convexity

We give the definition of directed discrete midpoint convexity and show its basic properties.

For an ordered pair (x, y) of $x, y \in \mathbb{Z}^n$, we define $\mu(x, y) \in \mathbb{Z}^n$ by

$$\mu(x, y)_i = \begin{cases} \left\lceil \frac{x_i + y_i}{2} \right\rceil & (x_i \geq y_i), \\ \left\lfloor \frac{x_i + y_i}{2} \right\rfloor & (x_i < y_i). \end{cases}$$

That is, each component $\mu(x, y)_i$ of $\mu(x, y)$ is defined by rounding up or rounding down $\frac{x_i + y_i}{2}$ to the integer in the direction of $x_i - y_i$. It is easy to show the next characterization of $\mu(x, y)$ and $\mu(y, x)$.

Proposition 1 *For $x, y, p, q \in \mathbb{Z}^n$, $p = \mu(x, y)$ and $q = \mu(y, x)$ hold if and only if the following conditions (a)~(c) hold:*

- $p + q = x + y$,
- $\|p - q\|_\infty \leq 1$, and
- for each $i = 1, \dots, n$, if $x_i \geq y_i$, then $p_i \geq q_i$; otherwise $p_i \leq q_i$.

For every $a, b \in \mathbb{R}^n$, let us denote the n -dimensional vector $(a_1 b_1, \dots, a_n b_n)$ by $a \odot b$. The next proposition gives fundamental properties of $\mu(\cdot, \cdot)$.

Proposition 2 *For every $x, y, d \in \mathbb{Z}^n$, the following properties hold.*

- (1) $x + y = \mu(x, y) + \mu(y, x)$.
- (2) If $x \geq y$, then $\mu(x, y) = \lceil \frac{x+y}{2} \rceil$ and $\mu(y, x) = \lfloor \frac{x+y}{2} \rfloor$.
- (3) If $\|x - y\|_\infty \leq 1$, then $\mu(x, y) = x$ and $\mu(y, x) = y$.
- (4) $\mu(x + d, y + d) = \mu(x, y) + d$.
- (5) For any permutation σ of $(1, \dots, n)$,

$$\mu((x_{\sigma(1)}, \dots, x_{\sigma(n)}), (y_{\sigma(1)}, \dots, y_{\sigma(n)})) = (\mu(x, y)_{\sigma(1)}, \dots, \mu(x, y)_{\sigma(n)}).$$

- (6) For any $\tau \in \{+1, -1\}^n$, $\mu(\tau \odot x, \tau \odot y) = \tau \odot \mu(x, y)$.

Proof Properties (1)–(5) are obvious by the definition of $\mu(\cdot, \cdot)$. Let us show (6). If $\tau_i = +1$,

$$\mu(\tau \odot x, \tau \odot y)_i = \begin{cases} \left\lceil \frac{x_i + y_i}{2} \right\rceil = \mu(x, y)_i & (x_i \geq y_i) \\ \left\lfloor \frac{x_i + y_i}{2} \right\rfloor = \mu(x, y)_i & (x_i < y_i) \end{cases}$$

holds, and if $\tau_i = -1$,

$$\mu(\tau \odot x, \tau \odot y)_i = \begin{cases} \left\lfloor \frac{-x_i - y_i}{2} \right\rfloor = -\left\lceil \frac{x_i + y_i}{2} \right\rceil = -\mu(x, y)_i & (x_i \geq y_i) \\ \left\lceil \frac{-x_i - y_i}{2} \right\rceil = -\left\lfloor \frac{x_i + y_i}{2} \right\rfloor = -\mu(x, y)_i & (x_i < y_i) \end{cases}$$

holds. □

By using the introduced $\mu(\cdot, \cdot)$, we propose new classes of functions and sets. We say that a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfies *directed discrete midpoint convexity* (DDM-convexity) or is a *directed discrete midpoint convex function* (DDM-convex function) if

$$f(x) + f(y) \geq f(\mu(x, y)) + f(\mu(y, x)) \quad (2.1)$$

for all $x, y \in \mathbb{Z}^n$. We call $S \subseteq \mathbb{Z}^n$ a *directed discrete midpoint convex set* (DDM-convex set) if its indicator function δ_S is DDM-convex, that is, if

$$x, y \in S \Rightarrow \mu(x, y), \mu(y, x) \in S$$

holds.

The next propositions are direct consequences of Proposition 2 and the definition (2.1).

Proposition 3 *The following statements hold:*

- (1) Any function defined on $\{0, 1\}^n$ is a DDM-convex function.
- (2) Any subset of $\{0, 1\}^n$ is a DDM-convex set.

- (3) For a DDM-convex function f , its effective domain $\text{dom} f$ and the set $\text{argmin} f$ of minimizers of f are DDM-convex sets, where $\text{argmin} f$ is defined by

$$\text{argmin} f = \{x \in \mathbb{Z}^n \mid f(x) \leq f(z) (\forall z \in \mathbb{Z}^n)\}.$$

Proposition 4 Let $f, f_1, f_2 : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be DDM-convex functions.

- (1) For any $d \in \mathbb{Z}^n$, $g(x) = f(x + d)$ is a DDM-convex function.
- (2) For any permutation σ of $(1, \dots, n)$, $g(x) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is a DDM-convex function.
- (3) For any $\tau \in \{+1, -1\}^n$, $g(x) = f(\tau \odot x)$ is a DDM-convex function.
- (4) For any $a_1, a_2 \geq 0$, $g(x) = a_1 f_1(x) + a_2 f_2(x)$ is a DDM-convex function. In particular, the sum of DDM-convex functions is DDM-convex.

Proposition 5 Let $S, S_1, S_2 \subseteq \mathbb{Z}^n$ be DDM-convex sets.

- (1) For any $d \in \mathbb{Z}^n$, $T = \{x + d \mid x \in S\}$ is a DDM-convex set.
- (2) For any permutation σ of $(1, \dots, n)$, $T = \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \mid (x_1, \dots, x_n) \in S\}$ is a DDM-convex set.
- (3) For any $\tau \in \{+1, -1\}^n$, $T = \{\tau \odot x \mid x \in S\}$ is a DDM-convex set.
- (4) $T = S_1 \cap S_2$ is a DDM-convex set.

3 Relationships with known discrete convexities

We discuss relationships between DDM-convexity and known discrete convexities, including integral convexity, $L^{\natural}$ -convexity, global/local discrete midpoint convexity and 2-separable convexity.

As mentioned in Sect. 1, the class of integrally convex functions is characterized by weak discrete midpoint convexity (1.1). Since DDM-convexity (2.1) trivially implies (1.1), any DDM-convex function is integrally convex. Therefore, DDM-convex functions inherit many properties of integrally convex functions. We introduce a good property of integrally convex functions as well as DDM-convex functions, *box-barrier property*.

Theorem 1 (Box-barrier property [17, Theorem 2.6]) Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function, and let $p \in (\mathbb{Z} \cup \{-\infty\})^n$ and $q \in (\mathbb{Z} \cup \{+\infty\})^n$ with $p \leq q$. Define

$$\begin{aligned}
S &= \{x \in \mathbb{Z}^n \mid p_i < x_i < q_i \ (i = 1, \dots, n)\}, \\
W_i^+ &= \{x \in \mathbb{Z}^n \mid x_i = q_i, p_j \leq x_j \leq q_j \ (j \neq i)\} \quad (i = 1, \dots, n), \\
W_i^- &= \{x \in \mathbb{Z}^n \mid x_i = p_i, p_j \leq x_j \leq q_j \ (j \neq i)\} \quad (i = 1, \dots, n), \\
W &= \bigcup_{i=1}^n (W_i^+ \cup W_i^-),
\end{aligned}$$

and $\hat{x} \in S \cap \text{dom} f$. If $f(\hat{x}) \leq f(y)$ for all $y \in W$, then $f(\hat{x}) \leq f(z)$ for all $z \in \mathbb{Z}^n \setminus S$.

By setting $p = \hat{x} - \mathbf{1}$ and $q = \hat{x} + \mathbf{1}$ where $\mathbf{1}$ denotes the vector of all ones, box-barrier property implies the minimality criterion of integrally convex functions.

Theorem 2 ([3, Proposition 3.1]; see also [21, Theorem 3.21]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an integrally convex function and $\hat{x} \in \text{dom} f$. Then $\hat{x}$ is a minimizer of f if and only if $f(\hat{x}) \leq f(\hat{x} + d)$ for all $d \in \{-1, 0, +1\}^n$.*

As a special case of Theorem 2, we have the minimality criterion of DDM-convex functions.

Corollary 1 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a DDM-convex function and $\hat{x} \in \text{dom} f$. Then $\hat{x}$ is a minimizer of f if and only if $f(\hat{x}) \leq f(\hat{x} + d)$ for all $d \in \{-1, 0, +1\}^n$.*

We next discuss the relationship between $L^{\natural}$ -convexity and DDM-convexity. $L^{\natural}$ -convex functions are originally defined by *translation-submodularity*:

$$f(x) + f(y) \geq f((x - \alpha \mathbf{1}) \vee y) + f(x \wedge (y + \alpha \mathbf{1})) \quad (3.1)$$

for all $x, y \in \mathbb{Z}^n$ and nonnegative integer α , where $p \vee q$ and $p \wedge q$ denote the componentwise maximum and minimum of the vectors p and q , respectively. Translation-submodularity is a generalization of *submodularity*:

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y). \quad (3.2)$$

$L^{\natural}$ -convexity has several equivalent characterizations as below.

Theorem 3 ([3, Corollary 5.2.2], [5, Theorem 3], [21, Theorem 7.7]) *For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the following properties are equivalent:*

- (1) *f is $L^{\natural}$ -convex, that is, (3.1) holds for all $x, y \in \mathbb{Z}^n$ and nonnegative integer α .*
- (2) *f satisfies discrete midpoint convexity (1.2) for all $x, y \in \mathbb{Z}^n$.*
- (3) *f is integrally convex and submodular.*
- (4) *For every $x, y \in \mathbb{Z}^n$ with $x \not\leq y$ and $A = \arg\max_i \{y_i - x_i\}$,*

$$f(x) + f(y) \geq f(x + \mathbf{1}_A) + f(y - \mathbf{1}_A),$$

where the i -component of $\mathbf{1}_A$ is 1 if $i \in A$; otherwise 0.

Theorem 3 yields the next property.

Proposition 6 Any $L^{\natural}$ -convex function is DDM-convex.

Proof Let $f : \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ be an $L^{\natural}$ -convex function. We arbitrarily fix $x, y \in \text{dom} f$ and show that (2.1) holds for x and y . By Proposition 2 (3), (2.1) holds if $\|x - y\|_{\infty} \leq 1$. Suppose that $m = \|x - y\|_{\infty} \geq 2$ and $\|y - x\|_{\infty} = \max_i \{y_i - x_i\}$ (by exchanging the role of x and y if necessary). Let $A = \text{argmax}_i \{y_i - x_i\}$, $p = x + \mathbf{1}_A$ and $q = y - \mathbf{1}_A$. The vectors p and q satisfy the properties (a) and (c) of Proposition 1 and $\|p - q\|_{\infty} \leq m$. Furthermore, by Theorem 3 (4), we have

$$f(x) + f(y) \geq f(p) + f(q). \quad (3.3)$$

If $\|p - q\|_{\infty} = m$, by applying the above process for the pair (q, p) again, we obtain p and q with $\|p - q\|_{\infty} < m$ preserving (a), (c) of Proposition 1 and (3.3). By repeating this argument, we finally obtain p and q having (a)~(c) of Proposition 1 and (3.3), which means that (2.1) holds for x and y . $\square$

A set $S \subseteq \mathbb{Z}^n$ is called an $L^{\natural}$ -convex set if its indicator function δ_S is $L^{\natural}$ -convex.

Corollary 2 Any $L^{\natural}$ -convex set is DDM-convex.

Example 1 ([18, Remark 1]) The class of $L^{\natural}$ -convex functions is a proper subclass of DDM-convex functions. For example,

$$S = \{(1, 0), (0, 1)\}$$

is a DDM-convex set, but for $x = (1, 0), y = (0, 1)$, $\lceil \frac{x+y}{2} \rceil = (1, 1) \notin S$ and $\lfloor \frac{x+y}{2} \rfloor = (0, 0) \notin S$, which means that S is not $L^{\natural}$ -convex. $\square$

Example 2 ([18, Remark 2]) A set $S \subseteq \mathbb{Z}^n$ is said to be $L_2^{\natural}$ -convex set if it is the Minkowski sum of two $L^{\natural}$ -convex sets. DDM-convexity and $L_2^{\natural}$ -convexity are mutually independent. For example,

$$\{(1, 0), (0, 1)\}$$

is a DDM-convex set, but is not $L_2^{\natural}$ -convex. On the other hand,

$$S = \{(0, 0, 0, 0), (0, 1, 1, 0), (1, 1, 0, 0), (1, 2, 1, 0)\}$$

is the Minkowski sum of two $L^{\natural}$ -convex sets $S_1 = \{(0, 0, 0, 0), (0, 1, 1, 0)\}$ and $S_2 = \{(0, 0, 0, 0), (1, 1, 0, 0)\}$, but S is not DDM-convex because for $x = (0, 0, 0, 0)$, $y = (1, 2, 1, 0)$, $\mu(x, y) = (0, 1, 0, 0) \notin S$ and $\mu(y, x) = (1, 1, 1, 0) \notin S$. $\square$

We next discuss the independence between global/local discrete midpoint convexity and DDM-convexity by showing the independence between discrete midpoint convex sets and DDM-convex sets.

Example 3 It is easy to show that the set S defined by

$$S = \{(0, 0, 0), (1, 1, 0), (1, 0, -1), (2, 1, -1)\}$$

is discrete midpoint convex, but S is not DDM-convex because for $x = (0, 0, 0)$ and $y = (2, 1, -1)$, we have $\mu(x, y) = (1, 0, 0) \notin S$ and $\mu(y, x) = (1, 1, -1) \notin S$.

On the other hand,

$$T = \{(0, 0, 0), (1, 0, 0), (1, 1, 1), (2, 1, 1), (1, 1, -1), (2, 1, -1), (1, 1, 0), (2, 1, 0)\}$$

is DDM-convex. However, T is not discrete midpoint convex, and moreover, for any $(\tau_1, \tau_2, \tau_3) \in \{-1, +1\}^3$, the modified set

$$\tau \odot T = \{(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3) \mid (x_1, x_2, x_3) \in T\}$$

is not discrete midpoint convex while it is DDM-convex by Proposition 5 (3). The reason is as follows. Since T is symmetric on the third component, we can assume $\tau_3 = +1$.

- In the case where $\tau = (\pm 1, +1, +1)$, for $x = (0, 0, 0) = \tau \odot (0, 0, 0)$ and $y = (\pm 2, 1, -1) = \tau \odot (2, 1, -1)$, $\lfloor (x+y)/2 \rfloor = (\pm 1, 0, -1) \notin \tau \odot T$.
- In the case where $\tau = (\pm 1, -1, +1)$, for $x = (0, 0, 0) = \tau \odot (0, 0, 0)$ and $y = (\pm 2, -1, 1) = \tau \odot (2, 1, 1)$, $\lceil (x+y)/2 \rceil = (\pm 1, 0, 1) \notin \tau \odot T$. $\square$

We finally show that 2-separable convex functions are DDM-convex. Let $\xi_i, \varphi_{ij}, \psi_{ij} : \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ ($i = 1, \dots, n; j \in \{1, \dots, n\} \setminus \{i\}$) be univariate discrete convex functions. A 2-separable convex function [7] is defined as a function represented as

$$f(x) = \sum_{i=1}^n \xi_i(x_i) + \sum_{i,j:j \neq i} \varphi_{ij}(x_i - x_j) + \sum_{i,j:j \neq i} \psi_{ij}(x_i + x_j) \quad (x \in \mathbb{Z}^n). \quad (3.4)$$

It is known that the function g defined by

$$g(x) = \sum_{i=1}^n \xi_i(x_i) + \sum_{i,j:j \neq i} \varphi_{ij}(x_i - x_j) \quad (x \in \mathbb{Z}^n)$$

is $\mathbb{L}^{\natural}$ -convex [21, Proposition 7.9]. By Propositions 4 (4) and 6, it is enough to show that each ψ_{ij} is DDM-convex in order to prove DDM-convexity of 2-separable convex function

$$f(x) = g(x) + \sum_{i,j:j \neq i} \psi_{ij}(x_i + x_j) \quad (x \in \mathbb{Z}^n).$$

Lemma 1 For a univariate discrete convex function $\psi : \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$, $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \psi(x_1 + x_2) \quad (x \in \mathbb{Z}^n)$$

is DDM-convex.

Proof For every $x, y \in \mathbb{Z}^n$, we show that

$$\psi(x_1 + x_2) + \psi(y_1 + y_2) \geq \psi(\mu(x, y)_1 + \mu(x, y)_2) + \psi(\mu(y, x)_1 + \mu(y, x)_2). \quad (3.5)$$

Suppose that $\min\{x_1 + x_2, y_1 + y_2\} = x_1 + x_2$ and $\max\{x_1 + x_2, y_1 + y_2\} = y_1 + y_2$ without loss of generality. By convexity of ψ , for every $a, b, p, q \in \mathbb{Z}$ such that (i) $a + b = p + q$, (ii) $a \leq p \leq b$ and (iii) $a \leq q \leq b$, we have $\psi(a) + \psi(b) \geq \psi(p) + \psi(q)$. Thus, it is enough to show that

$$(x_1 + x_2) + (y_1 + y_2) = (\mu(x, y)_1 + \mu(x, y)_2) + (\mu(y, x)_1 + \mu(y, x)_2), \quad (3.6)$$

$$x_1 + x_2 \leq \mu(x, y)_1 + \mu(x, y)_2 \leq y_1 + y_2, \quad (3.7)$$

$$x_1 + x_2 \leq \mu(y, x)_1 + \mu(y, x)_2 \leq y_1 + y_2. \quad (3.8)$$

Obviously, (3.6) holds by

$$x_1 + y_1 = \mu(x, y)_1 + \mu(y, x)_1, \quad x_2 + y_2 = \mu(x, y)_2 + \mu(y, x)_2.$$

To show (3.7) and (3.8) under $x_1 + x_2 \leq y_1 + y_2$, we consider the following three cases separately: Case 1: $x_1 \leq y_1$ and $x_2 \leq y_2$, Case 2: $x_1 > y_1$ and $x_2 < y_2$, Case 3: $x_1 < y_1$ and $x_2 > y_2$.

Case 1 ($x_1 \leq y_1$ and $x_2 \leq y_2$). In this case, we have $\mu(x, y)_1 = \lfloor \frac{x_1 + y_1}{2} \rfloor$, $\mu(x, y)_2 = \lfloor \frac{x_2 + y_2}{2} \rfloor$, $\mu(y, x)_1 = \lceil \frac{x_1 + y_1}{2} \rceil$ and $\mu(y, x)_2 = \lceil \frac{x_2 + y_2}{2} \rceil$, which imply

$$x_1 \leq \mu(x, y)_1 \leq \mu(y, x)_1 \leq y_1, \quad x_2 \leq \mu(x, y)_2 \leq \mu(y, x)_2 \leq y_2.$$

Conditions (3.7) and (3.8) are direct consequences of the above inequalities.

Case 2 ($x_1 > y_1$ and $x_2 < y_2$). In this case, under condition $x_1 + x_2 \leq y_1 + y_2$, $c_1 = x_1 - y_1$ and $c_2 = y_2 - x_2$ satisfy $c_2 \geq c_1 > 0$. By the following calculations:

$$\begin{aligned} \mu(x, y)_1 &= \left\lfloor \frac{x_1 + y_1}{2} \right\rfloor = x_1 - \left\lfloor \frac{c_1}{2} \right\rfloor = y_1 + \left\lceil \frac{c_1}{2} \right\rceil, \\ \mu(x, y)_2 &= \left\lfloor \frac{x_2 + y_2}{2} \right\rfloor = x_2 + \left\lfloor \frac{c_2}{2} \right\rfloor = y_2 - \left\lceil \frac{c_2}{2} \right\rceil, \\ \mu(y, x)_1 &= \left\lceil \frac{x_1 + y_1}{2} \right\rceil = x_1 - \left\lceil \frac{c_1}{2} \right\rceil = y_1 + \left\lfloor \frac{c_1}{2} \right\rfloor, \\ \mu(y, x)_2 &= \left\lceil \frac{x_2 + y_2}{2} \right\rceil = x_2 + \left\lceil \frac{c_2}{2} \right\rceil = y_2 - \left\lfloor \frac{c_2}{2} \right\rfloor, \end{aligned}$$

we have

$$\mu(x, y)_1 + \mu(x, y)_2 = x_1 + x_2 + \left(\left\lfloor \frac{c_2}{2} \right\rfloor - \left\lfloor \frac{c_1}{2} \right\rfloor \right) = y_1 + y_2 - \left(\left\lceil \frac{c_2}{2} \right\rceil - \left\lceil \frac{c_1}{2} \right\rceil \right),$$

$$\mu(y, x)_1 + \mu(y, x)_2 = x_1 + x_2 + \left(\left\lceil \frac{c_2}{2} \right\rceil - \left\lceil \frac{c_1}{2} \right\rceil \right) = y_1 + y_2 - \left(\left\lfloor \frac{c_2}{2} \right\rfloor - \left\lfloor \frac{c_1}{2} \right\rfloor \right).$$

Conditions (3.7) and (3.8) follow from $\left\lfloor \frac{c_2}{2} \right\rfloor - \left\lfloor \frac{c_1}{2} \right\rfloor \geq 0$ and $\left\lceil \frac{c_2}{2} \right\rceil - \left\lceil \frac{c_1}{2} \right\rceil \geq 0$.

Case 3 ($x_1 < y_1$ and $x_2 > y_2$). In this case, we can show (3.7) and (3.8) in the same way as Case 2. $\square$

By Proposition 4 (4), Proposition 6 and Lemma 1, we have the next property.

Theorem 4 Any 2-separable convex function is DDM-convex.

Hence, 2-separable convex function is integrally convex.

4 Parallelogram inequality

Parallelogram inequality was originally proposed in [18] for globally/locally discrete midpoint convex functions. By borrowing arguments from [18], we show that DDM-convex sets/functions have similar properties.

For every pair $(x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $\|y - x\|_\infty = m$, we consider sets defined by

$$A_k = \{i \mid y_i - x_i \geq k\}, \quad B_k = \{i \mid y_i - x_i \leq -k\} \quad (k = 1, \dots, m), \quad (4.1)$$

for which $A_1 \supseteq A_2 \supseteq \dots \supseteq A_m$, $B_1 \supseteq B_2 \supseteq \dots \supseteq B_m$, $A_1 \cap B_1 = \emptyset$ and $A_m \cup B_m \neq \emptyset$.

We first show the following property of DDM-convex sets.

Theorem 5 Let $S \subseteq \mathbb{Z}^n$ be a DDM-convex set, $x, y \in S$ with $\|y - x\|_\infty = m$, and $J \subseteq \{1, 2, \dots, m\}$. If $\{A_k\}$ and $\{B_k\}$ are defined by (4.1) and $d = \sum_{k \in J} (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$, we have $x + d \in S$ and $y - d \in S$.

To show this theorem, it is enough to verify

$$x + \sum_{k \in J} (\mathbf{1}_{A_k} - \mathbf{1}_{B_k}) \in S \quad (\forall J \subseteq \{1, 2, \dots, m\}). \quad (4.2)$$

We first show that the decomposition $\sum_{k=1}^m (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$ of $y - x$ can be constructed by using the operation $\mu(\cdot, \cdot)$.

For every $x \in \mathbb{Z}^n$, let us consider multiset $D(x)$ of vectors by the following recursive formula:

$$D(x) = \begin{cases} \emptyset & (x = \mathbf{0}), \\ \{x\} & (\|x\|_\infty = 1), \\ \{\mu(x, \mathbf{0}), \mu(\mathbf{0}, x)\} & (\|x\|_\infty = 2), \\ D(\mu(x, \mathbf{0})) \cup D(\mu(\mathbf{0}, x)) & (\|x\|_\infty \geq 3), \end{cases} \quad (4.3)$$

where $\mathbf{0}$ denotes the n -dimensional zero vector. We give several propositions.

Proposition 7 *If a multiset $\{d^k \in \{-1, 0, 1\}^n \setminus \{\mathbf{0}\} \mid k = 1, \dots, m\}$ satisfies*

$$1 \geq d_i^1 \geq d_i^2 \geq \dots \geq d_i^m \geq 0 \quad \text{or} \quad -1 \leq d_i^1 \leq d_i^2 \leq \dots \leq d_i^m \leq 0 \quad (4.4)$$

for each $i \in \{1, \dots, n\}$, then

$$D\left(\sum_{k=1}^m d^k\right) = \{d^k \mid k = 1, \dots, m\}. \quad (4.5)$$

Proof We prove the assertion by induction on m . The assertion obviously holds if $m \leq 1$.

Suppose that $m = 2$. By (4.4), we have

$$\|d^1 + d^2\|_\infty = 2, \quad \mu(d^1 + d^2, \mathbf{0}) = d^1, \quad \mu(\mathbf{0}, d^1 + d^2) = d^2,$$

which, together with (4.3), imply the assertion.

Suppose that $m \geq 3$. Let $K = \{1, 2, \dots, m\}$, $K^O = \{k \in K \mid k \text{ is odd}\}$ and $K^E = \{k \in K \mid k \text{ is even}\}$. By induction hypothesis together with $|K^O|, |K^E| < |K|$, we obtain

$$D\left(\sum_{k \in K^O} d^k\right) = \{d^k \mid k \in K^O\}, \quad D\left(\sum_{k \in K^E} d^k\right) = \{d^k \mid k \in K^E\}. \quad (4.6)$$

Furthermore, the claim below guarantees that

$$\mu\left(\sum_{k \in K} d^k, \mathbf{0}\right) = \sum_{k \in K^O} d^k, \quad \mu\left(\mathbf{0}, \sum_{k \in K} d^k\right) = \sum_{k \in K^E} d^k. \quad (4.7)$$

By combining (4.3), (4.6) and (4.7), we have

$$\begin{aligned} D\left(\sum_{k \in K} d^k\right) &= D\left(\mu\left(\sum_{k \in K} d^k, \mathbf{0}\right)\right) \cup D\left(\mu\left(\mathbf{0}, \sum_{k \in K} d^k\right)\right) \\ &= D\left(\sum_{k \in K^O} d^k\right) \cup D\left(\sum_{k \in K^E} d^k\right) \\ &= \{d^k \mid k \in K^O\} \cup \{d^k \mid k \in K^E\} \\ &= \{d^k \mid k \in K\}. \end{aligned}$$

Claim (i) $\mu\left(\sum_{k \in K} d^k, \mathbf{0}\right) = \sum_{k \in K^O} d^k$, and (ii) $\mu\left(\mathbf{0}, \sum_{k \in K} d^k\right) = \sum_{k \in K^E} d^k$.

Proof We show (i) (and can show (ii) in the same way). Let us fix $i \in \{1, \dots, n\}$. Assume that $\left(\sum_{k \in K} d^k\right)_i = l > 0$. Since $d_i^1 = \dots = d_i^l = 1$ and $d_i^{l+1} = \dots = d_i^m = 0$ by (4.4), we have

$$\mu\left(\sum_{k \in K} d^k, \mathbf{0}\right)_i = \left\lfloor \frac{l}{2} \right\rfloor = |\{1, \dots, l\} \cap \{k \mid k \text{ is odd}\}| = \left(\sum_{k \in K^0} d^k\right)_i.$$

In the case where $(\sum_{k \in K} d^k)_i = -l < 0$, we have $d_i^1 = \dots = d_i^l = -1$ and $d_i^{l+1} = \dots = d_i^m = 0$ by (4.4), and hence

$$\mu\left(\sum_{k \in K} d^k, \mathbf{0}\right)_i = \left\lfloor \frac{-l}{2} \right\rfloor = -\left\lceil \frac{l}{2} \right\rceil = -|\{1, \dots, l\} \cap \{k \mid k \text{ is odd}\}| = \left(\sum_{k \in K^0} d^k\right)_i.$$

If $(\sum_{k \in K} d^k)_i = 0$, by (4.4), we have $d_i^1 = \dots = d_i^m = 0$ and

$$\mu\left(\sum_{k \in K} d^k, \mathbf{0}\right)_i = 0 = \left(\sum_{k \in K^0} d^k\right)_i.$$

Thus, (i) holds. (End of the proof of Claim). $\square$

Proposition 8 $D(y-x) = \{\mathbf{1}_{A_k} - \mathbf{1}_{B_k} \mid k = 1, \dots, m\}$.

Proof By the construction (4.1), $y-x = \sum_{k=1}^m (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$. By defining $d^k = \mathbf{1}_{A_k} - \mathbf{1}_{B_k}$ ($k = 1, \dots, m$), condition (4.4) of Proposition 7 holds. The assertion is an immediate consequence of (4.5). $\square$

By Proposition 8, (4.2) can be rewritten as

$$x + \sum \{d \mid d \in E\} \in S \quad (\forall E \subseteq D(y-x)). \quad (4.8)$$

Therefore, Theorem 5 can be shown by the following proposition.

Proposition 9 For a DDM-convex set S and $x, y \in S$, (4.8) holds.

Proof We prove (4.8) for x, y by induction on $\|y-x\|_\infty$. If $\|y-x\|_\infty \leq 1$, (4.8) trivially holds. If $\|y-x\|_\infty = 2$, then $D(y-x) = \{\mu(y-x, \mathbf{0}), \mu(\mathbf{0}, y-x)\}$. Since S is DDM-convex, we have

$$\begin{aligned} x + \mu(y-x, \mathbf{0}) &= \mu(y, x) \in S, \\ x + \mu(\mathbf{0}, y-x) &= \mu(x, y) \in S, \end{aligned}$$

which guarantee that (4.8) holds.

Suppose that $\|y-x\|_\infty \geq 3$, and (4.8) holds for every $x'', y'' \in S$ with $\|x'' - y''\|_\infty < \|x-y\|_\infty$. We fix $E \subseteq D(y-x)$ arbitrarily. Let $x' = \mu(x, y)$ and $y' = \mu(y, x)$. Then we have $y' - x = y - x' = \mu(y-x, \mathbf{0})$. By DDM-convexity of S , x' and y' also belong to S . By Proposition 2 (4) and the assumption $\|y-x\|_\infty \geq 3$, we have

$$\|y' - x\|_\infty = \|y - x'\|_\infty = \|\mu(y-x, \mathbf{0})\|_\infty < \|y-x\|_\infty.$$

By induction hypothesis, (4.8) holds for (x, y') and (x', y) , and furthermore, by the equality $D(y' - x) = D(y - x') = D(\mu(y - x, \mathbf{0}))$, we have

$$\begin{aligned} u &= x + \sum \{d \mid d \in E \cap D(\mu(y - x, \mathbf{0}))\} \in S, \\ v &= x' + \sum \{d \mid d \in E \cap D(\mu(y - x, \mathbf{0}))\} \in S. \end{aligned} \quad (4.9)$$

We also have $v - u = x' - x = \mu(\mathbf{0}, y - x)$ and

$$\|v - u\|_\infty = \|x' - x\|_\infty = \|\mu(\mathbf{0}, y - x)\|_\infty < \|y - x\|_\infty,$$

which, together with the induction hypothesis, guarantee that (4.8) holds for (u, v) . Moreover, by (4.3), $D(y - x) = D(\mu(y - x, \mathbf{0})) \cup D(\mu(\mathbf{0}, y - x))$ includes E , and hence, $D(v - u) = D(\mu(\mathbf{0}, y - x))$ includes $E \setminus D(\mu(y - x, \mathbf{0}))$. Thus, we have

$$w = u + \sum \{d \mid d \in E \setminus D(\mu(y - x, \mathbf{0}))\} \in S. \quad (4.10)$$

By (4.9) and (4.10), we obtain

$$w = x + \sum \{d \mid d \in E\} \in S,$$

which implies (4.8). $\square$

We denote by $\text{DDMC}(k)$ and by $\text{DDMC}(\geq k)$ the classes of functions $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ that satisfy DDM-convexity (2.1) for all $x, y \in \mathbb{Z}^n$ with $\|x - y\|_\infty = k$ and for all $x, y \in \mathbb{Z}^n$ with $\|x - y\|_\infty \geq k$, respectively. Before presenting parallelogram inequality for DDM-convex functions, we give a useful property.

Theorem 6 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function in $\text{DDMC}(2)$ such that $\text{dom} f$ is DDM-convex. For $x \in \text{dom} f$ and $y \in \mathbb{Z}^n$ with $\|y - x\|_\infty = m$, and for any partition (I, J) of $\{1, \dots, m\}$, we consider*

$$d_1 = \sum_{i \in I} (\mathbf{1}_{A_i} - \mathbf{1}_{B_i}), \quad d_2 = \sum_{j \in J} (\mathbf{1}_{A_j} - \mathbf{1}_{B_j}),$$

where A_k, B_k ($k = 1, \dots, m$) are the sets defined by (4.1). Then we have

$$f(x) + f(x + d_1 + d_2) \geq f(x + d_1) + f(x + d_2). \quad (4.11)$$

Proof We note that $y = x + d_1 + d_2$. If $y \notin \text{dom} f$, by $f(y) = +\infty$, (4.11) trivially holds. In the sequel, we assume that $y \in \text{dom} f$. Let I be denoted by $\{i_1, i_2, \dots, i_{|I|}\}$ ($i_1 < i_2 < \dots < i_{|I|}$) and J by $\{j_1, j_2, \dots, j_{|J|}\}$ ($j_1 < j_2 < \dots < j_{|J|}$). For every $k = 1, \dots, |I|$ and $l = 1, \dots, |J|$, we denote $d_1^k = \mathbf{1}_{A_{i_k}} - \mathbf{1}_{B_{i_k}}$ and, similarly, $d_2^l = \mathbf{1}_{A_{j_l}} - \mathbf{1}_{B_{j_l}}$. For every $k = 0, 1, \dots, |I|$ and for every $l = 0, 1, \dots, |J|$, define

$$x(k, l) = x + \sum_{i=1}^k d_1^i + \sum_{j=1}^l d_2^j.$$

By Theorem 5, for every k, l , we have $x(k, l) \in \text{dom} f$. We note that (4.11) is equivalent to

$$f(x(0, 0)) + f(x(|I|, |J|)) \geq f(x(|I|, 0)) + f(x(0, |J|)). \quad (4.12)$$

Fix $k \in \{1, \dots, |I|\}$ and $l \in \{1, \dots, |J|\}$. According to whether $i_k > j_l$ or $i_k < j_l$, either

$$A_{i_k} \subseteq A_{j_l}, \quad B_{i_k} \subseteq B_{j_l}, \quad A_{j_l} \cap B_{j_l} = \emptyset, \quad A_{i_k} \cup B_{i_k} \neq \emptyset$$

or

$$A_{j_l} \subseteq A_{i_k}, \quad B_{j_l} \subseteq B_{i_k}, \quad A_{i_k} \cap B_{i_k} = \emptyset, \quad A_{j_l} \cup B_{j_l} \neq \emptyset.$$

Thus, in the case where $i_k > j_l$, we have

$$(d_1^k + d_2^l)_p = \begin{cases} 1 & (p \in A_{j_l} \setminus A_{i_k}), \\ 2 & (p \in A_{i_k}), \\ -2 & (p \in B_{i_k}), \\ -1 & (p \in B_{j_l} \setminus B_{i_k}), \\ 0 & (p \notin (A_{j_l} \cup B_{j_l})), \end{cases}$$

which implies $\|d_1^k + d_2^l\|_\infty = 2$, $\mu(d_1^k + d_2^l, \mathbf{0}) = \mathbf{1}_{A_{j_l}} - \mathbf{1}_{B_{j_l}} = d_2^l$ and $\mu(\mathbf{0}, d_1^k + d_2^l) = \mathbf{1}_{A_{i_k}} - \mathbf{1}_{B_{i_k}} = d_1^k$. Similarly, in the case where $i_k < j_l$, we have $\|d_1^k + d_2^l\|_\infty = 2$, $\mu(d_1^k + d_2^l, \mathbf{0}) = d_1^k$, $\mu(\mathbf{0}, d_1^k + d_2^l) = d_2^l$. In both cases, since $\|x(k, l) - x(k-1, l-1)\|_\infty = \|d_1^k + d_2^l\|_\infty = 2$ and $f \in \text{DDMC}(2)$, we obtain

$$\begin{aligned} f(x(k, l)) + f(x(k-1, l-1)) \\ \geq f(\mu(x(k, l), x(k-1, l-1))) + f(\mu(x(k-1, l-1), x(k, l))). \end{aligned} \quad (4.13)$$

On the other hand, the facts

$$\begin{aligned} \mu(x(k, l), x(k-1, l-1)) &= x(k-1, l-1) + \mu(d_1^k + d_2^l, \mathbf{0}), \\ \mu(x(k-1, l-1), x(k, l)) &= x(k-1, l-1) + \mu(\mathbf{0}, d_1^k + d_2^l), \end{aligned}$$

together with $\{\mu(d_1^k + d_2^l, \mathbf{0}), \mu(\mathbf{0}, d_1^k + d_2^l)\} = \{d_1^k, d_2^l\}$ in the both cases where $i_k > j_l$ and $i_k < j_l$, yield the right-hand side of (4.13) is equal to $f(x(k, l-1)) + f(x(k-1, l))$. Therefore, we obtain

$$f(x(k, l)) + f(x(k-1, l-1)) \geq f(x(k, l-1)) + f(x(k-1, l)).$$

By adding the above inequalities for (k, l) with $1 \leq k \leq |I|$ and $1 \leq l \leq |J|$, we obtain (4.12). We emphasize that all the terms that are canceled in this addition of inequalities are finite valued because $x(k, l) \in \text{dom} f$ for all (k, l) with $0 \leq k \leq |I|$ and $0 \leq l \leq |J|$. $\square$

The next theorem is an immediate consequence of Theorem 6, because a DDM-convex function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ belongs to $\text{DDMC}(2)$ and $\text{dom} f$ is DDM-convex.

Theorem 7 Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a DDM-convex function. For every $x, y \in \text{dom} f$, let $\{A_k \mid k = 1, \dots, m\}$ and $\{B_k \mid k = 1, \dots, m\}$ be the families defined by (4.1), where $m = \|y - x\|_\infty$. For any subset $J \subseteq \{1, \dots, m\}$, let $d = \sum_{k \in J} (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$. Then we have

$$f(x) + f(y) \geq f(x + d) + f(y - d). \quad (4.14)$$

We call the inequality (4.14) *parallelogram inequality* of DDM-convex functions.

5 Characterizations

In this section, we give several equivalent conditions of DDM-convexity and a simple characterization of quadratic DDM-convex functions.

For every pair $(x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n$, we recall that the families $\{A_k \mid k = 1, \dots, m\}$ and $\{B_k \mid k = 1, \dots, m\}$ are defined by

$$A_k = \{i \mid y_i - x_i \geq k\}, \quad B_k = \{i \mid y_i - x_i \leq -k\} \quad (k = 1, \dots, m),$$

in (4.1), where $m = \|y - x\|_\infty$.

Theorem 8 For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the following properties are equivalent to each other.

- (1) f is DDM-convex, i.e., $f \in \text{DDMC}(\geq 1)$.
- (2) $\text{dom} f$ is DDM-convex and $f \in \text{DDMC}(2)$.
- (3) f satisfies parallelogram inequalities for every $x, y \in \text{dom} f$ and for any subset $J \subseteq \{1, \dots, m\}$ where $m = \|y - x\|_\infty$.
- (4) For every $x, y \in \text{dom} f$, we have

$$f(x) + f(y) \geq f(x + \mathbf{1}_{A_m} - \mathbf{1}_{B_m}) + f(y - \mathbf{1}_{A_m} + \mathbf{1}_{B_m}), \quad (5.1)$$

where $m = \|y - x\|_\infty$, and the sets A_m and B_m are defined by (4.1).

- (5) For every $x, y \in \text{dom} f$ and for any $\alpha \in \{1, \dots, m\}$, by defining $d = \sum_{k=1}^{\alpha} (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$, we have

$$f(x) + f(y) \geq f(x + d) + f(y - d),$$

where $m = \|y - x\|_\infty$, and the families $\{A_k \mid k = 1, \dots, m\}$ and $\{B_k \mid k = 1, \dots, m\}$ are defined by (4.1).

Proof The implication (1) $\Rightarrow$ (2) is obvious and the implications from (1) to (3), (4) and (5) follow from Theorem 7. To prove opposite implications, we show $f \in \text{DDMC}(m)$ for every $m \geq 1$ from (4), and show [(5) $\Rightarrow$ (4)], [(3) $\Rightarrow$ (4)] and [(2) $\Rightarrow$ (3)].

(4) $\Rightarrow$ (1): Since $f \in \text{DDMC}(1)$ always holds from Proposition 2, we assume $m \geq 2$ and fix $x, y \in \text{dom} f$ with $\|x - y\|_\infty = m$, arbitrarily. Let denote $p = x + \mathbf{1}_{A_m} - \mathbf{1}_{B_m}$

and $q = y - \mathbf{1}_{A_m} + \mathbf{1}_{B_m}$ in (5.1). Vectors p and q satisfy conditions (a) and (c) of Proposition 1, $\|p - q\|_\infty < m$ and $p, q \in \text{dom} f$. By repeating (5.1) for (p, q) until $\|p - q\|_\infty \leq 1$, the final p and q satisfy all conditions of Proposition 1, and hence, $p = \mu(x, y)$ and $q = \mu(y, x)$ are satisfied. This shows $f \in \text{DDMC}(m)$ holds.

(5) $\Rightarrow$ (4): By setting $\alpha = m - 1$ in (5), we obtain

$$x + d = y - \mathbf{1}_{A_m} + \mathbf{1}_{B_m}, \quad y - d = x + \mathbf{1}_{A_m} - \mathbf{1}_{B_m},$$

and hence, (4).

(3) $\Rightarrow$ (4): By setting $J = \{1, \dots, m - 1\}$ in (3), we obtain

$$x + d = y - \mathbf{1}_{A_m} + \mathbf{1}_{B_m}, \quad y - d = x + \mathbf{1}_{A_m} - \mathbf{1}_{B_m},$$

and hence, (4).

(2) $\Rightarrow$ (3): Property (2) and Theorem 6 imply (3). $\square$

Remark 1 Theorem 8 may be regarded as a generalization of Theorem 3.

Property (4) of Theorem 8 corresponds to (4) of Theorem 3. For an $L^{\natural}$ -convex function f , the following inequalities

$$f(x) + f(y) \geq f(x + \mathbf{1}_{A_m}) + f(y - \mathbf{1}_{A_m})$$

and

$$f(x) + f(y) \geq f(x - \mathbf{1}_{B_m}) + f(y + \mathbf{1}_{B_m})$$

hold. These inequalities imply (5.1). However, for a DDM-convex function f , the two parts on A_m and B_m must be combined to a single inequality (5.1). This demonstrates the difference between $L^{\natural}$ -convexity and DDM-convexity.

Property (5) of Theorem 8 corresponds to (1) of Theorem 3. In (5) of Theorem 8, two vectors $x + d = x + \sum_{k=1}^{\alpha} (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$ and $y - d = y - \sum_{k=1}^{\alpha} (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$ are expressed by

$$(x + d)_i = \begin{cases} x_i - \alpha & (x_i - y_i \geq \alpha), \\ x_i + \alpha & (y_i - x_i \geq \alpha), \\ y_i & (|x_i - y_i| < \alpha), \end{cases} \quad (y - d)_i = \begin{cases} y_i + \alpha & (x_i - y_i \geq \alpha), \\ y_i - \alpha & (y_i - x_i \geq \alpha), \\ x_i & (|x_i - y_i| < \alpha). \end{cases}$$

On the other hand, in (3.1), two vectors $(x - \alpha \mathbf{1}) \vee y$ and $x \wedge (y + \alpha \mathbf{1})$ can be rewritten as

$$((x - \alpha \mathbf{1}) \vee y)_i = \begin{cases} x_i - \alpha & (x_i - y_i \geq \alpha), \\ y_i & (x_i - y_i < \alpha), \end{cases} \\ (x \wedge (y + \alpha \mathbf{1}))_i = \begin{cases} y_i + \alpha & (x_i - y_i \geq \alpha), \\ x_i & (x_i - y_i < \alpha). \end{cases}$$

In the same way as the relation between (4) of Theorem 8 and (4) of Theorem 3, two separate operations for $L^{\natural}$ -convex functions must be executed simultaneously for DDM-convex functions. $\square$

For a quadratic function $f(x) = x^{\top} Qx$ ($x \in \mathbb{Z}^n$) with a symmetric matrix $Q = [q_{ij}]$, we show that f is DDM-convex if and only if Q is *diagonally dominant with non-negative diagonals*:

$$q_{ii} \geq \sum_{j:j \neq i} |q_{ij}| \quad (\forall i = 1, \dots, n). \quad (5.2)$$

For each $p \in \mathbb{R}$, let $p^+ = \max\{p, 0\}$ and $p^- = \max\{-p, 0\}$. Note that $|p| = p^+ + p^-$. Quadratic function $f(x) = x^{\top} Qx$ can be written as²

$$\begin{aligned} f(x) &= \sum_{i,j=1}^n q_{ij} x_i x_j \\ &= \sum_{i=1}^n \left(q_{ii} - \sum_{j:j \neq i} |q_{ij}| \right) x_i^2 + \frac{1}{2} \sum_{i,j:j \neq i} (q_{ij}^+ (x_i + x_j)^2 + q_{ij}^- (x_i - x_j)^2). \end{aligned} \quad (5.3)$$

In (5.3), the condition (5.2) of Q implies the nonnegativity of coefficients of x_i^2 . Thus, if Q is diagonally dominant with nonnegative diagonals, then f is 2-separable convex, and hence, DDM-convex by Theorem 4. By proving the opposite implication, we obtain the following property.

Theorem 9 *For a quadratic function $f(x) = x^{\top} Qx$ ($x \in \mathbb{Z}^n$) with a symmetric matrix $Q = [q_{ij}]$, f is DDM-convex if and only if Q is diagonally dominant with nonnegative diagonals.*

Proof It is enough to show that if f is DDM-convex, then Q is diagonally dominant with nonnegative diagonals. For each $i \in \{1, \dots, n\}$, define $z^i \in \mathbb{Z}^n$ by

$$z_j^i = \begin{cases} 2 & (j = i), \\ -1 & (j \neq i \text{ and } q_{ij} \geq 0), \\ 1 & (j \neq i \text{ and } q_{ij} < 0). \end{cases}$$

By DDM-convexity of f , the inequality

$$f(z^i) + f(\mathbf{0}) \geq f(\mu(z^i, \mathbf{0})) + f(\mu(\mathbf{0}, z^i))$$

must hold. Since $\mu(z^i, \mathbf{0}) = z^i - \mathbf{1}_i$ and $\mu(\mathbf{0}, z^i) = \mathbf{1}_i$, we have

$$(z^i)^{\top} Q z^i \geq (z^i - \mathbf{1}_i)^{\top} Q (z^i - \mathbf{1}_i) + \mathbf{1}_i^{\top} Q \mathbf{1}_i,$$

² Private communication with F. Tardella (2017).

which implies

$$\begin{aligned} 0 &\leq (z^i)^\top Q \mathbf{1}_i - \mathbf{1}_i^\top Q \mathbf{1}_i \\ &= q_{ii} - \sum_{j:j \neq i \wedge q_{ji} \geq 0} q_{ji} + \sum_{j:j \neq i \wedge q_{ji} < 0} q_{ji} \\ &= q_{ii} - \sum_{j:j \neq i} |q_{ji}|. \end{aligned}$$

By $Q^\top = Q$, we obtain the diagonal dominance with nonnegative diagonals of Q . $\square$

The minimizers of DDM-convex functions are DDM-convex sets, while the minimizers of $L^{\natural}$ -convex functions are $L^{\natural}$ -convex sets. The class of $L^{\natural}$ -convex functions has a characterization in terms of minimizers. For a function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $p \in \mathbb{R}^n$, we denote by $f - p$ the function given by

$$(f - p)(x) = f(x) - \sum_{i=1}^n p_i x_i \quad (\forall x \in \mathbb{Z}^n).$$

Theorem 10 ([21, 23]) *Under some regularity condition, a function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is $L^{\natural}$ -convex if and only if $\operatorname{argmin}(f - p)$ is an $L^{\natural}$ -convex set for every $p \in \mathbb{R}^n$ with $\inf(f - p) > -\infty$.*

Unfortunately, the class of DDM-convex functions does not have a similar characterization.

Example 4 Let us consider the function $f: \mathbb{Z}^3 \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$\begin{aligned} f(0, 0, 0) &= 0, & f(1, 0, 1) &= f(1, 1, 0) = 1, \\ f(1, 0, 0) &= f(1, 1, 1) = 2, & f(2, 1, 1) &= 3, \\ f(x) &= +\infty \quad (\text{otherwise}). \end{aligned}$$

The function f is not DDM-convex because

$$f(0, 0, 0) + f(2, 1, 1) = 3 < 4 = f(1, 0, 0) + f(1, 1, 1),$$

while $\operatorname{dom} f$ is a DDM-convex set (in fact, an $L^{\natural}$ -convex set). Furthermore, $\operatorname{argmin}(f - p)$ is a DDM-convex set for every $p \in \mathbb{R}^3$ as follows. There exists no $p \in \mathbb{R}^3$ such that $\{(0, 0, 0), (2, 1, 1)\} \subseteq \operatorname{argmin}(f - p)$, because we have $0 \leq 1 - p_1 - p_2$ from $(f - p)(0, 0, 0) \leq (f - p)(1, 1, 0)$, and $2 - p_1 - p_2 \leq 0$ from $(f - p)(2, 1, 1) \leq (f - p)(1, 0, 1)$. For any $p \in \mathbb{R}^3$ and for any $x, y \in \operatorname{argmin}(f - p)$, this fact implies that $\|x - y\|_{\infty} \leq 1$ must hold, and hence, $\operatorname{argmin}(f - p)$ is a DDM-convex set. We note that this example also shows that a similar characterization does not hold for the classes of globally/locally discrete midpoint convex functions. $\square$

6 Operations

We discuss several operations for discrete convex functions, including scaling operations [17, 18, 21], restriction, projection, direct sum and convolution operations [16, 21, 22].

6.1 Scaling operations

Scaling operations are useful techniques for designing efficient algorithms in discrete optimization. It is shown in [18] that global/local discrete midpoint convexity, including L^1 -convexity, is closed under scaling operations. We show that DDM-convexity is also closed under scaling operations.

Given a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a positive integer α , the α -scaling of f is the function f^α defined by

$$f^\alpha(x) = f(\alpha x) \quad (x \in \mathbb{Z}^n).$$

We also define the α -scaling S^α of a set $S \subseteq \mathbb{Z}^n$ by

$$S^\alpha = \{x \in \mathbb{Z}^n \mid \alpha x \in S\}.$$

Theorem 11 *Given a DDM-convex function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a positive integer α , the scaled function $f^\alpha : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is also DDM-convex.*

Proof By the equivalence between (1) and (4) of Theorem 8, it is sufficient to show that

$$f^\alpha(x) + f^\alpha(y) \geq f^\alpha(x + \mathbf{1}_{A_m} - \mathbf{1}_{B_m}) + f^\alpha(y - \mathbf{1}_{A_m} + \mathbf{1}_{B_m})$$

for every $x, y \in \mathbb{Z}^n$ with $\|x - y\|_\infty = m$ and for families $\{A_k \mid k = 1, \dots, m\}$ and $\{B_k \mid k = 1, \dots, m\}$ defined by (4.1). The above inequality is written as

$$f(\alpha x) + f(\alpha y) \geq f(\alpha(x + \mathbf{1}_{A_m} - \mathbf{1}_{B_m})) + f(\alpha(y - \mathbf{1}_{A_m} + \mathbf{1}_{B_m})). \quad (6.1)$$

For αx and αy with $\|\alpha x - \alpha y\|_\infty = \alpha m$, by defining

$$A_l^\alpha = \{i \mid \alpha y_i - \alpha x_i \geq l\}, \quad B_l^\alpha = \{i \mid \alpha y_i - \alpha x_i \leq -l\} \quad (l = 1, \dots, \alpha m),$$

we have $\alpha y - \alpha x = \sum_{l=1}^{\alpha m} (\mathbf{1}_{A_l^\alpha} - \mathbf{1}_{B_l^\alpha})$ and

$$A_k = A_{\alpha(k-1)+j}^\alpha, \quad B_k = B_{\alpha(k-1)+j}^\alpha \quad (k = 1, \dots, m; j = 1, \dots, \alpha).$$

By (5) of Theorem 8 for

$$d = \sum_{l=1}^{\alpha(m-1)} (\mathbf{1}_{A_l^\alpha} - \mathbf{1}_{B_l^\alpha}), \quad d' = \sum_{l=\alpha(m-1)+1}^{\alpha m} (\mathbf{1}_{A_l^\alpha} - \mathbf{1}_{B_l^\alpha}) = \alpha(\mathbf{1}_{A_m} - \mathbf{1}_{B_m})$$

we have

$$\begin{aligned} f(\alpha x) + f(\alpha y) &\geq f(\alpha x + d) + f(\alpha y - d) \\ &= f(\alpha y - d') + f(\alpha x + d') \\ &= f(\alpha(y - \mathbf{1}_{A_m} + \mathbf{1}_{B_m})) + f(\alpha(x + \mathbf{1}_{A_m} - \mathbf{1}_{B_m})), \end{aligned}$$

that is, (6.1). $\square$

Corollary 3 *For a DDM-convex set $S \subseteq \mathbb{Z}^n$ and a positive integer α , the α -scaled set S^α is also DDM-convex.*

6.2 Restrictions

For a function $f : \mathbb{Z}^{n+m} \rightarrow \mathbb{R} \cup \{+\infty\}$, the *restriction* of f on $\mathbb{Z}^n$ is the function g defined by

$$g(x) = f(x, \mathbf{0}) \quad (x \in \mathbb{Z}^n).$$

For a set $S \subseteq \mathbb{Z}^{n+m}$, the restriction of S on $\mathbb{Z}^n$ is also defined by

$$T = \{x \in \mathbb{Z}^n \mid (x, \mathbf{0}) \in S\}.$$

Obviously, the following properties hold.

Proposition 10 *For a DDM-convex function, its restrictions are also DDM-convex.*

Proposition 11 *For a DDM-convex set, its restrictions are also DDM-convex.*

6.3 Projections

For a function $f : \mathbb{Z}^{n+m} \rightarrow \mathbb{R} \cup \{+\infty\}$, the *projection* of f to $\mathbb{Z}^n$ is the function defined by

$$g(x) = \inf\{f(x, y) \mid y \in \mathbb{Z}^m\} \quad (x \in \mathbb{Z}^n), \quad (6.2)$$

where we assume that $g(x) > -\infty$ for all $x \in \mathbb{Z}^n$. For a set $S \subseteq \mathbb{Z}^{n+m}$, the projection of S to $\mathbb{Z}^n$ is also defined by

$$T = \{x \in \mathbb{Z}^n \mid \exists y \in \mathbb{Z}^m : (x, y) \in S\}.$$

In the same way as the proof for globally discrete midpoint convex functions in [16, Theorem 3.5] we can show the following property.

Proposition 12 *For a DDM-convex function, its projections are DDM-convex.*

Proof Let g be the projection defined by (6.2) of a DDM-convex function f . For every $x^{(1)}, x^{(2)} \in \text{dom } g$ and every $\varepsilon > 0$, by the definition of the projection, there exist $y^{(1)}, y^{(2)} \in \mathbb{Z}^m$ with $g(x^{(i)}) \geq f(x^{(i)}, y^{(i)}) - \varepsilon$ for $i = 1, 2$. Thus, we have

$$g(x^{(1)}) + g(x^{(2)}) \geq f(x^{(1)}, y^{(1)}) + f(x^{(2)}, y^{(2)}) - 2\varepsilon. \quad (6.3)$$

By DDM-convexity of f and the definition of the projection, we have

$$\begin{aligned} f(x^{(1)}, y^{(1)}) + f(x^{(2)}, y^{(2)}) \\ \geq f(\mu((x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}))) + f(\mu((x^{(2)}, y^{(2)}), (x^{(1)}, y^{(1)}))) \\ \geq g(\mu(x^{(1)}, x^{(2)})) + g(\mu(x^{(2)}, x^{(1)})). \end{aligned} \quad (6.4)$$

By (6.3) and (6.4), we obtain

$$g(x^{(1)}) + g(x^{(2)}) \geq g(\mu(x^{(1)}, x^{(2)})) + g(\mu(x^{(2)}, x^{(1)})) - 2\varepsilon$$

for any $\varepsilon > 0$, which guarantees DDM-convexity of g . $\square$

Corollary 4 For a DDM-convex set, its projections are also DDM-convex.

6.4 Direct sums

For two functions $f_1 : \mathbb{Z}^{n_1} \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{Z}^{n_2} \rightarrow \mathbb{R} \cup \{+\infty\}$, the *direct sum* of f_1 and f_2 is the function $f_1 \oplus f_2 : \mathbb{Z}^{n_1+n_2} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$(f_1 \oplus f_2)(x, y) = f_1(x) + f_2(y) \quad (x \in \mathbb{Z}^{n_1}, y \in \mathbb{Z}^{n_2}).$$

For two sets $S_1 \subseteq \mathbb{Z}^{n_1}$ and $S_2 \subseteq \mathbb{Z}^{n_2}$, the direct sum of S_1 and S_2 is defined by

$$S_1 \oplus S_2 = \{(x, y) \mid x \in S_1, y \in S_2\}.$$

DDM-convexity is closed under direct sums as below.

Proposition 13 For two DDM-convex functions $f_1 : \mathbb{Z}^{n_1} \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{Z}^{n_2} \rightarrow \mathbb{R} \cup \{+\infty\}$, the direct sum $f_1 \oplus f_2$ is also DDM-convex.

Proof For every $x^{(1)}, y^{(1)} \in \mathbb{Z}^{n_1}$ and $x^{(2)}, y^{(2)} \in \mathbb{Z}^{n_2}$, it follows from DDM-convexity of f_1 and f_2 that

$$f_1(x^{(1)}) + f_1(y^{(1)}) \geq f_1(\mu(x^{(1)}, y^{(1)})) + f_1(\mu(y^{(1)}, x^{(1)})), \quad (6.5)$$

$$f_2(x^{(2)}) + f_2(y^{(2)}) \geq f_2(\mu(x^{(2)}, y^{(2)})) + f_2(\mu(y^{(2)}, x^{(2)})). \quad (6.6)$$

By the following relations

$$\begin{aligned}\mu((x^{(1)}, x^{(2)}), (y^{(1)}, y^{(2)})) &= (\mu(x^{(1)}, y^{(1)}), \mu(x^{(2)}, y^{(2)})), \\ \mu((y^{(1)}, y^{(2)}), (x^{(1)}, x^{(2)})) &= (\mu(y^{(1)}, x^{(1)}), \mu(y^{(2)}, x^{(2)})),\end{aligned}$$

and by (6.5) and (6.6), we obtain

$$\begin{aligned}(f_1 \oplus f_2)(x^{(1)}, x^{(2)}) + (f_1 \oplus f_2)(y^{(1)}, y^{(2)}) \\ &= [f_1(x^{(1)}) + f_2(x^{(2)})] + [f_1(y^{(1)}) + f_2(y^{(2)})] \\ &\geq [f_1(\mu(x^{(1)}, y^{(1)})) + f_2(\mu(x^{(2)}, y^{(2)}))] + [f_1(\mu(y^{(1)}, x^{(1)})) + f_2(\mu(y^{(2)}, x^{(2)}))] \\ &= (f_1 \oplus f_2)((\mu(x^{(1)}, y^{(1)}), \mu(x^{(2)}, y^{(2)}))) + (f_1 \oplus f_2)((\mu(y^{(1)}, x^{(1)}), \mu(y^{(2)}, x^{(2)}))) \\ &= (f_1 \oplus f_2)(\mu((x^{(1)}, x^{(2)}), (y^{(1)}, y^{(2)}))) + (f_1 \oplus f_2)(\mu((y^{(1)}, y^{(2)}), (x^{(1)}, x^{(2)}))),\end{aligned}$$

which says DDM-convexity of $f_1 \oplus f_2$. $\square$

Corollary 5 For two DDM-convex sets $S_1 \subseteq \mathbb{Z}^{n_1}$ and $S_2 \subseteq \mathbb{Z}^{n_2}$, $S_1 \oplus S_2$ is also DDM-convex.

6.5 Convolutions

For two functions $f_1, f_2 : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the *convolution* $f_1 \square f_2$ is the function defined by

$$(f_1 \square f_2)(x) = \inf\{f_1(y) + f_2(z) \mid x = y + z, \ y, z \in \mathbb{Z}^n\} \quad (x \in \mathbb{Z}^n),$$

where we assume $(f_1 \square f_2)(x) > -\infty$ for every $x \in \mathbb{Z}^n$. For two sets $S_1, S_2 \subseteq \mathbb{Z}^n$, the *Minkowski sum* $S_1 + S_2$ defined by

$$S_1 + S_2 = \{x + y \mid x \in S_1, y \in S_2\}$$

corresponds to the convolution of indicator functions δ_{S_1} and δ_{S_2} . The next example shows that the Minkowski sum of two DDM-convex sets may not be DDM-convex, and hence, DDM-convexity is not closed under the convolutions.

Example 5 We borrow the example in [16, Example 4.2], which shows that $L^{\natural}$ -convexity and global/local discrete midpoint convexity may not be closed under the convolutions.

Let $S_1 = \{(0, 0, 0), (1, 1, 0)\}$ and $S_2 = \{(0, 0, 0), (0, 1, 1)\}$ which are DDM-convex. The Minkowski sum of S_1 and S_2

$$S_1 + S_2 = \{(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 2, 1)\}$$

is not DDM-convex, because for $x = (0, 0, 0)$ and $y = (1, 2, 1)$, we have $\mu(x, y) = (0, 1, 0) \notin S_1 + S_2$ and $\mu(y, x) = (1, 1, 1) \notin S_1 + S_2$. $\square$

It is known that the convolution of an $L^{\natural}$ -convex function and a separable convex function is also $L^{\natural}$ -convex, where a separable convex function $\varphi : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is given by

$$\varphi(x) = \sum_{i=1}^n \varphi_i(x_i) \quad (x \in \mathbb{Z}^n)$$

for univariate discrete convex functions $\varphi_i (i = 1, \dots, n)$. By the same arguments of [16, Proposition 4.7], this can be extended to DDM-convexity.

Proposition 14 *The convolution of a DDM-convex function and a separable convex function is also DDM-convex.*

Proof Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a DDM-convex function, $\varphi : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ a separable convex function represented as $\sum_{i=1}^n \varphi_i$ and let $g = f \square \varphi$. For every $x^{(1)}, x^{(2)} \in \text{dom } g$ and $\varepsilon > 0$, by the definition of convolutions, there exist $y^{(i)}, z^{(i)} (i = 1, 2)$ such that

$$g(x^{(i)}) \geq f(y^{(i)}) + \varphi(z^{(i)}) - \varepsilon, \quad x^{(i)} = y^{(i)} + z^{(i)} \quad (i = 1, 2). \quad (6.7)$$

It follows from DDM-convexity of f that

$$f(y^{(1)}) + f(y^{(2)}) \geq f(\mu(y^{(1)}, y^{(2)})) + f(\mu(y^{(2)}, y^{(1)})). \quad (6.8)$$

Let

$$z' = \mu(x^{(1)}, x^{(2)}) - \mu(y^{(1)}, y^{(2)}), \quad z'' = \mu(x^{(2)}, x^{(1)}) - \mu(y^{(2)}, y^{(1)}).$$

By $g = f \square \varphi$, we have

$$f(\mu(y^{(1)}, y^{(2)})) + \varphi(z') \geq g(\mu(x^{(1)}, x^{(2)})), \quad (6.9)$$

$$f(\mu(y^{(2)}, y^{(1)})) + \varphi(z'') \geq g(\mu(x^{(2)}, x^{(1)})). \quad (6.10)$$

The claim, below, states that

$$\varphi_i(z_i^{(1)}) + \varphi_i(z_i^{(2)}) \geq \varphi_i(z'_i) + \varphi_i(z''_i) \quad (6.11)$$

for each $i = 1, \dots, n$. Thus, we have

$$\varphi(z^{(1)}) + \varphi(z^{(2)}) \geq \varphi(z') + \varphi(z''). \quad (6.12)$$

By summing up (6.7), (6.8), (6.9), (6.10) and (6.12), we obtain

$$g(x^{(1)}) + g(x^{(2)}) \geq g(\mu(x^{(1)}, x^{(2)})) + g(\mu(x^{(2)}, x^{(1)})) - 2\varepsilon$$

for any $\varepsilon > 0$, which guarantees DDM-convexity of g .

Claim (6.11) holds.

Proof Since φ_i is univariate discrete convex, for every $a, b \in \mathbb{Z}$ with $a \leq b$ and for every $p, q \in \mathbb{Z}$ such that (i) $a + b = p + q$, (ii) $a \leq p \leq b$ and (iii) $a \leq q \leq b$, we have $\varphi_i(a) + \varphi_i(b) \geq \varphi_i(p) + \varphi_i(q)$. Thus, it is enough to show the following relations:

$$z_i^{(1)} + z_i^{(2)} = z_i' + z_i'', \quad (6.13)$$

$$\min\{z_i^{(1)}, z_i^{(2)}\} \leq z_i' \leq \max\{z_i^{(1)}, z_i^{(2)}\}, \quad (6.14)$$

$$\min\{z_i^{(1)}, z_i^{(2)}\} \leq z_i'' \leq \max\{z_i^{(1)}, z_i^{(2)}\}. \quad (6.15)$$

Condition (6.13) follows from

$$\begin{aligned} \mu(x^{(1)}, x^{(2)})_i + \mu(x^{(2)}, x^{(1)})_i &= x_i^{(1)} + x_i^{(2)}, \\ \mu(y^{(1)}, y^{(2)})_i + \mu(y^{(2)}, y^{(1)})_i &= y_i^{(1)} + y_i^{(2)}, \end{aligned}$$

and

$$\begin{aligned} z_i' + z_i'' &= (\mu(x^{(1)}, x^{(2)})_i - \mu(y^{(1)}, y^{(2)})_i) + (\mu(x^{(2)}, x^{(1)})_i - \mu(y^{(2)}, y^{(1)})_i) \\ &= (\mu(x^{(1)}, x^{(2)})_i + \mu(x^{(2)}, x^{(1)})_i) - (\mu(y^{(1)}, y^{(2)})_i + \mu(y^{(2)}, y^{(1)})_i) \\ &= (x_i^{(1)} + x_i^{(2)}) - (y_i^{(1)} + y_i^{(2)}) = (x_i^{(1)} - y_i^{(1)}) + (x_i^{(2)} - y_i^{(2)}) \\ &= z_i^{(1)} + z_i^{(2)}. \end{aligned}$$

To show (6.14) and (6.15), we consider the following cases: Case 1: $x_i^{(1)} \geq x_i^{(2)}$ and $y_i^{(1)} \geq y_i^{(2)}$, Case 2: $x_i^{(1)} < x_i^{(2)}$ and $y_i^{(1)} < y_i^{(2)}$, Case 3: $x_i^{(1)} \geq x_i^{(2)}$ and $y_i^{(1)} < y_i^{(2)}$, Case 4: $x_i^{(1)} < x_i^{(2)}$ and $y_i^{(1)} \geq y_i^{(2)}$.

Case 1 ($x_i^{(1)} \geq x_i^{(2)}$ and $y_i^{(1)} \geq y_i^{(2)}$). In this case, we have

$$z_i' = \left\lceil \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rceil - \left\lceil \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rceil, \quad z_i'' = \left\lfloor \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rfloor - \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor.$$

By substituting $z_i^{(1)} + z_i^{(2)} = (x_i^{(1)} + x_i^{(2)}) - (y_i^{(1)} + y_i^{(2)})$ into

$$\min\{z_i^{(1)}, z_i^{(2)}\} \leq \frac{z_i^{(1)} + z_i^{(2)}}{2} \leq \max\{z_i^{(1)}, z_i^{(2)}\},$$

we obtain

$$\min\{z_i^{(1)}, z_i^{(2)}\} + \frac{y_i^{(1)} + y_i^{(2)}}{2} \leq \frac{x_i^{(1)} + x_i^{(2)}}{2} \leq \max\{z_i^{(1)}, z_i^{(2)}\} + \frac{y_i^{(1)} + y_i^{(2)}}{2}.$$

In the above inequalities, we round up and round down every terms, to obtain

$$\min\{z_i^{(1)}, z_i^{(2)}\} + \left\lceil \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rceil \leq \left\lceil \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rceil \leq \max\{z_i^{(1)}, z_i^{(2)}\} + \left\lceil \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rceil,$$

$$\min\{z_i^{(1)}, z_i^{(2)}\} + \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor \leq \left\lfloor \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rfloor \leq \max\{z_i^{(1)}, z_i^{(2)}\} + \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor.$$

Thus, (6.14) and (6.15) are satisfied.

Case 2 ($x_i^{(1)} < x_i^{(2)}$ and $y_i^{(1)} < y_i^{(2)}$). In the same way as Case 1, we can show (6.14) and (6.15).

Case 3 ($x_i^{(1)} \geq x_i^{(2)}$ and $y_i^{(1)} < y_i^{(2)}$). In this case, we have

$$z'_i = \left\lceil \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rceil - \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor, \quad z''_i = \left\lfloor \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rfloor - \left\lceil \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rceil.$$

Moreover, we obtain

$$\begin{aligned} z_i^{(1)} = x_i^{(1)} - y_i^{(1)} &\geq \left\lceil \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rceil - \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor = z'_i \\ &\geq \left\lfloor \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rfloor - \left\lceil \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rceil = z''_i \\ &\geq x_i^{(2)} - y_i^{(2)} = z_i^{(2)}, \end{aligned}$$

and hence, (6.14) and (6.15).

Case 4: ($x_i^{(1)} < x_i^{(2)}$ and $y_i^{(1)} \geq y_i^{(2)}$). In the same way as Case 3, we can show (6.14) and (6.15). (End of the proof of Claim). $\square$

Corollary 6 *Minkowski sum of a DDM-convex set and an integral box is also DDM-convex, where an integral box is the set defined by $\{x \in \mathbb{Z}^n \mid a \leq x \leq b\}$ for some $a \in (\mathbb{Z} \cup \{-\infty\})^n$ and $b \in (\mathbb{Z} \cup \{+\infty\})^n$ with $a \leq b$.*

7 Proximity theorems

For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a positive integer α , a proximity theorem estimates the distance between a given local minimizer x° of the α -scaled function f^α and a minimizer x^* of f . For instance, the following proximity theorems for $L^\natural$ -convex functions and globally/locally discrete midpoint convex functions are known.

Theorem 12 ([11, 21]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an $L^\natural$ -convex function, α be a positive integer and $x^\circ \in \text{dom } f$. If $f(x^\circ) \leq f(x^\circ + \alpha d)$ for all $d \in \{0, +1\}^n \cup \{0, -1\}^n$, then there exists $x^* \in \text{argmin } f$ with $\|x^\circ - x^*\|_\infty \leq n(\alpha - 1)$.*

Theorem 13 ([18]) *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a globally/locally discrete midpoint convex function, α be a positive integer and $x^\circ \in \text{dom} f$. If $f(x^\circ) \leq f(x^\circ + \alpha d)$ for all $d \in \{-1, 0, +1\}^n$, then there exists $x^* \in \text{argmin} f$ with $\|x^\circ - x^*\|_\infty \leq n(\alpha - 1)$.*

In the same way as the arguments in [18], we can show the following proximity theorem for DDM-convex functions.

Theorem 14 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a DDM-convex function, α be a positive integer and $x^\circ \in \text{dom} f$. If $f(x^\circ) \leq f(x^\circ + \alpha d)$ for all $d \in \{-1, 0, +1\}^n$, then there exists $x^* \in \text{argmin} f$ with $\|x^\circ - x^*\|_\infty \leq n(\alpha - 1)$.*

We note that f^α is also DDM-convex by Theorem 11 and x° corresponds to a minimizer $\mathbf{0}$ of $f^\alpha(y) = f(x^\circ + \alpha y)$ by Corollary 1. We emphasize that the bound $n(\alpha - 1)$ for DDM-convex functions is the same as that for $L^{\natural}$ -convex functions and globally/locally discrete midpoint convex functions.

To prove Theorem 14, we assume $x^\circ = \mathbf{0}$ without loss of generality. Let $S = \{x \in \mathbb{Z}^n \mid \|x\|_\infty \leq n(\alpha - 1)\}$, $W = \{x \in \mathbb{Z}^n \mid \|x\|_\infty = n(\alpha - 1) + 1\}$ and let $\gamma = \text{argmin}\{f(x) \mid x \in S\}$. We show that

$$f(y) \geq \gamma \quad (\forall y \in W). \quad (7.1)$$

Then Theorem 1 (box-barrier property) implies that $f(z) \geq \gamma$ for all $z \in \mathbb{Z}^n$.

Fix $y = (y_1, \dots, y_n) \in W$, and let $\|y\|_\infty = m (= n(\alpha - 1) + 1)$. By using

$$A_k = \{i \mid y_i \geq k\}, \quad B_k = \{i \mid y_i \leq -k\} \quad (k = 1, \dots, m),$$

we can write y as

$$y = \sum_{k=1}^m (\mathbf{1}_{A_k} - \mathbf{1}_{B_k}),$$

where $A_1 \supseteq \dots \supseteq A_m$, $B_1 \supseteq \dots \supseteq B_m$, $A_1 \cap B_1 = \emptyset$ and $A_m \cup B_m \neq \emptyset$.

Lemma 2 *There exists some $k_0 \in \{1, \dots, m - \alpha + 1\}$ with $(A_{k_0}, B_{k_0}) = (A_{k_0+j}, B_{k_0+j})$ for $j = 1, \dots, \alpha - 1$.*

Proof By $A_m \cup B_m \neq \emptyset$, we may assume $A_m \neq \emptyset$. Let $s = |A_1|$ and $(a_k, b_k) = (|A_k|, |B_k| + s)$ for $k = 1, \dots, m$. Since $A_1 \supseteq \dots \supseteq A_m \neq \emptyset$, $B_1 \supseteq \dots \supseteq B_m$ and $A_1 \cap B_1 = \emptyset$, we have $n - s \geq |B_1|$, $s = a_1 \geq \dots \geq a_m \geq 1$ and $n \geq b_1 \geq \dots \geq b_m \geq s$. Therefore, $(s, n) \geq (a_1, b_1) \geq \dots \geq (a_m, b_m) \geq (1, s)$. Because $m = n(\alpha - 1) + 1$ and the length of a strictly decreasing chain connecting (s, n) to $(1, s)$ in $\mathbb{Z}^2$ is bounded by n , there exists a constant subsequence of length $\geq \alpha$ in the sequence $\{(a_k, b_k)\}_{k=1, \dots, m}$ by the pigeonhole principle. Hence the assertion holds. $\square$

By using k_0 in Lemma 2, we define a subset J of $\{1, \dots, m\}$ by $J = \{k_0, \dots, k_0 + \alpha - 1\}$. By the parallelogram inequality (4.14) in Theorem 7, where $d_0 = \mathbf{1}_{A_{k_0}} - \mathbf{1}_{B_{k_0}}$ and $d = \sum_{j \in J} (\mathbf{1}_{A_j} - \mathbf{1}_{B_j}) = \alpha d_0$, we obtain

$$f(\mathbf{0}) + f(y) \geq f(\alpha d_0) + f(y - \alpha d_0).$$

By the assumption, we have $f(\alpha d_0) \geq f(x^\circ) = f(\mathbf{0})$. We also have $y - \alpha d_0 \in S$ because

$$\|y - \alpha d_0\|_\infty = m - \alpha = (n-1)(\alpha-1) \leq n(\alpha-1).$$

By the definition of γ , $f(y - \alpha d_0) \geq \gamma$ must hold. Therefore,

$$f(y) \geq f(y - \alpha d_0) + [f(\alpha d_0) - f(\mathbf{0})] \geq \gamma + 0 = \gamma,$$

which implies (7.1), completing the proof of Theorem 14.

8 Minimization algorithms

In this section, we propose two algorithms for DDM-convex function minimization.

8.1 The 1-neighborhood steepest descent algorithm

We first propose a variant of steepest descent algorithm for DDM-convex function minimization problem.

Let $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a DDM-convex function with $\operatorname{argmin} f \neq \emptyset$. We suppose that an initial point

$$x^{(0)} \in \operatorname{dom} f \setminus \operatorname{argmin} f$$

is given. Let L denote the minimum l_∞ -distance between $x^{(0)}$ and a minimizer of f , that is, L is defined by

$$L = \min\{\|x - x^{(0)}\|_\infty \mid x \in \operatorname{argmin} f\}.$$

For all $k = 0, 1, \dots, L$ we define sets S_k by

$$S_k = S_k(x^{(0)}) = \{x \in \mathbb{Z}^n \mid \|x - x^{(0)}\|_\infty \leq k\}.$$

The idea of our algorithm is to generate a sequence of minimizers in S_k for $k = 1, \dots, L$. The next proposition guarantees that consecutive minimizers can be chosen to be close to each other.

Proposition 15 *For each $k = 1, \dots, L$ and for any $x^{(k-1)} \in \operatorname{argmin}\{f(x) \mid x \in S_{k-1}\}$, there exists $x^{(k)} \in \operatorname{argmin}\{f(x) \mid x \in S_k\}$ with $\|x^{(k)} - x^{(k-1)}\|_\infty \leq 1$.*

Proof If $k = 1$, the assertion is obvious. Suppose that $k \geq 2$ and y is any point in S_k . By (2.1) for $x^{(k-1)}$ and y , we have

$$f(x^{(k-1)}) + f(y) \geq f(\mu(x^{(k-1)}, y)) + f(\mu(y, x^{(k-1)})). \quad (8.1)$$

Since $x^{(k-1)}, y \in S_k$ and S_k is a DDM-convex set, we also have

$$\mu(y, x^{(k-1)}) \in S_k. \quad (8.2)$$

Next, we show

$$\mu(x^{(k-1)}, y) \in S_{k-1}. \quad (8.3)$$

To show this we arbitrarily fix $i \in \{1, \dots, n\}$, and consider the two cases: Case 1: $x_i^{(k-1)} - y_i = l$ ($l \geq 1$) and Case 2: $x_i^{(k-1)} - y_i = -l$ ($l \geq 1$).

Case 1 ($x_i^{(k-1)} - y_i = l$ ($l \geq 1$)). In this case, $\mu(x^{(k-1)}, y)_i = x_i^{(k-1)} - \left\lfloor \frac{l}{2} \right\rfloor$ and $\left\lfloor \frac{l}{2} \right\rfloor \leq l - 1$. Thus, we have

$$\begin{aligned} x_i^{(0)} + (k-1) &\geq x_i^{(k-1)} \\ &\geq \mu(x^{(k-1)}, y)_i = x_i^{(k-1)} - \left\lfloor \frac{l}{2} \right\rfloor \\ &\geq x_i^{(k-1)} - (l-1) = y_i + 1 \geq x_i^{(0)} - (k-1). \end{aligned}$$

Case 2 ($x_i^{(k-1)} - y_i = -l$ ($l \geq 1$)). In this case, $\mu(x^{(k-1)}, y)_i = x_i^{(k-1)} + \left\lfloor \frac{l}{2} \right\rfloor$ and $\left\lfloor \frac{l}{2} \right\rfloor \leq l - 1$. Thus, we have

$$\begin{aligned} x_i^{(0)} - (k-1) &\leq x_i^{(k-1)} \\ &\leq \mu(x^{(k-1)}, y)_i = x_i^{(k-1)} + \left\lfloor \frac{l}{2} \right\rfloor \\ &\leq x_i^{(k-1)} + (l-1) = y_i - 1 \leq x_i^{(0)} + (k-1). \end{aligned}$$

By the above arguments, (8.3) holds.

Let y^* be a point y in $\operatorname{argmin}\{f(x) \mid x \in S_k\}$ minimizing $\|y - x^{(k-1)}\|_\infty$. To prove $\|y^* - x^{(k-1)}\|_\infty \leq 1$ by contradiction, suppose that $\|y^* - x^{(k-1)}\|_\infty \geq 2$, which yields $\|y^* - x^{(k-1)}\|_\infty > \|\mu(y^*, x^{(k-1)}) - x^{(k-1)}\|_\infty$. Since $\mu(y^*, x^{(k-1)}) \in S_k$ by (8.2), this implies $f(y^*) < f(\mu(y^*, x^{(k-1)}))$. Moreover, by (8.3), we have $f(x^{(k-1)}) \leq f(\mu(x^{(k-1)}, y^*))$. These two inequalities contradict (8.1) for $x^{(k-1)}$ and y^* . Hence $\|y^* - x^{(k-1)}\|_\infty \leq 1$ must hold. $\square$

By Proposition 15, it seems be natural to assume that we can find a minimizer of f within the l -neighborhood $N_l(x)$ of x defined by

$$N_l(x) = \{y \in \mathbb{Z}^n \mid \|z - x\|_\infty \leq 1\}.$$

With the use of a l -neighborhood minimization oracle, which finds a point minimizing f in $N_l(x)$ for any $x \in \operatorname{dom} f$, our algorithm can be described as below.

The 1-neighborhood steepest descent algorithm

- D0: Find $x^{(0)} \in \text{dom} f$, and set $k := 1$.
 D1: Find $x^{(k)}$ that minimizes f in $N_1(x^{(k-1)})$.
 D2: If $f(x^{(k)}) = f(x^{(k-1)})$, then output $x^{(k-1)}$ and stop.
 D3: Set $k := k + 1$, and go to D1.

Theorem 15 *For a DDM-convex function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\text{argmin} f \neq \emptyset$, the 1-neighborhood steepest descent algorithm finds a minimizer of f exactly in $(L+1)$ iterations, that is, exactly in $(L+1)$ calls of the 1-neighborhood minimization oracles.*

Proof By Proposition 15, the sequence $\{x^{(k)}\}$ generated by the 1-neighborhood steepest descent algorithm satisfy

$$x^{(k)} \in \text{argmin}\{f(x) \mid x \in S_k\} \quad (k = 1, 2, \dots). \quad (8.4)$$

Claim *If $f(x^{(k)}) = f(x^{(k-1)})$ at Step D2, then $x^{(k-1)} \in \text{argmin} f$.*

Proof For any $d \in \{+1, 0, -1\}^n$, $x^{(k-1)} + d$ belongs to S_k , and hence, $f(x^{(k)}) \leq f(x^{(k-1)} + d)$ by (8.4). Therefore, if $f(x^{(k)}) = f(x^{(k-1)})$, then $f(x^{(k-1)}) \leq f(x^{(k-1)} + d)$ for any d . Corollary 1 in Sect. 3 guarantees $x^{(k-1)} \in \text{argmin} f$. (End of the proof of Claim).

By the definition of L , $x^{(k)} \neq x^{(k-1)}$ if $k \leq L$, and $x^{(L)} = x^{(L+1)}$. Therefore, our algorithm stops in $(L + 1)$ iterations. $\square$

Remark 2 Theorem 15 says that the sequence of points generated by the 1-neighborhood steepest descent algorithm is bounded by the ℓ_∞ -distance between an initial point and the nearest minimizer. Similar facts are pointed out for $L^{\natural}$ -convex function minimization [13, 24, 26] and globally/locally discrete midpoint convex function minimization [18]. $\square$

Remark 3 Let $F(n)$ denote the number of function evaluations in the 1-neighborhood minimization oracle. Since any function defined on $\{0, 1\}^n$ is DDM-convex, the 1-neighborhood minimization problem is NP-hard. In almost cases, $F(n)$ seems to be $\Theta(3^n)$ by a brute-force calculation, because $|N_1(\cdot)| = 3^n$. Fortunately, for $L^{\natural}$ -convex functions, $F(n)$ is bounded by a polynomial in n . Another hopeful case is a fixed parameter tractable case, that is, the case where there exists some parameter k such that the number of function evaluations $F(n, k)$ in n and k is bounded by a polynomial $p(n)$ in n times any function $g(k)$ in k (see the next remark). $\square$

Remark 4 Let us consider the following problem:

$$\begin{cases} \text{minimize} & \frac{1}{2}x^\top Qx + c^\top x \\ \text{subject to} & x \in \mathbb{Z}^n, \end{cases}$$

where a symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is nonsingular and diagonally dominant with nonnegative diagonals, and $c \in \mathbb{R}^n$. Since Q is nonsingular, the (convex) continuous relaxation problem has a unique minimizer $-Q^{-1}c$. Furthermore, because the objective function is 2-separable convex, it follows from Theorem 18 in the next section that there exists an optimal solution in the box:

$$B = \{x \in \mathbb{Z}^n \mid -Q^{-1}c - n\mathbf{1} \leq x \leq -Q^{-1}c + n\mathbf{1}\}.$$

Therefore, the 1-neighborhood steepest descent algorithm with an initial point $\lfloor -Q^{-1}c \rfloor$ find an optimal solution in $O(n)$ iterations. Furthermore, if $Q = [q_{ij}]$ is $(2k+1)$ -diagonal, that is,

$$q_{ij} = 0 \quad (i = 1, \dots, n; j : |j - i| > k),$$

then $F(n, k) = O(n3^{k+1})$ as in [6]. $\square$

8.2 Scaling algorithm

In the same way as the scaling algorithm for minimization of globally/locally discrete midpoint convex functions in [18], the scaling property (Theorem 11) and the proximity theorem (Theorem 14) enable us to design a scaling algorithm for the minimization of DDM-convex functions with bounded effective domains.

Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a DDM-convex function with bounded effective domain. We suppose that $K_\infty = \max\{\|x - y\|_\infty \mid x, y \in \text{dom} f\}$ ($K_\infty < +\infty$) and an initial point $x \in \text{dom} f$ are given. Our algorithm can be described as follows.

Scaling algorithm for DDM-convex functions

S0: Let $x \in \text{dom} f$ and $\alpha := 2^{\lceil \log_2(K_\infty + 1) \rceil}$.

S1: Find a vector y that minimizes $f^\alpha(y) = f(x + \alpha y)$ subject to $\|y\|_\infty \leq n$ (by the 1-neighborhood steepest descent algorithm), and set $x := x + \alpha y$.

S2: If $\alpha = 1$, then stop (x is a minimizer of f).

S3: Set $\alpha := \alpha/2$, and go to S1.

Theorem 16 *For a DDM-convex function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with bounded effective domain, the scaling algorithm finds a minimizer of f in $O(n \log_2 K_\infty)$ calls of the 1-neighborhood minimization oracles.*

Proof The correctness of the algorithm can be shown by induction on α . If $\alpha = 2^{\lceil \log_2(K_\infty + 1) \rceil}$, then x is a unique point of $\text{dom} f^\alpha$ because $\alpha = 2^{\lceil \log_2(K_\infty + 1) \rceil} > K_\infty$, that is, a minimizer of f^α . Let x^{2^α} denote the point x at the beginning of S1 for α and assume that x^{2^α} is a minimizer of f^{2^α} . The function $f^\alpha(y) = f(x^{2^\alpha} + \alpha y)$ is DDM-convex by Theorem 11. Let $y^\alpha = \arg\min\{f^\alpha(y) \mid \|y\|_\infty \leq n\}$ and $x^\alpha = x^{2^\alpha} + \alpha y^\alpha$. Theorem 14 guarantees that x^α is a minimizer of f^α because of $x^{2^\alpha} \in \arg\min f^{2^\alpha}$. At the termination of the algorithm, we have $\alpha = 1$ and $f^\alpha = f$. The output of the algorithm,

which is computed by the 1-neighborhood steepest descent algorithm, satisfies the condition of Corollary 1, and hence, the output is indeed a minimizer of f .

The time complexity of the algorithm can be analyzed as follows: by Theorem 15, S1 terminates in $O(n)$ calls of the 1-neighborhood minimization oracles in each iteration. The number of iterations is $O(\log_2 K_\infty)$. Hence, the assertion holds. $\square$

9 DDM-convex functions in continuous variables

In [19], proximity theorems between $L^{\natural}$ -convex functions and their continuous relaxations are proposed. We extend these results to DDM-convexity.

It is known that the continuous version of $L^{\natural}$ -convexity can naturally be defined by using translation-submodularity (3.1). In this section, we define DDM-convexity in continuous variables in a different way. We call a continuous convex function $F : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ a *directed discrete midpoint convex function in continuous variables* ($\mathbb{R}$ -DDM-convex function) if for any positive integer α , the function $f^{1/\alpha} : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f^{1/\alpha}(x) = F\left(\frac{x}{\alpha}\right) \quad (x \in \mathbb{Z}^n) \quad (9.1)$$

is DDM-convex. We denote by f the DDM-convex function $f^{1/1}$ which is nothing but the restriction of F to $\mathbb{Z}^n$.

An example of an $\mathbb{R}$ -DDM-convex function is a continuous 2-separable convex function F which is defined as

$$F(x) = \sum_{i=1}^n \xi_i(x_i) + \sum_{i,j:j \neq i} \varphi_{ij}(x_i - x_j) + \sum_{i,j:j \neq i} \psi_{ij}(x_i + x_j) \quad (x \in \mathbb{R}^n)$$

for univariate continuous convex functions $\xi_i, \varphi_{ij}, \psi_{ij} : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ ($i = 1, \dots, n$; $j \in \{1, \dots, n\} \setminus \{i\}$) as below. The restriction f of F to $\mathbb{Z}^n$ is trivially a 2-separable convex function on $\mathbb{Z}^n$ defined by (3.4). Furthermore, the function $F^{1/\alpha} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$F^{1/\alpha}(x) = F\left(\frac{x}{\alpha}\right) \quad (x \in \mathbb{R}^n)$$

is also a continuous 2-separable convex function, and hence, the restriction $f^{1/\alpha}$ of $F^{1/\alpha}$ to $\mathbb{Z}^n$ is also a 2-separable convex function on $\mathbb{Z}^n$.

We have the following proximity theorems between an $\mathbb{R}$ -DDM-convex function F and its restriction f to $\mathbb{Z}^n$.

Theorem 17 *Let $F : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an $\mathbb{R}$ -DDM-convex function. For each $x^* \in \operatorname{argmin} f (= \operatorname{argmin} f^{1/1})$, there exists $\bar{x} \in \operatorname{argmin} F$ with $\|x^* - \bar{x}\|_\infty \leq n$.*

Proof Since f is DDM-convex, by Corollary 1 in Sect. 3, we have

$$f(x^*) \leq f(x^* + d) \quad (\forall d \in \{-1, 0, +1\}^n).$$

Thus, for every integer $\alpha \geq 2$, by $f(x) = f^{1/\alpha}(\alpha x)$ ($x \in \mathbb{Z}^n$), we have

$$f^{1/\alpha}(\alpha x^*) \leq f^{1/\alpha}(\alpha x^* + \alpha d) \quad (\forall d \in \{-1, 0, +1\}^n).$$

By Theorem 14 for $f^{1/\alpha}$, there exists $x^{1/\alpha} \in \operatorname{argmin} f^{1/\alpha}$ with

$$\alpha x^* - (\alpha - 1)n\mathbf{1} \leq x^{1/\alpha} \leq \alpha x^* + (\alpha - 1)n\mathbf{1}. \quad (9.2)$$

By dividing all terms in (9.2) by α , we obtain

$$x^* - n\mathbf{1} \leq x^* - \frac{\alpha - 1}{\alpha}n\mathbf{1} \leq \frac{x^{1/\alpha}}{\alpha} \leq x^* + \frac{\alpha - 1}{\alpha}n\mathbf{1} \leq x^* + n\mathbf{1}.$$

Let $B = \{x \in \mathbb{R}^n \mid x^* - n\mathbf{1} \leq x \leq x^* + n\mathbf{1}\}$. For each integer $k \geq 1$, considering $\alpha_k = 2^k$ and $x^{1/\alpha_k} \in \operatorname{argmin} f^{1/\alpha_k}$, we have $\frac{x^{1/\alpha_k}}{\alpha_k} \in B$. Since B is compact, there exists a subsequence $\{\frac{x^{1/\alpha_{k_i}}}{\alpha_{k_i}}\}$ with

$$\lim_{i \rightarrow \infty} \frac{x^{1/\alpha_{k_i}}}{\alpha_{k_i}} = \bar{x} \in B.$$

Since F is continuous, we have

$$\lim_{i \rightarrow \infty} F\left(\frac{x^{1/\alpha_{k_i}}}{\alpha_{k_i}}\right) = F\left(\lim_{i \rightarrow \infty} \frac{x^{1/\alpha_{k_i}}}{\alpha_{k_i}}\right) = F(\bar{x}).$$

Since $x^{1/\alpha_{k_i}} \in \operatorname{dom} f^{1/\alpha_{k_i+1}}$ holds for each i by the definition (9.1), we have $F(\frac{x^{1/\alpha_{k_1}}}{\alpha_{k_1}}) \geq F(\frac{x^{1/\alpha_{k_2}}}{\alpha_{k_2}}) \geq \dots \geq F(\frac{x^{1/\alpha_{k_i}}}{\alpha_{k_i}}) \geq \dots$ which together with $x^{1/2^{k_i}} \in \operatorname{argmin} f^{1/2^{k_i}}$ for all i , guarantees that

$$F(\bar{x}) \leq F\left(\frac{x^{1/2^{k_i}}}{2^{k_i}}\right) = \min f^{1/2^{k_i}} \quad (i = 1, 2, \dots). \quad (9.3)$$

We finally show $F(\bar{x}) = \min F$, that is, $\bar{x} \in \operatorname{argmin} F$. Suppose to the contrary that there exists x' with $F(x') < F(\bar{x})$. Let $\varepsilon = F(\bar{x}) - F(x') > 0$. By the continuity of F , there exists δ_ε such that

$$\forall y \in \mathbb{R}^n, \|x' - y\|_\infty < \delta_\varepsilon \Rightarrow |F(x') - F(y)| < \varepsilon. \quad (9.4)$$

Because there exist $N \in \{k_i \mid i = 1, 2, \dots\}$ and $y \in \mathbb{R}^n$ such that $2^N y \in \mathbb{Z}^n$ and $\|x' - y\|_\infty < \delta_\varepsilon$, by (9.4), we have

$$\min f^{1/2^N} \leq F(y) < F(x') + \varepsilon = F(\bar{x}),$$

which contradicts (9.3). Therefore, $\bar{x}$ must be a minimizer of F . $\square$

If F has a unique minimizer, the converse of Theorem 17 also holds.

Theorem 18 *Let $F : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an $\mathbb{R}$ -DDM-convex function. If F has a unique minimizer $\bar{x}$, there exists $x^* \in \operatorname{argmin} f$ with $\|x^* - \bar{x}\|_\infty \leq n$.*

Proof Let $\bar{x}$ be a unique minimizer of F . If f has a minimizer x^* , then $\|x^* - \bar{x}\|_\infty \leq n$ must hold by Theorem 17. Thus, it is enough to show that f has a minimizer.

Suppose to the contrary that f has no minimizer and let $B = \{x \in \mathbb{R}^n \mid \bar{x} - n\mathbf{1} \leq x \leq \bar{x} + n\mathbf{1}\}$. Then, there exists $y \in \text{dom} f \setminus B$ such that $f(y) < f(x)$ for all $x \in \text{dom} f \cap B$. Let $\ell = \|y - \bar{x}\|_\infty$ and $B' = \{x \in \mathbb{R}^n \mid \bar{x} - \ell\mathbf{1} \leq x \leq \bar{x} + \ell\mathbf{1}\}$. Note that $\ell > n$ and $B' \supset B$. Let us consider the restriction G of F to B' defined by

$$G(x) = \begin{cases} F(x) & (x \in B') \\ +\infty & (x \notin B') \end{cases} \quad (x \in \mathbb{R}^n).$$

Obviously, G is $\mathbb{R}$ -DDM-convex and $\bar{x}$ is a unique minimizer of G . In particular, the restriction g of G to $\mathbb{Z}^n$ is DDM-convex and has a minimizer z since B' is bounded. This point z does not belong to B since $y \notin B$ and $f(y) < f(x)$ for all $x \in \text{dom} f \cap B$. However, this contradicts Theorem 17 for G and g . $\square$

If F has a bounded effective domain, a similar statement holds. Let $\bar{K}_\infty = \sup\{\|x - y\|_\infty \mid x, y \in \text{dom} F\}$.

Theorem 19 Let $F : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an $\mathbb{R}$ -DDM-convex function. If $\text{dom} F$ is bounded (i.e., $\bar{K}_\infty < \infty$), for each $\bar{x} \in \text{argmin} F$, there exists $x^* \in \text{argmin} f$ with $\|x^* - \bar{x}\|_\infty \leq n$.

Proof If $\text{dom} f = \text{argmin} f$, the assertion holds. In the sequel, we assume that $\text{dom} f \neq \text{argmin} f$. We fix a minimizer $\bar{x}$ of F , arbitrarily. For a sufficiently small $\varepsilon > 0$, let us consider functions $F_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f_\varepsilon : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} F_\varepsilon(x) &= F(x) + \varepsilon \sum_{i=1}^n (x_i - \bar{x}_i)^2 \quad (x \in \mathbb{R}^n), \\ f_\varepsilon(x) &= F_\varepsilon(x) \quad (x \in \mathbb{Z}^n). \end{aligned}$$

Function F_ε has the unique minimizer $\bar{x}$ and satisfies the conditions of Theorem 18, by Proposition 4 (4), because $f_\varepsilon^{1/\alpha}$ defined by (9.1) for F_ε is the sum of $f^{1/\alpha}$ and a separable convex function which are DDM-convex. Thus, by Theorem 18 for F_ε and f_ε , there exists $x^\varepsilon \in \text{argmin} f_\varepsilon$ with $\|x^\varepsilon - \bar{x}\|_\infty \leq n$.

Let $\beta = \min\{f(x) \mid x \in \text{dom} f \setminus \text{argmin} f\} > 0$. Note that β is well-defined by boundedness of $\text{dom} f$. We show that if $\varepsilon < (\beta - \min f)/(n\bar{K}_\infty^2)$, then $x^\varepsilon \in \text{argmin} f$. For any $x \in \text{argmin} f$, by $f_\varepsilon(x^\varepsilon) \leq f_\varepsilon(x)$, we have

$$f(x^\varepsilon) \leq f(x) + \varepsilon \sum_{i=1}^n \{(x_i - \bar{x}_i)^2 - (x_i^\varepsilon - \bar{x}_i)^2\}.$$

As $f(x^\varepsilon) \geq f(x)$, $\sum_{i=1}^n \{(x_i - \bar{x}_i)^2 - (x_i^\varepsilon - \bar{x}_i)^2\} \geq 0$ must hold. Therefore, we obtain

$$\begin{aligned} f(x^\varepsilon) &< f(x) + \frac{\beta - \min f}{n\bar{K}_\infty^2} \sum_{i=1}^n \{(x_i - \bar{x}_i)^2 - (x_i^\varepsilon - \bar{x}_i)^2\} \\ &\leq f(x) + \beta - \min f = \beta, \end{aligned}$$

which says $x^\varepsilon \in \operatorname{argmin} f$. $\square$

Remark 5 There is a convex function which is not $\mathbb{R}$ -DDM-convex. For example, for a positive definite matrix $Q = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, the function

$$F(x) = x^\top Q x \quad (x \in \mathbb{R}^2)$$

is convex, but the restriction f of F to $\mathbb{Z}^2$ is not DDM-convex by Theorem 9, and hence, F is not $\mathbb{R}$ -DDM-convex. $\square$

Remark 6 The convex extension of a DDM-convex function may not be $\mathbb{R}$ -DDM-convex. For example,

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

is a DDM-convex set, and hence, its indicator function $f = \delta_S$ is DDM-convex. We denote the convex hull of S by $\bar{S}$. Then the convex extension F of f is expressed by

$$F(x) = \begin{cases} 0 & (x \in \bar{S}) \\ +\infty & (x \notin \bar{S}) \end{cases} \quad (x \in \mathbb{R}^3),$$

and $f^{1/2}$ is given by

$$f^{1/2}(x) = \begin{cases} 0 & (x \in T) \\ +\infty & (x \notin T) \end{cases} \quad (x \in \mathbb{Z}^3),$$

where $T = \{(2, 0, 0), (1, 1, 0), (0, 2, 0), (0, 1, 1), (0, 0, 2), (1, 0, 1)\}$. The function $f^{1/2}$ is not DDM-convex, because for $x = (2, 0, 0)$ and $y = (0, 1, 1)$, we have $\mu(x, y) = (1, 0, 0) \notin T$. $\square$

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