Japan J. Indust. Appl. Math. (2017) 34:343–372
DOI 10.1007/s13160-017-0250-3

ORIGINAL PAPER

Area 1

# Bessel function type solutions of the ultradiscrete Painlevé III equation with parity variables

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Received: 18 November 2016 / Revised: 21 March 2017 / Published online: 9 June 2017


**Abstract** The discrete Painlevé III equation ($\mathrm{dP_{III}}$) possesses a class of special solutions with determinantal structure whose entries are written by the $q$-difference Bessel function. In this paper, an ultradiscrete analogue of the $q$-Bessel function is constructed by ultradiscretization with parity variables (p-ultradiscretization). Based on this result, special solutions for the p-ultradiscrete Painlevé III equation are derived from those of $\mathrm{dP_{III}}$. The ultradiscrete solutions capture oscillating behaviour of the (differential) Painlevé III equation.

**Keywords** Painlevé equation · Bessel function · Ultradiscretization · $q$-difference equation

**Mathematics Subject Classification** 34M55 · 33E30 · 39A13

## 1 Introduction

Ultradiscretization [23] is a limiting procedure for reducing a given difference equation to a piecewise linear equation that is written by addition, subtraction, and the max operation among dependent variables. In this procedure, we first transform a dependent variable $x_n$ in the given difference equation by

$$x_n = e^{\frac{X_n}{\varepsilon}}, \tag{1}$$

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where $\varepsilon > 0$ is a parameter. Then, we apply $\varepsilon \log$ to both sides of the equation and take the limit $\varepsilon \to +0$. By using the exponential laws and the identity

$$\lim_{\varepsilon \to +0} \varepsilon \log \left( e^{\frac{X}{\varepsilon}} + e^{\frac{Y}{\varepsilon}} \right) = \max(X, Y), \tag{2}$$

multiplication, division, and addition among $x_n$ are replaced by addition, subtraction, and the max operation among $X_n$, respectively. The resulting piecewise linear equation, which is called an ultradiscrete system, is often considered to be the time evolution rule of a cellular automaton [24] since its dependent variables as well as its independent variables will take discrete values.

We can regard the transformation (1) as an approximation of the solution $x_n$ by its leading term $e^{X_n/\varepsilon}$. In this context, the ultradiscrete system is the balancing relation among the exponents of leading terms, say $X_n$. In spite of this drastic approximation, an ultradiscrete system may inherit essential properties of the original equation. In particular, if we have an exact solution of the original equation, we can constract a solution of the corresponding ultradiscrete system. The box-and-ball system (BBS) [20], which is an ultradiscrete analogue of the Korteweg–de Vries (KdV) equation, is a good example to observe this feature of ultradiscrete systems. The BBS has an infinite number of preserving quantities and therefore keeps the integrability of the KdV equation. Moreover, the BBS admits the $N$-soliton solutions, which illustrate soliton interaction in a simple manner. The direct relationship between the soliton solutions of BBS and those of the KdV equation is also clarified through the limiting procedure. Because of this interesting feature, ultradiscrete systems are faithfully studied from viewpoints of mathematical interest and applied possibility (See, for example, [19]).

However, $x_n$ must be positive to apply (1). Moreover, the equation that we consider must be subtraction-free since it is not straightforward to find a meaningful limit of $\varepsilon \log \left( e^{X/\varepsilon} - e^{Y/\varepsilon} \right)$. These conditions strongly restrict a class of equations to which we can apply ultradiscretization procedure. For example, it is difficult to ultradiscretize a equation describing oscillation. Some attempts have been made to overcome this difficulty. Instead of (1), the ansatz $x_n = \sinh(X_n/\varepsilon)$ has been studied in [7,10]. Kasman and Lafortune investigated the limit of the form $\varepsilon \log \sum \gamma_i e^{w_i(\varepsilon)/\varepsilon}$ $(\gamma_i \in \mathbb{C})$ in detail [6]. An algebraic approach by means of a non-archimedean valuation was presented by Ormerod and it was applied to the discrete Painlevé III equation [16]. The present author proposed a new procedure called ultradiscretization with parity variables (p-ultradiscretization) with coworkers [14]. This procedure keeps track of the sign of the original variables by introducing the parity (sign) variable $\xi_n = x_n/|x_n|$. We can study ultradiscrete analogues of a wider class of equations by virtue of these attempts.

The Painlevé equations are important nonlinear differential equations that have rich mathematical structures, for example, they admit a class of special solutions written in terms of a determinant whose entries are given by a hypergeometric-type function. Their $q$-difference analogues [17] are simply called the $q$-Painlevé equations. They also have rich structures and are actively studied (See, for example, [4,18]). Ultra-discrete analogues of the $q$-Painlevé equations and their special solutions are also investigated vigourously. Ultradiscrete Painlevé equations by traditional procedure are studied in [13,15,21], and so on. In [16], as mentioned above, hypergeometric solutions of the ultradiscrete Painlevé III equation and an ultradiscrete Bessel function were constructed by an algebraic approach. The procedure of p-ultradiscretization has been applied to the $q$-Painlevé II equation [3] in [9] and some successive works have been reported in [8,11,12]. Furthermore, the p-ultradiscrete analogue of the Painlevé VI equation and its special solutions have been proposed and the asymptotic behaviour of the solutions was discussed in [22]. It is an interesting problem to construct p-ultradiscrete analogues of other Painlevé equations and study their mathematical structures including special solutions.

In this study, we focus on a discrete analogue of the Painlevé III equation ($\mathrm{dP_{III}}$), which is written as [5]

$$w(n+1)w(n-1) = \frac{\alpha w(n)^2 + \beta\lambda^n w(n) + \gamma\lambda^{2n}}{w(n)^2 + \delta w(n) + \alpha}, \tag{3}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, and $\lambda$ are parameters. For a special set of parameters

$$\begin{gathered} \alpha = -q^{4N}, \quad \beta = (q^{\nu+N} - q^{-\nu-N-2})q^{8N}(1-q)^2, \\ \gamma = q^{2(6N-1)}(1-q)^4, \quad \delta = (q^{\nu-N} - q^{-\nu+N})q^{2N}, \quad \lambda = q^2, \end{gathered} \tag{4}$$

(3) admits a class of special solutions as follows. A $q$-difference analogue of the Bessel equation is written as

$$J_\nu(q^2x) - (q^\nu + q^{-\nu})J_\nu(qx) + \{1 + (1-q)^2x^2\}J_\nu(x) = 0, \tag{5}$$

where $q$ is a multiplicative difference interval that satisfies $|q| < 1$. We write

$$J_\nu(q^n) = \mathrm{J}_\nu(n) \tag{6}$$

and consider a function with determinantal structure,

$$\tau_N^\nu(n) = \begin{vmatrix} \mathrm{J}_\nu(n) & \mathrm{J}_\nu(n+1) & \cdots & \mathrm{J}_\nu(n+N-1) \\ \mathrm{J}_\nu(n+2) & \mathrm{J}_\nu(n+3) & \cdots & \mathrm{J}_\nu(n+N+1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{J}_\nu(n+2N-2) & \mathrm{J}_\nu(n+2N-1) & \cdots & \mathrm{J}_\nu(n+3N-3) \end{vmatrix}. \tag{7}$$

Then, functions defined by

$$w_N^\nu(n) = \frac{\tau_{N+1}^\nu(n+1)\tau_N^{\nu+1}(n)}{\tau_{N+1}^\nu(n)\tau_N^{\nu+1}(n+1)} - q^{\nu+N} \tag{8}$$

solve (3) with parameters (4). Note that (7) satisfies the bilinear equations

$$\tau_{N+1}^\nu(n)\tau_N^{\nu+1}(n+1) - q^{-\nu-N}\tau_{N+1}^\nu(n+1)\tau_N^{\nu+1}(n) = -(1-q)q^{n+2N}\tau_{N+1}^{\nu+1}(n)\tau_N^\nu(n+1), \tag{9}$$

$$\tau_{N+1}^{\nu+1}(n)\tau_N^\nu(n+1) - q^{\nu-N+1}\tau_{N+1}^{\nu+1}(n+1)\tau_N^\nu(n) = (1-q)q^{n+2N}\tau_{N+1}^\nu(n)\tau_N^{\nu+1}(n+1), \tag{10}$$

$$\tau^{\nu}_{N+1}(n)\tau^{\nu+1}_{N}(n+3)-q^{-\nu-N}\tau^{\nu}_{N+1}(n+1)\tau^{\nu+1}_{N}(n+2)=-(1-q)q^{n}\tau^{\nu+1}_{N+1}(n)\tau^{\nu}_{N}(n+3), \tag{11}$$

$$\tau^{\nu+1}_{N+1}(n)\tau^{\nu}_{N}(n+3)-q^{\nu-N+1}\tau^{\nu+1}_{N+1}(n+1)\tau^{\nu}_{N}(n+2)=(1-q)q^{n}\tau^{\nu}_{N+1}(n)\tau^{\nu+1}_{N}(n+3). \tag{12}$$

Since (3) is subtraction-free, it is ultradiscretizable in the traditional procedure. However, the above special solutions may not be captured directly, because (5) shows oscillating behaviour and the determinant (7) itself includes subtraction. Hence, it is reasonable to introduce p-ultradiscretization. This is another approach to ultradiscretization of the Painlevé III equation by means of a different method from [16].

Our aim is to derive a class of special solutions for p-ultradiscrete analogues of (9)–(12) from (7). Moreover, we construct some solutions of p-ultradiscrete analogues of (3) and study their behaviour. This paper is organized as follows. In Sect. 2, we review the procedure of p-ultradiscretization. In Sect. 3, we evaluate a $q$-difference analogue of the Bessel function as preparation for the main content. In Sect. 4, we evaluate (7) into which we substitute the $q$-difference Bessel function. In Sect. 5, we construct an ultradiscrete solution for the p-ultradiscrete analogue of (3) and compare its behaviour with the corresponding special solution of the Painlevé III (differential) equation. Finally, concluding remarks are given in Sect. 6.

## 2 Ultradiscretization with parity variables

In this section, we review the procedure of p-ultradiscretization. As an example, we consider a simple difference equation

$$x_{n+1}=-ax_n+b \ \ (a,b>0,\ n\geq 0). \tag{13}$$

Note that (13) possesses the general solution

$$x_n=x_0(-a)^n+\frac{(-1)^{n-1}a^nb+b}{1+a}. \tag{14}$$

One can formally obtain an usual ultradiscrete analogue of (13) by assuming $x_n>0$. If we deform (13) as $x_{n+1}+ax_n=b$, all terms become positive. Hence, by putting $a=\exp(A/\varepsilon)$, $b=\exp(B/\varepsilon)$ and $x_n=\exp(X_n/\varepsilon)$, the deformed equation is ultradiscretized as

$$\max(X_{n+1},A+X_n)=B. \tag{15}$$

However, we find no solution of (15) for some specific values of $A$, $B$ and $X_n$. For example, when $A=1$, $B=0$ and $X_n=1$, (15) does not satisfied for any values of $X_{n+1}$. In this meaning, (15) is not well-defined. This failure of ultradiscretization trivially comes from unreasonable assumption $x_n>0$.

Introduction of p-ultradiscretization improves this situation. We assume $x_n\neq 0$ and introduce the parity (or sign) variable $\xi_n\in\{1,-1\}$ by

$$\xi_n=\frac{x_n}{|x_n|}, \tag{16}$$

which gives the sign number of $x_n$. We consider the following four cases for (13): (i) $\xi_{n+1} = 1$ and $\xi_n = 1$, (ii) $\xi_{n+1} = 1$ and $\xi_n = -1$, (iii) $\xi_{n+1} = -1$ and $\xi_n = 1$, (iv) $\xi_{n+1} = -1$ and $\xi_n = -1$. For each case, we deform (13) as

$$\begin{cases} \text{(i) } x_{n+1} + a x_n = b \\ \text{(ii) } x_{n+1} = -a x_n + b \\ \text{(iii) } a x_n = -x_{n+1} + b \\ \text{(iv) } 0 = -x_{n+1} - a x_n + b. \end{cases} \tag{17}$$

Since all terms are positive in (17), each equation is ultradiscretized by introducing the amplitude variable $X_n$ by $|x_n| = \exp(X_n/\varepsilon)$ and the formal replacement $0 = \exp(-\infty/\varepsilon)$ as

$$\begin{cases} \text{(i) } \max(X_{n+1}, A + X_n) = B \\ \text{(ii) } X_{n+1} = \max(A + X_n, B) \\ \text{(iii) } A + X_n = \max(X_{n+1}, B) \\ \text{(iv) } -\infty = \max(X_{n+1}, A + X_n, B), \end{cases} \tag{18}$$

respectively. Now, we regard the obtained set of equations (18) as an ultradiscrete analogue of (13). A pair $(\xi_n, X_n)$ includes richer information of $x_n$ than usual ultradiscrete variable. Let us study (18) with $A = 1$, $B = 0$ and $(\xi_n, X_n) = (1, 1)$. For these values, the cases (ii) and (iv) do not apply to $\xi_n = 1$ and the case (i) gives no solution. However, the case (iii) admits the unique solution $X_{n+1} = 2$ and this means $\xi_{n+1}$ should be $-1$. Hence, we have found the solution $(\xi_{n+1}, X_{n+1}) = (-1, 2)$, which we could not for (15).

We comment on indeterminacy in solving a p-ultradiscrete equation under specific conditions. Let us study (18) with $A = 1$, $B = 2$ and $(\xi_n, X_n) = (1, 1)$. Although the cases (ii) and (iv) do not apply again, the other two cases give the same equation $\max(X_{n+1}, 2) = 2$. From this equation, we must have the result that the parity is arbitrary ($\xi_{n+1} = \pm 1$) and the amplitude is not unique ($X_{n+1} \leq 2$). We proceed to the next step by choosing one of the indeterminate values. Therefore, we have an infinite number of solutions in this case. We note that a guideline to choose a 'nice value', whose meaning depends on property of the solution which one tries to capture, is still under study. Through this example, we find that uniqueness of solution is lost for some specific values of $A$, $B$ and $X_n$ but a solution always exists. Although this indeterminacy of solution looks troublesome, the situation becomes better comparing from (15), in which a solution may not exist. Noticing the existence of solution, we claim that (18) is well-defined. This example also shows that p-ultradiscretization does not make the max operation invertible. Hence, a p-ultradiscrete system is not equivalent to the original equation but its approximation.

In order to give 'simpler' representation of (18), we use the following notation. We define a function

$$s(\xi) = \begin{cases} 1 & (\xi = 1) \\ 0 & (\xi = -1). \end{cases} \tag{19}$$

Note that the sign $\xi$ is represented as $\xi = s(\xi) - s(-\xi)$. Then, we rewrite $x_n$ as

$$x_n = \{s(\xi_n) - s(-\xi_n)\}e^{\frac{X_n}{\varepsilon}} \tag{20}$$

and substitute it into (13). In other words, we employ (20) instead of (1). Then, transposing the negative terms to the other side of the equation, we have

$$s(\xi_{n+1})e^{\frac{X_{n+1}}{\varepsilon}} + s(-\xi_n)e^{\frac{X_n+A}{\varepsilon}} = s(-\xi_{n+1})e^{\frac{X_{n+1}}{\varepsilon}} + s(\xi_n)e^{\frac{X_n+A}{\varepsilon}} + e^{\frac{B}{\varepsilon}}. \tag{21}$$

We apply $\varepsilon \log$ to both sides of (21) and then take the limit $\varepsilon \to +0$. If we define a function $S$ by

$$S(\xi) = \begin{cases} 0 & (\xi = 1) \\ -\infty & (\xi = -1), \end{cases} \tag{22}$$

the following identity holds:

$$\lim_{\varepsilon \to +0} \varepsilon \log \left( s(\xi)e^{\frac{X}{\varepsilon}} + e^{\frac{Y}{\varepsilon}} \right) = \max(S(\xi) + X, Y). \tag{23}$$

Here, the term including $-\infty$ vanishes from max. We may regard $S(\xi)$ as the formal ultradiscrete analogue of $s(\xi)$ with $s(\xi) = e^{\frac{S(\xi)}{\varepsilon}}$. By utilizing this identity, (21) reduces to

$$\begin{aligned} &\max(S(\xi_{n+1}) + X_{n+1}, S(-\xi_n) + X_n + A) \\ &\quad = \max(S(-\xi_{n+1}) + X_{n+1}, S(\xi_n) + X_n + A, B). \end{aligned} \tag{24}$$

By considering all cases for the parity variables, we recover the explicit equations (18) from the implicit equation (24). That is, we represent the four cases (i)–(iv) in terms of the function $S$. We call (24) as well as (18) a p-ultradiscrete analogue of (13). We hereafter present p-ultradiscrete equations in the implicit expression by the following reasons. Firstly, the implicit expression has shorter form than its explicit counterpart. Secondly, one can construct both of forward and backward schemes from the implicit expression, if necessary. We comment that it is not always reasonable to treat the implicit expression as an equation on max-plus algebra for characters $X$ and $S(\xi)$. For example, the distributive law $\max(S(\xi) + X, Y) = S(\xi) + \max(X, Y - S(\xi))$ is not well-defined for $\xi = -1$, which directly corresponds to zero-dividing for the original variable. We should treat (24) as equations on max-plus algebra with cases represented by $S(\xi)$, not on a new algebra.

In closing this section, we review the following lemma [12], which provides a useful sufficient condition for obtaining a solution of a p-ultradiscrete equation.

**Lemma 1** *If a solution of a given difference equation $x_n(\varepsilon)$, where $\varepsilon$ is an arbitrary positive parameter, is evaluated as*

$$x_n(\varepsilon) = (-1)^{\hat{\xi}_n} e^{\frac{X_n}{\varepsilon}} (c_n + O(\varepsilon)), \quad c_n > 0, \tag{25}$$

*the pair of the sign* $(-1)^{\hat{\xi}_n}$ *and the amplitude* $X_n$ *solve the corresponding p-ultradiscrete equation.*

We may simply refer to the pair $((-1)^{\hat{\xi}_n}, X_n)$ obtained by this lemma as p-ultradiscrete limit of $x_n$. For example, we consider (14) and assume $n \geq 1$, $A > 0$, $x_0 = e^{X_0/\varepsilon} > 0$ and $X_0 > B - A$ for simplicity. Then, we find that $((-1)^n, nA + X_0)$ is the p-ultradiscrete limit of (14).

## 3 *q*-Bessel function and its ultradiscrete limit

In this section, we study a special solution of (5) with $\nu \in \mathbb{Z}_{\geq 0}$, which we refer to as the $q$-Bessel function. It is given by

$$J_\nu(x) = (1-q)^\nu x^\nu \sum_{j=0}^{\infty} \frac{(-1)^j (1-q)^{2j}}{(q^2;q^2)_j (q^2;q^2)_{\nu+j}} x^{2j}, \tag{26}$$

where

$$(a;q)_k = \begin{cases} 1 & (k=0) \\ (1-a)(1-aq)\dots(1-aq^{k-1}) & (k \in \mathbb{Z}_{>0}). \end{cases} \tag{27}$$

Jackson's $q$-Bessel function, which is defined even for $\nu \in \mathbb{C}$,

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\frac{x}{2}\right)^\nu {}_2\phi_1\left({0,0 \atop q^{\nu+1}}; q, -\frac{x^2}{4}\right) \tag{28}$$

is well known, where ${}_r\phi_s$ is the basic hypergeometric series

$${}_r\phi_s\left({a_1,\dots,a_r \atop b_1,\dots,b_s}; q, z\right) = \sum_{j=0}^{\infty} \frac{(a_1;q)_j \dots (a_r;q)_j}{(q;q)_j (b_1;q)_j \dots (b_s;q)_j} [(-1)^j q^{\frac{j(j-1)}{2}}]^{1-r+s} z^j. \tag{29}$$

The relationship between (26) and (28) for $\nu \in \mathbb{Z}_{\geq 0}$

$$J_\nu(x) = J_\nu^{(1)}(2(1-q)x; q^2) \tag{30}$$

is readily found.

### 3.1 Evaluation of the *q*-Bessel function

Our aim in this subsection is to prove the following proposition.

**Proposition 1** *(i) The* $q$*-Bessel function* $J_\nu(x)$ *is deformed as follows:*

$$J_\nu(x) = \frac{(1-q)^\nu x^\nu}{(-(1-q)^2 x^2; q^2)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k(k+\nu)} (1-q)^{2k}}{(q^2;q^2)_k (q^2;q^2)_{k+\nu}} x^{2k}. \tag{31}$$

*(ii) The evaluation as $q \to 0$*

$$J_\nu(q^n) = \begin{cases} q^{n\nu}\,(1+O(q)) & (n \geq 1) \\ q^{n(n+\nu-1)}\left(\frac{1}{2}+O(q)\right) & (0 \geq n \geq -\nu) \\ (-1)^{\frac{n+\nu}{2}} q^{\frac{n(n-2)-\nu^2}{2}}\left(\frac{1}{2}+O(q)\right) & (n \leq -\nu-1, n+\nu : even) \\ (-1)^{\frac{n+\nu+1}{2}} q^{\frac{n(n-2)-\nu^2+3}{2}}\,(1+O(q)) & (n \leq -\nu-1, n+\nu : odd) \end{cases} \tag{32}$$

*holds for $n \in \mathbb{Z}$.*

To prove Proposition 1 (i), we introduce the well known formula [2]

$$\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} x^k \tag{33}$$

$$(x;q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(q;q)_k} x^k \tag{34}$$

$$(a;q)_{\nu+k} = (a;q)_\nu (aq^\nu;q)_k \tag{35}$$

and the following lemma.

**Lemma 2** (i) *The ‘q-Euler transformation’ [1]*

$$\sum_{j=0}^{\infty} c_j d_j x^j = \sum_{k=0}^{\infty} \frac{(\hat{D}^k c_0)}{[k]!} x^k \hat{B}^k f(x) \tag{36}$$

*holds, where*

$$f(x) := \sum_{j=0}^{\infty} d_j x^j \tag{37}$$

$$\hat{B} f(x) := \frac{f(x) - f(qx)}{(1-q)x} \tag{38}$$

$$[k] := \frac{1-q^k}{1-q}, \quad [k]! := \begin{cases} 1 & (k=0) \\ [k]\,[k-1]\ldots[1] & (k \in \mathbb{Z}_{>0}) \end{cases} \tag{39}$$

$$\hat{E} c_j := c_{j+1}\ (j \in \mathbb{Z}_{\geq 0}), \quad \hat{D}^k := (\hat{E}-1)(\hat{E}-q)\ldots(\hat{E}-q^{k-1}). \tag{40}$$

(ii) *The formula*

$$\hat{D}^k c_0 = \sum_{j=0}^{k} (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} k \\ j \end{bmatrix} c_{k-j} \tag{41}$$

*holds, where*

$$\begin{bmatrix} k \\ j \end{bmatrix} := \frac{[k]!}{[k-j]![j]!} = \frac{(q;q)_k}{(q;q)_{k-j}(q;q)_j}. \tag{42}$$

*Proof of Proposition 1 (i)* We consider a function

$$\hat{J}_\nu^{(1)}(t;q) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(q;q)_{j+\nu}(q;q)_j} t^j. \tag{43}$$

Note that

$$\left(\frac{x}{2}\right)^\nu \hat{J}_\nu^{(1)}((x/2)^2;q) = J_\nu^{(1)}(x;q). \tag{44}$$

If we put

$$c_j = \frac{1}{(q;q)_{\nu+j}}, \qquad d_j = \frac{(-1)^j}{(q;q)_j}, \tag{45}$$

we have $\hat{J}_\nu^{(1)}(t;q) = \sum c_j d_j t^j$. Then, $f$ defined by (37) becomes

$$f(t) = \frac{1}{(-t;q)_\infty} \tag{46}$$

from (33). Hence, we obtain

$$\hat{B}^k f(t) = \frac{(-1)^k}{(1-q)^k} f(t). \tag{47}$$

Next, we prove

$$\hat{D}^k c_0 = \frac{q^{k(k+\nu)}}{(q;q)_{k+\nu}}. \tag{48}$$

We introduce an identity

$$\frac{(x;q)_\infty}{(q;q)_\nu} {}_2\phi_1\left({0,0 \atop q^{\nu+1}};q,x\right) = \frac{1}{(q;q)_\nu} {}_0\phi_1\left({- \atop q^{\nu+1}};q,q^{\nu+1}x\right), \tag{49}$$

which appears in [2], Exercises 3.2 (iii). Note that

$$(q^{\nu+1};q)_{k-l}(q;q)_\nu = (q;q)_{\nu+k-l} = \frac{1}{c_{k-l}}. \tag{50}$$

The left hand side of (49) is deformed as follows:

$$\frac{1}{(q;q)_\nu}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(q;q)_k} x^k\right\}\left\{\sum_{k=0}^{\infty} \frac{1}{(q;q)_k (q^{\nu+1};q)_k} x^k\right\}$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{l=0}^{k} \frac{(-1)^l q^{\frac{l(l-1)}{2}}}{(q;q)_l (q;q)_{k-l} \left( (q^{\nu+1};q)_{k-l} (q;q)_\nu \right)} \right\} x^k$$
$$= \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} \left\{ \sum_{l=0}^{k} (-1)^l q^{\frac{l(l-1)}{2}} \begin{bmatrix} k \\ l \end{bmatrix} c_{k-l} \right\} x^k$$
$$= \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} \left( \hat{D}^k c_0 \right) x^k. \tag{51}$$

The right hand side of (49) is rewritten as follows:

$$\frac{1}{(q;q)_\nu} \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q;q)_k (q^{\nu+1};q)_k} (q^{\nu+1} x)^k = \sum_{k=0}^{\infty} \frac{q^{k(k+\nu)}}{(q;q)_k \left( (q^{\nu+1};q)_k (q;q)_\nu \right)} x^k$$
$$= \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} \frac{q^{k(k+\nu)}}{(q;q)_{\nu+k}} x^k. \tag{52}$$

We have (48) by comparing these two series.

Now, applying the $q$-Euler transformation, $\hat{J}_\nu^{(1)}$ is deformed as follows:

$$\hat{J}_\nu^{(1)}(t;q) = \sum_{k=0}^{\infty} \frac{1}{[k]!} \frac{q^{k(k+\nu)}}{(q;q)_{k+\nu}} t^k \frac{(-1)^k}{(1-q)^k} f(t) = f(t) \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+\nu)}}{(q;q)_k (q;q)_{k+\nu}} t^k, \tag{53}$$

which gives

$$J_\nu^{(1)}(x;q) = \left( \frac{x}{2} \right)^\nu \frac{1}{(-x^2/4;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+\nu)}}{(q;q)_k (q;q)_{k+\nu}} \left( \frac{x}{2} \right)^{2k} \tag{54}$$

by (44). Moreover, (54) reduces to (31) using (30). □

*Proof of Proposition 1 (ii)* Substituting $x = q^n$ into (26) and (31), we obtain

$$J_\nu(q^n) = (1-q)^\nu q^{n\nu} \sum_{j=0}^{\infty} \frac{(-1)^j (1-q)^{2j}}{(q^2;q^2)_j (q^2;q^2)_{j+\nu}} q^{2nj} \tag{55}$$

$$J_\nu(q^n) = \frac{(1-q)^\nu q^{\nu n}}{(-(1-q)^2 q^{2n};q^2)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (1-q)^{2k}}{(q^2;q^2)_k (q^2;q^2)_{k+\nu}} q^{\tilde{f}_{\nu,n}(k)}, \tag{56}$$

respectively, where

$$\tilde{f}_{\nu,n}(k) := 2k(k+\nu+n) = 2\left( k + \frac{\nu+n}{2} \right)^2 - \frac{(\nu+n)^2}{2}. \tag{57}$$

Note that

$$\frac{1}{(q^2;q^2)_m} = 1 + O(q^2) \quad (q \to 0) \tag{58}$$

$$\frac{1}{(-(1-q)^2q^{2n};q^2)_\infty} = q^{n(n-1)}\left(\frac{1}{2} + O(q)\right) \quad (q \to 0,\ n \leq 0). \tag{59}$$

If $n \geq 1$, the term with $j = 0$ in (55) is dominant. Hence, we readily obtain

$$J_\nu(q^n) = q^{\nu n}\,(1 + O(q)) \quad (q \to 0). \tag{60}$$

Since it is difficult to evaluate (55) for $n \in \mathbb{Z}_{\leq 0}$, we focus on the other expression (56) and study the minimum value of $\tilde{f}_{\nu,n}(k)$ for $k \in \mathbb{Z}_{\geq 0}$. If $0 \geq n \geq -\nu$, we find that $\min \tilde{f}_{\nu,n}(k) = \tilde{f}_{\nu,n}(0) = 0$ and that the term with $k = 0$ in the summation in (56) is dominant. Hence, we obtain

$$J_\nu(q^n) = q^{\nu n + n(n-1)}\left(\frac{1}{2} + O(q)\right) \quad (q \to 0). \tag{61}$$

Next, if $n < -\nu$ and $\nu + n$ is even, we find that

$$\min \tilde{f}_{\nu,n}(k) = \tilde{f}_{\nu,n}\left(-\frac{\nu+n}{2}\right) = -\frac{(\nu+n)^2}{2}. \tag{62}$$

Hence, the term with $k = -\frac{\nu+n}{2}$ is dominant and then

$$J_\nu(q^n) = (-1)^{\frac{\nu+n}{2}}\, q^{\frac{n(n-2)-\nu^2}{2}}\left(\frac{1}{2} + O(q)\right) \quad (q \to 0) \tag{63}$$

holds. Finally, we consider the case where $n < -\nu$ and $\nu + n$ is odd. We put $\nu + n = 2\mu + 1$ ($\mu = -1, -2, \ldots$). We find that

$$\min \tilde{f}_{\nu,n}(k) = \tilde{f}_{\nu,n}(-\mu) = \tilde{f}_{\nu,n}(-\mu - 1) = -2\mu(\mu + 1) =: \tilde{m}. \tag{64}$$

Then, the leading term in the summation in (56) may be given by the sum of the terms with $k = -\mu$ and $-\mu - 1$,

$$\begin{aligned}&(-1)^{-\mu}(1-q)^{-2\mu}q^{\tilde{m}}(1 + O(q^2)) + (-1)^{-\mu-1}(1-q)^{-2\mu-2}q^{\tilde{m}}(1 + O(q^2))\\ &\quad = (-1)^{-\mu-1}q^{1+\tilde{m}}(2 + O(q)).\end{aligned} \tag{65}$$

We also find from $|\tilde{f}_{\nu,n}(-\mu) - \tilde{f}_{\nu,n}(-\mu+1)| = 4$ that the other terms do not contribute to the leading term. Therefore, we have

$$
\begin{aligned}
J_\nu(q^n) &= q^{\nu n+n(n-1)} \cdot (-1)^{-\mu-1} q^{1+\tilde{m}} \,(1+O(q)) \\
&= (-1)^{\frac{\nu+n+1}{2}} q^{\frac{n(n-2)-\nu^2+3}{2}} (1+O(q)) \quad (q \to 0).
\end{aligned} \tag{66}
$$

Now, (32) has been proved. □

### 3.2 Ultradiscrete solution and supplementary result

If we put $x = q^n$, $q = e^{\frac{Q}{\varepsilon}}$ $(Q < 0)$, and

$$
J_\nu(q^n) = \{s(\beta_n^\nu) - s(-\beta_n^\nu)\} e^{\frac{B_n^\nu}{\varepsilon}}, \tag{67}
$$

(5) reduces to the p-ultradiscrete Bessel equation,

$$
\begin{aligned}
&\max[S(\beta_{n+1}^\nu)+B_{n+1}^\nu,\, S(-\beta_n^\nu)+B_n^\nu-\nu Q,\, S(\beta_{n-1}^\nu)+B_{n-1}^\nu+\max(0, (2n-2)Q)] \\
&\quad = \max[S(-\beta_{n+1}^\nu)+B_{n+1}^\nu,\, S(\beta_n^\nu)+B_n^\nu-\nu Q,\, S(-\beta_{n-1}^\nu) \\
&\qquad +B_{n-1}^\nu+\max(0, (2n-2)Q)].
\end{aligned} \tag{68}
$$

Applying Lemma 1 to (32), we readily obtain an explicit expression for the p-ultradiscrete analogue of the $q$-Bessel function

$$
\begin{aligned}
\mathcal{B}_n^\nu &= \left(\beta_n^\nu, B_n^\nu\right) \\
&= \begin{cases}
(1, n\nu Q) & (n \geq 1) \\
(1, n(n+\nu-1)Q) & (0 \geq n \geq -\nu) \\
\left((-1)^{\frac{n+\nu}{2}}, \frac{n(n-2)-\nu^2}{2} Q\right) & (n \leq -\nu-1, n+\nu : \text{even}) \\
\left((-1)^{\frac{n+\nu+1}{2}}, \frac{n(n-2)-\nu^2+3}{2} Q\right) & (n \leq -\nu-1, n+\nu : \text{odd}).
\end{cases}
\end{aligned} \tag{69}
$$

Note that this result is identical to the ultradiscrete Bessel function by Narasaki (see Appendix 1), unless $n \leq -\nu - 1$ and $n + \nu$ is odd.

Another supplementary result can be obtained from Proposition 1. It is concerned with the number of restricted partitions $p_n(k)$ defined by the generating function

$$
\frac{1}{(q;q)_n} = \sum_{k=0}^{\infty} p_n(k) q^k. \tag{70}
$$

We have the series expression of (43) by utilizing (70),

$$
\begin{aligned}
\hat{J}_\nu^{(1)}(q^n; q) &= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{j=0}^{\infty} p_k(j) q^j\right) \left(\sum_{j=0}^{\infty} p_{k+\nu}(j) q^j\right) q^{kn} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k \sum_{l=0}^{j} p_k(j-l) p_{k+\nu}(l) q^{j+kn}.
\end{aligned} \tag{71}
$$

From (53), we also obtain

$$\hat{J}_\nu^{(1)}(q^n;q)=\begin{cases}1+O(q) & (n\geq 1)\\ q^{\frac{n(n-1)}{2}}(\frac{1}{2}+O(q)) & (0\geq n\geq -\nu)\\ (-1)^{-\frac{n+\nu}{2}}q^{\frac{n(n-1)}{2}-(\frac{n+\nu}{2})^2}(\frac{1}{2}+O(q)) & (n<-\nu,\ n+\nu:\text{even})\\ (-1)^{-\frac{n+\nu-1}{2}}q^{\frac{n(n-1)}{2}-\frac{(n+\nu-1)(n+\nu+3)}{4}}(\frac{1}{2}+O(q)) & (n<-\nu,\ n+\nu:\text{odd}).\end{cases} \tag{72}$$

Comparing these two expressions, we find that (71) has an infinite number of extra terms for a negative integer $n$. From this fact, we have the following formula.

**Proposition 2** *For given $\nu\in\mathbb{Z}_{\geq 0}$ and $n\in\mathbb{Z}_{<0}$, fix $m\in\mathbb{Z}$ such that*

$$m<\begin{cases}\frac{n(n-1)}{2} & (0>n\geq -\nu)\\ \frac{n(n-1)}{2}-\left(\frac{n+\nu}{2}\right)^2 & (n<-\nu,\ n+\nu:\textit{even})\\ \frac{n(n-1)}{2}-\frac{(n+\nu-1)(n+\nu+3)}{4} & (n<-\nu,\ n+\nu:\textit{odd}).\end{cases} \tag{73}$$

*Then*

$$\sum_{j,k}(-1)^k\sum_{l=0}^{j}p_k(j-l)p_{k+\nu}(l)=0 \tag{74}$$

*holds, where $j$ and $k$ run over all pairs satisfying $j+kn=m$ in the summation.*

We do not know a combinatorial meaning of this (formal) formula yet.

## 4 Evaluation for special solutions of the discrete Painlevé III equation

In this section, we substitute the $q$-Bessel function into (7) and study the p-ultradiscrete analogue of the resulting function. If $\tau_N^\nu(n)$ is written in the form of (25), its p-ultradiscrete analogue is readily obtained by Lemma 1. We first explain a useful notation and then summarize the main result.

We rewrite (32) in a simpler form since we often use it. We introduce

$$\psi(n):=n(n+\nu-1) \tag{75}$$

$$p_1(n):=\frac{3\{1+(-1)^{n+1}\}}{4}=\begin{cases}0 & (n\text{: even})\\ \frac{3}{2} & (n\text{: odd})\end{cases} \tag{76}$$

$$p_2(n):=\frac{3+(-1)^{n+1}}{4}=\begin{cases}\frac{1}{2} & (n\text{: even})\\ 1 & (n\text{: odd})\end{cases} \tag{77}$$

$$\varphi_\nu(n):=\frac{n(n-2)}{2}-\frac{\nu^2}{2} \tag{78}$$

and the binomial coefficient $\binom{n}{2}=(n-1)n/2$. Using (6) and these notations, (32) can be rewritten as follows:

$$\mathrm{J}_\nu(n)=q^{n\nu}(1+O(q))\qquad (n\geq 1) \tag{79}$$

$$\mathrm{J}_\nu(n) = q^{\psi(n)}\left(\frac{1}{2} + O(q)\right) \qquad (0 \geq n \geq -\nu) \tag{80}$$

$$\mathrm{J}_\nu(n) = (-1)^{\binom{n+\nu+1}{2}} q^{\varphi_\nu(n)+p_1(n+\nu)}\left(p_2(n+\nu) + O(q)\right) \qquad (n \leq -\nu-1). \tag{81}$$

For $\tau_N^\nu(n)$, we define the sign $y_{N,n}^\nu$ and the amplitude $Y_{N,n}^\nu$ by

$$y_{N,n}^\nu = \frac{\tau_N^\nu(n)}{|\tau_N^\nu(n)|}, \quad |\tau_N^\nu(n)| = e^{\frac{Y_{N,n}^\nu}{\varepsilon}}, \tag{82}$$

respectively. Moreover, we use the formula [22]

$$s(\xi)s(\zeta) + s(-\xi)s(-\zeta) = s(\xi\zeta) \tag{83}$$

to obtain simpler expression for a multiplicative term. Then, p-ultradiscrete analogues of (9)–(12) are respectively written as

$$\begin{aligned}
&\max[S(y_{N+1,n}^\nu y_{N,n+1}^{\nu+1}) + Y_{N+1,n}^\nu + Y_{N,n+1}^{\nu+1},\\
&\quad S(-y_{N+1,n+1}^\nu y_{N,n}^{\nu+1}) + Y_{N+1,n+1}^\nu + Y_{N,n}^{\nu+1} - (\nu+N)Q,\\
&\quad S(y_{N+1,n}^{\nu+1} y_{N,n+1}^\nu) + Y_{N+1,n}^{\nu+1} + Y_{N,n+1}^\nu + (n+2N)Q,\\
&\quad S(-y_{N+1,n}^{\nu+1} y_{N,n+1}^\nu) + Y_{N+1,n}^{\nu+1} + Y_{N,n+1}^\nu + (n+2N+1)Q]\\
&\ = \max[S(-y_{N+1,n}^\nu y_{N,n+1}^{\nu+1}) + Y_{N+1,n}^\nu + Y_{N,n+1}^{\nu+1},\\
&\quad S(y_{N+1,n+1}^\nu y_{N,n}^{\nu+1}) + Y_{N+1,n+1}^\nu + Y_{N,n}^{\nu+1} - (\nu+N)Q,\\
&\quad S(-y_{N+1,n}^{\nu+1} y_{N,n+1}^\nu) + Y_{N+1,n}^{\nu+1} + Y_{N,n+1}^\nu + (n+2N)Q,\\
&\quad S(y_{N+1,n}^{\nu+1} y_{N,n+1}^\nu) + Y_{N+1,n}^{\nu+1} + Y_{N,n+1}^\nu + (n+2N+1)Q],
\end{aligned} \tag{84}$$

$$\begin{aligned}
&\max[S(y_{N+1,n}^{\nu+1} y_{N,n+1}^\nu) + Y_{N+1,n}^{\nu+1} + Y_{N,n+1}^\nu,\\
&\quad S(-y_{N+1,n+1}^{\nu+1} y_{N,n}^\nu) + Y_{N+1,n+1}^{\nu+1} + Y_{N,n}^\nu + (\nu-N+1)Q,\\
&\quad S(-y_{N+1,n}^\nu y_{N,n+1}^{\nu+1}) + Y_{N+1,n}^\nu + Y_{N,n+1}^{\nu+1} + (n+2N)Q,\\
&\quad S(y_{N+1,n}^\nu y_{N,n+1}^{\nu+1}) + Y_{N+1,n}^\nu + Y_{N,n+1}^{\nu+1} + (n+2N+1)Q]\\
&\ = \max[S(-y_{N+1,n}^{\nu+1} y_{N,n+1}^\nu) + Y_{N+1,n}^{\nu+1} + Y_{N,n+1}^\nu,\\
&\quad S(y_{N+1,n+1}^{\nu+1} y_{N,n}^\nu) + Y_{N+1,n+1}^{\nu+1} + Y_{N,n}^\nu + (\nu-N+1)Q,\\
&\quad S(y_{N+1,n}^\nu y_{N,n+1}^{\nu+1}) + Y_{N+1,n}^\nu + Y_{N,n+1}^{\nu+1} + (n+2N)Q,\\
&\quad S(-y_{N+1,n}^\nu y_{N,n+1}^{\nu+1}) + Y_{N+1,n}^\nu + Y_{N,n+1}^{\nu+1} + (n+2N+1)Q],
\end{aligned} \tag{85}$$

$$\begin{aligned}
&\max[S(y_{N+1,n}^\nu y_{N,n+3}^{\nu+1}) + Y_{N+1,n}^\nu + Y_{N,n+3}^{\nu+1},\\
&\quad S(-y_{N+1,n+1}^\nu y_{N,n+2}^{\nu+1}) + Y_{N+1,n+1}^\nu + Y_{N,n+2}^{\nu+1} - (\nu+N)Q,\\
&\quad S(y_{N+1,n}^{\nu+1} y_{N,n+3}^\nu) + Y_{N+1,n}^{\nu+1} + Y_{N,n+3}^\nu + nQ,
\end{aligned}$$

$$
\begin{aligned}
&S(-y^{\nu+1}_{N+1,n}y^{\nu}_{N,n+3}) + Y^{\nu+1}_{N+1,n} + Y^{\nu}_{N,n+3} + (n+1)Q]\\
&= \max[S(-y^{\nu}_{N+1,n}y^{\nu+1}_{N,n+3}) + Y^{\nu}_{N+1,n} + Y^{\nu+1}_{N,n+3},\\
&S(y^{\nu}_{N+1,n+1}y^{\nu+1}_{N,n+2}) + Y^{\nu}_{N+1,n+1} + Y^{\nu+1}_{N,n+2} - (\nu+N)Q,\\
&S(-y^{\nu+1}_{N+1,n}y^{\nu}_{N,n+3}) + Y^{\nu+1}_{N+1,n} + Y^{\nu}_{N,n+3} + nQ,\\
&S(y^{\nu+1}_{N+1,n}y^{\nu}_{N,n+3}) + Y^{\nu+1}_{N+1,n} + Y^{\nu}_{N,n+3} + (n+1)Q],
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
&\max[S(y^{\nu+1}_{N+1,n}y^{\nu}_{N,n+3}) + Y^{\nu+1}_{N+1,n} + Y^{\nu}_{N,n+3},\\
&S(-y^{\nu+1}_{N+1,n+1}y^{\nu}_{N,n+2}) + Y^{\nu+1}_{N+1,n+1} + Y^{\nu}_{N,n+2} + (\nu-N+1)Q,\\
&S(-y^{\nu}_{N+1,n}y^{\nu+1}_{N,n+3}) + Y^{\nu}_{N+1,n} + Y^{\nu+1}_{N,n+3} + nQ,\\
&S(y^{\nu}_{N+1,n}y^{\nu+1}_{N,n+3}) + Y^{\nu}_{N+1,n} + Y^{\nu+1}_{N,n+3} + (n+1)Q]\\
&= \max[S(-y^{\nu+1}_{N+1,n}y^{\nu}_{N,n+3}) + Y^{\nu+1}_{N+1,n} + Y^{\nu}_{N,n+3},\\
&S(y^{\nu+1}_{N+1,n+1}y^{\nu}_{N,n+2}) + Y^{\nu+1}_{N+1,n+1} + Y^{\nu}_{N,n+2} + (\nu-N+1)Q,\\
&S(y^{\nu}_{N+1,n}y^{\nu+1}_{N,n+3}) + Y^{\nu}_{N+1,n} + Y^{\nu+1}_{N,n+3} + nQ,\\
&S(-y^{\nu}_{N+1,n}y^{\nu+1}_{N,n+3}) + Y^{\nu}_{N+1,n} + Y^{\nu+1}_{N,n+3} + (n+1)Q].
\end{aligned}
\tag{87}
$$

These p-ultradiscrete equations are solved by $y$ and $Y$, as presented below, which are obtained as the p-ultradiscrete analogue of (7). To give explicit functional forms, we introduce the following six cases:

(A) $n \geq 1$,
(B) $0 \geq n \geq \max(2-2N, -\nu-N)+1$,
(C-a) $\nu \geq N-1$ and $2-2N \geq n \geq 1-\nu-N$,
(C-b) $\nu \leq N-2$ and $-\nu-N \geq n \geq 3-2N$,
(D) $\min(2-2N, -\nu-N) \geq n \geq 2-2N-\nu$,
(E) $n \leq 1-2N-\nu$.

Note that there exist values of $n$ such that all antidiagonal elements in (7) are of the type (80) for $\nu \geq N-1$, but such values of $n$ do not exist for $\nu \leq N-2$. We also introduce

$$A_{N,n} := 2n + 3N - 3 \tag{88}$$

$$B_\nu(k) := \varphi_\nu(k) + p_1(k+\nu) \tag{89}$$

$$M = \min(\mathrm{floor}(|n|/2)+1, N), \tag{90}$$

where floor$(x)$ denotes the integer part of $x$. Now, the p-ultradiscrete analogue of (7) is written as follows.
Case (A):

$$y^{\nu}_{N,n} = (-1)^{\binom{N}{2}} \tag{91}$$

$$Y^{\nu}_{N,n} = Q\left\{\frac{\nu N}{2}A_{N,n} + 2\binom{N}{2}(n+N-2)\right\}. \tag{92}$$

Case (B):

$$y^{\nu}_{N,n} = (-1)^{\binom{N}{2}} \tag{93}$$

$$\begin{aligned} Y^{\nu}_{N,n} = Q\Bigg\{ &\frac{\nu N}{2} A_{N,n} + 2\binom{N-M}{2}(n+N+M-2) \\ &+ \frac{1}{2}\binom{M+1}{3} + 2M\binom{n+N+\frac{M-3}{2}}{2}\Bigg\}. \end{aligned} \tag{94}$$

Case (C-a):

$$y^{\nu}_{N,n} = (-1)^{\binom{N}{2}} \tag{95}$$

$$Y^{\nu}_{N,n} = Q\left\{\frac{N}{4} A_{N,n}(A_{N,n} + 2\nu - 2) + \frac{1}{2}\binom{N+1}{3}\right\}. \tag{96}$$

Case (C-b):

$$y^{\nu}_{N,n} = (-1)^{\binom{N}{2}} \prod_{k=n+N-1}^{-\nu-1} (-1)^{\binom{k+\nu+1}{2}} \tag{97}$$

$$\begin{aligned} Y^{\nu}_{N,n} = Q\Bigg\{ &\frac{\nu N}{2} A_{N,n} + 2\binom{N-M}{2}(n+N+M-2) \\ &+ \sum_{k=-\nu}^{n+N+M-2} k(k-1) + \sum_{k=n+N-1}^{-\nu-1} (B_{\nu}(k) - \nu k)\Bigg\}. \end{aligned} \tag{98}$$

Case (D):

$$y^{\nu}_{N,n} = (-1)^{\binom{N}{2}} \prod_{k=n+N-1}^{-\nu-1} (-1)^{\binom{k+\nu+1}{2}} \tag{99}$$

$$Y^{\nu}_{N,n} = Q\left\{\sum_{k=n+N-1}^{-\nu-1} B_{\nu}(k) + \sum_{k=-\nu}^{n+2N-2} \psi(k)\right\}. \tag{100}$$

Case (E):

$$y^{\nu}_{N,n} = (-1)^{\binom{N}{2}} \prod_{k=n+N-1}^{n+2N-2} (-1)^{\binom{k+\nu+1}{2}} \tag{101}$$

$$Y^{\nu}_{N,n} = Q \sum_{k=n+N-1}^{n+2N-2} B_{\nu}(k). \tag{102}$$

We study each case in the following subsections.

### 4.1 Case (A)

In this case, all arguments of $\mathrm{J}_\nu(n)$ in (7) are positive. Substituting (55) into (7) and using multi-linearity of the determinant, we obtain

$$\tau_N^\nu(n) = (1-q)^{\nu N}\left\{\prod_{k=1}^{N} q^{(n+3k-3)\nu}\right\}\sum_{j_1,\dots,j_N} P_N^\nu(\boldsymbol{j})\left\{\prod_{k=1}^{N} q^{2(n+2k-2)j_k}\right\} \times \begin{vmatrix} 1 & q^{2j_1} & \dots & q^{2(N-1)j_1} \\ 1 & q^{2j_2} & \dots & q^{2(N-1)j_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q^{2j_N} & \dots & q^{2(N-1)j_N} \end{vmatrix}, \tag{103}$$

where

$$P_N^\nu(\boldsymbol{j}) := \prod_{k=1}^{N} \frac{(-1)^{j_k}(1-q)^{2j_k}}{(q^2;q^2)_{j_k+\nu}(q^2;q^2)_{j_k}} = \left\{\prod_{k=1}^{N}(-1)^{j_k}\right\}(1+O(q)) \tag{104}$$

$$\sum_{j_1,\dots,j_N} := \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_N=0}^{\infty}. \tag{105}$$

Since the determinant in (103) is the Vandermonde determinant, the terms with $j_k = j_l$ $(k \neq l)$ disappear. Moreover, under any permutation of $(j_1, j_2, \dots, j_N)$, the absolute values of all elements but the factor $\prod_{k=1}^{N} q^{2(n+2k-2)j_k}$ in (103) are invariant. Therefore, noting $0 < q < 1$, the largest absolute value of the monomial in (103) is achieved by the $j_k$'s with

$$0 \le j_N < \cdots < j_2 < j_1, \tag{106}$$

which maximize $\prod_{k=1}^{N} q^{2(n+2k-2)j_k}$. Then, in the Vandermonde determinant, the product of diagonal elements $\prod_{k=1}^{N} q^{2(k-1)j_k}$ contributes to the evaluation. Hence, we study $j_k$'s that maximize $\prod_{k=1}^{N} q^{2(n+2k-2)j_k+2(k-1)j_k}$; i.e., $j_k$'s that minimize

$$\sum_{k=1}^{N} 2(n+3k-3)\,j_k \tag{107}$$

under (106). We readily find that such $j_k$'s are given by

$$j_k = N-k, \quad k = 1, 2, \dots, N. \tag{108}$$

Moreover, further contributions by the factor $\prod_{k=1}^{N} q^{(n+3k-3)\nu}$ and sign $\prod_{i=1}^{N}(-1)^{N-i} = (-1)^{\binom{N}{2}}$ from $P_N^\nu(\boldsymbol{j})$ must be considered. Accordingly, the leading term of $\tau_N^\nu(n)$

is given by

$$\tau_N^\nu(n) \sim (-1)^{\binom{N}{2}} \prod_{k=1}^{N} q^{(n+3k-3)(2N-2k+\nu)}$$
$$= (-1)^{\binom{N}{2}} q^{\frac{\nu N}{2}(2n+3N-3)+N(N-1)(n+N-2)}. \tag{109}$$

### 4.2 Cases (C-a), (D), and (E)

In these cases, all arguments of $\mathrm{J}_\nu(n)$ in anti-diagonal elements of (7) are negative. The following proposition plays an important role in the evaluation.

**Proposition 3** *Write the absolute value of the leading term of* $\mathrm{J}_\nu(n)$ *as* $q^{g_\nu(n)}$. *For* $n \in \mathbb{Z}$ *and* $k, l \in \mathbb{Z}_{>0}$,

$$\tilde{g} := g_\nu(n) + g_\nu(n+2k+l) - g_\nu(n+2k) - g_\nu(n+l) \begin{cases} = 0 & (n \geq 0) \\ > 0 & (n \leq -1) \end{cases} \tag{110}$$

*holds.*

*Proof* This inequality is proved by direct calculation by considering all 46 possible cases of $g_\nu$'s. Here, we illustrate three typical cases. First, if $n + 2k + l \leq -\nu - 1$ and both $n + \nu$ and $n + l + \nu$ are even, we have

$$g_\nu(n) = \varphi_\nu(n), \qquad g_\nu(n+2k+l) = \varphi_\nu(n+2k+l),$$
$$g_\nu(n+l) = \varphi_\nu(n+l), \qquad g_\nu(n+2k) = \varphi_\nu(n+2k)$$

and $\tilde{g} = 2kl \geq 2 > 0$. Second, if $n + 2k \leq -\nu - 1$, $n$ is even, $0 \geq n + l \geq -\nu$, and $n + 2k + l \geq 1$, then we have

$$g_\nu(n) = \varphi_\nu(n), \qquad g_\nu(n+2k+l) = \nu(n+2k+l),$$
$$g_\nu(n+l) = \psi(n+l), \qquad g_\nu(n+2k) = \varphi_\nu(n+2k).$$

Since $n+l+\nu-1 \geq -1$, we consider two cases, $n+l+\nu-1 \geq 0$ and $n+l+\nu = 0$. When $n + l + \nu - 1 \geq 0$, $g_\nu(n + l) \leq 0$ and therefore $\tilde{g} \geq g_\nu(n) + g_\nu(n + 2k + l) - g_\nu(n + 2k) = -2k(n + k) + \nu(n + 2k + l) + 2k > 0$. When $n + l + \nu = 0$, we obtain $2k - \nu \geq 1$ and therefore $\tilde{g} = -2k(n + k) + \nu(2k - \nu) + 2k > 0$. Finally, if $n \leq -\nu - 1$ and $n + \nu$ is even and $0 \geq n + 2k, n + l, n + 2k + l \geq -\nu$, then the $g_\nu$'s are given by

$$g_\nu(n) = \varphi_\nu(n), \qquad g_\nu(n+2k+l) = \psi(n+2k+l),$$
$$g_\nu(n+l) = \psi(n+l), \qquad g_\nu(n+2k) = \psi(n+2k)$$

and $\tilde{g} = \{8kl - (n + \nu)^2\}/2$. Noting that $|n + \nu| \leq l$ and $|n + \nu| \leq 2k$, we obtain $(n + \nu)^2 \leq 2kl < 8kl$. Hence, $\tilde{g} > 0$ holds. □

From Proposition 3, there follows this inequality: $|\mathrm{J}_\nu(n)\mathrm{J}_\nu(n+2k+l)| < |\mathrm{J}_\nu(n+2k)\mathrm{J}_\nu(n+l)|$ for $n < 0$ as $q \to 0$. This inequality implies that the product of anti-diagonal elements is dominant in (7) for $-n \gg 1$ (See (111)).

$$\begin{vmatrix} \cdots & \mathrm{J}_\nu(n) & \cdots & \mathrm{J}_\nu(n+l) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \mathrm{J}_\nu(n+2k) & \cdots & \mathrm{J}_\nu(n+2k+l) & \cdots \end{vmatrix}. \tag{111}$$

Moreover, this is true even for the case in which all anti-diagonal elements have nonpositive arguments.

We first study the case (E). In this case, all anti-diagonal elements of (7) are of the type (81), and therefore we have

$$\tau_N^\nu(n) \sim (-1)^{\binom{N}{2}} \prod_{k=n+N-1}^{n+2N-2} (-1)^{\binom{k+\nu+1}{2}} p_2(k+\nu) q^{\varphi_\nu(k)+p_1(k+\nu)}, \tag{112}$$

where we have used the notation (76)–(78).

For the case (C-a), all anti-diagonal elements in (7) are of type (80). The leading term of (7) is given by

$$\begin{aligned} \tau_N^\nu(n) &\sim (-1)^{\binom{N}{2}} \prod_{k=n+N-1}^{n+2N-2} \frac{1}{2} q^{\psi(k)} \\ &= (-1)^{\binom{N}{2}} 2^{-N} q^{\frac{\nu N}{2}(2n+3N-3)+\frac{N}{4}(2n+3N-3)(2n+3N-5)+\frac{(N-1)N(N+1)}{12}} \\ &= (-1)^{\binom{N}{2}} 2^{-N} q^{\frac{N}{4}(2n+3N-3)(2n+3N+2\nu-5)+\frac{1}{2}\binom{N+1}{3}}, \end{aligned} \tag{113}$$

where $\psi$ is defined by (75).

The case (D) gives a 'mixed' situation. The anti-diagonal elements in (7) are of type (80) or (81). Using (75)–(78), the leading term can be represented as follows:

$$\tau_N^\nu(n) \sim (-1)^{\binom{N}{2}} \prod_{k=n+N-1}^{-\nu-1} (-1)^{\binom{k+\nu+1}{2}} p_2(k+\nu) q^{\varphi_\nu(k)+p_1(k+\nu)} \prod_{k=-\nu}^{n+2N-2} \frac{1}{2} q^{\psi(k)}. \tag{114}$$

### 4.3 Case (B)

In this case, $\mathrm{J}_\nu(n)$'s of types (79) and (80) appear as elements in (7). Moreover, there exist some rows (or a row) that comprise only $\mathrm{J}_\nu(n)$'s with positive arguments. The product of anti-diagonal elements is no longer dominant. We consider $M$ defined by (90). Then, the $i$th row ($M+1 \le i \le N$) comprises only $\mathrm{J}_\nu(n)$'s with positive arguments. When $n$ is even, (7) has the form

$$
(M\text{th}) \begin{vmatrix} \mathrm{J}_\nu(n) & \mathrm{J}_\nu(n+1) & \cdots & \mathrm{J}_\nu(n+N-1) \\ \vdots & \vdots & & \vdots \\ \mathrm{J}_\nu(0) & \mathrm{J}_\nu(1) & \cdots & \mathrm{J}_\nu(N-1) \\ \mathrm{J}_\nu(2) & \mathrm{J}_\nu(3) & \cdots & \mathrm{J}_\nu(N+1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{J}_\nu(n+2N-2) & \mathrm{J}_\nu(n+2N-1) & \cdots & \mathrm{J}_\nu(n+3N-3) \end{vmatrix}. \tag{115}
$$

We present the procedure for calculating its leading term. Although we illustrate the procedure only for the case in which some anti-diagonal elements are of type (80), this procedure applies, in a similar manner, for the case in which all anti-diagonal elements are of type (79) or for odd $n$. We use the notation $\{n\} := (-(1-q)^2 q^{2n}; q^2)_\infty$ for simplicity. Substituting the series expressions of $\mathrm{J}_\nu(q^n)$ and employing multi-linearity of the determinant, we obtain

$$
\begin{aligned}
\tau_N^\nu(n) =& (1-q)^{\nu N} \left\{ \prod_{k=1}^{N} q^{(n+3k-3)\nu} \right\} \sum_{j_1,\ldots,j_N} P_N^\nu(\boldsymbol{j}) \left\{ \prod_{k=M+1}^{N} q^{2(n+2k-2)j_k} \right\} \\
&\times \begin{vmatrix} \frac{q^{2j_1(j_1+\nu+n)}}{\{n\}} & \frac{q^{2j_1(j_1+\nu+n+1)}}{\{n+1\}} & \cdots & \frac{q^{2j_1(j_1+\nu+n+N-1)}}{\{n+N-1\}} \\ \vdots & \vdots & \frac{q^{2j_\kappa(j_\kappa+\nu+n+2\kappa+\lambda-3)}}{\{n+2\kappa+\lambda-3\}} & \vdots \\ \frac{q^{2j_M(j_M+\nu)}}{\{0\}} & q^{2j_M} & \cdots & q^{2(N-1)j_M} \\ 1 & q^{2j_{M+1}} & \cdots & q^{2(N-1)j_{M+1}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q^{2j_N} & \cdots & q^{2(N-1)j_N} \end{vmatrix},
\end{aligned} \tag{116}
$$

where we have used (104) and (105). Let the $(\kappa, \lambda)$-element for the determinant in (116) be of type (80). It has the form $q^{2j_\kappa(j_\kappa+\nu+n+2\kappa+\lambda-3)}/\{n+2\kappa+\lambda-3\}$ and $\nu+n+2\kappa+\lambda-3 \geq 0$ (see Sect. 2). Hence, the minimum of $j(j+\nu+n+2\kappa+\lambda-3)$ occurs at $j=0$. Note that the determinants with $j_k = j_l$ $(1 \leq k, l \leq M)$ do not disappear, unlike those in case (A). Therefore, the largest absolute value of the monomial in this determinant occurs with the $j_k$'s given by

$$
j_k = \begin{cases} 0 & (1 \leq k \leq M, k = N) \\ N-k & (M+1 \leq k \leq N-1). \end{cases} \tag{117}
$$

Although other sets of $j_k$'s actually produce the largest absolute value, we discuss them later. Noting that $(1-q)^k \sim 1$ and $P_N^\nu(\boldsymbol{j}) \sim (-1)^{\sum_{k=M+1}^{N-1}(N-k)} = (-1)^{\binom{N-M}{2}}$ under (117), our aim turns into evaluating

$$
\tau_N^\nu(n) \sim (-1)^{\binom{N-M}{2}} \left\{ \prod_{k=1}^{N} q^{(n+3k-3)\nu} \right\} \left\{ \prod_{k=M+1}^{N} q^{2(n+2k-2)(N-k)} \right\} \tilde{\tau}_N^\nu(n), \tag{118}
$$

where

$$\tilde{\tau}_N^{\nu}(n) := \begin{vmatrix} \langle n \rangle & \langle n+1 \rangle & \cdots & \langle n+N-1 \rangle \\ \vdots & \vdots & \langle n+2\kappa+\lambda-3 \rangle & \vdots \\ \langle 0 \rangle & 1 & \cdots & 1 \\ 1 & q^{2(N-M-1)} & \cdots & q^{2(N-1)(N-M-1)} \\ 1 & q^{2(N-M-2)} & \cdots & q^{2(N-1)(N-M-2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} \tag{119}$$

$$\langle n \rangle := 1/\{n\} \quad (n \le 0). \tag{120}$$

We deform $\tilde{\tau}_N^{\nu}(n)$ according to the *Procedure* presented in Appendix 1. To represent the resulting expression, we introduce additional notation. For $x_k = q^{2(N-M-k)}$, $k = 1, 2, \ldots, N-M-1$, we consider the fundamental symmetric expression $\tilde{a}_k$ among them, i.e.,

$$\tilde{a}_0 \equiv 1, \quad \tilde{a}_1 = -(x_1 + \cdots + x_{N-M-1}), \quad \tilde{a}_2 = x_1x_2 + \cdots + x_{N-M-2}x_{N-M-1},$$
$$\ldots, \quad \tilde{a}_k = (-1)^k \sum x_{j_1} \cdots x_{j_k}, \quad \ldots,$$
$$\tilde{a}_{N-M-1} = (-1)^{N-M-1} x_1x_2 \cdots x_{N-M-1}, \quad \tilde{a}_{N-M} \equiv 0.$$

Note that

$$\tilde{a}_k \sim (-1)^k q^{2+4+\cdots+2k} = (-1)^k q^{k(k+1)} \tag{121}$$

as $q \to 0$. Furthermore, we add a new variable $x_{N-M} = 1$ to $x_1, \ldots, x_{N-M-1}$ and write the fundamental symmetric expression among them as $a_k$ $(k = 1, 2, \ldots, N-M)$ and set $a_0 = 1$. Then, the relations

$$a_k = \tilde{a}_k - \tilde{a}_{k-1} \quad (k = 1, 2, \ldots, N-M) \tag{122}$$

hold. For convenience, we extend the definition of $\langle n \rangle$ in (120) to

$$\langle n \rangle = \begin{cases} 1/\{n\} & (n \le 0) \\ 1 & (n \ge 1). \end{cases} \tag{123}$$

Using this notation, we define a function

$$T(n) = \sum_{k=0}^{N-M} a_k(-1 + \langle n-k \rangle). \tag{124}$$

Then, after applying the *Procedure,* $\tilde{\tau}_N^{\nu}(n)$ is written as follows:

$$\tilde{\tau}_N^{\nu}(n) =(-1)^{M(N-M)} \left\{ \prod_{k=1}^{N-M-1} q^{2k(N-M-1-k)} \right\} \left\{ \prod_{k=1}^{N-M-1} (q^2;q^2)_k \right\} \\ \times \det\left(T(n+2\kappa+\lambda-3)\right)_{\substack{1\le\kappa\le M \\ N-M+1\le\lambda\le N}}, \tag{125}$$

where, for simplicity, we have used the trivial formula $(-1)^{2(N-M)} = 1$. Since $(q^2;q^2)_k \sim 1$, our aim reduces to calculating the leading term of $\det(T(n+2\kappa+\lambda-3))$, which, from the following lemma, is given by the product of anti-diagonal elements.

**Lemma 3** (i) $T(n)$ *is evaluated as follows:*

$$T(n) \sim \begin{cases} \frac{1}{2}q^{n(n-1)} & (n\le 0) \\ \frac{(-1)^{n-1}}{2}q^{n(n-1)} & (n\ge 1). \end{cases} \tag{126}$$

(ii) *Denote* $\psi_0(n) = n(n-1)$. *For* $n \in \mathbb{Z}$ *and* $k, l \in \mathbb{Z}_{>0}$,

$$\psi_0(n) + \psi_0(n+k+l) - \psi_0(n+k) - \psi_0(n+l) > 0 \tag{127}$$

*holds.*

*Proof* We first prove (i). For $n \le 0$, we have

$$T(n) = -\sum_{k=0}^{N-M} a_k + \sum_{k=0}^{N-M} a_k\langle n-k\rangle. \tag{128}$$

The first summation becomes zero by (122) and we find from (59) that $T \sim a_0\langle n\rangle \sim \frac{1}{2}q^{n(n-1)}$. For $n \ge 1$, employing (122), (121), and (59), we obtain

$$\begin{aligned} T(n) &= a_n(-1+\langle 0\rangle) + a_{n+1}(-1+\langle -1\rangle) + \cdots \\ &= (\tilde{a}_n - \tilde{a}_{n-1})\left(-\frac{1}{2}+o(1)\right) + \cdots \\ &\sim \frac{1}{2}\tilde{a}_{n-1} \sim \frac{(-1)^{n-1}}{2}q^{n(n-1)}. \end{aligned} \tag{129}$$

We can prove (ii) by direct calculation. □

Therefore, the leading term of $\det\left(T(n+2\kappa+\lambda-3)\right)$ is written as follows:

$$(-1)^{\binom{M}{2}} \prod_{k=n+N-1}^{\min(0,n+N+M-2)} \frac{1}{2}q^{k(k-1)} \prod_{k=\max(1,n+N-1)}^{n+N+M-2} (-1)^{k-1}\frac{1}{2}q^{k(k-1)} \tag{130}$$

and then, the leading term of $\tilde{\tau}_n^\nu(n)$ is obtained from (125). However, when $n+N+M-2 \geq 1$, that is, when $T(n)$ with $n \geq 2$ appears in the anti-diagonal elements of $\det(T(n+2\kappa+\lambda-3))$, the leading term of $\tau_n^\nu(n)$ is not obtained only from $\tilde{\tau}_n^\nu(n)$. We must consider other determinants that have the same order of leading terms, in other words, other sets of $j_k$'s. We present the following lemma:

**Lemma 4** *Suppose that an anti-diagonal element in* $\det(T(n+2\kappa+\lambda-3))$ *is* $T(\tilde{n})$ *with* $\tilde{n} \geq 2$ *and it is in the* $i$*th row* ($1 \leq i \leq M$)*. In* (116)*, consider the determinants given by the sets of arguments*

$$j_k = \begin{cases} 1, 2, \ldots, \tilde{n}-1 & (k=i) \\ 0 & (1 \leq k \leq i,\ i+1 \leq k \leq M) \\ N-k & (M+1 \leq k \leq N). \end{cases} \tag{131}$$

*The order of leading terms of these determinants is identical to that obtained from* (117).

*Proof* Since we find by observation that other sets of arguments give a larger order of leading term than that of (117), the cases that we should study are only (131). We write $j_i = m$ $(1 \leq m \leq \tilde{n}-1)$, adopt (131) for (116), and then apply the *Procedure* but replace "step 0-$i$" with "Add the $(N-m)$th row $\times\ (-q^{-2m(2-\tilde{n}-2i)})$". Then, we obtain an expression similar to (125) but with $T$'s in the $i$th row replaced by $T^{(m)}$ defined by

$$T^{(m)}(n) = \sum_{k=n}^{N-M} a_k(-q^{2m(n-k)} + \langle n-k\rangle q^{2m(m+\nu+n-k)}). \tag{132}$$

Since $q^{2m(n-k)} \gg \langle n-k\rangle q^{2m(m+\nu+n-k)}$, our interest is in evaluating $\sum -a_k q^{2m(n-k)}$. For $\tilde{n} \geq 1$, we find from (122) and (121) that

$$a_k(-q^{2m(\tilde{n}-k)}) \sim \tilde{a}_{k-1} q^{2m(\tilde{n}-k)} \sim (-1)^{k-1} q^{k(k-1)+2m(\tilde{n}-k)}. \tag{133}$$

We put $h(k) = k(k-1) + 2m(\tilde{n}-k)$ and consider $k = \tilde{n}+1, \tilde{n}+2, \ldots, N-M$. From $m \leq \tilde{n}-1$ and $k \leq \tilde{n}+1$, we readily obtain $m-k+1 < 0$ and therefore $h(k-1) - h(k) = 2(m-k+1) < 0$. This inequality means that

$$T^{(m)}(\tilde{n}) \sim \tilde{a}_{\tilde{n}-1} \sim (-1)^{\tilde{n}-1} q^{\tilde{n}(\tilde{n}-1)} \sim 2T(\tilde{n}), \tag{134}$$

which is independent of $m$ and moreover of $i$. □

If we change the original set of $j_k$'s (117) into that of (131), an extra sign $(-1)^{j_i}$ arises from $P_N^\nu(\boldsymbol{j})$. Hence, for $\tilde{n} \geq 2$, we may replace (129) with

$$T(\tilde{n}) \sim (-1)^{\tilde{n}-1}\frac{1}{2}q^{\tilde{n}(\tilde{n}-1)} + (-1)^{\tilde{n}-1}q^{\tilde{n}(\tilde{n}-1)}\sum_{j_i=1}^{\tilde{n}-1}(-1)^{j_i} = \frac{1}{2}q^{\tilde{n}(\tilde{n}-1)}. \tag{135}$$

We accordingly obtain

$$\begin{aligned}
\tau_N^\nu &\sim (-1)^{M(N-M)+\binom{M}{2}+\binom{N-M}{2}} \\
&\quad \times \prod_{k=1}^{N} q^{(n+3k-3)\nu} \prod_{k=M+1}^{N} q^{2(n+2k-2)(N-k)} \prod_{k=1}^{N-M-1} q^{2k(N-M-1-k)} \prod_{k=n+N-1}^{n+N+M-2} \frac{1}{2} q^{k(k-1)} \\
&= \frac{(-1)^{\binom{N}{2}}}{2^M} q^{\frac{\nu N}{2}(2n+3N-3)+\frac{2}{3}\binom{N-M}{2}(3n+2N+4M-4)+2\binom{N-M}{3}+\frac{1}{2}\binom{M+1}{3}+2M\binom{n+N+\frac{M-3}{2}}{2}} \\
&= \frac{(-1)^{\binom{N}{2}}}{2^M} q^{\frac{\nu N}{2}(2n+3N-3)+2\binom{N-M}{2}(n+N+M-2)+\frac{1}{2}\binom{M+1}{3}+2M\binom{n+N+\frac{M-3}{2}}{2}}. \qquad (136)
\end{aligned}$$

### 4.4 Case (C-b)

In this case, all types of $\mathrm{J}_\nu(n)$'s appear as anti-diagonal elements in (7). Although our discussion is similar to that for case (B), elements of type (81) contribute to the dominant term. The result is represented as follows:

$$\begin{aligned}
\tau_N^\nu &\sim (-1)^{\binom{N}{2}} q^{\frac{\nu N}{2}(2n+3N-3)+2\binom{N-M}{2}(n+N+M-2)} \prod_{k=-\nu}^{n+N+M-2} \frac{1}{2} q^{k(k-1)} \\
&\quad \times \prod_{k=n+N-1}^{-\nu-1} \left\{(-1)^{\binom{k+\nu+1}{2}} p_2(k+\nu) q^{\varphi_\nu(k)+p_1(k+\nu)-\nu k}\right\}. \qquad (137)
\end{aligned}$$

## 5 Behaviour of the special solution

The Painlevé III equation ($\mathrm{P_{III}}$) is written as follows:

$$\frac{d^2u}{dx^2} = \frac{1}{u}\left(\frac{du}{dx}\right)^2 - \frac{1}{x}\frac{du}{dx} + \frac{1}{x}(au^2+d) + cu^3 + \frac{b}{u}, \qquad (138)$$

where $a, b, c, d$ are parameters. Equation (138) can be derived from (3) through the following continuous limit. That is, (3) reduces to (138) if we introduce a parameter $\epsilon < 0$ and put $u(n) = \lambda^{-n/2} w(n)$, $n\epsilon = z$, $\alpha = -1/c\epsilon^2$, $\beta = -d/c$, $\gamma = -b/c$, $\delta = a/c$, and $\lambda = 1+2\epsilon$; then we take the limit $\epsilon \to -0$ and transform the independent variable by $z = e^x$. For

$$a = 2(N-\nu), b = -1, c = 1, d = 2(\nu+N+1) \quad (N \in \mathbb{Z}_{\geq 0}), \qquad (139)$$

(138) admits special solutions written in terms of a solution of the Bessel equation

$$\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)v = 0. \qquad (140)$$

That is, the function $u_N^\nu(x)$ defined by

$$u_N^\nu(x) = \frac{d}{dx}\left(\log \frac{\tau_N^{\nu+1}}{\tau_{N+1}^\nu}\right) + \frac{\nu+N}{x} \tag{141}$$

$$\tau_N^\nu = \left| D^{i+j-2} v_\nu(x) \right|_{1\le i,j\le N} \tag{142}$$

$$D = x\frac{d}{dx} \tag{143}$$

solves (138) with (139), where $v_\nu(x)$ is a solution of (140).

We consider (3) with (4) and construct its p-ultradiscrete analogue by introducing $w_N^\nu(n) = \omega_n e^{W_n/\varepsilon}$ and $q = e^{Q/\varepsilon}$. Applying the procedure of p-ultradiscretization, we obtain (the special case of) the p-ultradiscrete Painlevé III equation (p-uP$_{\mathrm{III}}$)

$$\begin{aligned}
\max\big[&S(\omega_{n+1}\omega_n\omega_{n-1}) + W_{n+1} + W_n + W_{n-1} + (\nu+N)Q,\\
&S(-\omega_{n+1}\omega_n\omega_{n-1}) + W_{n+1} + W_n + W_{n-1} + (-\nu+3N)Q,\\
&S(\omega_{n+1}\omega_{n-1}) + W_{n+1} + 2W_n + W_{n-1},\\
&S(-\omega_{n+1}\omega_{n-1}) + W_{n+1} + W_{n-1} + 4NQ,\\
&S(\omega_n) + W_n + (7N-\nu-2+2n)Q,\\
&S(-\omega_n) + W_n + (9N+\nu+2n)Q,\ 2W_n + 4NQ\big]\\
= \max\big[&S(-\omega_{n+1}\omega_n\omega_{n-1}) + W_{n+1} + W_n + W_{n-1} + (\nu+N)Q,\\
&S(\omega_{n+1}\omega_n\omega_{n-1}) + W_{n+1} + W_n + W_{n-1} + (-\nu+3N)Q,\\
&S(-\omega_{n+1}\omega_{n-1}) + W_{n+1} + 2W_n + W_{n-1},\\
&S(\omega_{n+1}\omega_{n-1}) + W_{n+1} + W_{n-1} + 4NQ,\\
&S(-\omega_n) + W_n + (7N-\nu-2+2n)Q,\\
&S(\omega_n) + W_n + (9N+\nu+2n)Q,\ 2(6N-1+2n)Q\big].
\end{aligned} \tag{144}$$

Moreover, the p-ultradiscrete analogue of (8) is written as follows:

$$\mathbf{Y} := Y_{N+1,n+1}^\nu + Y_{N,n}^{\nu+1} - Y_{N+1,n}^\nu - Y_{N,n+1}^{\nu+1} \tag{145}$$

$$\mathbf{y} := y_{N+1,n+1}^\nu y_{N,n}^{\nu+1} y_{N+1,n}^\nu y_{N,n+1}^{\nu+1} \tag{146}$$

$$\max[S(\omega_n) + W_n, S(-\mathbf{y}) + \mathbf{Y}, (\nu+N)Q] = \max[S(-\omega_n) + W_n, S(\mathbf{y}) + \mathbf{Y}]. \tag{147}$$

We regard this relation as the equation for $\omega_n$ and $W_n$ and solve this equation. Then for almost all cases, we obtain the unique transformation

$$W_n = \max[\mathbf{Y}, (\nu+N)Q] \tag{148}$$

$$\omega_n = \begin{cases} \mathbf{y} & (\mathbf{Y} > (\nu+N)Q) \\ -1 & (\mathbf{Y} < (\nu+N)Q \text{ or, } \mathbf{Y} = (\nu+N)Q \text{ and } \mathbf{y} = -1) \end{cases} \tag{149}$$

but for the special case in which $\mathbf{Y} = (\nu+N)Q$ and $\mathbf{y} = 1$, we obtain the indeterminate transformation

$$W_n \leq (\nu + N)Q, \quad \omega_n = \pm 1. \tag{150}$$

Employing this transformation, we construct a special solution to (144) from $(y^{\nu}_{N,n}, Y^{\nu}_{N,n})$ obtained in Section 4. Here, we study the case in which $N = 1$ and $\nu = 1$. For $n \leq -3$, we uniquely obtain

$$(\omega_n, W_n) = \begin{cases} (1, 0) & (n = -3) \\ (1, (n+1)Q) & (n < -3, n \equiv 0 \mod 2) \\ (-1, (n+4)Q) & (n < -3, n \equiv 1 \mod 2). \end{cases} \tag{151}$$

For $n \geq -2$, we encounter the indeterminate transformation

$$W_n \leq 2Q, \quad \omega_n = \pm 1. \tag{152}$$

Therefore, we substitute $(\omega_{-3}, W_{-3}) = (1, 0)$ and $(\omega_{-4}, W_{-4}) = (1, -3Q)$ into (144) with $n = -3$ and solve the resulting equation for $(\omega_{-2}, W_{-2})$. Then, we have an indeterminate solution

$$W_{-2} \leq Q, \quad \omega_{-2} = \pm 1. \tag{153}$$

Let us choose $(\omega_{-2}, W_{-2}) = (1, KQ)$ with $K > 2$, which satisfies both of (152) and (153). On substituting $(\omega_{-2}, W_{-2}) = (1, KQ)$ and $(\omega_{-3}, W_{-3}) = (1, 0)$ into (144) with $n = -2$ and solving it, we obtain the unique solution $(\omega_{-1}, W_{-1}) = (-1, -2Q)$, which does not satisfy (152). Another choice $(\omega_{-2}, W_{-2}) = (-1, KQ)$ with $K > 2$ gives the same result. Hence, we consider $(\omega_{-2}, W_{-2}) = (1, 2Q)$ and $(-1, 2Q)$. In both cases, we have the indeterminate solution

$$\omega_{-1} = \pm 1, \quad W_{-1} \leq -2Q. \tag{154}$$

If we choose $W_{-1} \leq 2Q$, the solution is consistent with (152). Hence, $(\omega_{-2}, W_{-2})$ should be $(1, 2Q)$ or $(-1, 2Q)$ in order to obtain $(\omega_{-1}, W_{-1})$ to be compatible with (152). Next, we substitute $(\omega_{-2}, W_{-2}) = (\pm 1, 2Q)$ and $(\omega_{-1}, W_{-1}) = (\pm 1, KQ)$ with $K \geq 2$ into (144) with $n = -1$ and search $(\omega_0, W_0)$ which solves the equation and satisfies (152). Though we omit details, we find that $\omega_{-2}$ is arbitrary but $(\omega_{-1}, W_{-1}) = (1, 4Q)$ is required to obtain such $(\omega_0, W_0)$. Continuing these experiments, we guess a special solution

$$(\omega_n, W_n) = (1, 2(n+3)Q) \quad (n \geq -2). \tag{155}$$

We can check that this function satisfies (144) by direct substitution. Now, we have obtained a special solution given by (151) and (155). From this example, we find that (152) does not give a unique solution to (144), but it does give important information about the solution, which we want to consider.

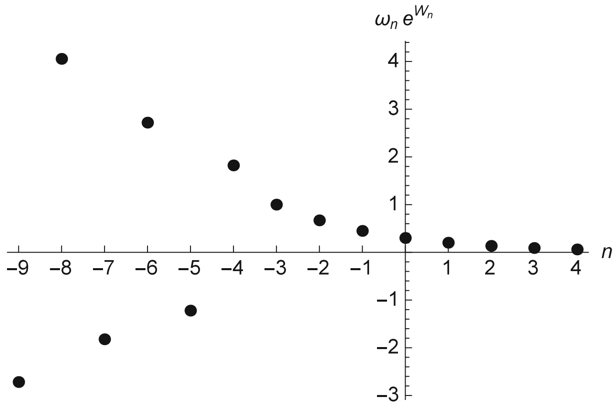


**Fig. 1** Special solution of p-uP$_{\text{III}}$ with $N = 1$, $\nu = 1$ and $Q = -1/5$

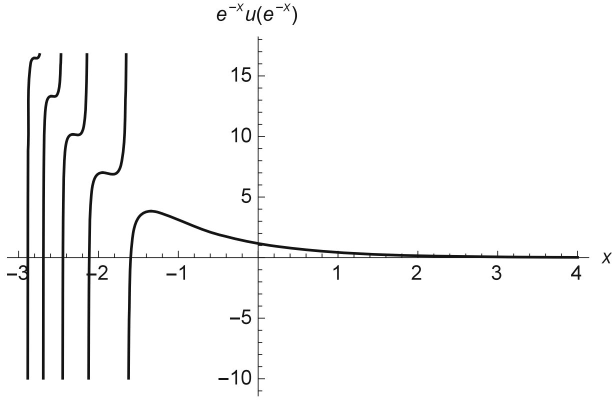


**Fig. 2** Special solution of P$_{\text{III}}$ with $N = 1$ and $\nu = 1$

We study the behaviour of this ultradiscrete solution by introducing the variable $\omega_n e^{W_n}$. This variable shows qualitative behavior similar to that of $w(n)$ since it is obtained from $w_N^\nu(n) = \omega_n e^{W_n/\varepsilon}$ with $\varepsilon = 1$. Moreover, it corresponds to the continuous variable $e^{-x}u_N^\nu(e^{-x})$, where $u_N^\nu(x)$ is defined by (141). We find this correspondence from variable transformations in the continuous limit. Figure 1 shows a plot of $\omega_n e^{W_n}$. Its behaviour is similar to that shown in Fig. 2 for $e^{-x}u_N^\nu(e^{-x})$. That is, both solutions tend to zero as $n$ or $x \to \infty$, and both show oscillating behaviour for negative values of the independent variables. Since only discrete points are considered in the ultradiscrete equation and scaling by $\varepsilon$ is ignored, the divergence in Fig. 2 is unclear in Fig. 1.

## 6 Concluding remarks

We have obtained the p-ultradiscrete Bessel function by evaluating the $q$-Bessel function $J_\nu(q^n)$. Although the function $J_\nu(x)$ may be studied in the context of $q$-analysis, the evaluation has become non-trivial because of the special independent variable $x = q^n$. The key to the evaluation for $n \leq 0$ is to deform the known series expression for $J_\nu(q^n)$ by the $q$-difference Euler transformation.

Based on the evaluation for $J_\nu(q^n)$, we have derived the explicit functional forms of special solutions for the p-ultradiscrete analogues of the bilinear equations for $\mathrm{dP_{III}}$, for general values of system parameters $N \in \mathbb{Z}_{\geq 1}$ and $\nu \in \mathbb{Z}_{\geq 0}$. These special solutions are obtained by evaluating those of $\mathrm{dP_{III}}$ represented by the Casorati determinants whose elements are given by the $q$-Bessel function. The evaluation for determinantal solutions of the $q$-Painlevé II equation was discussed in [12]. Compared with its calculation, we have needed a new technique to evaluate the determinant for $\mathrm{dP_{III}}$ as shown in Sects. 4.3 and 4.4.

We have also constructed a special solution of p-$\mathrm{uP_{III}}$ through the variable transformation. We have found that the behaviour of the obtained ultradiscrete solution is qualitatively similar to the corresponding continuous solution. Although the transformation locally takes the indeterminate form (150), it enables us to find a special solution in a fairly specified form by considering its global behaviour. Such application of indeterminate form has newly observed in this research.

It is a future problem to give ultradiscrete special solutions corresponding to the $q$-Neumann function and to the general solution as studied in [8]. Another future problem is to construct ultradiscrete analogues of other Painlevé equations and to investigate their mathematical structures, such as degeneration structures or Bäcklund transformations. We hope that the results in this paper will contribute to future analysis of discrete Painlevé equations.

**Acknowledgements** This work was supported by JSPS KAKENHI Grant Number 26790082. The author is grateful to anonymous reviewers for many fruitful comments.



## Appendix 1

The p-ultradiscrete Bessel function $(\beta_n^\nu, B_n^\nu)$ for $\nu > 0$ in Mr. Narasaki's Master thesis (Aoyama Gakuin University (2011), in Japanese) is written as

$$
\begin{aligned}
\beta_n^\nu &= \begin{cases} \beta_0^\nu & (n \geq -\nu) \\ (-1)^{\frac{n+\nu}{2}} \beta_0^\nu & (n \leq -\nu - 1,\ n+\nu : \text{even}) \\ (-1)^{\frac{n+\nu-1}{2}} \beta_0^\nu & (n \leq -\nu - 1,\ n+\nu : \text{odd}) \end{cases} \\
B_n^\nu &= \begin{cases} B_0^\nu + n\nu Q & (n \geq 0) \\ B_0^\nu + n(n+\nu-1)Q & (-1 \geq n \geq -\nu) \\ B_0^\nu + \frac{n(n-2)-\nu^2}{2} Q & (n \leq -\nu - 1,\ n+\nu : \text{even}) \\ B_0^\nu + \frac{n(n-4)-(\nu+3)(\nu-1)}{2} Q & (n \leq -\nu - 1,\ n+\nu : \text{odd}) \end{cases}
\end{aligned}
\tag{156}
$$

and that for $\nu = 0$ is

$$\begin{aligned}\beta_n^0 &= \begin{cases}\beta_0^0 & (n \geq 1)\\ (-1)^{\frac{n}{2}}\beta_0^0 & (n \leq -1 : \text{even})\\ (-1)^{\frac{n-1}{2}}\beta_0^0 & (n \leq -1 : \text{odd})\end{cases}\\ B_n^0 &= \begin{cases}B_0^0 & (n \geq 1)\\ B_0^0 + \frac{n(n-2)}{2}Q & (n \leq -1 : \text{even})\\ B_0^0 + \frac{(n-1)(n-3)}{2}Q & (n \leq -1 : \text{odd}).\end{cases}\end{aligned} \tag{157}$$

This solution was constructed as a solution to the specific initial value problem for (68).

## Appendix 2

We present the procedure used to deform $\tilde{\tau}_N^{\nu}(n)$ in Sect. 4.3.
*Procedure*:

step 0-1 : Add the $N$th row $\times$ $(-1)$ to the first row.
step 0-2 : Add the $N$th row $\times$ $(-1)$ to the second row.

$\vdots$

step 0-$M$ : Add the $N$th row $\times$ $(-1)$ to the $M$th row.
step 1-1 : Add the $(N-1)$th column $\times$ $-q^{2(N-M-1)}$ to the $N$th column.
step 1-2 : Add the $(N-2)$th column $\times$ $-q^{2(N-M-1)}$ to the $(N-1)$th column.

$\vdots$

step 1-$(N-1)$ : Add the first column $\times$ $-q^{2(N-M-1)}$ to the second column.
step 1-Ex. : Expand the determinant with the $(M+1)$th row. We obtain a determinant of size $(N-1)\times(N-1)$.
step 2-1 : For the resulting determinant, add the $(N-2)$th column $\times$ $-q^{2(N-M-2)}$ to the $(N-1)$th column.
step 2-2 : Add the $(N-3)$th column $\times$ $-q^{2(N-M-2)}$ to the $(N-2)$th column.

$\vdots$

step 2-$(N-2)$ : Add the first column $\times$ $-q^{2(N-M-2)}$ to the second column.
step 2-Ex. : Expand the determinant with the $(M+1)$th row. We obtain a minor of size $(N-2)\times(N-2)$.

$\vdots$

step $(N-M)$-1 : For the resulting determinant of size $(M+1)\times(M+1)$, add the $M$th column $\times$ $(-1)$ to the $(M+1)$th column.
step $(N-M)$-2 : Add the $(M-1)$th column $\times$ $(-1)$ to the $M$th column.

$\vdots$

step $(N-M)$-$M$ : Add the first column $\times$ $(-1)$ to the second column.
step $(N-M)$-Ex. : Expand the determinant with the $(M+1)$th row. We obtain a minor of size $M\times M$.

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