Vortex Merger in Shallow Water Model

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Abstract

An inviscid, nonlinear shallow water model is applied to study the formation of filaments and merging of potential vorticity (PV, Π). We confirm that misalignment of vorticity with the streamlines is crucial to merger. Flow creates positive PV tendency (Π_t) on the lee side of Π ridge, which increases the angle between streamlines and PV contours and the efficiency of vorticity transport along the parallel but opposite direction in the positive quadrants of Π_t . So, the vortices move closer while rotating around each other. Filamentation starts as weak vorticity shatters from the outer edge of vortex core. The filament grows and rotates around the cores, but shows little effects to positive PV advection or vortex merging in the inner core, evidenced by the steady core area integrated PV trends. Consecutive PV transfer from lower to higher interval levels are observed during merger process while elongated filamentation is dissolving into the lower level PV regime. The vortices never merge when negative Π_t prevails between two vortices. Distance between the vortex cores show direct correspondence to positive Π_t between them. Hence, the advance of positive Π_t provides a simple mechanism of merging. The Rossby radius of deformation (L_D) is confirmed to be another strong indicator for vortex pair merger where L_D comparable or smaller than the initial vortex core separation length scale makes merger more likely due to geostrophic adjustment. The Rossby number (Ro) affected the overall flow interaction speed when L_D is fixed.

Keywords Vortex · Potential vorticity · Merger · Filament · Inviscid · Advection

1 Introduction

Typhoons, hurricanes, cyclones, and anticyclones are prominent vortices in nature. Vortices also frequently develop on the lee of mountains and islands. They grow, merge, then shed away from the source regions (Sun et al. 1991; Sun and Chern 1993; 1994; Haines et al. 1997; Oh 2003; 2007). Vortex merger and sheets are crucial to scale interactions, energy cascade, and dissipation in geofluid

dynamics (Kolmogorov et al. 1991a; 1991b; Hopfinger and van Heijst 1993; McWilliams 2006). Implications to real life is enormous as pairs of cyclones often exhibit Fujiwhara effect making storm track prediction more difficult. In case of merger, tropical cyclones intensify which can potentially cause devastating damage in the affected region. Thus, understanding the underlying dynamics of the vortex pair interaction has drawn attention from researchers. Many scientists (McWilliams 1984; Melander et al. 1987; Polvani et al. 1989; Cerretelli and Williamson 2003; Brandt and Nomura 2006; 2007) have applied 2D, Quasi-Geostrophic models, or laboratory experiments to study the phenomena. They observed that merging consists of three phases: the first viscous/diffusive phase, convective phase, and the second diffusive stage. Waugh (1992) studied the efficiency of merging based on the radius of vortices and the distance between them. Melander et al. (1987) proposed axisymmetrization principle that long spiral vorticity filaments occur from the outer region of the vortex core (low vorticity contour region) by getting advected away from the vortex core. The formation of asymmetric filaments carries away vorticity and breaks the

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elliptical symmetry of the vortex core. Hence, the elliptical-shaped vorticity is oriented at some angle with respect to the approximately elliptic streamlines. Kimura and Herring (2001), Huang (2005), Brandt and Nomura (2006), and Brandt and Nomura (2007) proposed that the tilting effect occurs when filamentation (generation of filamentary structures of vorticity during the merger process) initiates, and the positive production of palinstrophy increases the vorticity gradient and misalignment of vorticity with the streamlines. This causes vorticity to enter the outer region and filamentation to occur. The associated vorticity acts to advect the vorticities towards each other (Brandt and Nomura 2007). However, Velasco Fuentes (2001) and Huang (2005) found that merger is possible even if the vorticity distribution does not extend to regions outside the exchange band defined by Melander et al. (1988), and concluded that the filamentation is not the cause of merger but one of its effects (Huang 2005). Le Dizès and Laporte (2002) and Meunier et al. (2005) reported that short-wave instability arising in the initial co-rotating vortex flow may play an important role in merging. Meunier et al. (2005) also suggested that vorticity arms could be ejected far from the vortices which lead to vortex merging, although they acknowledged there is no experiment or numerical evidence to confirm it.

Indeed, the misalignment of the major axes of the vortex and the connecting line between the vortex centers is crucial to merger, but it is unclear whether the filaments are ejected from the vortex core, or they really carry vorticity away and cause vortices to merge. Hence, we revisit vortex merging in a constant rotating frame (i.e., using constant Coriolis parameter f) by using an inviscid, nonlinear shallow water model (Sun 2011) in which the potential vorticity (PV) is conserved. Without diffusion, the system retains the fundamental advective phase of merger and filament formation as in the 2D models. We find that the advection of potential vorticity provides a simple but eloquent explanation of vortex merger and filament formation. The simulated filaments move around and the vortex core, but they are not ejected from the core or carry away vorticity directly from the inner core as proposed previously. Long spiral filament formation is not necessary for merger either. Meanwhile, merging may not create long filaments in the outer regions. We also confirm that where the Rossby radius of deformation (L_D) is comparable or smaller than the initial vortex core separation length scale, vortex merger is more likely to occur due to geostrophic adjustment (Rossby 1938; Gill 1982; Sun 2007). The Rossby number (Ro) affected the overall flow interaction speed when L_D is fixed. The circulations near vortex centers remain near circular and rotate around the individual center with little interactions.

2 Governing Equations, Numerical Method, and Experimental Design

2.1 Governing Equations and Numerical Method

The shallow water equations are

$$\frac{\partial h}{\partial t} = -\frac{\partial uh}{\partial x} - \frac{\partial vh}{\partial y} \quad (1a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -g \frac{\partial h}{\partial x} + \mu \nabla^2 u \quad (1b)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -g \frac{\partial h}{\partial y} + \mu \nabla^2 v \quad (1c)$$

where h is the water depth, u and v are x and y component velocities, μ is viscosity. They can be combined into the potential vorticity (Π) equation:

$$\frac{d}{dt} \ln(\Pi) = \frac{d}{dt} \ln \left[\frac{(\zeta + f)}{h} \right] = \frac{\mu \nabla^2 \zeta}{(\zeta + f)} \quad (2)$$

where relative vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = v_x - u_y$ and $PV \equiv \Pi = \frac{(\zeta + f)}{h}$ is conserved in an inviscid fluid, i.e.,

$$\frac{d\Pi}{dt} = \frac{d}{dt} \left[\frac{(\zeta + f)}{h} \right] = 0 \quad (3a)$$

or

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{(\zeta + f)}{h} \right] &= -\mathbf{v} \cdot \nabla \left[\frac{(\zeta + f)}{h} \right] \\ &= -u \frac{\partial}{\partial x} \left[\frac{(\zeta + f)}{h} \right] - v \frac{\partial}{\partial y} \left[\frac{(\zeta + f)}{h} \right]. \end{aligned} \quad (3b)$$

If f and h are constants, Eq. (2) becomes:

$$\frac{d}{dt} \zeta = \mu \nabla^2 \zeta \quad (4a)$$

and

$$\begin{aligned} \frac{d}{dt} \left(\left(\frac{\partial \zeta}{\partial x} \right)^2 + \left(\frac{\partial \zeta}{\partial y} \right)^2 \right) &= [-2\zeta_x \zeta_y u_x - 2\zeta_x \zeta_y (u_y + v_x) - 2\zeta_y \zeta_y v_y] \\ &\quad + 2\mu [\zeta_x (\nabla^2 \zeta)_x + \zeta_y (\nabla^2 \zeta)_y]. \end{aligned} \quad (4b)$$

The terms inside the first bracket of Eq. (4b), the production of palinstrophy (the squared vorticity gradient), was derived by Kimura and Herring (2001) for a 2D flow. Our pattern of Π is similar to ζ in 2D. Scientists usually integrate ζ in Eq. (4a) or the Quasi-Geostrophic-Potential Vorticity in a Quasi-Geostrophic model (von Hardenberg et al. 2000), then solve the Poisson's equation for velocity and pressure. In this study, we integrate the inviscid, flux form equations:

$$\frac{\partial h}{\partial t} = -\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \quad (5a)$$

Fig. 1 Parameter space of test case experiments in terms of Ro and L_D (both in log scale). Red (blue) dot indicates merger (non-merger) case

$$\frac{\partial \psi}{\partial t} + \frac{\partial(u\psi)}{\partial x} + \frac{\partial(v\psi)}{\partial y} - f\phi = -\frac{g}{2} \frac{\partial h^2}{\partial x} \quad (5b)$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial(u\phi)}{\partial x} + \frac{\partial(v\phi)}{\partial y} + f\psi = -\frac{g}{2} \frac{\partial h^2}{\partial y} \quad (5c)$$

where $\psi = hu$ and $\phi = hv$. Eqs. (5a–5c) are solved by the finite volume scheme (Sun 2011), which is applied to simulate the shock wave in dam break, the soliton Rossby wave (Sun and Sun 2013), and vortex moving over complex terrain in Taiwan (Sun 2016), etc.

2.2 Experimental Design

The domain consists of 450×450 grids, with $dx = dy = 0.025 m$. Following Waugh (1992) and Oh (2007), we set $g = 1 m s^{-2}$, and the initial surface perturbation is

$$h' = h_0 / \left[\left(\frac{x - x_0}{r_0} \right)^2 + \left(\frac{y - y_0}{r_0} \right)^2 + 1 \right]^{1.5} \quad (6)$$

where $y_0 = 0$, $x_0 = \pm 0.5 \times d_0$ ($\pm 40 dx$), and $r_0 = 1.2 m$ ($48 dx$). The initial momentum field is set by assuming geostrophic balance. Open boundary condition is applied at the lateral boundary where disturbances and waves can freely pass through (Sun 2018) to avoid disturbances coming back from the lateral boundary with periodic boundary condition. We designed the model domain to be sufficiently large enough so that vortex core interactions are barely affected by the boundary condition with the exception of case E52 and E53 as vortex cores diverge from each other for these cases. No viscosity is applied as shown in Eq. (5) and no form of any filter is used for simulating the experiments.

The solutions are influenced by the Rossby number $Ro = U/fL$, the Rossby radius of Deformation $L_D = \sqrt{gH}/f$, initial distance of the vortex centers d_0 , width of the vortex r_0 , and the amplitude of surface perturbation h_0 , where U is velocity scale, L is horizontal length scale, and the undisturbed depth H is depth scale. For simplicity, we set

Table 1 Setting for each cases at $t = 0$

Case (Unit)	H m	f s^{-1}	d_0 m	h_0 m	$\max(U)^*$ m/s	Ro —	L_D m	Time** s
A	1.0	1.0	2	-0.2	0.16	0.16	1.0	130
B	0.6	2.0	2	-0.2	0.08	0.04	0.4	160
C	0.6	2.0	2	-0.4	0.16	0.08	0.4	160
E11	1.0	0.5	2	-0.025	0.04	0.08	2.0	320
E12	2.0	0.5	2	-0.025	0.04	0.08	2.8	400
E13	4.0	0.5	2	-0.025	0.04	0.08	4.0	320
E21	1.0	0.5	2	-0.05	0.08	0.16	2.0	160
E22	2.0	0.5	2	-0.05	0.08	0.16	2.8	130
E23	1.0	0.25	2	-0.0125	0.04	0.16	4.0	320
E31	1.0	0.5	2	-0.1	0.16	0.32	2.0	130
E32	2.0	0.5	2	-0.1	0.16	0.32	2.8	130
E33	4.0	0.5	2	-0.1	0.16	0.32	4.0	130
E41	1.0	0.5	2	-0.2	0.32	0.64	2.0	130
E42	2.0	0.5	2	-0.2	0.32	0.64	2.8	130
E43	4.0	0.5	2	-0.2	0.32	0.64	4.0	80
E51	0.25	0.25	2	-0.1	0.32	1.28	2.0	35
E52	2.0	0.5	2	-0.4	0.64	1.28	2.8	100
E53	4.0	0.5	2	-0.4	0.64	1.28	4.0	50

* As the initial momentum field is assuming geostrophic balance, $\max(U)$ is determined by the initial profile of h_0 and f . (**) Total integration time

Fig. 2 PV (thick black), h (green), streamline (thin black), and speed (shaded color) for case A at $t = 0$ s (left) and at $t = 20$ s (right)

U as the maximum speed, H as the undisturbed height, and $L = d_0/2$ at $t = 0$ s.

We conducted 18 test case experiments (case A, B, C, and E[1..5][1..3], see Fig. 1) to study various types of vortex merger situations by observing how different Ro and L_D value affect vortex pair merger process where parameters used for each case are described in Table 1. Case A setting use the same setting used in Oh (2007) for comparison which shows a classical case for vortex pair merger. Settings for merging case (very small $L_D = 0.4m$) with different Ro is compared in case B and C. Case E32 serves as a base configuration as it is neutral as its vortex pair neither merges nor separates. For test cases with E[I][J] notation, I index indicates the Ro level (0.08, 0.16, 0.32, 0.64 and 1.28) while J index indicates the L_D level (2, 2.8, and 4 m) where larger indices correspond to higher values. Thus, for example, comparison of case E31, E32 and E33 shows how sensitive vortex pair respond with respect to L_D . Similarly, comparison of case E22, E32 and E42 shows Ro sensitivity. Comparison and analyses are mainly done for select cases (A, B, C, E[22, 31, 32, 33, 42 53]) as many flow features overlap for the rest of the tested cases.

3 Numerical Results and Discussion

3.1 Case A: Classical Vortex Merger Case

Figure 2 shows the initial PV, surface height h , streamlines, and speed. The strong wind zones (red zone) transports fluid downstream at $t = 20$ s, which keeps the same initial vorticity and rotates slowly, but the velocity is larger at point 1 than point 2 (Fig. 2, right). Hence, the development of the filaments is due to differential velocity across the PV contours. The streamlines also show that the vorticity

of filaments are not ejected from the vortex cores as proposed by Kimura and Herring (2001) and Cerretelli and Williamson (2003), and Melander et al. (1987).

Kimura and Herring (2001) found a quadrupole structure of palinstrophy production in vortex merging. The first and third quadrants have negative value of production, while the second and fourth quadrants have positive values. As the ellipse rotates anti-clockwise, the vorticity contour lines inside get closer in the positive quadrants. At the same time the regions are stretched and begin to eject vortex filaments, while the negative quadrants are pushed concentrically and contracted. They also found a significant positive production at the outer edge of the spirals, which carry the vorticity from the inner core, as discussed in Melander et al. (1987).

The asymmetric structure of quadrants does exist in our advection of PV at $t = 20$ s in Fig. 3 and other merging cases. Advection generates positive Π_t on the lee side of the PV ridge, where PV increases, and contours become more convex. The angle between PV contours and streamlines increases in this region. Hence, in addition to the anti-clockwise rotation, positive PV can be effectively transported along two parallel but opposite paths, from point 1 to point 2 and from point 3 to point 4 shown in in Fig. 5 (case A, $t = 20$ s). Hence, the vortices move closer while rotating around each other as shown by the trajectories of the vortex cores in Fig. 6.

The major axis of individual vortex is different from the line of vortex centers. The mechanism is different from the axisymmetry due to filaments generation (Melander et al. 1987; Melander et al. 1988). On the windward side of PV ridge, advection creates $\Pi_t < 0$, and PV contours are pushed toward the vortex center and become concave. The contraction of PV lengthens the filaments in the negative quadrants of Π_t . Positive Π_t advances along parallel but opposite direction, which

Fig. 3 PV (thick black), h (green), streamline (thin black), and Π_t (shaded color) for case A, B, C and E22

is more directly related to merging, and is a simpler predictor than the axisymmetrization principle, the angle between vortex major axis and vortex-centers-line (Huang 2005), interactions between vorticity and rate of strain

(Moore et al. 1975; Brandt and Nomura 2007), or production of palinstrophy (Cerretelli and Williamson 2003; Kimura and Herring 2001). The corresponding dynamics between the distance of vortex cores and Π_t value at their

Fig. 4 PV (thick black), h (green), streamline (thin black), and Π , (shaded color) for case E31, E32, E33, E42 and E53

Fig. 5 Enlarged plot of Fig. 3 for Case A at $t = 20$ s with point 1–4 annotation

midpoint location is shown in Fig. 8. For case A, we observe that vortex cores respond with a slight lag and starts to draw towards each other with sustained large Π_t value.

The contours of h and PV at $t = 60$ s in Fig. 3 are in good agreement with the characteristic based Semi-Lagrangian simulations (Oh 2007). A stronger flow cuts deeply near the base of filaments and make the filaments look longer. Wind blows across filaments and form positive Π_t on the lee side and a negative Π_t on the windward side. The maximum speed, 0.18 ms^{-1} , occurs along the inner core boundary. The velocity at the vortex centers, 0.09 ms^{-1} , is identical to the angular velocity, $\omega = v/r = 0.09 \text{ ms}^{-1}/0.36 \text{ m} = 0.25 \text{ s}^{-1} \approx 14^\circ \text{ s}^{-1}$. They rotate around the middle of two vortices. As time increases, the filaments stretch longer and wrap around the shrinking inner core at $t = 100$ s (Fig. 3, case A, $t = 100$ s). The shape of inner core changes from elliptic to circular and the change slows down with time. The vorticities are still distinguishable at the end of integration, $t = 130$ s (Fig. 6).

From Fig. 6 we find the angular velocity of vortices around $t = 60$ s is approximately 14° s^{-1} , consistent with the velocity at vortex centers, 0.09 ms^{-1} . The angular velocity slightly increases with time as two vortices distance decreases. Moore et al. (1975) and Tsai and Widnall (1976), and Meunier et al. (2005) proposed that Kelvin mode and shear instability may cause the formation of filaments and vortex merge. In a linearized axisymmetric flow, the speed of Kelvin wave is $U(r) + \sqrt{gh(r)}$, where $U(r)$ and $h(r)$ are the velocity and height at distance r from the center of rotation. Our simulated vortices advect with $U(r)$ along h -contours, which is like the observed cold front advected southward along the eastern coasts of Taiwan and China

with the observed wind U instead of $U + \sqrt{gh}$ (Sun and Chern 2006). We do not see any instability, because shear is not strong in our simulations. Waugh (1992) also found very little sign of instability during the merger process in his 2D simulation. It is noted that the vortex system changes with time before the final convective stage, which is different from a steady, axisymmetric background used in most stability analyses. Instability can also be induced by adding diffusivity in the equations (Sun 2010).

Here, we define domain summed potential vorticity $\text{sPV}(t) = \sum_{\Omega} \Pi(t)$, mass $\text{sM}(t) = \sum_{\Omega} h(t)$, kinetic energy $\text{sKE}(t) = 0.5 \sum_{\Omega} h(u^2 + v^2)$, potential energy $\text{sPE}(t) = 0.5 \sum_{\Omega} gh^2$, and total energy $\text{sTE}(t) = \text{sKE}(t) + \text{sPE}(t)$, where summation notation indicates that each corresponding variables are summed throughout the whole model domain gridpoint-wise. Figure 9 shows $\text{sM}(t)/\text{sM}(t=0)$ in green, $\text{sPV}(t)/\text{sPV}(t=0)$ in orange, $\text{sPE}(t)/\text{sPE}(t=0)$ in light blue, and $\text{sTE}(t)/\text{sTE}(t=0)$ in red for case A, B, C, E[22, 31, 32, 33, 42, 53]. Normalized total mass, energy, and potential vorticity show good conservation property (with the exception of case E33 and E53), even though open boundary condition is applied at the lateral boundary. Furthermore, near identical results of the the vortex merger case of Oh (2007) and this study's case A results at $t = 20, 60, 100$ s (Fig. 3) show that boundary condition has negligible effects as major circulation occurs far away from the lateral boundary. For case E53, vortex cores diverge from each other and approach model boundary with time which will subsequently affect the conservation property as open boundary condition is applied (See Section 3.6).

3.2 Case E32: Neutral Case

The initial maximum speed 0.16 ms^{-1} is the same as previous case, but $Ro = 0.32$ and $L_D = 2.8 \text{ m}$ are larger than case A where $Ro = 0.16$ and $L_D = 1 \text{ m}$. According to geostrophic adjustment (Gill 1982; Sun 2007), the pressure will adjust to wind field for length scale less than L_D . Hence, circulation rotating around its vortex center remains close to the initial wind field compared with case A. Figure 4 (case E32, $t = 60$ s) shows that flows rotate around the individual vortex center with weak interactions at $t = 60$ s. The sign of Π_t oscillates with time between two vortices (Fig. 8). Hence, the vortices just rotate around each other, remain separate but do not move away, as shown at $t = 100$ s in Fig. 4 and in Fig. 7.

3.3 Case E31 and E33: L_D Sensitivity

In case E31 (E33), the undisturbed water height is $H = 1 \text{ m}$ (4 m) while $H = 2 \text{ m}$ for case E32. This consequently changes L_D to 2 m (4 m) while $Ro = 0.32$ remains the same as case E32. For both case E31 and E33, the PV

Fig. 6 Trajectories of the vortex cores for case A, B, C, E11 – E23 where orange (blue) line is the trajectory of the left (right) initial vortex core. Circle and triangle marker denote start and ending point. Square markers denote trajectories at every 20 s interval

field at $t = 20$ s in Fig. 4 shows similar patterns to case E32 while PV magnitude is different. According to geostrophic adjustment, the smaller L/L_D where L is the horizontal length scale, the stronger the influence of velocity on the circulation. Hence, case E31 (E33) shows more (less) distortion on the PV contour field at $t = 60$ s compared to the initial field while case E32 is in between E31 and E33. Case E31 exhibits large positive Π_t (Fig. 8) and shows much more interactions between two vortex cores such as acceleration of vortex pair rotation (Fig. 7) and decreasing

distance between the cores (Fig. 8) which lead to merger. Eventually, vortices merge (Fig. 7) and long spiral filaments develop at $t = 100$ s. By comparing case E31 and E33, our simulations may also explain that a larger L_D make merging more difficult as reported by Waugh (1992). For case E33, we anticipate that merger will be less efficient than case E31 and E32. Results indicate that for case E33 at $t = 100$ s, vortex core distance is slightly larger than case E32 with slightly slower rotation (Figs. 4 and 7). Overall Π_t magnitude is smaller than case E32 and vortex

Fig. 7 Same as Fig. 6 for case E31 – E53

core distance is larger as shown in Fig. 8, even showing signs of vortex core separation in the end of simulation. This reconfirms that larger L_D makes vortex merger less likely (Fig. 5).

3.4 Case E22 and E42: Ro Sensitivity

In case E22 and E42, we change the initial perturbation height h_0 to -0.05 m and -0.2 m for case E22 and E42, respectively, while keeping H the same as case E32 which will consequently change Ro (case E22: 0.16, case E32:

0.32, case E42: 0.64) while leaving L_D constant at 2.8 m to see how Ro affects vortex merger. Comparing case E32 and E42, we observe that doubling Ro roughly doubles vortex core rotation speed by checking the corresponding vortex core location at $t = 20, 60, 100\text{ s}$ for case E32 (Figs. 4 and 7) and E22 (Fig. 3 and 6). Conversely, we observe that reducing Ro to half makes vortex core rotation speed slower when we compare case E32 and E42. Decreasing Ro (case E22, Fig. 6) looks as if the vortex evolution is slower than case E32 without signs of inducing vortex separation as shown in case E33.

Fig. 8 Time series of Π_t at the midpoint of the two vortex cores (upper panel for each cases) and distance between two vortex cores (lower panel for each cases)

For this particular comparison, increasing Ro appears to make merger more likely as shown by the vortex core trajectory in Figs. 6 and 7. However, if we consider all test cases (Fig. 1), we observe that L_D is a much better indicator for vortex merger than Ro . All cases for $L_D \leq 2m$ merged and $L_D \geq 4m$ did not merge. Case E12, E22, E32, E42, E52 all corresponds to $L_D = 2.8m$ which may be considered a borderline value for vortex merger as L_D is in a comparable range of the initial vorticity separation distance $2m$. Π_t analysis from Fig. 8 for case E42 shows that signs of Π_t is largely positive from $t = 50s$ while signals of vortex core attraction starts to show roughly at $t = 90s$ (Figs. 7 and 8), indicating a lead time of $40s$ predicting merger when L_D is in the grey zone between $2 - 4m$ (Fig. 9).

3.5 Case B and C: Small L_D and Ro case

Due to small Ro value, the flow follows geostrophic wind for both case B and C (Fig. 3). The length of filaments is quite short due to small wind shear. The maximum speed $0.16ms^{-1}$ in case C is greater than $0.08ms^{-1}$ in case B. Because of stronger shear, the filaments are longer in case B. Small L_D value of $0.4m$ facilitated efficient vortex core merger. Similar flow evolution pattern with different time scale is quite noticeable by judging from Figs. 3 and 6.

3.6 Case E53: Vortex Pair Separation Case

Long and wide filaments coexist with two vortices at $t = 10s$ (Fig. 4). They do not carry away vorticity from the

Fig. 9 Time series of $sM(t)/M(t=0)$ (green), $sPV(t)/sPV(t=0)$ (orange), $sPE(t)/sTE(t=0)$ (red), and $sTE(t)/sTE(t=0)$ (blue)

inner core to trigger merging according to velocity field. Negative Π_t occupies most area between two vortices and cut down their interactions (Fig. 8, $t = 12$ s). Those two vortices never merge. The upper vortex moves westward but the lower one moves eastward, because positive Π_t faces different direction (Fig. 7). Eventually they depart from each other as time increase. This case shows that Π_t is a good predictor for the movement of vortex as well as vortex merging in the previous cases. Comparing case E33 and E53 shows that both cases have $L_D = 4$ m while Ro is four times

larger in case E53 and increasing Ro in this comparison seems to facilitate separation instead of merger.

3.7 Effects of Filamentation on the Vortex Core Integrated PV Value

To examine the effect of spiral band filamentation on PV in the vortex core, we compute time series of area integrated PV between predefined PV contour intervals (Fig. 10). PV interval values are determined by computing

Fig. 10 Time series of iPV for case A, B, C, E[22, 31, 32, 33, 42, 53]. PV contour level interval values are noted in the legend for each cases. All series show differences from each corresponding initial value, normalized by initial total PV value

minimum and maximum PV value during each case simulations. Each intervals are equal except for the last interval which is double than the others. The last interval with highest PV limits may be regarded as the integrated PV for the vortex core area which we are interested in. For case A, the minimum PV is $0.965 \text{ m}^{-1} \text{ s}^{-1}$ and maximum PV is $2.347 \text{ m}^{-1} \text{ s}^{-1}$. The time series of integrated PV is normalized by subtracting its own initial

value, subsequently divided by the initial total PV value, i.e.,

$$iPV(t) \equiv \frac{\int_{\Omega_i} \Pi(t) - \int_{\Omega_i} \Pi(t=0)}{(\text{Total PV at } t=0)} \quad (7)$$

where $\Omega_i \in a \leq \Pi(t) < b$ and a, b are PV interval limits noted as [a,b] in Fig. 10. With this setup, we can learn if PV core values are being conserved and how PV is being transferred between other PV interval levels. Total PV

Fig. 11 Time series of iPV for case A with more granular intervals. All series show differences from each corresponding initial value, normalized by initial total PV value

change is also drawn for reference which is the net sum of all other interval levels.

Judging from Figs. 6 and 7, among A, B, C, E[22, 31, 32, 33, 42, 53] cases, case A, B, C, E31, and E42 vortex cores are merging (Fig. 1). As merger is in progress until the end of the simulation for case B, E42, and PV is not being conserved well for case C, we limit our attention to case A and E31. Results indicate that there are subsequent PV transfers between each intervals where lower iPV levels are being transferred to higher iPV levels. We also notice that for case A, initially up to approximately $t = 35$ s, PV core is weakening probably due to structural dilation of the core area (red line in Fig. 10). From $t = 70 - 80$ s, major PV transfer from orange line (PV interval [1.241, 1.518]) to blue line (PV interval [0.954, 1.241]) occurs which indicates that PV is being transferred across the PV=1.241 contour line while higher level iPVs stay steady. From Fig. 3 (Case A, $t = 60$ s and $t = 100$ s), we see that this is where PV filament is elongating where lower level iPVs are possibly being dissolved to the background. After $t = 80$ s, all PV interval levels stay steady. There is an obvious contrast between lower and higher level iPV trends. Similar behavior is observed for case E31.

For case A, we investigate the PV transfer with finer iPV interval levels in Fig. 11. From $t = 15 - 70$ s, we observe PV transfer occurring from smaller to larger iPV levels (notice small bumps occurring around $t = 15 - 30$ s) except for the largest iPV interval [2.135, 2.347] (red fine dot) which shows slight decreasing trend overall with time. This is probably because intervals at this level is too small to be a representative of a vortex core area. By aggregating the last couple intervals, the loss at the largest interval is expected to be compensated. PV transfer between PV interval levels from lower to higher occur during the merger process, when the vortex core pair is getting closer to each other, explaining PV intensification mechanism. However,

at the background level, examining Fig. 10 case A, E31, E42 indicates that all show major PV transfer to the lower background (orange to blue line) distinctively at the latter part of the simulation ($t = 70 - 80$ s for case A, $t = 90 - 100$ s for case E31, $t = 110 - 120$ s for case E42). Figs. 6 and 7 (time can be estimated by counting the square marks as they mark the 20 second interval) and Fig. 8 show that these timeframe corresponds to when vortex core pair transitions from spiral attraction mode to rotation mode. As vortex core path becomes circular, elongated PV filaments in the outer surrounding area gets dissolved to the background while PV core area is barely affected by filamentation. This process is believed to be responsible for the latter part major PV transfer. Area representing the outer area is relatively large compared to the vortex pair core region and hence the large PV transfer via diffusion. Although we noted that no explicit diffusion or filters are used in our model, still, there are extremely small inherent numerical diffusion introduced by discretization such as space discretization or time marching scheme which will eventually cause small scale PV filaments to diffuse into the background.

4 Summary

An inviscid, nonlinear shallow water model is applied to study the formation of filaments and merging of vortices, which have been investigated using numerical models and laboratory experiments. We confirm that misalignment of PV with the streamlines is crucial to merger. However the angle between vortex major axis and vortex-centers-line is not created by the formation of filaments. The merging cases show that flow creates positive Π_t on the lee side of PV ridge, and this is where PV contours becomes more convex, which increases the angle between streamlines

and PV contours and efficient transport of PV along the parallel but opposite direction in the positive quadrants of Π_t . Thus, vortices move closer while rotating around each other. On the windward side of PV ridge, the PV is pushed inward, especially, near the base of filaments, which lengthens the filaments. However, velocity indicates that the filaments are shattered instead of being ejected away from the core, which is also evidenced by the steady core area integrated PV trends. Furthermore, consecutive PV transfer from lower to higher interval levels are observed by the merger process. On the other hand, elongated filaments get dissolved in the lower level PV regime. The vortices never merge when negative Π_t prevails between two vortices. Conversely, large Π_t with corresponding decreasing vortex core distance is seen for merger cases. Hence, the advance of positive Π_t provides a simple yet eloquent mechanism for vortex merging. Vortices are more difficult to merge for a large L_D , because the circulation proceeds more circular around the individual vortex which hinders the interactions between vortices according to geostrophic adjustment. All test cases with $L_D \geq 4m$ did not merge while all cases with $L_D \leq 2m$ did merge regardless of Ro . Cases with $L_D = 2.8m$ showed mixed results indicating that this may be considered a borderline value for vortex merger as L_D is in a comparable range of the initial vorticity separation distance $2m$. We have defined L as $d_0/2$ but other criteria may be used such as the combination of vortex width, vortex pair separation distance and others to define the length scale. In that case, the L_D/L decision number (or range) would change by some factor. As for our study, the L_D/L decision range seems to be in between 2 and 4. If we redefine L as d_0 instead of $d_0/2$, the decision range would corresponded to a range between 1 and 2, meaning that vortex pair length scales smaller than L_D tends to merge while vortex pair length scales that are larger than L_D by a factor of (roughly) 2 tend not to merge. As for the grey area of L_D/L range between 1 and 2, L_D is not conclusive but Π_t analysis proved to be a useful diagnostic tool for assessing vortex merger cases providing insight with a lead time of 40 seconds as shown in case E42. Ro tended to dictate the speed of flow interaction given the same L_D while its role on vortex merger was not as significant as L_D in this study. The analysis of this work can be applied for improving cyclone path prediction or for natural hazard risk assessment to support decision making.

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