Journal für Mathematik-Didaktik (2023) 44:533–561
https://doi.org/10.1007/s13138-023-00223-3

ORIGINALARBEIT/ORIGINAL ARTICLE

Posing and Solving Modelling Problems—Extending the Modelling Process from a Problem Posing Perspective

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Received: 16 May 2022 / Accepted: 20 May 2023 / Published online: 5 July 2023

Abstract In mathematics education, pre-formulated modelling problems are used to teach mathematical modelling. However, in out-of-school scenarios problems have to be identified and posed often first before they can be solved. Despite the ongoing emphasis on the activities involved in solving given modelling problems, little is known about the activities involved in developing and solving own modelling problems and the connection between these activities. To help fill this gap, we explored the modelling process from a problem posing perspective by asking the questions: (1) What activities are involved in developing modelling problems? and (2) What activities are involved in solving self-generated modelling problems? To answer these research questions, we conducted a qualitative study with seven pre-service teachers. The pre-service teachers were asked to pose problems that were based on given real-world situations and to solve their self-generated problems while thinking aloud. We analyzed pre-service teachers' developing and subsequent solving phases with respect to the problem posing and modelling activities they were engaged in. Based on theories of problem posing and modelling, we developed an integrated process-model of posing and solving own modelling problems and validated it in the present study. The results indicate that posing own modelling problems might foster important modelling activities. The integrated process-model of developing and solving own modelling problems provides the basis for future research on modelling problems from a problem posing perspective.

Keywords Problem posing · Modelling · Real-world situations · Cognitive processes · (Pre-service) teachers

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Die Entwicklung und Lösung von Modellierungsaufgaben – Eine Erweiterung des Modellierungsprozesses aus einer Problem Posing-Perspektive

Zusammenfassung Im Mathematikunterricht werden vorformulierte Modellierungsprobleme genutzt, um mathematisches Modellieren zu unterrichten. In außerschulischen Situationen müssen mathematische Probleme jedoch oft zunächst identifiziert und entwickelt werden, bevor sie anschließend gelöst werden können. Trotz der anhaltenden Fokussierung der Aktivitäten, die mit der Lösung vorgegebener Modellierungsaufgaben verbunden sind, ist wenig über die Aktivitäten bekannt, die mit der Entwicklung und Lösung eigener Modellierungsaufgaben einhergehen, und über die Verbindung zwischen diesen Aktivitäten. Um einen Beitrag zur Schließung dieser Forschungslücke zu leisten, wurde in der vorliegenden Studie der Modellierungsprozess aus einer Problem Posing-Perspektive untersucht, indem die folgenden Forschungsfragen fokussiert wurden: (1) Welche Aktivitäten sind an der Entwicklung von Modellierungsaufgaben beteiligt? und (2) Welche Aktivitäten sind an der Lösung von selbst-entwickelten Modellierungsaufgaben beteiligt? Zur Beantwortung dieser Forschungsfragen wurde eine qualitative Studie mit sieben angehenden Lehrkräften durchgeführt. Die angehenden Lehrkräfte wurden gebeten, Aufgaben basierend auf vorgegebenen realweltlichen Situationen zu entwickeln und ihre selbst-entwickelten Aufgaben zu lösen. Dabei wurde die Methode des Lauten Denkens angewendet. Die Entwicklungs- und Bearbeitungsprozesse wurden anschließend im Hinblick auf die stattfindenden Problem Posing- und Modellierungsaktivitäten analysiert. Aus einer theoretischen Perspektive wurde ein integriertes Prozessmodell der Entwicklung und Lösung eigener Modellierungsaufgaben entwickelt, das basierend auf den Ergebnissen validiert wurde. Die Ergebnisse deuten darauf hin, dass das Aufstellen eigener Modellierungsaufgaben wichtige Modellierungsaktivitäten fördern kann. Das integrierte Prozessmodell der Entwicklung und Lösung eigener Modellierungsaufgaben bietet eine Grundlage für zukünftige Forschung zum mathematischen Modellieren aus einer Problem Posing-Perspektive.

Schlüsselwörter Problem Posing · Modellieren · Realweltliche Situationen · Kognitive Prozesse · (Angehende) Lehrkräfte

1 Introduction

Mathematical modelling is an essential part of teaching and learning mathematics as it enables to use mathematics to deal with situations that occur in everyday life (Niss and Blum 2020). In modelling research the problem itself and it's development are considered as an important aspect of modelling (Pollak 2015). Problems encountered in the real world have to be identified and generated first before a solution can be found. However, in mathematics class and also in theoretical models that describe the activities taking place in modelling (Blum and Leiss 2007), the modelling process starts with a given problem. An open question is, what activities take place, if modelers develop problems themselves and how does the solution of the modelling problem afterwards look like. The present study addresses this research gap by investigating what activities take place when people pose problems that are based on given real-world situations and how the problem posing activities are connected to the subsequent modelling activities. The identification of problem posing activities, modelling activities, and the co-occurrence of both helps to understand the activities that take place in developing and solving a modelling problem. The overarching goal is to extent the modelling process from a problem posing perspective by developing an integrated process-model of posing and solving modelling problems from a theoretical perspective and validating it based on empirical findings.

2 Theoretical and Empirical Background

2.1 Mathematical Modelling

Mathematical modelling can be defined as the development of a mathematical model for dealing with real-world situations (Niss and Blum 2020) and is enclosed in the German school curriculum (e.g., Kultusministerkonferenz [KMK] 2003) as well as in curricula all over the world (e.g., National Council of Teachers of Mathematics [NCTM] 2000). In mathematics class, a specific type of problem—called a modelling problem—is used to teach mathematical modelling. Modelling problems can vary greatly in scope and complexity, from highly complex authentic modelling problems encountered in industry to less complex modelling problems that can be solved within a mathematics lesson in school (e.g., Greefrath et al. 2022; Humenberger 2021; Maaß 2010) to. In our study we focus the second type of modelling problems.

For solving modelling problems, demanding translation processes between the rest of the world and the mathematics are needed. The activities involved in solving a modelling problem can be described in a circular model. These activities were intensively investigated in research on modelling under a cognitive perspective (see the overview in Schukajlow et al. 2021). One description of solving a modelling problem that is widely accepted in the field is the modelling cycle by Blum and Leiss (2007) including the seven activities understanding, simplifying and structuring, mathematizing, working mathematically, interpreting, validating, and exposing. The modelling cycle presents an idealized model of solving a modelling problem with the aim of describing the cognitive processes of a modeler. It is not meant to be a description of the order of activities that a modeler actually passes through (Niss and Blum 2020). Modelling research has demonstrated the existence of individual modelling routes, starting at different points in the cycle and repeating or skipping steps (Borromeo Ferri 2010; Leiss 2007; Matos and Carreira 1997). The starting point for authentic modelling is not always a pre-formulated problem as in the real world the problem has to be identified and posed first before it can be solved (Pollak 2015). For example Galbraith et al. (2010) instructed school students for authentic modelling to choose a real-world scenario, to pose a problem that was based on this scenario, and afterwards to solve the self-generated problem. However, in contrast to knowledge about the modelling routes and activities that are involved in solving given modelling problems, not much is known about how the activities involved in modelling look like when the starting point is a situation without a given problem.

2.2 Problem Posing

The origin of problem posing begins far in the past by introducing problem posing as an overall pedagogic teaching approach (Freire 1970) as well as a strategy for exploring problems through variation and generating further problems (*What-if-not-strategy*) (Brown and Walter 2005). In mathematics education, problem posing has received increased attention in the last decade as an important mathematical process on its own as well as an approach to foster other competencies (Cai and Leikin 2020). Overall, the term problem posing in mathematics education refers to two processes: the process of developing new problems and the process of reformulating given problems. Problem posing can be initiated by different stimuli. For stimulating problem posing different situations as well as prompts can be used (Cai et al. 2022). Regarding the prompts that initiate problem posing, unstructured and structured problem posing prompts can be differentiated (Baumanns and Rott 2021; Stoyanova 1997). Structured problem posing prompts are based on an initial problem and refer to the process of reformulating a problem. The problem poser is instructed to reformulate the initial problem or to pose further problems that are based on the initial problem. Unstructured prompts comprise problem posing situations with fewer restrictions and refer to the process of generating a new problem. They can encompass mathematical calculations, descriptions of situations, or pictures. Analogous to the classification of problems with respect to their connection to reality (Blum and Niss 1991), problem posing situations can be classified according to whether or not the given stimulus refers to aspects of the real world. Problem posing based on mathematical situations starts in the mathematical domain and refers to intra-mathematical situations including given mathematical graphs, tables, or conjectures (e.g., Christou et al. 2005), whereas problem posing based on real-world situations starts in the extra-mathematical domain and comprises objects or descriptions of real-world situations. An example of problem posing based on given real-world objects can be found in the study by Bonotto and Santo (2015). In this study, researchers presented school students with objects from real-world situations (e.g., restaurant menus, supermarket bills) and instructed them to use the given objects to pose problems. In our study, we focus on problem posing in context with modelling, and will refer to this as modelling-related problem posing. Problem posing can take place before, during, or after the problem is solved (Silver 1994). On the one hand, problem posing can be used as an explicit prompt to develop modelling problems based on a given real-world situation before solving them. On the other hand, solving modelling problems can naturally involve problem posing activities without an explicit prompt (Hansen and Hana 2015). In the presented study, we focus on the generation of new problems based on given descriptions of real-world situations (unstructured situations) as an explicit prompt before solving them. An exemplary stimulus for initiating modelling-related problem posing is presented in Fig. 1. The result is a modelling problem that can be solved subsequently.

Cable Car

For more than 90 years, the *Nebelhorn* cable car has taken numerous guests up into the heights. Now it can go into well-earned retirement. Beginning in the summer of 2021, a new cable car will transport enthusiastic outdoor fans to the Nebelhorn mountain. The aim of the project is to avoid long waiting times, provide seated transportation with an optimal view from every seat, and increase carrying capacity.

Technical data for the old cable car:

Model:	Large-cabin aerial tramway
Weight empty cabin:	1600 kg
Weight full cabin:	3900 kg
Height valley station:	1933 m
Height top station:	2214.2 m
Horizontal difference:	905.77 m
Speed:	8 m/s
Carrying capacity:	500 people/h
Power Unit:	120 PS

Pose a mathematical problem based on the given real-world situation.

Fig. 1 Modelling-related problem posing stimulus Cable Car

In order to pose a problem that is based on a given situation, creative mathematical thinking processes are needed (Bonotto and Santo 2015). The problem is created on the basis of an individual's mathematical experiences and interpretation of the given situation (Stoyanova 1997). To describe the creative process of mathematical thinking in mathematics education research, numerous models have been developed on the basis of Wallas (1926) four phase model (for an overview, see Pitta-Pantazi et al. 2018). Wallas (1926) four phase model describes creative mathematical thinking as a linear process that consists of the four activities *preparation (exploration), incubation, illumination*, and *verification*. The process begins with a problematic situation. During preparation, the given situation has to be understood and explored. Then, through incubation, the idea subconsciously matures, and then through illumination, the AHA! experience occurs as a decision emerges. Finally, in the context of verification, the idea that has been raised is examined, and, if necessary, adjustments to it are made or new ideas are developed.

In contrast to the description of creative mathematical thinking, there is no agreed upon model for describing the activities that take place when posing a problem (Cai et al. 2015). However, previous research has assumed (e.g., Bonotto and Santo 2015) and empirically identified (e.g., Baumanns and Rott 2022b; Christou et al. 2005; Pelczer and Gamboa 2009) some activities that are important for successful problem posing. In studies on structured problem posing that starts in the mathematical world overall the cognitive activities exploring, generating and evaluating have been identified to be part of the problem posing process (Baumanns and Rott 2022b; Bonotto and Santo 2015; Pelczer and Gamboa 2009). Further, some studies indicated that a planning activity aiming at solving the problem already seems to be involved in the problem-posing process (Baumanns and Rott 2022b) as the problem-solving strategies that were typically employed were found to guide the problem-posing process (Cai and Hwang 2002), and problem posers were found to pose problems that they knew how to solve (Chen et al. 2007). Empirical research on the cognitive process of problem posing indicated that these activities run off by no means linear and are instead characterized by jumping back and forth, thus leading to individual posing processes (Baumanns and Rott 2022b; Pelczer and Gamboa 2009). However, previous research has primarily focused the problem posing process based on structured prompts starting in the mathematical world. As the starting point for modelling-related problem posing is in the real world and the prompt is unstructured, the open question that remains is whether the activities are the same for modelling-related problem posing.

2.3 Modelling, Problem Solving and Problem Posing

In mathematics education research, problem posing is considered to be closely related to problem solving (Cai and Hwang 2002; Chen et al. 2013; Silver and Cai 1996) and that interventions with a focus on problem posing have a positive effect on problem solving (Cai and Leikin 2020; Chen and Cai 2020; Voica et al. 2020). A reason for this is that posing a problem already involves activities that are needed for subsequent problem solving (Baumanns and Rott 2022b). Therefore, to get insights into the modelling process from a problem posing perspective, problem posing and solving have to be situated in the modelling process first.

The modelling process can be divided into a developing and a solving phase. The developing phase begins with a given real-world situation. To develop a modelling problem, problem posing activities (Sect. 2.2) are needed and the phase ends in a real-world situation plus question. Then the subsequent solving phase can start. The solving phase can be seen as a specific problem solving process that begins in the extra-mathematical domain (Zawojewski 2013). Therefore, findings from the connection of the activities involved in problem posing and problem solving might be transferable to the connection of the activities involved in problem posing and modelling. However, the connection between problem posing and modelling might be even stronger as both start in the extra-mathematical domain and translation processes between the extra-mathematical domain and the mathematical domain are needed. Giving problem posing based on given real-world situation as an explicit prompt might already engage the following modelling activities: Developing an understanding as well as engaging in activities (e.g., simplifying and structuring) that are needed to construct an adequate situation model and a real model could be aided by exploring the given situation in an in-depth manner. Initial indications for the involvement of these activities come from empirical research on modelling-related problem posing that revealed that after problem posing relevant real-world aspects and requirements are enclosed in the solution (Bonotto 2006). Further, mathematizing and working mathematically could be stimulated during the development of a possible plan for solving the self-generated problems. Despite the connection between the processes that is expected from a theoretical viewpoint, the open questions that remain are: What modelling activities are involved in the phase of developing and solving own modelling problems?

3 Theoretical Model on Developing and Solving Modelling Problems

We extended the modelling cycle by developing an idealized process-model for developing and solving modelling problems. The process-model is based on the seven step modelling cycle described by Blum and Leiss (2007) for solving modelling problems (see Sect. 2.1) and theoretical considerations on the connection between problem posing and modelling (see Sects. 2.2 and 2.3). The integrated process-model is presented in Fig. 2.

In the circular model, two different areas which are called “rest of the world” and “mathematics” are distinguished. The process begins in the rest of the world with a given real-world situation. Modelers first have to *understand* the given real-world situation by reading the text about the cable car conversion and comprehending the given information to construct a mental model of the given situation called the situation model. Then the modelers have to explore the given situation in an in-depth manner by focusing important information in the situation for generating a possible problem leading to a real model. For example, the technical data of the old cable car can be identified as important information that can be used for generating a problem. Based on this exploration, problems can be generated. A possible problem based on this information could be how long the cable of the old cable car was. Then, in the course of the evaluation, the self-generated problem can be assessed on the basis of individual criteria. For example, the evaluation can include the assessment of solvability and appropriateness in relation to the given real-world situation. If the problem posed is not deemed appropriate, the problem can be adapted or a new problem is generated. If the problem is deemed appropriate, the information can be translated from the rest of the world into the mathematical world. The relevant information has to be *mathematized* by transforming the information into a mathematical model. In this situation, a rectangular triangle with a missing hypotenuse can be

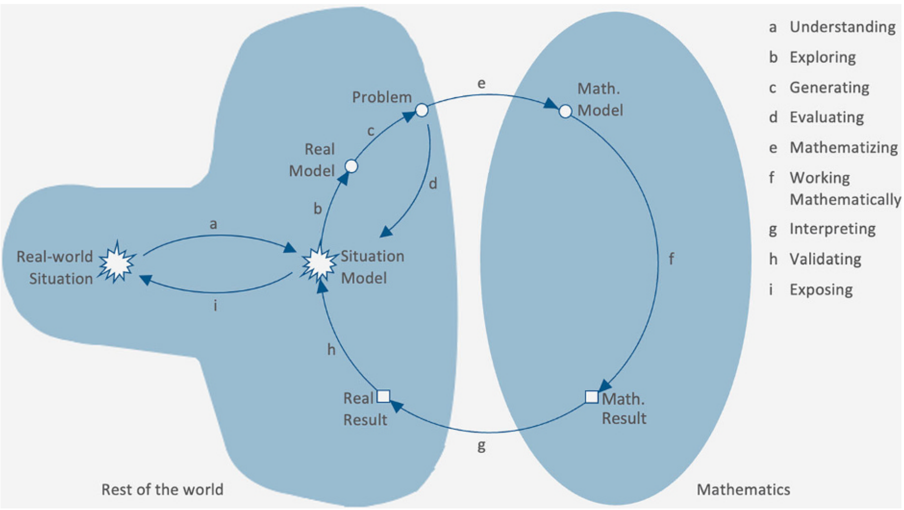


Fig. 2 Idealized process-model for developing and solving modelling problems

used as a mathematical model. By working mathematically, the hypotenuse can be calculated by applying the Pythagorean theorem. Then this mathematical result has to be translated back to the rest of the world by *interpreting* it. The real result may show that the steel cable is around 5500 m long. By *validating,* the modeler has to check whether or not the models that were used (e.g., the rectangular triangle, the use of Pythagorean theorem) and the results (e.g., the length of 5500 m) are appropriate for capturing the real-world situation. If they are not appropriate, the modeler will need to modify the models and will have to start the process over again. The activity *exposing* is idealized described as a transfer between the situation model and the real-world situation taking place continuously during the solving phase by documenting and communicating the solution.

In accordance with the modelling cycle described by Blum and Leiss (2007) for solving modelling problems, the model presents an idealized model of the development and subsequent solution of own modelling problems with the aim to describe the cognitive process and is not meant to be a description of the order in which the activities actually occur.

4 Research Questions

The main goal of the study is to extend the theoretical model of modelling from a problem posing perspective. For this purpose, it is important to investigate what cognitive activities take place during the phase of developing and solving own modelling problems and the connection between these activities. The activities are analyzed from two different perspectives, the perspective of problem posing and the perspective of mathematical modelling emerging from the two different theoretical backgrounds. The results are used to find empirical evidence for the theoretically developed process-model of developing and solving own modelling problems. Thus, the research questions are the following:

1. What activities are involved in the phase of *developing* own modelling problems?
    a. What problem-posing activities are involved in the phase of developing modelling problems?
    b. What modelling activities are involved in the phase of developing modelling problems?
    c. With which problem-posing activities do the modelling activities co-occur?
2. What modelling activities are involved in the phase of *solving* own modelling problems?

5 Method

5.1 Sample

Our sample comprised seven pre-service teachers from a large university in Germany (three women and four men). We selected pre-service teachers (and not school stu-

Table 1 Overview of the sample

Name	Age	Program	Bachelor/ Master	Math grade	Modelling experience	Problem posing experience
Max	20	HRSGe	Bachelor	13 (A)	+	+
Lea	24	GymGes	Bachelor	13 (A)	+	+
Lisa	22	HRSGe	Bachelor	8 (C)	+	–
Theo	22	GymGes	Master	14 (A)	+	+
Leon	24	GymGes	Master	12 (B)	+	+
Fabian	22	GymGes	Master	14 (A)	+	+
Nina	26	GymGes	Master	11 (B)	+	+

dents) for this study because we were interested in analyzing the activities involved in posing and solving modelling problems and we found in a prior study that school students have difficulties in developing and solving own modelling problems (Hartmann et al. 2021). A sample with experience in modelling was chosen in order to collect data that would be rich in modelling activities and provides the most information about the phenomenon. Table 1 presents an overview of the participants. Five of the pre-service teachers participated in a program offering a higher track secondary school teachers' degree (GymGes: German Gymnasium) and two of them in a middle track secondary school teachers' degree program (HRSGe: German Haupt-, Real-, Sekundar-, and Gesamtschule). Four pre-service teachers were enrolled in the master's degree program, as they had already completed their bachelor's degree, whereas three were still participating in the bachelor's degree program. They were between 20 and 26 years of age, and their mathematics grade in school was between 8 and 14 points (grades C to A). According to the interview data, all of them already had experience in modelling and six of them in problem posing. We used a purposeful sample selection with heterogeneity sampling (Patton 2015). The sampling criteria were that the sample had to cover different levels of mathematical achievement (grades) and different levels of prior knowledge, assessed by participation in different university programs (Table 1). Based on the decision of an ethics committee, there was no ethical disregard for participants in this study as they participated voluntarily, and they were not expected to suffer from any adverse effects or damage. To protect participants' privacy, we used pseudonyms for their names.

5.2 Procedure and Instruments

To collect the data, we used a three-step qualitative design that included thinking aloud, stimulated recall, and interviews as recommended by Busse and Borromeo Ferri (2003) for analyzing cognitive processes. We instructed the participants to think aloud, and we videotaped these processes. The participants were taught about the think aloud method and engaged in a brief exercise with the aim to become familiar with the method. Next, the pre-service teachers received three real-world situations along with the following instructions:

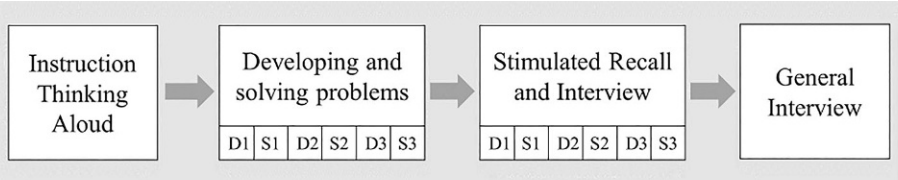


Fig. 3 Overview of the data collection. (*D* developing problems, *S* solving problems)

> You will now develop problems that are based on three different situations, one after another. On the first task sheet, you will find a first situation from the real world. Unlike the problems you usually encounter, however, the situations do not include a mathematical question. Today, you get to develop the problem yourself. You should proceed as follows: First, read the description of the situation aloud. Then think about what mathematical question you can ask yourself about the situation.

After the participant finished developing a problem for the first situation that was given, the interviewer asked the participant to solve their self-generated problem. This procedure was repeated for the second and third situation. We used the recorded videos, including participants' writing, thinking aloud, gestures, and facial expressions as stimuli for the subsequent Stimulated Recall Interviews (SRIs). Further, we conducted an interview after every SRI to address any unresolved aspects. Figure 3 represents an overview of the data collection.

As we were especially interested in the cognitive activities involved and the connection between developing and solving own modelling problems, we adapted real-world situations from modelling problems that were used in previous studies (Schukajlow et al. 2015). We deleted the questions about the problems and extended the descriptions to include additional authentic information to allow pre-service teachers to pose a variety of problems. In order to capture the process of modelling, the situations were designed to stimulate the development of modelling problems. We selected three situations on the basis of Hartmann et al.'s (2021) results. These situations allowed school students to pose a large number of diverse self-generated modelling problems. The situations we selected were the real-world situations titled Cable Car (see Fig. 1), Fire brigade (see appendix, Fig. 8), and Chopsticks (see appendix, Fig. 9).

5.3 Data Analysis

To analyze the data, we first transcribed the 960 min of video material that included developing and solving phases, the stimulated recall, and the interview. Then, we paraphrased the transcripts with regard to content-bearing semantic elements (sequences). A new sequence was set whenever the participants picked up a new thematic thought or engaged in a new action. Further, we included a time marker and divided the data into developing phases, solving phases, and general interview phases. The developing phases started with a given real-world situation and ended

Table 2 Problem posing activities

Category	Description
Understanding	Comprehending and understanding the given real-world situation and the information that was given in the description of the situation
Exploring	Discovering and gathering relevant information to develop possible problems and organizing the information
Generating	Raising and formulating possible problems and defining a problem
Planning of a solution	Planning a more or less concrete solution for the self-generated question
Evaluating	Evaluating possible problems on the basis of individual criteria (solvable, meaningful, complete, appropriate formulation, difficulty, suitable for a particular target group)

Table 3 Modelling activities

Category	Description
Understanding	Comprehending and understanding the given real-world situation and the information that was given in the description of the situation
Simplifying and Structuring	Simplifying and structuring the given real-world situation by differentiating between important and unimportant information, identifying missing information, making assumptions about this information, and identifying possible solution steps
Mathematizing	Translating the selected information into a mathematical model (e.g., table, term, equation, diagram)
Working mathematically	Performing the mathematical operations to generate a mathematical result
Interpreting	Interpreting the mathematical result with respect to the real-world situation and the self-generated question
Validating	Checking models and results for plausibility and appropriateness by referring back to the real-world situation

in self-generated questions. The solving phases started with the given real-world situation plus question and ended with a solution to the question. Then, we analyzed the developing phases to address Research Question 1 (developing modelling problems) and the solving phases to address Research Question 2 (solving modelling problems) by using Mayring's (2015) content analysis. To answer Research Questions 1a and 1c, we developed a coding scheme on the basis of the first indications from research on problem posing and creative mathematical thinking involving the activities *exploring, generating, planning a solution*, and *evaluating* (see Sect. 2.2) and extended it inductively on the basis of the developing phases. The coding scheme is presented in Table 2.

To address Research Questions 1b, 1c, and 2, we developed a coding scheme on the basis of the modelling activities described in models for solving given modelling problems (Blum and Leiss 2007). The coding scheme is presented in Table 3. For Research Questions 1b and 1c, we analyzed the developing phases and for Research Question 2, we analyzed the solving phases on the basis of the coding scheme. The first author coded the data. To test interrater reliability, about 50% of the data were coded by a second well-trained rater. Interrater reliability measured as Cohen's Kappa was at least moderate (Cohens 1960) for the problem posing activities (ranging from $\kappa = 0.81$ to $\kappa = 0.95$), the modelling activities during the developing phase (ranging from $\kappa = 0.84$ to $\kappa = 0.97$), and the modelling activities during the solving phase (ranging from $\kappa = 0.76$ to $\kappa = 0.92$).

To gain deeper insight into the activities involved in developing and solving own modelling problems and the connection between these activities, we conducted an in-depth analysis of exemplary developing and solving phases. We chose Theo's developing and solving phases as an example of the interaction between problem posing and modelling. Theo's developing and solving phases provide the most information about the phenomenon because Theo showed the richest posing and solving processes.

6 Research Findings

6.1 Developing Phase

Regarding the developing phase, we first focus on the problem posing activities (RQ 1a) and modelling activities (RQ 1b) that were involved and afterwards on the co-occurrence of the problem posing and modelling activities (RQ 1c).

6.1.1 Problem Posing Activities

The analysis of pre-service teachers' developing phases confirmed the involvement of the problem posing activities exploring, generating, planning a solution, and evaluating found in prior research. Further, we extended the description of the activities inductively and identified an additional problem posing activity called understanding in pre-service teachers' developing phases.

Understanding comprised building an understanding of the given real-world situation. For this purpose, pre-service teachers read the given situation, summarized the information, asked comprehension questions, and related the given information to their personal experiences. In the following excerpt, Lisa questioned her understanding of the definition of the horizontal distance.

> Um theoretically, I'm wondering right now if the horizontal distance really means that it's sort of between the valley station and the top station.

Exploring was aimed at discovering the given real-world situation to generate ideas for possible problems. Exploring primarily involved gathering information and identifying relationships and constraints. Exemplarily, Lisa identified the information about the height of the top and valley stations and the horizontal difference as relevant information and was trying to relate these different pieces of information to each other by making a drawing in the following excerpt.

> So, I'm sort of making a drawing (Fig. 4) for this because I know that I have here, let's say the (draws in a first point), the um top ah the valley station and the valley station here (draws in a second point). And I know that the height here at the valley station (labels one point) is 1933 m and the top station (labels the other point) is 2214.2 m.

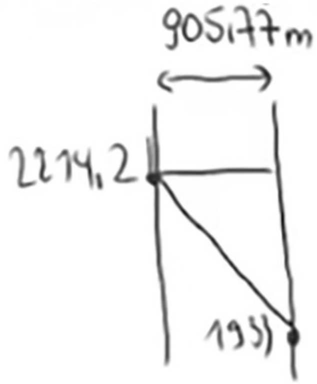


Fig. 4 Lisa's drawing

Generating included all activities related to posing and writing down a problem. For this purpose, possible problems were posed. From these, one question was then selected, formulated, and written down. For example, Theo identified the goal of the cable car project (enabling seated transportation with an optimal view) as relevant information and generated an idea for a possible problem in the following excerpt.

> The goal of the project is to avoid long waiting times, seated transportation with an optimal view. Ok there you can perhaps consider how many people can realistically fit into such a cabin, so that each person sits at the window and has an optimal view and then consider whether you will exceed the weight of a full cabin or not.

Planning a solution involved devising a more or less concrete plan for solving the self-generated problem. For this purpose, mathematical operations for the solution or more or less detailed solution steps were specified. For example, Max posed a problem about the number of people that could fit into the old cable car. In the following excerpt, he described a rather less detailed way to solve the problem.

> You have to work through different steps bit by bit in order to solve it because I don't think you can come up with the solution directly in a calculation.

Evaluating was aimed at assessing the posed problems on the basis of individual criteria. The evaluations that were carried out referred primarily to the assessment of appropriateness in the given situation, solvability, and the formulation of the problem. In the following excerpt, Lea raised a question about the weight of an old cable car cabin and then she evaluated the appropriateness of the question on the basis of the given situation.

> So, you could somehow ask something about the weight in any case. But then the information that we need a new one is not relevant.

To assess the extent to which the individual problem posing activities were involved in the developing phases, we analyzed the frequency and duration of the individual problem posing activities for 7 pre-service teachers based on the three given real-world situations. The frequency of the activities differed decisively. Most sequences were assigned to the generating activity (114 sequences) and slightly fewer to the activities exploring (91 sequences) and evaluating (80 sequences). The smallest numbers of sequences were assigned to the activities understanding (49 sequences) and planning a solution (30 sequences). The descending frequency of the activities in the order generating, exploring, evaluating, understanding and planning a solution was observed for all three given real-world situations. Regarding the duration of the individual activities, especially with regard to the activities understanding and evaluating, a different distribution was observed. Overall, the most time was spent on generating (40 min, 30% of the overall developing time), exploring (39 min, 29% of the overall developing time), and understanding (35 min, 26% of the overall developing time). Participants engaged in planning a solution (11 min, 8% of the overall developing time) and evaluating (11 min, 8% of the overall developing time) for only short periods of time. The descending duration of the activities in the order generating, exploring, understanding, planning a solution and evaluating was observed for all three given real-world situations.

6.1.2 Modelling Activities

Regarding the modelling activities involved in the developing phases, our analysis of pre-service teachers' developing phases revealed that all modelling activities were involved except validating.

Understanding was conceptualized in the same way as understanding in the problem posing activity.

Simplifying and structuring were aimed at organizing the given real-world situation and primarily involved identifying important and missing data, making assumptions about missing data, and identifying possible solution steps. In the following excerpt, Max distinguished between relevant and irrelevant information for the self-generated problem.

> I now find Großkabinen-Pendelbahn to be rather irrelevant. Weight empty cabin 1600 kg and full cabin 3900 kg. I think I should narrow that down more, what information should go out and not be used.

Mathematizing included formulating a more or less concrete mathematical model to solve the self-generated problem. For example, Lea figured out that the Pythagorean theorem could be used as a mathematical model to solve the self-generated problem in the following excerpt.

> Then once again the Pythagorean theorem can be used.

Working mathematically comprised performing mathematical operations. In the following excerpt, Leon calculated the difference between the valley station and the top station.

> First, very briefly, how large is the actual difference? So $3900 - 1600 = 2300$.

Interpreting was aimed at connecting the mathematical result that had been calculated with the given real-world situation by considering what the result meant in the context of the given real-world situation.

Table 4 Co-occurrence of activities in the developing phases

		Problem posing activities					
		Under-standing	Exploring	Generating	Planning a solution	Evaluating	Σ
Modelling activities	*Understanding*	112	0	0	0	0	*112*
	Simplifying/ Structuring	0	144	7	24	22	*197*
	Mathematizing	0	0	0	12	0	*12*
	Working math	0	2	0	2	0	*4*
	Interpreting	0	1	0	1	0	*2*
	Validating	0	0	0	0	0	*0*

To assess the extent to which the modelling activities occurred in the developing phases, we analyzed the frequency of the sequences and the duration of the modelling activities for 7 pre-service teachers based on the three given real-world situations. Regarding the frequency and duration, there were large differences between the individual activities. Simplifying and structuring was the most common modelling activity identified in the developing phase on which the most time was spent (135 sequences, 45 min) followed by the activity of understanding (49 sequences, 35 min). Mathematizing (12 sequences, 2 min), working mathematically (4 sequences, 1 min), and interpreting (2 sequences, 12 s) occurred rather rarely and for only short periods of time. The descending frequency and duration of the activities in the order simplifying and structuring, understanding, mathematizing, working mathematically, and interpreting occurred for all three given real-world situations.

To gain an overall picture of the occurrence of modelling activities during developing a modelling problem, it is important to consider the problem posing activities in which the individual modelling activities occurred. Table 4 presents an overview of the co-occurrence of the activities in the developing phases.

Understanding as a problem posing and a modelling activity exclusively occurred together. Simplifying and structuring usually co-occurred with the problem posing activity exploring. This is exemplified in the following excerpt, in which Max explored the given real-world situation by filtering relevant from irrelevant data.

> Let's take another look at the data for the old Nebelhornbahn. Um, I find the Großkabinen-Pendelbahn rather irrelevant now. Weight empty cabin 1600 kg and full cabin 3900 kg.

Further, simplifying and structuring occurred along with the problem posing activities generating, planning a solution, and evaluating. During generating, simplifying and structuring was identified as making assumptions or mentioning information that was relevant for solving the problem in the formulation of the self-generated problem. In the following excerpt, Nina added the information that the number of people and the speed should be taken into account when solving the problem in her self-generated problem (*What is the best way to shorten the waiting time for the new cable car?*).

(Supplements the problem) Consider the number of people and speed. Something like this.

In the context of planning a solution, simplifying and structuring was identified as planning the steps that would be needed to solve the self-generated problem. For example, Max planned a possible solution for his self-generated problem in the following excerpt.

But then we also have the travel speed of 8 m/s. This means that one could theoretically also determine the travel time if we have the length of the route. How long the cable car needs from one station to the next. That would be the next solution step so to speak.

During evaluation, the self-generated problem was evaluated with regard to solvability by checking whether all the information necessary to solve the problem was given in the situation. In the following excerpt, Lea evaluated her self-generated problem by identifying the relevant information and checking whether all the information that was necessary to solve the problem was given.

Because we know how fast it is, we know where it starts, we know how it's going, and we can say that it's just going straight, so it's kind of going up as a linear function; Then you could ... This is a nice question.

Mathematizing came up exclusively with the problem posing activity planning a solution. This included identifying the mathematical operations that could be applied to solve the self-generated problem. For example, in the following excerpt, Max identified the Pythagorean theorem as an equation that could be used to solve his self-generated problem.

Um yes. For example, one could theoretically use the Pythagorean theorem again.

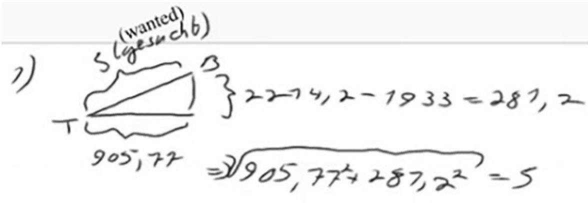
Working mathematically as well as interpreting occurred during exploration and planning a solution. Working mathematically during exploration included the use of mathematical operations to explore the situation. For example, in the following excerpt, Leon subtracted two numbers and was looking for a number by which the difference could be divided.

First, very briefly, how big is the difference? So $3900-1600=2300$. Can that be divided by anything nice? 23 not really. Um. (types something into the calculator).

6.2 Solving Phase

Regarding the solving phase, we focus the modelling activities (RQ 1b) that were involved in the solving phase (RQ 2). All modelling activities (understanding, simplifying/structuring, mathematizing, working mathematically, interpreting, validating) occurred during the solving phase.

Fig. 5 Leon's drawing



Understanding was rarely involved in the solving phase. It was conceptualized in the same way as understanding in the developing phase (see Sect. 6.1). Indications for understanding in the solving phase have been repeating of the self-generated problems and questioning the meaning of the given information in the given situation. Exemplarily, in the following excerpt, Max asked what exactly was meant by the given transport capacity.

> Um, 500 persons/h conveyor capacity actually means. Ah, okay, means um that in one hour 500 people rode it at all, not there and back, so that's the question now, but probably a distance.

Simplifying and structuring involved making assumptions, structuring the given information, identifying possible solution steps, and rather rarely differentiating between relevant and irrelevant information. In the following excerpt, Max made an assumption about how to simplify the given real-world situation about the cable car.

> In any case, if I want to model the cable as—um, well, the cable somehow as a line, that is, the cable as a line. Then I have to assume that the cable doesn't have any dents or sags because the cabin is attached to it or that, for example, we have some kind of supports so that the cable is no longer a stretch, no longer a straight line.

Mathematizing included both identifying mathematical objects or operations that could be used to solve the self-generated problem and formulating a specific mathematical formula to solve it. In the following excerpt, Leon realized that the Pythagorean theorem was needed to solve the problem and wrote down the specific formula.

> So, and now again the Pythagorean theorem is: $a^2 + b^2 = c^2$. So, are 900 ... (writes) $905.77^2 + 281.2^2$. The square root of ... And already we have that equal to our length of cable. Let's make S. Um, then this is S (labels the drawing) which is what we want (Fig. 5).

Working mathematically involved working with the previously defined mathematical model. For this purpose, mathematical operations that led to either an intermediate or the final mathematical result were applied. Further, this activity included recalculations of mathematical results. Pre-service teachers did the calculations either in their heads or with the help of the calculator. Exemplarily, in the following excerpt, Nina calculated the difference between the weight of a full and an empty cabin in order to subsequently infer the number of people in a cabin.

And, namely, the weight difference is, I could calculate it in my head now (enters something into the calculator), 3900 – 1600, so 2300 kg is the weight difference.

Interpreting included relating the intermediate or main mathematical result back to the given real-world situation. In the following excerpt, Lisa wrote down the answer to her self-generated problem in the form of an answer sentence.

Um, and can now answer my problem or my question and say that the old cable car (writes) is 948.42 m long.
[A: The old cable car was 948.42 m long.]

Validating involved evaluating the models that were used and results that were achieved with respect to whether or not they were appropriate for the given real-world situation. Max validated the real model he used in the following excerpt by recognizing that he assumed that the gondola runs without any stops.

Yes, well, ah, what of course now also—what was assumed is that the cabin, when it arrives at the top, also travels directly back again because it does not take a break in between. Because this is probably per hour ... so the performance of this conveyor probably also had breaks calculated in.

To assess the extent to which the individual modelling activities were involved in the solving phases, we analyzed the frequency and duration of the individual modelling activities for 7 pre-service teachers based on the 3 given real-world situations. The frequencies of the modelling activities differed decisively in the solving phase. Most sequences were assigned to the activity working mathematically (115 sequences), somewhat fewer to the activities of simplifying and structuring (109 sequences) and mathematizing (100 sequences), and the smallest number of sequences to the activities interpreting (61 sequences), validating (39 sequences), and understanding (23 sequences). The descending frequency of the activities in the order working mathematically, simplifying and structuring, mathematizing, interpreting, validating and understanding was observed for all three given real-world situations. Regarding the duration of the individual activities, nearly the same tendencies could be observed. A different order was observed for only the three most frequently identified activities. Overall, in the solving phase, the most time was spent simplifying and structuring (63 min, 36% of the overall solving time), followed by working mathematically (42 min, 24% of the overall solving time) and mathematizing (33 min, 19% of the overall solving time). Less time was spent on interpreting (18 min, 10% of the overall solving time) and validating (15 min, 9% of the overall solving time), and only a few minutes were spent on understanding (3 min, 2% of the overall solving time). The descending duration of the activities in the order simplifying and structuring, working mathematically, mathematizing, interpreting, validating and understanding was observed for all three given real-world situations.

6.3 In-Depth Analysis of Theo's Developing and Solving Phases

To gain deeper insight into the connection between problem posing and modelling activities, we want to present Theo's developing and solving phases. For the given real-world situation Cable Car (see Fig. 1), he developed the modelling problem *How many people are transported per hour if only window seats are used?*

6.3.1 Developing Phase

In the beginning, Theo reads the given description of the real-world situation and then he went through the information that was most important to him again.

> Ok, that was too much of the data at once, um, therefore I'm going through them again to get the most important data: So, the type is ... I personally do not care now first. The weight of an empty cabin is 1600 kg and the weight of a full cabin is 3900 kg.

On the basis of the information that he identified as most important, he developed his first problem.

> That is, you could ask yourself how much weight can a cabin transport at all?

On the basis of the first question he generated, he tried to identify what he could calculate next with the answer to the first question and generated additional problems.

> And then estimate how much a person weighs and calculate accordingly how many people per hour.

Then he evaluated that the answer to the problem he posed was already given in the description of the real-world situation.

> Well, whereas no, that's what it says, the number of people per hour, can be transported. Therefore, we already have that information.

Therefore, he read the given situation again, this time focusing on the information about how the aim of the project was to avoid long waiting times and to provide seated transportation with an optimal view, and he developed a new problem that was based on the given information.

> So, my question now would be, how many people are transported with the ... how many people per hour if each person has a window seat.

Finally, he wrote his self-generated problem down and concluded that his problem fit the given real-world situation, as every person had an optimal view.

6.3.2 Solving Phase

The solving phase began directly with Theo identifying the information needed to solve the problem and, based on the given picture, making an assumption about the number of people that would fit in a cabin if only window seats were available.

Fig. 6 Theo's drawing

> The type is irrelevant, weight is now also not relevant for me; the height of the valley and top station are also irrelevant, horizontal distance is also not relevant, travel speed is not relevant, transport capacity. I have to make sure that I do not exceed 500 people per hour in the end, and the power unit is also irrelevant. [...] So, if I make a drawing (made a drawing) (Fig. 6), I could say that the people with distance ... 5 people fit here, and on the other side, exactly the same, 5 people. On the left and on the right, I would do two, with one being an entry, so also really maximum two. That is ... that makes 4 left and right plus 10 total makes 14 people.

After making two initial mathematical models and validating that these would not be appropriate for answering his self-generated problem, he developed a mathematical model that began with him calculating the length of the cable by using the Pythagorean theorem and then using the carrying capacity.

> Horizontal distance (made a drawing) is 905.77; height top station minus (types something into calculator) height valley station 2214.2 − 1133 = 281.2 m. So, using the Pythagorean theorem, it is $\sqrt{905.77^2 + 281.2^2}$ (types something into the calculator). So, it's about 948.4 m. Namely, it manages 8 meters in one second. 948.4 m : 8 (types something into calculator). So it manages to go 948.4 m in 118.55 s. I will recalculate that by 60 and that's about 2 min.

After finishing his calculation, he validated whether his result would fit the given real-world situation and added an assumption about the time needed in the top and valley stations.

> At the valley station, all the people get in, and it takes 2 min. Then it goes for 2 min and then everybody gets off at the top again. 6 min. And the same for the way down. So, it needs 6 min per pass and can always take 14 people.

Then, he calculated how many people per hour could be transported and validated his result with respect to the given real-world situation.

> And I can take a quick look: The carrying capacity is 500 people per hour, so it is quite a bit below that, but you probably could have filled the space in between. When all the people are standing, you can then fit a lot more people in there. So, from that, I would say that that's realistic.

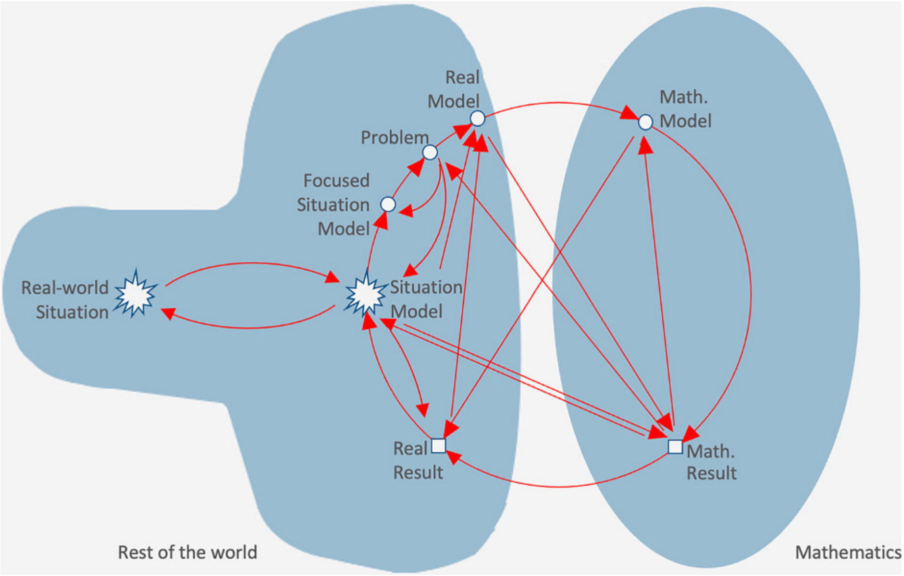


Fig. 7 Theo's individual modelling route

Finally, he answered his self-generated problem. Figure 7 presents Theo's individual modelling route.

In the following, the findings from the in-depth analysis are summarized with regard to the research questions: Regarding Research Question 1 (developing modelling problems), Theo's developing phase involved the problem posing activities understanding, exploring, generating, evaluating, and planning a solution. The activities were by no means linear, as after he evaluated his initial idea as inappropriate, he began exploring again. Theo had already built a situation model during the developing phase by understanding the given situation. Further, he started simplifying and structuring the situation by identifying relevant information that he wanted to focus on for his problem.

Regarding Research Question 2 (solving modelling problems), all modelling activities except understanding were involved in Theo's solving phase. Therefore, his solving phase directly began with him simplifying and structuring the situation model. Further, his solving phase was characterized by jumping back and forth between the individual modelling activities, especially due to his validation of the mathematical models he created.

Overall, a strong interaction between problem posing and modelling activities could be observed. Already in the developing phase, Theo created a situation model by understanding the given situation. As the solving phase directly began with simplifying and structuring, the construction of the situation model might be fully outsourced in the developing phase. Further, Theo began simplifying and structuring his situation model during the developing phase. However, this activity was also addressed in the solving phase for a long period of time. Therefore, Theo may have modified his situation model by focusing important information for posing a problem leading to a focused situation model during the development of a modelling problem and built on this model in the subsequent solving phase to construct a real model (see Fig. 7). In the solving phase, Theo discarded his mathematical model several times. Therefore, it seemed that he did not consider a possible solution strategy while developing his problem.

7 Discussion

7.1 Developing Phase

Based on the empirical findings, we validated the theoretically developed integrated process-model for developing and solving modelling problem. We observed the following activities: understanding, exploring, generating, and evaluating. Identifying the activities exploring, generating, and evaluating is in line with prior studies on posing word- and intra-mathematical problems (Baumanns and Rott 2022b; Christou et al. 2005; Pelczer and Gamboa 2009). The results revealed that some of the activities (exploring and evaluating) described in Wallas (1926) model of creative mathematical thinking could also be observed in the processes involved in modelling-related problem posing. This finding contributes to the assumption that problem posing is a creative process (Bonotto and Santo 2015). Further, we were able to proof the existence of an activity aiming at planning a possible solution in the developing phase supporting the assumption that problem posers think about a possible solution when posing a problem (Cai and Hwang 2002). This activity is not explicitly included in the theoretical integrated process-model as planning a solution is a metacognitive activity and the model describes the cognitive processes of a modeler. Metacognitive activities play a key role in posing and solving modelling problems (Baumanns and Rott 2022a; Stillman 2011), but they are mostly not included in models describing the cognitive processes. In addition to the activities found in prior studies, modelling-related problem posing involved an understanding activity. Understanding is an important activity in solving modelling problems (Blum and Leiss 2007) and is essential for modelling-related problem posing, too.

Regarding the sequence of activities, the integrated process-model presented in Sect. 3 involves an ideal-typical sequence in which the activities take place. The process begins with understanding, which is followed by exploring, generating, and evaluating. However, our empirical findings revealed—as is known from modelling research (Borromeo Ferri 2010) and as has also been described by Baumanns and Rott (2022b) and Pelczer and Gamboa (2009) for problem posing—that problem posers are jumping back and forth between the individual activities leading to individual problem posing routes. This can be seen in Theo's problem posing phase.

Regarding the modelling activities that took place in the developing phase, our analysis revealed the involvement of nearly all modelling activities in developing modelling problems based on given real-world situations. Primarily the modelling activities understanding, simplifying and structuring were already involved in the developing phase. The modelling activity mathematizing rarely occurred in the developing phase. The other modelling activities, working mathematically, interpreting, and validating, were only very rarely or not at all involved in developing own modelling problems. Therefore, problem posing based on given real-world situations might not trigger these activities. Our results indicate that especially the modelling activities that occur in the beginning of solving modelling problems and are located in the rest of the world (i.e., understanding, simplifying, and structuring) are involved in developing a modelling problem when real-world situations are used as problem posing stimuli.

To gather information about the interaction between problem posing and modelling activities, we analyzed the co-occurrence of the activities in the developing phase. The modelling activities understanding, simplifying, and structuring were found to mainly co-occur with the problem posing activities understanding and exploring. This co-occurrence can be explained by the similarity of the activities, as by applying these activities, pre-service teachers aimed to analyze the given situation in an in-depth manner. This proves the assumption involved in the integrated process-model that situation models based on the given real-world situation might already be built up during developing a modelling problem. This can be seen in Theo's developing phase. In his developing phase, understanding the given situation was included. Further, some simplifying and structuring activities were included. The modelling activity mathematizing typically co-occurred with the problem posing activity planning a solution. Therefore, planning activities that occur while developing a modelling problem may help problem posers plan possible solution steps by creating a partial mathematical model. Working mathematically and interpreting co-occurred with the problem posing activities exploring and planning a solution. However, these activities were only rarely involved in the developing phase. Overall, modelling-related problem posing involves modelling activities that are located in the rest of the world and prepare the solving phase. When solving a modelling problem, these modelling activities in particular are known to be difficult (Krawitz et al. 2018; Verschaffel et al. 2020). Therefore, problem posing might stimulate a deeper understanding and examination of the given situation and may help to overcome potential cognitive barriers. This has to be investigated in future studies.

7.2 Solving Phase

An important part of the study was that pre-service teachers had to solve their self-generated problems after posing them. Our analysis of seven pre-service teachers' solving phases revealed that all of the modelling activities known from theory were involved when the participants solved their self-generated problems. However, participants spent only 2% of the time from the solving phase on the activity understanding. This finding is not in line with the results from modelling research indicating that a high proportion of solution time was spent on understanding the given situation when solving a given modelling problem (Leiss et al. 2019; Stillman and Galbraith 1998). A possible explanation could be that a situation model was already developed during the developing phase. This explanation was also supported by the finding that understanding occurred for a long period of time in the developing phase, and around 26% of time was spent on understanding the given situation. Also, Theo focused on understanding in his developing phase for a long period of time and immediately began with simplifying and structuring in his solving phase. Hence, understanding seems to have been primarily outsourced to the developing phase as presented in the integrated process-model (see Fig. 2), and therefore, when participants solved their self-generated problem, the situation model had already been developed and they had to spend only a brief period of time recalling the situation model. By contrast, all other modelling activities were involved for a longer period of time in the solving phase. Hence, problem posing seems to trigger some of these activities (especially simplifying and structuring), but they are again addressed at length during the solving phase. This finding provides evidence that the problem posing activity exploring and the modelling activity simplifying and structuring are two distinct activities that overlap strongly in pre-service teachers' modelling processes. Therefore, it seems not appropriate to assume that based on the exploration a real model is developed as presented in the integrated process-model (see Fig. 2). Rather we collected indications that based on the exploration modelers develop a focused situation model as presented in Theo's modelling route (see Fig. 7). Students use this focused situation model as a basis for building the real model during their solution processes. In contrast to results from modelling research (Blum and Borromeo Ferri 2009), validation activities could be observed in solving the self-generated problem. This finding is important given the value of validation activities for modelling (Czocher 2018). A possible explanation could be that by posing their own problem, modelers feel responsible for their solution, including verifying and checking their results. The effect of problem posing on validation and metacognitive activities, such as planning, monitoring, and regulation, should be investigated in future studies. Along with prompting students to develop multiple solutions (Krug and Schukajlow 2020) and stimulating students' metacognition (Vorhölter 2021), problem posing might be a promising approach for improving students' performance in solving modelling problems. To gain deeper insight into the effect of problem posing on modelling, a comparison of the occurrence of the individual modelling activities when solving a self-generated versus a given problem is needed and should be investigated in future studies. Further, future studies should investigate whether problem posing can affect the performance in solving modelling problems.

8 Strengths and Limitations of the Present Approach

The qualitative research design allowed us to perform an in-depth analysis of pre-service teachers' posing and solving own modelling problems, which should be verified in future studies. However, our study has some limitations that readers should keep in mind when interpreting the results. As we already know from earlier studies that school students find it challenging to pose modelling problems, we conducted our study with pre-service teachers who already had experience with modelling, as we felt that using an experienced sample would help us gather data that would be rich in modelling activities (Hartmann et al. 2021). Due to the specific sample and the small sample size, our results cannot be generalized to other groups. Future studies have to show whether or not our hypothesized theoretical model can be transferred to other samples (e.g., high school or middle school students) and especially to novices in mathematical modelling.

For data collection, we used thinking aloud and SRI. This method is one of the few data collection methods that allows insights into the participants' cognitive processes. However, it is still possible that the participants did not externalize all relevant processes. For example, preservice teachers might have considered intuitively whether and how a problem is solvable before they have verbalized it. In future studies, further data collection methods, such as eye-tracking, should be used in order to collect more information about the processes that take place.

Further, we focused on a specific type of modelling. For this purpose, we used three real-world situations involving different authentic real-world events (e.g., cable car conversion or fire department operations). We used the written descriptions of the situations to ensure that the situations were standardized. Another limitation of this study is taking real-world situations that can be used in school mathematics classes. In the real world, however, people can face more complex situations, which can affect problem posing and modelling processes. Additionally, we focused on problem posing based on an explicit prompt before solving the problems. However, problem posing can also be involved implicitly in the solving phase by specifying the given problem and identifying sub-problems and has a great potential to support the solution of a modelling problem (Hansen and Hana 2015). Overall, it is possible that different activities and connections will be observed if other situations (e.g., experiencing the situation in the real world or using artefacts from the real world) and prompts are used.

Another limitation results from the used instruction. We decided to first instruct the participants to generate a problem and afterwards to solve their self-generated problem. Instructing pre-service teachers to solve their self-generated problems after they have finished the developing phase was necessary as we were interested in the activities involved in developing and solving modelling problems. In order to minimize the effects of the information about the subsequent solving phase, pre-service teachers received this instruction after finishing the developing phase. However, when posing a problem based on the second and third situation, they already knew that they should solve their self-generated subsequently. Thus, planning a solution can appear more often during the developing phase based on the second and third situation.

9 Conclusion

The present study contributes to research on modelling and problem posing by qualitatively exploring the problem posing and modelling activities involved in posing and solving modelling problems and the connection between them. The development of an integrated process-model is a theoretical contribution of our study that supplements the knowledge about the activities involved in solving modelling problems. This model can serve as a vehicle for communicating the activities involved in developing and subsequent solving of own modelling problems and can be used in future studies as an instrument for diagnosing and analyzing the processes of modelling as well as the concomitant difficulties. In addition, this study can help to improve teaching of mathematical modelling as it informs teachers about the cognitive processes involved and teachers with higher knowledge can better support school students' learning (Cai et al. 2015). Our results offer new insights into the modelling process from a problem posing perspective. Modelling-related problem posing includes the activities understanding, exploring, generating, evaluating, and planning a solution and especially involves activities known from solving modelling problems that are located in the real world. The empirical findings provided first evidence for the activities involved in the integrated process-model of developing and solving modelling problems, but also revealed that these activities ran off by no mean linear as presented in the model and are rather characterized by jumping back and forth. The integrated process-model has to be validated in future studies with empirical data from integrated developing and solving processes. On the basis of the findings from this study, we conclude that there is a strong connection between developing and solving modelling problems and that therefore the development of one's own modelling problems has a great deal of potential to foster modelling. The close connection between the problem posing and modelling should be kept in mind when teaching mathematical modelling through modelling-related problem posing.

10 Appendix

Enclosed you will find the real-world situations used in this study called Fire brigade and Chopsticks.

Fire brigade
The Muenster fire department has a total of 16 locations in downtown Muenster so that there is a maximum distance of 6 km from a burning house. On average, a truck can drive about 40 km/h in Muenster city traffic.

A central component of the Muenster fire brigade is a fire engine with a turn-ladder. The dimensions of such a fire engine with a 30 m turn-ladder are specified in the fire department's guidelines.

Dimensions of a fire engine with a turn-ladder:

Length	11.0 m
Width	2.55 m
Height	3.3 m
Weight	16000 kg

By using a fire engine with a turn-ladder, the fire brigade can rescue people from great heights when flames and smoke make it impossible for them to escape from a burning house. The rescue is carried out via a cage attached to the end of the ladder.
To use the fire engine with a turn-ladder, so called HAUS rules, which specify minimum distances for the vehicle, are applied.

Distances for a fire engine with a turn-ladder in operation:

- **1.50 m** distance from objects located to the side of the vehicle when extending the lateral supports
- **7 m** distance from the burning house
- **10 m** distance from objects at the end of the vehicle so that rescued people can leave the rescue cage unhindered.

Fig. 8 The real-world situation called Fire brigade

Chopsticks

Lisa is looking online for a gift for her mother's birthday. Since her mother likes to eat Chinese food, Lisa decided to buy her mother chopsticks. She finds the following offer on Amazon:

Lisa also finds a nice storage box to go with the chopsticks.

Product information

Technical Details

Product Dimensions	27.5 x 7 x 4.5; 499 g
Item weight	499 gram
Quantity	1

To save some money, Lisa searches online for a promotional discount. She discovers a promotional discount where she gets a 10% discount on her entire purchase if her purchase is worth 20€ or more and a 20% discount if her purchase is worth 30€ or more.

Fig. 9 The real-world situation called Chopsticks

Acknowledgements The authors express their appreciation to the editors and the anonymous reviewers for their relevant feedback and recommendations, which significantly enhanced the paper's quality.

Funding Open Access funding enabled and organized by Projekt DEAL.

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