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ORIGINAL ARTICLE

Generalized fuzzy variable precision rough sets based on bisimulations and the corresponding decision-making

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Abstract
Recently, the classical rough set has been extended in many ways. However, some of them are based on binary relations which only excavate "one step" information to distinguish objects. The "one step" in the binary relation means that the ordered pair of the starting and end points of the step belongs to the relation. Faced with some complex data sets, the "one step" information may be not feasible. Motivated by the notion of bisimulation in computer science, three types of bisimulation-based generalized fuzzy variable precision rough set (BGFVPRS) models are constructed. Different from many existed rough set models which are based on binary relations, the BGFVPRS models can distinguish objects by excavating the "multi-step" information of underlying relations. The related properties and relationships of BGFVPRS models are investigated. The uncertainty measure of BGFVPRS models and the reduction of fuzzy bisimulations are also discussed. Furthermore, learning from the PROMETHEE II method and combining it with our presented BGFVPRS models, a novel multiple-attribute decision-making method is provided. This method can effectively deal with complex problems including attribute data and relational data. The flexibility and effectiveness of our decision-making method are illustrated by comparative analysis and sensitivity analysis in the Zachary karate club network.

Keywords Multi-attribute decision-making · Bisimulation · PROMETHEE II method · Relational data · Fuzzy variable precision rough set · Fuzzy logical operator

1 Introduction

Multi-attribute decision-making (MADM) methods are well-known decision-making methods that aim to select the best alternative from the ranking order for all alternatives under the multiple attributes data tables. With the increasing complexity of the research problem, decision-makers (DMs) have to consider the relational data among the studied objects. For example, during the outbreak of a novel coronavirus, experts must consider whether people infected with the virus come into contact with others. In this case, some popular MADM methods only considering the attribute data seem unavailable. In view of this, an approach based on the generalized fuzzy variable precision rough set models with fuzzy bisimulations is presented in this paper. In the following, the development of MADM methods and the classical rough set (RS) model is briefly reviewed, respectively.

The MADM methods are widely used in many fields, such as investment decision-making and project evaluation. The essence of MADM method in [16] is that using the existing decision-making information to sort a group of (finite) schemes or select the optimal object in a certain way. It consists of two parts: obtaining decision information; gathering decision information in a certain way, and ranking the alternatives and selecting the best alternative. In [3, 19, 31, 33, 34, 45], many MADM methods are provided. The outranking approaches are a widely used class of MADM methods. These approaches are taken to identify whether the alternatives under consideration are preferable, neutral, or incomparable with attributes than other alternatives.

It is well known that ELECTRE [49, 59] and PROMETHEE [6, 29] methods are two main methods of outranking approaches. The PROMETHEE method, as a new sorting method in multi-attribute analysis, is stable, simple, and clear. Brans [6] first introduced the concept of PROMETHEE. Hereafter, many scholars developed it in many

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ways [4, 8]. The PROMETHEE I and PROMETHEE II methods are two versions of the PROMETHEE method. Through a comparison between them, PROMETHEE II method gets a complete ranking of all alternatives while PROMETHEE I obtains a partial sort of all alternatives. Then PROMETHEE II method [51] has been widely applied in practical applications. Because of the complexity of the environment in real life, the data tends to be vague or imprecise. It is hard to calculate or rank directly by means of the classical PROMETHEE II method based on the crisp sets for dealing with the vague or imprecise data. In addition, combining the ideas of fuzzy set theory (FST) [61] and the classical PROMETHEE II method [5], scholars have proposed some fuzzy PROMETHEE II methods [28, 51, 57, 66] to solve complex problems with vague information or imprecise data. A study on the applications of some MADM methods (such as PROMETHEE II method) shows that they may not be enough to solve the complicated problems involving relational data. For example, these MADM methods in [6, 20, 59, 63, 64] seem invalid for solving social network problems. The reason is that they only consider the attribute data and ignore the relational data between objects.

Figure 1 shows the development of the RS model. For effectively extracting useful information and making a clear decision from much uncertain knowledge in information systems, the RS model was early put forward by Pawlak [40]. Although this model can handle many uncertain problems well, it is hindered by some shortcomings in the development process. For example, the equivalence relation of RS model is too idealistic to be applied in real life. Because most of the relations in real life are general, they cannot always meet the harsh conditions of equivalence relations. At present, RS model is developed from the aspects of universe, equivalence relation, intersection and union, division

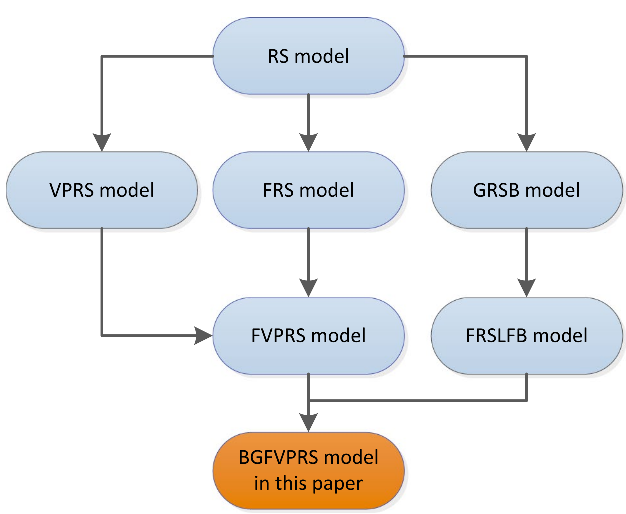

Fig. 1 A simple introduction about the development of RS models

and so on. The equivalence relations are extended to many forms such as neighborhood relations [32, 53, 55, 58], tolerance relations [18, 26, 43], similarity relations [1, 44], and even arbitrary binary relations [48, 54]. Based on the extensions of equivalence relations, some generalized RS models were born in [62–65]. On the foundation of the predecessor or successor neighborhoods, Yao [55] discussed the significant properties of the lower and upper approximations. Based on the similarity relations, two definitions of rough approximations were compared in [41]. In a relational structure, Fan [14] considered three key concepts of indiscernibility: congruence, bisimulation, and exact equivalence. By these indiscernibility relations, she also investigated rough approximations and knowledge reduction. Concerning a fuzzy relational structure, Du and Zhu took the bisimulation in [36] as indiscernibility and constructed a pair of fuzzy approximation operators in [12]. Zhu et al. pointed out that many generalized rough approximations which are on the basis of binary relations, only rely on “one step” information of potential relations in [68]. The “ one step” information could be not effective for discerning objects in some complex environments. Inspired by the concept of bisimulation, they put forward a generalized rough set based on bisimulation (GRSB). Subsequently, Du and Zhu [13] focused on the labeled fuzzy approximation space and took the bisimulation into consideration, and they provided a new pair of approximation operators in the fuzzy rough set based on the largest fuzzy bisimulation (FRSLFB).

A large number of researches on existed RS models find that some of them are sensitive to misclassification and perturbations. In order to make up this defect, the variable precision rough set (VPRS) in [35, 69] was proposed. Since then, many researchers have expanded the VPRS models and applied them in many complicated areas. The measure, which the VPRS is based on, can be written as the form of the conditional probabilities in a decision-theoretic rough set (DTRS) [56]. In this time, the VPRS is an exception to DTRS. The VPRS has been further studied under the fuzzy environment, and some achievements have been made. Combining the fuzzy rough set (FRS) [46] and VPRS [69], a powerful tool named fuzzy variable precision rough set (FVPRS) was constructed in [67]. It cannot only deal with numerical data but also is insensitive to misclassification and disturbance. In 2020, Zhan et al. [62] modified Ma’s fuzzy neighborhood operator and proposed a novel β-neighborhood operator that satisfies the reflexivity. Based on their defined neighborhood operator, a covering-based variable precision fuzzy rough set (CVPFRS) was presented. Meanwhile, Jiang et al. [22] presented a new type of CVPFRS models by means of fuzzy neighborhoods and applied it in the context of medical diagnosis.

From the above discussions, combing the ideas of bisimulation and VPRS models, some types of bisimulations-based generalized fuzzy variable precision rough set (BGFVPRS) models are constructed in the paper. Considering PROMETHEE II method and on the basis of BGFVPRS models, we hope to design a decision-making method for solving some complicated problems including not only the attribute data (or relational data) but also the mixed data including both the attribute data and relational data.

Next, the motivations of the paper are briefly introduced as follows.

- The classical PROMETHEE methods (including PROMETHEE I and PROMETHEE II) [5, 6] aim at dealing with the problems which only include the attribute data. However, they are invalid for solving the problems which contain relational data. For example, faced with the social network analysis problems, the classical PROMETHEE II method may be infeasible. In view of this, motivated by the idea of the PROMETHEE II method, an approach is constructed for settling these complex problems.
- Through observations, some MADM methods in [20, 62–64, 66] based on the extension of RS are effective for the multiple-attribute problems. However, faced with the problems involving relational data, they may be impossible to make a feasible decision, or even choose an optimal object. For this reason, a novel decision-making method based on BGFVPRS models and PROMETHEE II method is provided.
- On the basis of bisimulations relations, some approximation operators were constructed in [12–14]. It is worth noting that these models are only deeply studied in theory, and rarely studied in application. Inspired by this, based on our proposed BGFVPRS models, we hope to put forward an approach and apply it in real life.
- Some extended rough set models based on bisimulation [14, 68] and other generalized rough set models are sensitive to disturbance or misclassification, and no errors are allowed. To enrich the theoretical research of the RS models and apply it in wider field, three types of BGFVPRS models are raised. These models are combing the VPRS models, FRS models, and bisimulations that can help DMs extract "multi-step" information.
- Because of the different requirements of different DMs during different periods, the decision-making method is required to have high flexibility. Through the research of some algorithms, we find that they cannot change flexibly with the change of DMs' preferences. Then, it forces us to search for a flexible decision-making method to help DMs make a clear decision. DMs can adopt different fuzzy logical operators, negator operators, and variable precision values, according to the actual situation and their preferences. It also shows the flexibility of our decision-making method.

Furthermore, the main contribution of the paper is shown as follows.

- Our proposed BGFVPRS models can depend on the "multi-step" information of underlying relationship and can be applied in the complex environment including complex databases.
- By adopting different fuzzy logic operators and changing the value of θ, the decision-making method proposed in the paper can help DMs make a clear decision according to the actual situation and their own preferences.
- Compared with many existing RS models, the BGFVPRS models can be used as a powerful tool. They can solve the relational data or mixed data which contains relational data and attribute data. Besides, they are less sensitive for misclassification and disturbance.
- The method based on BGFVPRS models extends the application range of some classical MADM methods, such as the classical PROMETHEE II method which is ineffective for social network analysis and other relational data issues.

The remainder of the paper is as follows. Some correlated preliminary concepts are given in Sect. 2. Section 3 defines three types of BGFVPRS models and investigates the relationships among these models and some existing RS models. Besides, this section also discusses the related properties of BGFVPRS models. In Sect. 4, the uncertainty measure of BGFVPRS models and the reduction of fuzzy bisimulations are put forward. Section 5 gives a decision-making method based on the BGFVPRS models and the principle of PROMETHEE II. Through an example of selecting the best alternative in Zachary karate club network, from the comparative analysis and sensitivity analysis, the flexibility and effectiveness of our raised method are illustrated. In Sect. 6, a conclusion of our work is made and some further researches are given.

2 Preliminary

Some related preliminary concepts and properties of fuzzy logical operators, fuzzy relations, and fuzzy bisimulations are briefly introduced in this section.

2.1 Fuzzy logical operators

In the paper, $\mathcal{S}$ denotes the universe of the discourse, $\mathcal{F}(\mathcal{S})$ represents the set of all fuzzy sets, and $\hat{\gamma}$ indicates a constant fuzzy set in which $\hat{\gamma}(a) = \gamma$ for each $a \in \mathcal{S}$, $\hat{\gamma} \in \mathcal{F}(\mathcal{S})$ and $\gamma \in [0, 1]$. Without special explanation, $\mathcal{S}$ is finite.

Firstly, some significant fuzzy logical operators in [41] and the related properties of them are introduced as follows.

Definition 1 [41] If a mapping $T : [0,1] \times [0,1] \rightarrow [0,1]$ satisfies the following conditions:

(1) $T(\mu_1, \mu_2) = T(\mu_2, \mu_1)$;
(2) $T(T(\mu_1, \mu_2), \mu_3) = T(\mu_1, T(\mu_2, \mu_3))$;
(3) if $\mu_1 \leq \mu_3, \mu_2 \leq \mu_4$, then $T(\mu_1, \mu_2) \leq T(\mu_3, \mu_4)$;
(4) $T(1, \mu_1) = \mu_1$;

for all $\mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$, then T is named as a t-norm on $[0, 1]$. Three widely known continuous t-norms are the min operator T_M, the algebraic product T_P and the ˇukasiewicz t-norm T_L where $T_M(\mu_1, \mu_2) = \mu_1 \wedge \mu_2$, $T_P(\mu_1, \mu_2) = \mu_1 \cdot \mu_2$ and $T_L(\mu_1, \mu_2) = 0 \vee (\mu_1 + \mu_2 - 1)$ for each $\mu_1, \mu_2 \in [0, 1]$.

Definition 2 [41] Let a mapping $\mathfrak{T} : [0,1] \times [0,1] \rightarrow [0,1]$. If $\mathfrak{T}$ satisfies the following conditions:

(1) $\mathfrak{T}(\mu_1, \mu_2) = \mathfrak{T}(\mu_2, \mu_1)$;
(2) $\mathfrak{T}(\mathfrak{T}(\mu_1, \mu_2), \mu_3) = \mathfrak{T}(\mu_1, \mathfrak{T}(\mu_2, \mu_3))$;
(3) if $\mu_1 \leq \mu_3, \mu_2 \leq \mu_4$, then $\mathfrak{T}(\mu_1, \mu_2) \leq \mathfrak{T}(\mu_3, \mu_4)$;
(4) $\mathfrak{T}(0, \mu_1) = \mu_1$;

for any $\mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$, then $\mathfrak{T}$ is called an S-norm on $[0, 1]$.

The three most popular S-norms are the standard max operator $\mathfrak{T}_M$, the probabilistic sum $\mathfrak{T}_P$ and the bounded sum $\mathfrak{T}_L$ where $\mathfrak{T}_M(\mu_1, \mu_2) = \mu_1 \vee \mu_2$, $\mathfrak{T}_P(\mu_1, \mu_2) = \mu_1 + \mu_2 - \mu_1 \cdot \mu_2$, and $\mathfrak{T}_L(\mu_1, \mu_2) = 1 \wedge (\mu_1 + \mu_2)$ for each $\mu_1, \mu_2 \in [0, 1]$. It is worth noting that when T-norm and S-norm are two continuous functions on $[0, 1]$, they are called the continuous T-norm and continuous S-norm, respectively.

Definition 3 [41] Suppose that $\mathcal{N} : [0,1] \times [0,1] \longrightarrow [0,1]$ is a decreasing mapping. If $\mathcal{N}$ meets the condition that $\mathcal{N}(1) = 0$ and $\mathcal{N}(0) = 1$, then it is a negator operator. For each $\mu_1 \in [0, 1]$, $\mathcal{N}$ is a standard negator operator if $\mathcal{N}(\mu_1) = 1 - \mu_1$. In the following, $\mathcal{N}_s$ is used to represent the standard negator operator. Furthermore, $\mathcal{N}$ is involutive when $\mathcal{N}(\mathcal{N}(\mu_1)) = \mu_1$.

In the paper, using symbol $co_{\mathcal{N}}$ denotes the fuzzy complement. In other words, $co_{\mathcal{N}}(K)(t_i) = \mathcal{N}(K(t_i))$ for each $K \in \mathcal{F}(\mathcal{S}), t_i \in \mathcal{S}$.

Definition 4 [41] Given a mapping $I : [0,1] \times [0,1] \longrightarrow [0,1]$, if I contents the requirement that $I(0,0) = I(0,1) = I(1,1) = 1$ as well as $I(1,0) = 0$, then it is referred as a fuzzy implicator operator.

A border implicator is an implicator I if it satisfies $I(1, \mu_1) = \mu_1$, for each $\mu_1 \in [0, 1]$. For any $\mu_1 \in [0, 1]$, if implicator I meets that $I(\cdot, \mu_1)$ is decreasing (resp. $I(\mu_1, \cdot)$ is increasing), I is left monotonic (resp. right monotonic). An implicator I is hybrid monotonic if it is both left monotonic and right monotonic.

Three most popular fuzzy implicators are R-implicator, S-implicator and QL-implicator in [41]. An R-implicator I based on a continuous t-norm T iff for each $\mu_1, \mu_2 \in [0, 1]$, $I(\mu_1, \mu_2) = \sup\{\gamma \in [0, 1] : T(\mu_1, \gamma) \leq \mu_2\}$. An S-implicator I based on $\mathfrak{T}$ and $\mathcal{N}$ iff $I(\mu_1, \mu_2) = \mathfrak{T}(\mathcal{N}(\mu_1), \mu_2)$ for every $\mu_1, \mu_2 \in [0, 1]$. A QL-implicator I based on T, $\mathfrak{T}$ and $\mathcal{N}$ iff for each $\mu_1, \mu_2 \in [0, 1]$, $I(\mu_1, \mu_2) = \mathfrak{T}(\mathcal{N}(\mu_1), T(\mu_1, \mu_2))$.

2.2 Fuzzy relations

In this section, some important concepts and properties of fuzzy relations are introduced. Firstly, we show the concept of a binary fuzzy relation and the corresponding properties. A binary relation is a subset of $\mathcal{S} \times \mathcal{S}$ where $\mathcal{S} \times \mathcal{S}$ is the product set of $\mathcal{S}$ and $\mathcal{S}$. The binary fuzzy relation is the natural extension of the binary relation. Compared with binary relation, the application of binary fuzzy relation is wider. With the help of binary fuzzy relation, some problems in a fuzzy environment can be solved well.

Definition 5 [24, 39] A binary fuzzy relation is a mapping $\psi: \mathcal{S} \times \mathcal{S} \longrightarrow [0, 1]$. The binary fuzzy relation is called:

(1) reflexive if for each $t_i \in \mathcal{S}, \psi(t_i, t_i) = 1$;
(2) symmetric if for each $t_i, t_j \in \mathcal{S}, \psi(t_i, t_j) = \psi(t_j, t_i)$;
(3) T-transitive if for each $t_i, t_j, t_k \in \mathcal{S}, T(\psi(t_i, t_k), \psi(t_k, t_j)) \leq \psi(t_i, t_j)$.

For each $t, z \in \mathcal{S}$, when $\psi(t, z) = 1$ or $\psi(t, z) = 0$, the binary fuzzy relation is a binary relation. In other words, the binary relation is a special case of the binary fuzzy relation. In Definition 5, the ψ is a T-similarity relation if it satisfies the conditions (1), (2), and (3). If T=$\wedge$ and ψ satisfies the condition (3), then ψ is transitive. In this case, if ψ meets the above three conditions, ψ is a fuzzy equivalence relation.

The n-ary fuzzy relation is a natural extension of the binary fuzzy relation. A fuzzy set $S_1 \times S_2 \times \cdots \times S_n$ is named as n-ary fuzzy relation between $S_1, S_2, \ldots, S_n$. If $S_1 = S_2 = \cdots = S_n = S$, $S_1 \times S_2 \times \cdots \times S_n$ is an n-ary fuzzy relation on a set S. For simplicity, the set of all n-ary fuzzy relation on S is denoted by S^n. In the paper, the n-ary fuzzy relation is called the fuzzy relation.

In [9, 25], the composition of binary fuzzy relations is given. For non-empty sets S_1, S_2, S_3 and two binary fuzzy relations $\varphi_1 \in \mathcal{F}(S_1 \times S_2)$, $\varphi_2 \in \mathcal{F}(S_2 \times S_3)$, their composition $\varphi_1 \circ \varphi_2$ is a binary fuzzy relation from $\mathcal{F}(S_1 \times S_3)$ defined by $(\varphi_1 \circ \varphi_2)(a, c) = \bigvee_{b \in S_2} \varphi_1(a, b) \otimes \varphi_2(b, c)$, where $\otimes$ (called multiplication) is applied to model the conjunction. Subsequently, the composition of n-ary fuzzy relations is provided.

Assume that ψ_1 is a (h_1+h)-ary fuzzy relation and ψ_2 is a $(h+h_2)$-ary fuzzy relation where $\psi_1 \in S^{(h_1+h)}$ and $\psi_2 \in S^{(h+h_2)}$. Then, the h-composition of ψ_1 and ψ_2 is a (h_1+h_2)-ary fuzzy relation defined as:

$$(\psi_1 \circ_h \psi_2)(\mathbf{a},\mathbf{c}) = \bigvee_{\mathbf{b}=(b_1,b_2,\ldots,b_h)\in S^h} \psi_1(\mathbf{a},\mathbf{b}) \otimes \psi_2(\mathbf{b},\mathbf{c}),$$

for each $(\mathbf{a},\mathbf{c}) \in S^{h_1+h_2}$ where $\mathbf{a}=(a_1,a_2,\ldots,a_{h_1})$ and $\mathbf{c}=(c_1,c_2,\ldots,c_{h_2})$. Especially, if h=1, the subscript is omitted and the formula is written as $\psi_1 \circ \psi_2$.

2.3 Fuzzy bisimulations

Before the introduction of fuzzy bisimulation, we firstly bring into the concept of bisimulation. The fuzzy-language equivalence is a tool to compare the behavior of fuzzy systems. However, at this level, the comparison is too coarse. By means of this tool, the behavior of some fuzzy systems cannot be compared well. In view of this, bisimulation as a finer behavior measure has been introduced in fuzzy-finite automata. Cao et al. [7] considered this bisimulation for general fuzzy systems and obtained some results which are useful to compare the behavior of these systems. In the following, the concept of bisimulation defined in [68] is introduced.

Definition 6 [68] Assume that $(\mathcal{S}, R)$ is a generalized approximation space. A binary relation $B \subseteq \mathcal{S} \times \mathcal{S}$ is called a bisimulation if for all $(z,w) \in B$

(1) $(z,z') \in R$ implies that $(w,w') \in R$ for some $w \in \mathcal{S}$ with $(z',w') \in B$,
(2) $(w,w') \in R$ implies that $(z,z') \in R$ for some $w' \in \mathcal{S}$ with $(z',w') \in B$.

If B is a bisimulation and $(z,w) \in B$, then states z and w are bisimilar, denoted by $z \sim w$. In this paper, the relation $\sim$ is named as bisimilarity.

A bisimulation is a binary relation between discrete event systems. It relates systems with the same behavior. That is, one system simulates another, and vice versa. Intuitively, if the actions of the two systems match, they are bisimilar. In order to understand this relation well, an example is given as follows.

Example 1 Figure 2 represents a generalized approximation space where $\mathcal{S}=\{s_1,s_2,s_3,s_4,t_1,t_2,t_3\}$ and the solid arrows represent the relation R. Assume that a binary relation $R_1=\{(s_2,t_2),(s_4,t_3)\}$. From the Fig. 2, we can find that when the state s_2 moves one step, then the state t_2 moves one step. Similarly, when the state t_2 moves one step, the state s_2 also moves one step. After moving one step, the subsequent states of s_2 and t_2 are s_4 and t_3, respectively. In this time, $(s_4,t_3) \in R_1$. Furthermore, state s_4 stops when state t_3 stops, and vice versa. Then, the relation R_1 is called a bisimulation.

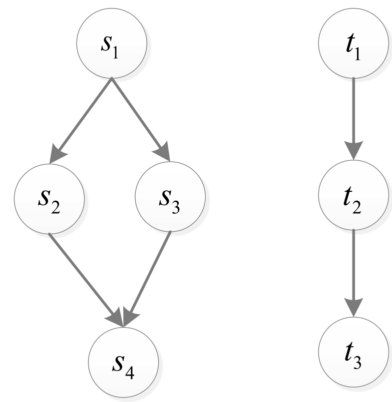

Fig. 2 The generalized approximation space $(\mathcal{S}, R)$

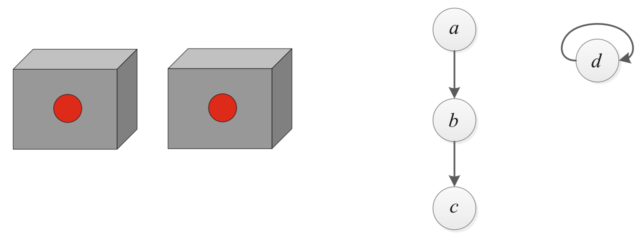

Fig. 3 Two black boxes and their behaviors

Compared with some binary relations which only obtain "one step" information, the bisimulation can excavate "multi-step" information. It is worth noting that the "one step" information could be not effective for discerning objects in some complex environments. In the following, an example is given to illustrate this. This example is adapted from [68].

Example 2 Assume that two reactive systems as two black boxes with one red button each, shown as Fig. 3. Through pressing the buttons, the interaction between persons and black boxes has been done. Sometimes the red button goes down, and sometimes it does not. The button going down means the persons succeed while the button not going down means the persons don't succeed. By this way, the difference between the two black boxes can be told. Assume that the button of the first black box can successively go down two times from its initial state and after that, it does not continue; for the second one, it can always go down. The right state transition graph in Fig. 3 shows their behaviors.

Given a generalized approximation space $(\mathcal{S}, R)$ where $\mathcal{S} = \{a, b, c, d\}$, $R = \{(a, b), (b, c), (d, d)\}$. Assume that the successor neighborhood $R_s(z) = \{w \in \mathcal{S} \mid (z, w) \in R\}$ of z. Furthermore, let us approximate a concept $W = \{a, d\}$. Using the approximation operators based on the successor neighborhoods, we have that

$$\underline{apr}_{R_s}(W) = \{z \in \mathcal{S} \mid R_s(z) \subseteq W\} = \{d\},$$
$$\overline{apr}_{R_s}(W) = \{z \in \mathcal{S} \mid R_s(z) \cap W \neq \emptyset\} = \{d\}.$$

It is known to us that the basic idea of rough set is that using a pair precise concept to describe the incomplete or inexact concepts. Through our research, the four states are different from one another. For example, in state a the left red button can successively go down two times, while in state d the right red button can only go down one time. Therefore, each state should be regarded as a granule of knowledge. In view of this, approximating the concept W by the union of two knowledge granules $\{a\}$ and $\{d\}$ i.e., $\{a, d\}$ is much better than using $\{d\}$ and $\{d\}$.

But, considering the approximations based on bisimilarity $\sim$ which is the largest bisimulation, we have that:

$$\sim = \{(a, a), (b, b), (c, c), (d, d)\}, [t]_\sim = \{y \in \mathcal{S} \mid (t, y) \in \sim\},$$
$$\underline{apr}_\sim(W) = \{t \in \mathcal{S} \mid [t]_\sim \subseteq W\} = \{a, d\},$$
$$\overline{apr}_\sim(W) = \{t \in \mathcal{S} \mid [t]_\sim \cap W \neq \emptyset\} = \{a, d\}.$$

In this case, by the basic idea of rough set, we should using $\{a, d\}$ to describe the concept W. Through the comparison between the two rough models, we find that the results of the second model does follow our intuition. This shows that the bisimulation plays an important role in some situations.

However, the bisimulation just can distinguish whether two states belong to bisimulation relation. If the relation between states is vague or fuzzy, it seems infeasible. Then, extending this relation in a fuzzy environment is necessary. In the following, a concept of fuzzy bisimulation which is the natural extension of bisimulation is given.

According to [14], a fuzzy relational structure is a binary tuple $(\mathcal{S}, (R_i)_{i\in\Lambda})$ where $\mathcal{S}$ is a universe, Λ is an index set and R_i is a fuzzy relation on $\mathcal{S}$. Based on this structure, the concept of fuzzy bisimulation is defined as follows.

Definition 7 [12] Assume that $(\mathcal{S}, (R_i)_{i\in\Lambda})$ is a fuzzy relational structure. A binary fuzzy relation ψ is referred to as a fuzzy bisimulation on $\mathcal{S}$, for every $(R_i)_{i\in\Lambda}$ if it meets the following conditions:

(1) $\psi^- \circ R_i \subseteq R_i \circ \psi^-$,
(2) $\psi \circ R_i \subseteq R_i \circ \psi$,

where ψ^- is the reverse of the fuzzy relation ψ. For each $s_1, s_2 \in \mathcal{S}$, $\psi^-(s_1, s_2) = \psi(s_2, s_1)$.

Note that if a fuzzy bisimulation φ satisfies the condition (1) in Definition 5, it is a reflexive fuzzy bisimulation. If φ meets the condition (2) in Definition 5, φ is called a symmetric fuzzy bisimulation. When φ fits the condition (3) in Definition 5, φ is a T-transitive fuzzy bisimulation. Especially, if φ meets the conditions (1), (2), and (3) in Definition 5, φ is a fuzzy T-similarity bisimulation relation. Similarly, if φ is reflective, symmetric, and transitive, it is a fuzzy equivalence bisimulation relation. The largest bisimulation [36] is an equivalence relation which is a special fuzzy equivalence bisimulation relation.

For better understand the concept of the fuzzy bisimulation, an example is given as follows.

Example 3 Let $(\mathcal{S}, R_1 \cup R_2)$ be a fuzzy relational structure and $\otimes = \wedge$. Here $\mathcal{S} = \{t_0, t_1\}$ and $R_1 = \frac{0.5}{(t_0,t_0)} + \frac{0.5}{(t_1,t_0)}$ and $R_2 = \frac{0.8}{(t_0,t_0,t_1)} + \frac{0.6}{(t_0,t_1,t_1)} + \frac{0.5}{(t_1,t_0,t_0)} + \frac{0.5}{(t_1,t_1,t_0)}$. Assume that a binary fuzzy relation $\varphi = \frac{1}{(t_0,t_0)} + \frac{0.5}{(t_0,t_1)} + \frac{0.5}{(t_1,t_0)} + \frac{1}{(t_1,t_1)}$. By Definition 7, it is easy to verify that φ is a fuzzy bisimulation.

The bisimulation is a special case of fuzzy bisimulation. If the fuzzy bisimulation φ is a bisimulation on $\mathcal{S} \times \mathcal{S}$, $\varphi(a, b) \in \{0, 1\}$ for each $a, b \in \mathcal{S}$. Compared with bisimulation, the application of fuzzy bisimulation is wider than the bisimulation. For example, if the relation between states of objects is vague, the bisimulation seems infeasible. But, the fuzzy bisimulation is still effective. In a word, the fuzzy bisimulation can be applied for dealing with vague relations in a fuzzy environment while the bisimulation cannot.

3 Generalized fuzzy variable precision rough sets based on bisimulations

The fuzzy logic operator [41] plays an important role in fuzzy set theory. In order to overcome some defects of the classical RS model, Radzikowska and Kerre proposed a general FRS model [41] based on the fuzzy logical operators. However, the FRS models and the classical RS model [40] are easily disturbed by noise data sets. To make up for this shortcoming, the VP model [69] is proposed. Based on logical operators, some extensions of FVPRS models [20, 21, 62, 67] are constructed. Their work further improves the theoretical basis of classical RS model. A study shows that these models are based on the binary relations which are only dependent on "one step" information. In the face of some more complex data sets, it is not enough to distinguish objects only by "one step" information. Therefore, we need an effective tools to obtain "multi-step" information for distinguishing objects. The fuzzy bisimulations can excavate "multi-step" information for underlying relations. By means of the fuzzy bisimulations, objects can be well distinguished. At present, there is rare research on RS models based on fuzzy bisimulations. Then, we want to develop the RS model by means of these relations.

To improve the RS model, enrich its theoretical research, and make it applicable to a wider range of fields, we want to present some generalized RS models. Motivated by fuzzy bisimulations and the concept of VP models, three types of BGFVPRS models are defined in this section. The relationships among these BGFVPRS models and other types of existing RS models are investigated. Besides, some feasible properties of these BGFVPRS models are discussed. In the paper, since some properties are easy to obtain, then we omit some trivial proof.

Definition 8 Assume $(\mathcal{S}, (R_i)_{i\in\Lambda})$ is a fuzzy relational structure and φ is a fuzzy bisimulation. Suppose that I_1, I_2 are two fuzzy implicators and T_1, T_2 are two continuous t-norms. For each $K \in \mathcal{F}(\mathcal{S})$, the first type of bisimulation-based generalized fuzzy variable precision I_1T_2-lower approximation (1-BGFVPI_1T_2A) $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)$ and T_1I_2-upper approximation (1-BGFVPT_1I_2UA) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)$ of K, with negator $\mathcal{N}$ and the variable precision $\theta \in [0,1)$, are defined as:

$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), \theta \vee K(c))), \tag{1}$$

$$\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta) \wedge K(c))). \tag{2}$$

If $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \neq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)$, we call K the first type of bisimulation-based generalized fuzzy variable precision rough set (1-BGFVPRS), otherwise we call it definable.

To better understand Definition 8, an example is given to illustrate it in the following. It is noted that the labeled (or unlabeled) fuzzy transition systems [23] are special fuzzy relational structures.

Example 4 Let $(\mathcal{S}, R)$ be a fuzzy relational structure. Assume that $S_1 \subseteq S$, and $\mathcal{S}_1 = \{t_0, t_1, t_2, t_3, t_4\}$. Figure 4a is an unlabeled transition system. In the figure, $t_i \xrightarrow{\alpha} t_j$ represents $R(t_i, t_j) = \alpha$ where $(i, j = 0, 1, 2, 3, 4)$ and $\alpha \in [0, 1]$. Here, $R(t_i, t_j) = \alpha$ denotes that the membership degree of t_i and t_j belonging to the binary fuzzy relation R is α. Assume that φ is a binary fuzzy relation where $\varphi(t_i, t_i) = 1$ and $\varphi(t_i, t_j) = 0$ $(i \neq j)$.

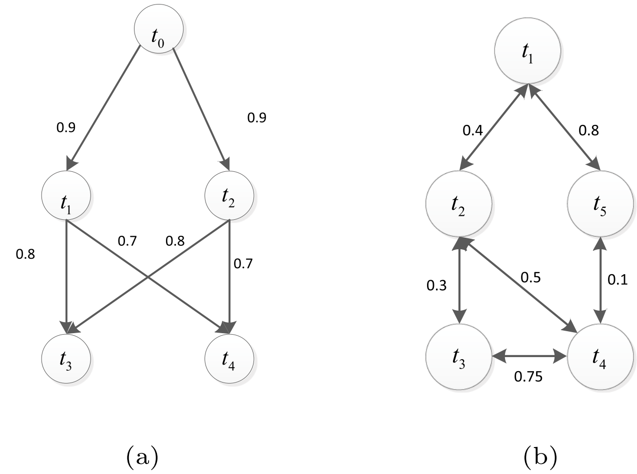

Fig. 4 **a** Is a fuzzy transition system and **b** is a social network for 5 persons

According to Fig. 4a, $R = \frac{0.9}{(t_0,t_1)} + \frac{0.9}{(t_0,t_2)} + \frac{0.8}{(t_1,t_3)} + \frac{0.7}{(t_1,t_4)} + \frac{0.8}{(t_2,t_3)} + \frac{0.7}{(t_2,t_4)}$ can be got. Let $\otimes = \wedge$. Since $\varphi^- \circ R \subseteq R \circ \varphi^-$ and $\varphi \circ R \subseteq R \circ \varphi$, then φ is a fuzzy bisimulation by Definition 7. Furthermore, assume that $I_L(k_1, k_2) = \min(1, 1 - k_1 + k_2)$, for each $k_1, k_2 \in [0, 1]$. Let $I_1 = I_2 = I_L, T_1 = T_2 = T_L$ and $\mathcal{N} = \mathcal{N}_s$. Suppose that θ =0.6 and $K = \frac{0.8}{t_0} + \frac{0.3}{t_1} + \frac{0.5}{t_2} + \frac{0.6}{t_3} + \frac{0.8}{t_4}$, and by Definition 8, the following results are got:

$$\underline{apr}^{\varphi,0.6}_{I_L,T_L}(K) = \frac{0.8}{t_0} + \frac{0.6}{t_1} + \frac{0.6}{t_2} + \frac{0.6}{t_3} + \frac{0.8}{t_4},$$
$$\overline{apr}^{\varphi,0.6}_{T_L,I_L}(K) = \frac{0.4}{t_0} + \frac{0.3}{t_1} + \frac{0.4}{t_2} + \frac{0.4}{t_3} + \frac{0.4}{t_4}.$$

Thus K is a 1-BGFVPRS.

In the following, another example is given to help us understand Definition 8 better.

Example 5 The center for disease control and prevention gets a message that a man has a novel coronavirus pneumonia test and the result of the test is positive. The institution immediately finds that the person t_1 who had dinner together John H. recently. A social network map for 5 persons is shown as Fig. 4b. In the figure, $t_i \overset{\alpha}{\longleftrightarrow} t_j$ means the contact degree between person t_i and t_j is $\alpha \in [0, 1]$ where $(i, j = 1, 2, 3, 4, 5)$. Here, $t_i \overset{\alpha}{\longleftrightarrow} t_j$ is equivalent to $R(t_i, t_j) = \alpha$. Note that the contact relationship is bidirectional or symmetric. Then the binary fuzzy relation R is symmetric. Even though the first testing results of 5 persons are negative, the experts need to observe and treat them further. Through observation, they obtain an assessment result as $K = \frac{0.75}{t_1} + \frac{0.3}{t_2} + \frac{0.5}{t_3} + \frac{0.55}{t_4} + \frac{0.8}{t_5}$. The value of $K(t_i)$ is the possibility of infection for person t_i. Based on experience, experts choose a regulatory threshold $\theta = 0.6$, a fuzzy relation $\varphi = R$, and $\otimes = \wedge$. The φ is a fuzzy bisimulation by Definition 7. According to experts' preferences and the complexity of the environment, they choose $I_1 = I_2 = I_L, T_1 = T_2 = T_L, \mathcal{N} = \mathcal{N}_s$. Then, by Definition 8, some results can be obtained as follows:

$$\underline{apr}^{\varphi,0.6}_{I_L,T_L}(K) = \frac{0.75}{t_1} + \frac{0.4}{t_2} + \frac{0.6}{t_3} + \frac{0.6}{t_4} + \frac{0.8}{t_5},$$
$$\overline{apr}^{\varphi,0.6}_{T_L,I_L}(K) = \frac{0.4}{t_1} + \frac{0.65}{t_2} + \frac{0.4}{t_3} + \frac{0.4}{t_4} + \frac{0.4}{t_5}.$$

Here K is a 1-BGFVPRS due to $\underline{apr}^{\varphi,0.6}_{I_l,T_l}(K) \neq \overline{apr}^{\varphi,0.6}_{T_L,I_L}(K)$.

The institution uses $\frac{\underline{apr}^{\varphi,0.6}_{I_L,T_L}(K)(t_i)+\overline{apr}^{\varphi,0.6}_{T_L,I_L}(K)(t_i)}{2}$ to evaluate the comprehensive disease degree of person t_i. Furthermore, according to the comprehensive disease degree of each person, $t_5 > t_1 > t_2 > t_3 = t_4$ is obtained. Thus, t_5 is most likely to get sick and should be treated further.

Furthermore, the relations among some existed RS models in [10, 20, 27, 41, 55, 58, 62, 63, 65] and 1-BGFVPRS model are discussed as follows.

Remark 1 (1) When $\theta = 0$, formulas (1) and (2) become the following forms:

$$\underline{apr}^{\varphi,0}_{I_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c),K(c))), \tag{3}$$

$$\overline{apr}^{\varphi,0}_{T_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c),K(c)). \tag{4}$$

(i) Since the fuzzy bisimulation φ is a special fuzzy relation which can extract "multi-step" information, then formulas (3) and (4) become the models defined in [10]. Formulas (3) and (4) can help DMs solve more complex uncertain problems that some RS models in [55, 58, 62, 63] cannot solve.

(ii) If φ is a fuzzy β-neighborhood $\tilde{N}$, the above model turns into the model defined in [63]:

$$\underline{apr}^{\tilde{N},0}_{I_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\tilde{N}(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\tilde{N}(b,c),K(c))), \tag{5}$$

$$\overline{apr}^{\tilde{N},0}_{T_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\tilde{N}(b,a), \bigvee_{c\in\mathcal{S}} T_1(\tilde{N}(b,c),K(c)). \tag{6}$$

(iii) Let $\varphi(b,a) = \varphi_b(a)$. If for each $a \in \mathcal{S}$, there at least exist one element $t \in \mathcal{S}$ s.t. $\varphi_b(t) = 1$ i.e., $\mathcal{S}$ forms a fuzzy covering, models (5) and (6) boil down the models built in [27].

(iv) If φ is a T-similarity relation on $\mathcal{S}$ and I_1, I_2 are two R-implicators, formulas (3) and (4) degenerate into the models in [41]. Note that with the condition given by [63], the two pairs of generalized approximation operators in [63] cannot degenerate into the models defined in [41]. But, if the fuzzy relation is a T-similarity relation, both pairs operators in [63] degenerate into the models in [41]. The reference [65] gives examples and proof to show why the equality relationship in Theorem 5.3 in [27] does not hold. But it does not give the condition for the equality relationship in Theorem 5.3 in [27]. Here we give the condition of the equality relationship in the theorem. The condition is that if the fuzzy relation is a T-similarity relation. The proof is easy to prove and thus we omit it.

(v) Assume that I_1, I_2 are R-implicators and T_1, T_2 are continuous t-norms while φ is a fuzzy equivalence relation. If φ is a fuzzy β-neighborhood, formulas (3) and (4) degenerate into the models defined in [20].

(2) When $\theta \neq 0$, if for each $c \in \mathcal{S}, \theta \le K(c) \le \mathcal{N}(\theta)$, the following model is obtained:

$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c),K(c))), \tag{7}$$

$$\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c),K(c)). \tag{8}$$

(i) When φ is replaced by the covering-neighborhood defined in [63], formulas (7) and (8) turn into the models provided in [63].

(ii) If φ is a general fuzzy relation, formulas (7) and (8) become the formulas defined in [10].

(3) If $\theta \neq 0$, φ is a fuzzy equivalence relation, using the new fuzzy β-neighborhood defined in [62] to substitute φ, then formulas (1) and (2) turn into the formulas built in [62].

(4) If K is a crisp set, formulae (1) and (2) become the following forms:

$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S},c\notin K} I_1(\varphi(b,c),\theta)), \tag{9}$$

$$\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S},c\in K} T_1(\varphi(b,c),\mathcal{N}(\theta))). \tag{10}$$

Furthermore, another type of BGFVPRS model is defined in the following.

Definition 9 Under the condition of Definition 8, for each $K \in \mathcal{F}(\mathcal{S})$, the second type of bisimulation-based generalized

Fig. 5 An unlabeled fuzzy transition system

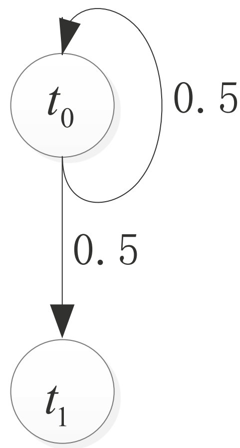

Table 1 A comparison between 1-BGFVPRS and 2-BGFVPRS

	$\underline{apr}^{\varphi,0.58}_{I_L,T_L}(K)$	$\underline{apr}^{\varphi,0.58}_{I_L,I_{KD}}(K)$	$\overline{apr}^{\varphi,0.58}_{T_L,I_{KD}}(K)$	$\overline{apr}^{\varphi,0.58}_{T_L,T_L}(K)$
t_0	0.8	0.7	0.42	0.42
t_1	0.58	0.58	0.42	0.42

fuzzy variable precision I_1I_2-lower approximation (2-BGFVPI_1I_2LA) $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)$ and T_1T_2-upper approximation (2-BGFVPT_1T_2UA) $\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)$ of K, with negator $\mathcal{N}$ and the variable precision $\theta \in [0,1)$, are defined as:

$$\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), \theta \vee K(c))), \tag{11}$$

$$\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta) \wedge K(c))). \tag{12}$$

If $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \neq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)$, we call K the second type of bisimulation-based generalized fuzzy variable precision rough set (2-BGFVPRS), otherwise we call it definable.

Example 6 An unlabeled fuzzy transition system is shown as Fig. 5 where $\mathcal{S}_1 = \{t_0, t_1\} \subseteq \mathcal{S}$. From this figure, we have that $R = \frac{0.5}{(t_0,t_0)} + \frac{0.5}{(t_0,t_1)}$. Assume that φ is a binary fuzzy relation in which $\varphi = \frac{1}{(t_0,t_0)} + \frac{0.3}{(t_0,t_1)} + \frac{0.3}{(t_1,t_0)} + \frac{1}{(t_1,t_1)}$. When $\otimes = \wedge$, we have that $\varphi^{-} \circ R \subseteq R \circ \varphi^{-}$ and $\varphi \circ R \subseteq R \circ \varphi$. Then φ is a fuzzy bisimulation. Furthermore, suppose that $I_1 = I_2 = I_L$, $T_1 = T_2 = T_L$, $\mathcal{N} = \mathcal{N}_s$, $\theta = 0.58$ and $K = \frac{0.8}{t_0} + \frac{0.45}{t_1}$. By Definitions 8 and 9, some results will be obtained as follows:

$$\underline{apr}^{\varphi,0.58}_{I_L,T_L}(K) = \underline{apr}^{\varphi,0.58}_{I_L,I_L}(K) = \frac{0.8}{t_0} + \frac{0.58}{t_1},$$
$$\overline{apr}^{\varphi,0.58}_{T_L,I_L}(K) = \overline{apr}^{\varphi,0.58}_{T_L,T_L}(K) = \frac{0.42}{t_0} + \frac{0.42}{t_1}.$$

Obviously, K is both 1-BGFVPRS and 2-BGFVPRS.

Let $I_{KD}(k_1, k_2) = \max(1 - k_1, k_2)$, for each $k_1, k_2 \in [0,1]$. When $I_1 = I_L, I_2 = I_{KD}, T_1 = T_2 = T_L$, a comparison between 1-BGFVPRS and 2-BGFVPRS is given as Table 1.

Through Table 1, we find that $\underline{apr}^{\varphi,0.58}_{I_L,I_{KD}}(K) \subseteq \underline{apr}^{\varphi,0.58}_{I_L,T_L}(K)$, $\overline{apr}^{\varphi,0.58}_{T_L,I_{KD}}(K) = \overline{apr}^{\varphi,0.58}_{T_L,T_L}(K)$. The relationship between 1-BGFVPRS and 2-BGFVPRS will be investigated deeply later, and thus it is not discussed here.

Besides, the the relations among some existing RS models in [27, 41, 50, 62, 63] and 2-BGFVPRS models are discussed as follows.

Remark 2 (1) When $\theta = 0$, formulas (11) and (12) will be the following forms:

$$\underline{apr}^{\varphi,0}_{I_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), K(c))), \tag{13}$$

$$\overline{apr}^{\varphi,0}_{T_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), K(c))). \tag{14}$$

(i) If $I_1 = I_2$ is an R-implicator, $T_1 = T_2$ is a continuous t-norm, φ is replaced by the fuzzy covering-based fuzzy neighborhood operator defined in [63], formulae (13) and (14) turn into the models defined in [63].

(ii) If $I_1 = I_2$ is an R-implicator, $T_1 = T_2$ is a continuous t-norm, φ is a fuzzy equivalence relation, models (13) and (14) degenerate into the models built in [41, 50].

(iii) Let φ be a successor fuzzy relation. For each $a, b \in \mathcal{S}$, using $\varphi_b(a)$ denotes $\varphi(b,a)$. For each $b \in \mathcal{S}$, if there at least exists one element $t \in \mathcal{S}$ s.t. $\varphi_b(t) = 1$ i.e., $\mathcal{S}$ forms a fuzzy covering. In this situation, formulae (13) and (3sps14) boil down the models built in [27].

(2) When $\theta \neq 0$, if for each $c \in \mathcal{S}, \theta \leq K(c) \leq \mathcal{N}(\theta)$, the following model is obtained:

$$\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), K(c))), \tag{15}$$

$$\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), K(c))). \tag{16}$$

Especially, when φ is the covering-neighborhood defined in [63], formulas (15) and (16) become the models defined in [63].

(3) Suppose that $\theta \neq 0, \varphi$ is a fuzzy equivalence relation, I_1, I_2 are R-implicators and T_1, T_2 are continuous t-norms. If using the new fuzzy β-neighborhood defined in [62] to substitute φ, then formulas (11) and (12) turn into the models built in [62].

(4) If K is a crisp set, then formulae (11) and (12) become the following forms:

$$\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S},c\notin K} I_1(\varphi(b,c),\theta)), \tag{17}$$

$$\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)(a) = \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigvee_{c\in\mathcal{S},c\in K} T_1(\varphi(b,c),\mathcal{N}(\theta))). \tag{18}$$

To better grasp 1-BGFVPRS and 2-BGFVPRS models, the related axiomatic properties of them are investigated in the following.

Proposition 1 *For the approximation operators of 1-BGFVPRS and 2-BGFVPRS models, some important properties are listed as follows.*

(1) $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(\mathcal{S}) = \mathcal{S}$, *if* φ *is a reflexive fuzzy bisimulation and* I_1 *is a border implicator which is left monotonic.*
(2) $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(\mathcal{S}) = \mathcal{S}$, *when* φ *is a reflexive fuzzy bisimulation and* I_1, I_2 *are border implicators which are left monotonic.*
(3) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(\emptyset) = \emptyset, \overline{apr}^{\varphi,\theta}_{T_1,I_2}(\mathcal{S}) = co_{\mathcal{N}}(\hat{\theta})$, *if* φ *is a reflexive fuzzy bisimulation and* I_2 *is a left monotonic and border implicator.*
(4) $\overline{apr}^{\varphi,\theta}_{T_1,T_2}(\emptyset) = \emptyset$, $\overline{apr}^{\varphi,\theta}_{T_1,T_2}(\mathcal{S}) = co_{\mathcal{N}}(\hat{\theta})$, *while* φ *is a reflexive fuzzy bisimulation.*
(5) *Let* I_1, I_2 *be R-implicators. The* $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \subseteq K \cup \hat{\theta}$ *and* $K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)$, *for each* $K \in \mathcal{F}(\mathcal{S})$.
(6) *If* I_1, I_2 *are two border implicators which satisfy the hybrid monotonicity, then* $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq K \cup \hat{\theta}$.
(7) $K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)$, *when* φ *is a reflexive fuzzy bisimulation.*
(8) *If* $K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta})$ *and* I_1, I_2 *are two R-implicators, then*

$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \subseteq K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K),$$
$$\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K).$$

(9) *When* I_1, I_2 *are two R-implicators with* $\theta = 0$, *then*

$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \subseteq K \subseteq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K),$$
$$\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq K \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K).$$

(10) $\forall K_1, K_2 \in \mathcal{F}(\mathcal{S})$, *if* $K_1 \subseteq K_2$ *and* I_1, I_2 *are two right monotonic implicators, then*

$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K_1) \cup \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K_2) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K_1 \cup K_2);$$
$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K_1) \cap \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K_2) \supseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K_1 \cap K_2);$$
$$\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K_1) \cup \underline{apr}^{\varphi,\theta}_{T_1,T_2}(K_2) \subseteq \underline{apr}^{\varphi,\theta}_{T_1,T_2}(K_1 \cup K_2);$$
$$\underline{apr}^{\varphi,\theta}_{T_1,T_2}(K_1) \cap \underline{apr}^{\varphi,\theta}_{T_1,T_2}(K_2) \supseteq \underline{apr}^{\varphi,\theta}_{T_1,T_2}(K_1 \cap K_2).$$

It is noted that the upper approximation operators of the 1-BGFVPRS and 2-BGFVPRS models have the same properties with the assertion (10) *and thus we omit them.*

(11) *When* I_1, I_2 *are two border implicators satisfying the left monotonicity, for each* $\hat{\gamma} \in \mathcal{F}(\mathcal{S})$, *we have*

$$\underline{apr}^{\varphi,\theta}_{I_1,T_2}(\hat{\gamma}) = \widehat{\gamma \vee \theta}, \overline{apr}^{\varphi,\theta}_{T_1,I_2}(\hat{\gamma}) = co_{\mathcal{N}}(\widehat{\theta \wedge \gamma});$$
$$\underline{apr}^{\varphi,\theta}_{I_1,I_2}(\hat{\gamma}) = \widehat{\gamma \vee \theta}, \overline{apr}^{\varphi,\theta}_{T_1,T_2}(\hat{\gamma}) = co_{\mathcal{N}}(\widehat{\theta \wedge \gamma}).$$

(12) *If* $\theta_1 < \theta_2$, *implicators* I_1, I_2 *are right monotonic, then for each* $K \in \mathcal{F}(\mathcal{S})$, *we have*

$$\underline{apr}^{\varphi,\theta_1}_{I_1,T_2}(K) \subseteq \underline{apr}^{\varphi,\theta_2}_{I_1,T_2}(K), \overline{apr}^{\varphi,\theta_1}_{T_1,I_2}(K) \supseteq \overline{apr}^{\varphi,\theta_2}_{T_1,I_2}(K);$$
$$\underline{apr}^{\varphi,\theta_1}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,\theta_2}_{I_1,I_2}(K), \overline{apr}^{\varphi,\theta_1}_{T_1,T_2}(K) \supseteq \overline{apr}^{\varphi,\theta_2}_{T_1,T_2}(K).$$

Proof In the paper, we only prove (5), (11), and (12). The proof of other assertions is similar to them and thus we omit it.

(5) Suppose that I_1, I_2 are R-implicators. Due to Lemma 2.1 in [27], for each $K \in \mathcal{F}(\mathcal{S}), \theta \in [0, 1)$ and $a \in S$, we have

$$\begin{aligned}
&\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a) \\
&= \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c),\theta \vee K(c))) \\
&\leq \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), I_1(\varphi(b,a),\theta \vee K(a))) \\
&\leq \bigvee_{b\in\mathcal{S}} (\theta \vee K(a)) \\
&= \theta \vee K(a).
\end{aligned}$$

That is, $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \subseteq K \cup \hat{\theta}$.

In the similar way, due to Lemma 2.1 in [27], for each $K \in \mathcal{F}(\mathcal{S}), \theta \in [0, 1)$ and $a \in S$, we have

$$\begin{aligned}
&\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)(a)\\
&\quad= \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c),\mathcal{N}(\theta)\wedge K(c)))\\
&\quad\geq \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), T_1(\varphi(b,a),\mathcal{N}(\theta)\wedge K(a)))\\
&\quad\geq \bigwedge_{b\in\mathcal{S}} (\mathcal{N}(\theta)\wedge K(a))\\
&\quad\geq \mathcal{N}(\theta)\wedge K(a).
\end{aligned}$$

Thus $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) \supseteq K \cap co_{\mathcal{N}}(\hat{\theta})$ is proved.

(11) Assume that I_1, I_2 are two border implicators satisfying the left monotonicity. For each $a \in \mathcal{S}$, we have

$$\begin{aligned}
&\underline{apr}^{\varphi,\theta}_{I_1,T_2}(\hat{\gamma})(a)\\
&\quad= \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c),\theta\vee\hat{\gamma}(c)))\\
&\quad= \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,a),\theta\vee\gamma))\\
&\quad= \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a),\theta\vee\gamma))\\
&\quad= \theta\vee\gamma.
\end{aligned}$$

That is, $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(\hat{\gamma}) = \widehat{\gamma\vee\theta}$. The other equations can be proved similarly, and their proof is omitted here. □

Note that the assertions (1)–(4) and (7) in Proposition 1 for the general binary fuzzy relation cannot be always true. In Example 4, for the Early Zadeh implicator I_Z (a border implicator) in [41], $\underline{apr}^{R,0.6}_{I_Z,I_Z}(\mathcal{S}) = \frac{1}{t_0} + \frac{0.9}{t_1} + \frac{0.8}{t_2} + \frac{0.7}{t_3} + \frac{1}{t_4}$. In Example 6, for the general fuzzy relation R, through computation we have that $\underline{apr}^{R,0.58}_{I_L,T_L}(\mathcal{S}) = \overline{apr}^{R,0.58}_{T_L,I_L}(\mathcal{S}) = \overline{apr}^{R,0.58}_{T_L,I_L}(\emptyset) = \underline{apr}^{R,0.58}_{I_L,T_L}(\emptyset) = \frac{0.5}{t_0} + \frac{0.5}{t_1}, \overline{apr}^{\varphi,0.58}_{T_L,T_L}(\mathcal{S}) = \emptyset$ and $\overline{apr}^{R,0.58}_{T_L,T_L}(K) = \emptyset \not\supseteq K \cap co_{\mathcal{N}}(\widehat{0.58})$. This phenomenon shows that for the general binary fuzzy relation R, the assertions (1)–(4) and (7) cannot hold. However, the above proposition for the largest fuzzy bisimulation φ^b i.e., the fuzzy bisimilarity holds.

It is worth noting that the S-implicator, R-implicator and QL-implicator are three known and widely used logical operators. Because of their characteristic properties, our provided approximation operators also have some feasible properties which are listed in the following.

Corollary 10

(1) *If φ is a reflexive fuzzy bisimulation and I_1 is an S-implicator or R-implicator, then Proposition 1(1) holds.*

(2) *If φ is a reflexive fuzzy bisimulation, I_1 is an S-implicator or R-implicator while I_2 is an S-implicator or R-implicator, then Propositions 1(2), (6) and (11) are true.*

(3) *If φ is reflexive, and I_2 is an S-implicator or R-implicator, then Proposition 1(3) is true.*

(4) *Assume that I_1, I_2 are two QL-implicators (or R-implicators or S-implicators), Propositions 1(10) and (12) hold.*

Note that the largest fuzzy bisimulation is a special binary fuzzy relation that satisfies the reflexivity. Under the conditions of Corollary 10, the same properties can be obtained similarly. For a general binary fuzzy relation, the assertion (4) in Corollary 10 is true. But, the assertions (1)–(3) cannot always be true for the general binary fuzzy relation.

We have studied the properties of approximation operators of 1-BGFVPRS and 2-BGFVPRS models. Furthermore, the relationships among $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K), \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K), \underline{apr}^{\varphi,\theta}_{I_1,I_2}(K), \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)$ are discussed in the following. Through the relations of these approximation operators, the DMs according to their needs choose the satisfied RS model and apply it to make a clear decision.

Proposition 2 *For each $K \in \mathcal{F}(\mathcal{S})$ and $\theta \in [0,1)$, if φ is a reflexive fuzzy bisimulation and I_2 is a border implicator, then we get the following results.*

(1) $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)$,

(2) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)$.

Proof

(1) Assume that φ is a reflexive fuzzy bisimulation and I_2 is a border implicator. For each $a \in \mathcal{S}$ and $\theta \in [0,1)$,

$$\begin{aligned}
&\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a)\\
&\quad= \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c),\theta\vee K(c)))\\
&\quad\geq T_2(\varphi(a,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(a,c),\theta\vee K(c)))\\
&\quad= \bigwedge_{c\in\mathcal{S}} I_1(\varphi(a,c),\theta\vee K(c)),
\end{aligned}$$

and

$$\begin{aligned}
&\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)(a)\\
&\quad= \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c),K(c)))\\
&\quad\leq I_2(\varphi(a,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(a,c),\theta\vee K(c)))\\
&\quad= \bigwedge_{c\in\mathcal{S}} I_1(\varphi(a,c),\theta\vee K(c)),
\end{aligned}$$

we have $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)(a) \leq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a)$.

Thus $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)$ for each $K \in \mathcal{F}(\mathcal{S})$ and $\theta \in [0,1)$.

(2) If φ is a reflexive fuzzy bisimulation and I_2 is a border implicator, for any $a \in \mathcal{S}$ and $\theta \in [0,1)$,

$$\begin{aligned}
&\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)(a)\\
&\quad= \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&\quad\leq I_2(\varphi(a,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(a,c), \mathcal{N}(\theta)\wedge K(c)))\\
&\quad= \bigvee_{c\in\mathcal{S}} T_1(\varphi(a,c), \mathcal{N}(\theta)\wedge K(c)),
\end{aligned}$$

and

$$\begin{aligned}
&\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)(a)\\
&\quad= \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&\quad\geq T_2(\varphi(a,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(a,c), \mathcal{N}(\theta)\wedge K(c)))\\
&\quad= \bigvee_{c\in\mathcal{S}} T_1(\varphi(a,c), \theta\vee K(c)).
\end{aligned}$$

Thus $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)$ is proved. □

Corollary 11 *If I_2 is an S-implicator (or R-implicator or QL-implicator), then the following results are obtained.*

(1) *When φ is a reflexive fuzzy bisimulation, $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)$ and $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)$;*
(2) *$\underline{apr}^{\varphi^b,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi^b,\theta}_{I_1,T_2}(K)$ and $\overline{apr}^{\varphi^b,\theta}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi^b,\theta}_{T_1,T_2}(K)$, if $\varphi = \varphi^b$ which is the largest fuzzy bisimulation.*

Proof Since R-implicator, S-implicator and QL-implicator are three border implicators and φ^b the largest fuzzy bisimulation is reflexive, the assertions (1) and (2) in Corollary 11 can be easily obtained by Proposition 2. □

Proposition 3 *Assume that φ is a reflexive fuzzy bisimulation, I_1, I_2 are two R-implicators and $K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta})$, we have that*

$$\begin{aligned}
&\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \subseteq K \cup \hat{\theta}\\
&\quad\subseteq K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K).
\end{aligned}$$

Proof Since an R-implicator is a special border implicator that is hybrid monotonic, by Proposition 1(8) and Proposition 2, the proof of this proposition is easily verified. □

When the value of fuzzy variable precision is 0, the relationship between 1-BGFVPRS and 2-BGFVPRS models is discussed in the following.

Proposition 4 *Under the condition in Proposition 3, if $\theta = 0$, for each $K \in \mathcal{F}(\mathcal{S})$, we have*

$$\begin{aligned}
&\underline{apr}^{\varphi,0}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,0}_{I_1,T_2}(K) \subseteq K\\
&\quad\subseteq \overline{apr}^{\varphi,0}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi,0}_{T_1,T_2}(K).
\end{aligned}$$

The identity relation $\chi_{\mathcal{S}} = \{(a,a) \mid a \in \mathcal{S}\}$ and empty relation $\phi_{\mathcal{S}}$ are two bisimulations. In the paper, $\chi_{\mathcal{S}}$ is a special fuzzy set in which $\chi_{\mathcal{S}}(a,a) = 1, \forall\, a \in \mathcal{S}$. The $\phi_{\mathcal{S}}$ is also a special fuzzy set where $\phi_{\mathcal{S}}(a,b) = 0$ for each $a, b \in \mathcal{S}$.

Proposition 5 *If I_2 is an R-implicator and φ is a fuzzy bisimulation, the following statements are true.*

(1) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) = \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K) = \emptyset$, *if* $\varphi = \phi_{\mathcal{S}}$.
(2) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) = \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K) = K \cap co_{\mathcal{N}}(\hat{\theta})$, *if* $\varphi = \chi_{\mathcal{S}}$.

From the above propositions, $(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K), \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K))$ is tighter than $(\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K), \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K))$. As we all know, the equivalence relation acts a pivotal part in the RS model. The effects of fuzzy bisimulation φ on 1-BGFVPRS and 2-BGFVPRS models are studied as follows. As the reflexivity of φ has already been mentioned above, we don't discuss it next.

Proposition 6 *If φ is a symmetric fuzzy bisimulation, then for each $K \in \mathcal{F}(\mathcal{S})$, the following statements are got.*

(1) $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) = \underline{apr}^{\varphi^-,\theta}_{I_1,I_2}(K), \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) = \underline{apr}^{\varphi^-,\theta}_{I_1,T_2}(K)$.
(2) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) = \overline{apr}^{\varphi^-,\theta}_{T_1,I_2}(K), \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K) = \overline{apr}^{\varphi^-,\theta}_{T_1,T_2}(K)$.

Corollary 12 *If $\varphi = \varphi^b$, for each $K \in \mathcal{F}(\mathcal{S})$, the following statements are obtained.*

(1) $\underline{apr}^{\varphi^b,\theta}_{I_1,I_2}(K) = \underline{apr}^{(\varphi^b)^-,\theta}_{I_1,I_2}(K), \underline{apr}^{\varphi^b,\theta}_{I_1,T_2}(K) = \underline{apr}^{(\varphi^b)^-,\theta}_{I_1,T_2}(K)$.
(2) $\overline{apr}^{\varphi^b,\theta}_{T_1,I_2}(K) = \overline{apr}^{(\varphi^b)^-,\theta}_{T_1,I_2}(K), \overline{apr}^{\varphi^b,\theta}_{T_1,T_2}(K) = \overline{apr}^{(\varphi^b)^-,\theta}_{T_1,T_2}(K)$.

Remark 3 The models defined in [62] are generalized into the following forms with fuzzy bisimulation relation φ as follows:

$$\underline{\varphi}^{\theta}_{I_3}(K)(a) = \bigwedge_{b\in\mathcal{S}} I_3(\varphi(a,b), \theta\vee K(b)), \tag{19}$$

$$\overline{\varphi}^{\theta}_{T_3}(K)(a) = \bigvee_{b\in\mathcal{S}} T_3(\varphi(a,b), \mathcal{N}(\theta) \wedge K(b)). \tag{20}$$

Here $\underline{\varphi}^{\theta}_{I_3}(K), \overline{\varphi}^{\theta}_{T_3}(K)$ are called the third type of bisimulation-based generalized fuzzy variable precision I_3-lower approximation (3-BGFVPI_3LA) and T_3-fuzzy upper approximation (3-BGFVPT_3UA) of K. When $\underline{\varphi}^{\theta}_{I_3}(K) \neq \overline{\varphi}^{\theta}_{T_3}(K)$, K is called the third type of bisimulation-based generalized fuzzy variable precision rough set (3-BGFVPRS). Otherwise it is definable.

(1) It is remarkable that if fuzzy bisimulation φ is replaced by the largest fuzzy bisimulation φ^b, $\theta = 0$ and I_3 is an S-implicator [41] based on the involutive negator $\mathcal{N}$, the 3-BGFVPRS model degenerates into the model defined in [12].

(2) When φ is a fuzzy β-neighborhood relation, the 3-BGFVPRS model turns out to be the model constructed in [20].

(3) If θ=0, φ is a general binary fuzzy relation and I_3 is an S-implicator, formulas (19) and (20) will be the S-lower and T-upper approximation operators in [67], respectively.

Let $\varphi_i (i \in \Lambda)$ be a fuzzy bisimulation. Then, $\varphi_i^-, \bigcup_{i\in\Lambda}\varphi_i$ and $\bigcup_{i\in\Lambda}\varphi_i^-$ are also fuzzy bisimulations. The union of all fuzzy bisimulations is the largest fuzzy bisimulation φ^b. Based on the properties of fuzzy bisimulations, some results of our models are listed as follows.

Proposition 7 *Let $\varphi_i (i \in \Lambda)$ be a fuzzy bisimulation. For any $\theta \in [0,1)$ and $K \in \mathcal{F}(\mathcal{S})$, the following statements are true.*

(1) $\underline{apr}^{\varphi^b,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{(\bigcup_{i\in\Lambda}\varphi_i),\theta}_{I_1,I_2}(K) \subseteq \bigcup_{i\in\Lambda} \underline{apr}^{\varphi_i,\theta}_{I_1,I_2}(K)$ *with two left monotonic implicators I_1, I_2.*

(2) $\underline{apr}^{\varphi^b,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{(\bigcup_{i\in\Lambda}\varphi_i),\theta}_{I_1,I_2}(K) = \bigcap_{i\in\Lambda} \underline{apr}^{\varphi_i,\theta}_{I_1,I_2}(K)$ *with two R-implicators I_1, I_2.*

(3) $\overline{apr}^{(\bigcup_{i\in\Lambda}\varphi_i),\theta}_{T_1,T_2}(K) = \bigcup_{i\in\Lambda} \overline{apr}^{\varphi_i,\theta}_{T_1,T_2}(K) \subseteq \overline{apr}^{\varphi^b,\theta}_{T_1,T_2}(K)$.

(4) $\underline{(\varphi^b)}^{\theta}_{I_3}(K) \subseteq \underline{(\bigcup_{i\in\Lambda}\varphi_i)}^{\theta}_{I_3}(K) \subseteq \bigcup_{i\in\Lambda} \underline{(\varphi_i)}^{\theta}_{I_3}(K)$ *where I_3 is a left monotonic implicator.*

(5) $\underline{(\varphi^b)}^{\theta}_{I_3}(K) \subseteq \underline{(\bigcup_{i\in\Lambda}\varphi_i)}^{\theta}_{I_3}(K) = \bigcap_{i\in\Lambda} \underline{(\varphi_i)}^{\theta}_{I_3}(K)$ *where I_3 is an R-implicator.*

(6) $\overline{(\bigcup_{i\in\Lambda}\varphi_i)}^{\theta}_{T_3}(K) = \bigcup_{i\in\Lambda} \overline{(\varphi_i)}^{\theta}_{T_3}(K) \subseteq \overline{(\varphi^b)}^{\theta}_{T_3}(K)$.

If φ_1, φ_2 are two fuzzy bisimulations, then $\varphi_1 \circ \varphi_2$ is a fuzzy bisimulation. Due to $(\varphi_1\circ\varphi_2)^- = \varphi_2^- \circ \varphi_1^-$ and $(\bigcup_{i\in\Lambda}\varphi_i)^- = \bigcup_{i\in\Lambda}(\varphi_i^-)$, the following statements are true.

Proposition 8 *Assume that $\varphi_i (i \in \Lambda)$ is a fuzzy bisimulation. For any $\theta \in [0,1)$ and $K \in \mathcal{F}(\mathcal{S})$, the following statements are true.*

(1) $\underline{apr}^{(\varphi_1\circ\varphi_2)^-,\theta}_{I_1,T_2}(K) = \underline{apr}^{(\varphi_1^-)\circ(\varphi_2^-),\theta}_{I_1,T_2}(K)$,
$\overline{apr}^{(\varphi_1\circ\varphi_2)^-,\theta}_{T_1,I_2}(K) = \overline{apr}^{(\varphi_1^-)\circ(\varphi_2^-),\theta}_{T_1,I_2}(K)$.

(2) $\underline{apr}^{(\bigcup_{i\in\Lambda}\varphi_i)^-,\theta}_{I_1,T_2}(K) = \underline{apr}^{\bigcup_{i\in\Lambda}(\varphi_i^-),\theta}_{I_1,T_2}(K)$,
$\overline{apr}^{(\bigcup_{i\in\Lambda}\varphi_i)^-,\theta}_{T_1,I_2}(K) = \overline{apr}^{\bigcup_{i\in\Lambda}(\varphi_i^-),\theta}_{T_1,I_2}(K)$.

Since the BGFVPRS models have the same properties of Proposition 8, only the relevant properties of 1-BGFVPRS model are shown.

Next, the relation between 2-BGFVPRS and 3-BGFVPRS models is researched as follows.

Proposition 9 *Let $I_1 = I_2 = I_3 = I$ and $T_1 = T_2 = T_3 = T$. If φ is a T-transitive and symmetric fuzzy bisimulation, I is an R-implicator based on continuous t-norm T, for each $K \in \mathcal{F}(\mathcal{S})$ with $\theta \in [0,1)$, the following statements are true.*

(1) $\underline{\varphi}^{\theta}_{I}(K) \subseteq \underline{apr}^{\varphi,\theta}_{I,I}(K) \subseteq K \cup \hat{\theta}$.

(2) $K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T,T}(K) \subseteq \overline{\varphi}^{\theta}_{T}(K)$.

Proof

(1) Due to φ is a T-transitive and symmetric fuzzy bisimulation, by Lemma 2.1 in [27], for each $a \in \mathcal{S}$ and $\theta \in [0,1)$, we have

$$\begin{aligned}
&\underline{apr}^{\varphi,\theta}_{I,I}(K)(a)\\
&= \bigwedge_{b\in\mathcal{S}} I(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I(\varphi(b,c), \theta \vee K(c)))\\
&= \bigwedge_{b\in\mathcal{S}}\bigwedge_{c\in\mathcal{S}} I(\varphi(b,a), I(\varphi(b,c), \theta \vee K(c)))\\
&= \bigwedge_{c\in\mathcal{S}}\bigwedge_{b\in\mathcal{S}} I(\varphi(b,a), I(\varphi(b,c), \theta \vee K(c)))\\
&= \bigwedge_{c\in\mathcal{S}}\bigwedge_{b\in\mathcal{S}} I(T(\varphi(b,a), \varphi(b,c)), \theta \vee K(c))\\
&= \bigwedge_{c\in\mathcal{S}} I(\bigvee_{b\in\mathcal{S}} T(\varphi(b,a), \varphi(b,c)), \theta \vee K(c))\\
&= \bigwedge_{c\in\mathcal{S}} I(\bigvee_{b\in\mathcal{S}} T(\varphi(a,b), \varphi(b,c)), \theta \vee K(c))\\
&\geq \bigwedge_{c\in\mathcal{S}} I(\varphi(a,c), \theta \vee K(c))\\
&= \underline{\varphi}_{I}(K)(a).
\end{aligned}$$

Thus, $\underline{\varphi}_I(K) \subseteq \underline{apr}^{\varphi,\theta}_{I,I}(K)$. Furthermore, by Proposition 1(6), $\underline{\varphi}_I(K) \subseteq \underline{apr}^{\varphi,\theta}_{I,I}(K) \subseteq K \cup \hat{\theta}$ is proved.

(2) For each $K \in \mathcal{F}(\mathcal{S}), \theta \in [0,1)$ and $a \in \mathcal{S}$,

$$\begin{aligned}
&\overline{apr}^{\varphi,\theta}_{T,T}(K)(a)\\
&= \bigvee_{b\in\mathcal{S}} T(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{b\in\mathcal{S}}\bigvee_{c\in\mathcal{S}} T(\varphi(b,a), T(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{c\in\mathcal{S}}\bigvee_{b\in\mathcal{S}} T(\varphi(b,a), T(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{c\in\mathcal{S}} T(\bigvee_{b\in\mathcal{S}} T(\varphi(b,a), (\varphi(b,c)), \mathcal{N}(\theta)\wedge K(c))\\
&= \bigvee_{c\in\mathcal{S}} T(\bigvee_{b\in\mathcal{S}} T(\varphi(a,b), (\varphi(b,c)), \mathcal{N}(\theta)\wedge K(c))\\
&\le \bigvee_{c\in\mathcal{S}} T(\varphi(a,c), \mathcal{N}(\theta)\wedge K(c))\\
&= \overline{\varphi}_T(K)(a),
\end{aligned}$$

$\overline{apr}^{\varphi,\theta}_{T,T}(K) \subseteq \overline{\varphi}_T(K)$ is obtained. In addition, by Proposition 1(7), $K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T,T}(K) \subseteq \overline{\varphi}_T(K)$ is obtained easily. □

The relationship among the approximation operators of 2-BGFVPRS and 3-BGFVPRS models is investigated in Proposition 9. Figure 6 gives the relationship of them clearly.

Proposition 10 *Let $I_1 = I_2 = I_3 = I, T_1 = T_2 = T_3 = T$ be an R-implicator, a continuous t-norm, respectively. For each $K \in \mathcal{F}(\mathcal{S})$ and $\theta \in [0,1)$, we have the following results.*

(1) $\underline{apr}^{\varphi,\theta}_{I,I}(K) = \underline{\varphi}^{\theta}_I(K)$, *if φ is a T-similarity fuzzy bisimulation.*

(2) $\overline{apr}^{\varphi,\theta}_{T,T}(K) = \overline{\varphi}^{\theta}_T(K)$, *when φ is a reflexive fuzzy bisimulation.*

(3) $\underline{apr}^{\varphi^b,\theta}_{I,I}(K) = \underline{(\varphi^b)}^{\theta}_I(K), \overline{apr}^{\varphi^b,\theta}_{T,T}(K) = \overline{(\varphi^b)}^{\theta}_T(K)$.

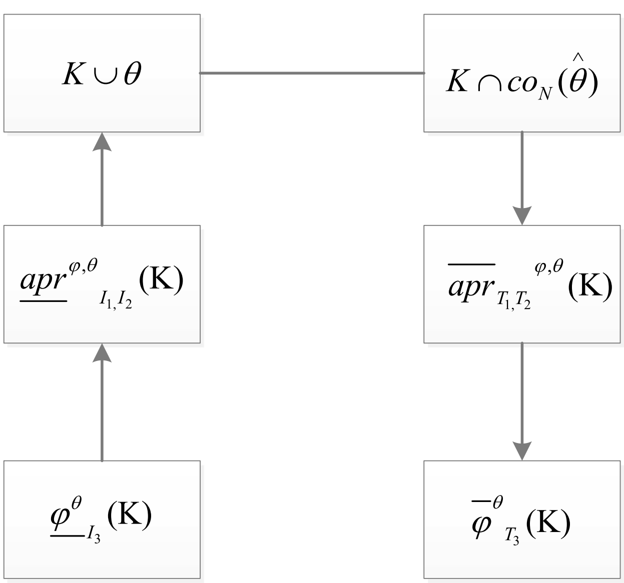

Fig. 6 The relationship between the 2-BGFVPRS and 3-BGFVPRS models

Proof (1) For each $a \in \mathcal{S}, \theta \in [0,1)$ and $K \in \mathcal{F}(\mathcal{S})$,

$$\begin{aligned}
&\underline{apr}^{\varphi,\theta}_{I,I}(K)(a)\\
&= \bigwedge_{b\in\mathcal{S}} I(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I(\varphi(b,c), \theta\vee K(c)))\\
&= \bigwedge_{b\in\mathcal{S}}\bigwedge_{c\in\mathcal{S}} I(\varphi(b,a), I(\varphi(b,c), \theta\vee K(c)))\\
&= \bigwedge_{c\in\mathcal{S}}\bigwedge_{b\in\mathcal{S}} I(\varphi(b,a), I(\varphi(b,c), \theta\vee K(c)))\\
&= \bigwedge_{c\in\mathcal{S}} I(\bigvee_{b\in\mathcal{S}} T(\varphi(b,a), \varphi(b,c)), \theta\vee K(c))\\
&= \bigwedge_{c\in\mathcal{S}} I(\bigvee_{b\in\mathcal{S}} T(\varphi(a,b), \varphi(b,c)), \theta\vee K(c))\\
&= \bigwedge_{c\in\mathcal{S}} I(\varphi(a,c), \theta\vee K(c)).\\
&= \underline{\varphi}^{\theta}_I(K)(a).
\end{aligned}$$

Thus $\underline{apr}^{\varphi,\theta}_{I,I}(K) = \underline{\varphi}^{\theta}_I(K)$ holds.

(2) For any $a \in \mathcal{S}, \theta \in [0,1)$ and $K \in \mathcal{F}(\mathcal{S})$,

$$\begin{aligned}
&\overline{apr}^{\varphi,\theta}_{T,T}(K)(a)\\
&= \bigvee_{b\in\mathcal{S}} T(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{b\in\mathcal{S}}\bigvee_{c\in\mathcal{S}} T(\varphi(b,a), T(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{c\in\mathcal{S}}\bigvee_{b\in\mathcal{S}} T(\varphi(b,a), T(\varphi(b,c), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{c\in\mathcal{S}} T(\bigvee_{b\in\mathcal{S}} T(\varphi(b,a), \varphi(b,c)), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{c\in\mathcal{S}} T(\bigvee_{b\in\mathcal{S}} T(\varphi(a,b), \varphi(b,c)), \mathcal{N}(\theta)\wedge K(c)))\\
&= \bigvee_{c\in\mathcal{S}} T(\varphi(a,c), \mathcal{N}(\theta)\wedge K(c))\\
&= \overline{\varphi}^{\theta}_T(K)(a).
\end{aligned}$$

That is, $\overline{apr}^{\varphi,\theta}_{T,T}(K) = \overline{\varphi}^{\theta}_T(K)$.

(3) Since φ^b is the largest fuzzy bisimulation, then it is reflexive. Through the assertions (1) and (2) in this proposition, the proof of (3) is easily proved. □

Remark 4 Under the condition in Proposition 9, if φ is a T-similarity fuzzy bisimulation, for each $K \in \mathcal{F}(\mathcal{S})$ and $\theta \in [0,1)$, some statements are obtained as follows.

(1) $\underline{\varphi}^{\theta}_{I_3}(K) = \underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \subseteq K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta}) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K) = \overline{\varphi}^{\theta}_{T_3}(K)$, if $K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta})$;

(2) $\underline{\varphi}^{\theta}_{I_3}(K) = \underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) \subseteq K \subseteq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(K) = \overline{\varphi}^{\theta}_{T_3}(K)$, if $\theta = 0$.

The statements in Propositions 7 and 8 for the general binary fuzzy relation are also true. It is worth noting that the assertions in Propositions 2, 3–6, 9, 10, in Corollaries 11, 12 and in Remark 4 cannot always hold if φ is a general binary fuzzy relation. But, for the largest fuzzy bisimulation φ^b, the above statements are always true.

According to Remark 4, under some special circumstances, the relationship among the approximation operators of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS models is obtained, as shown in Fig. 7. Through the research, the approximation operators of 1-BGFVPRS and 2-BGFVPRS models are tighter than the approximation operators of 3-BGFVPRS in some special cases. The approximation operator of 1-BGFVPRS is the most compact approximation operator among the three models.

It is noticed that the duality between rough approximation operators plays an important role in RS models. Many researchers usually take advantage of the duality to build a dual pair of approximation operators. Based on this, the duality of approximation operators provided in the paper is studied as follows.

Proposition 11 *Let I_1, I_2 be two S-implicators based on continuous t-norm T_1, T_2, respectively. If $\mathcal{N}$ is an involutive negator i.e., for any $k_1, k_2 \in [0,1]$, $I(k_1, k_2) = \mathcal{N}(T(k_1, \mathcal{N}(k_2)))$, then $\forall K \in \mathcal{F}(\mathcal{S})$ and $\theta \in [0,1)$, the following statements are true.*

(1) $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) = co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K)))$.

(2) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) = co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(co_{\mathcal{N}}(K)))$.

(3) $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K) = co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,T_2}(co_{\mathcal{N}}(K)))$.

(4) $\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K) = co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,I_2}(co_{\mathcal{N}}(K)))$.

Proof (1) Since $\mathcal{N}$ is an involutive negator, then for any $k_1, k_2 \in [0,1]$, $I(k_1, k_2) = \mathcal{N}(T(k_1, \mathcal{N}(k_2)))$. Due to I_1, I_2 are two S-implicators, for each $a \in \mathcal{S}$ and $\theta \in [0,1)$,

$$
\begin{aligned}
& co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K)))(a) \\
&= \mathcal{N}(\overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K))(a)) \\
&= \mathcal{N}(\bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta) \wedge \mathcal{N}(K(c))))) \\
&= \mathcal{N}(\bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta \vee K(c))))) \\
&= \mathcal{N}(\bigwedge_{b\in\mathcal{S}} (\mathcal{N}(T_2(\varphi(b,a), \mathcal{N}(\bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta \vee K(c)))))))) \\
&= \mathcal{N}(\bigwedge_{b\in\mathcal{S}} (\mathcal{N}(T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} \mathcal{N}(T_1(\varphi(b,c), \mathcal{N}(\theta \vee K(c)))))))) \\
&= \mathcal{N}(\bigwedge_{b\in\mathcal{S}} (\mathcal{N}(T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), \theta \vee K(c)))))) \\
&= \mathcal{N}(\mathcal{N}(\bigvee_{b\in\mathcal{S}} (T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), \theta \vee K(c)))))) \\
&= \bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), \theta \vee K(c))) \\
&= \underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a).
\end{aligned}
$$

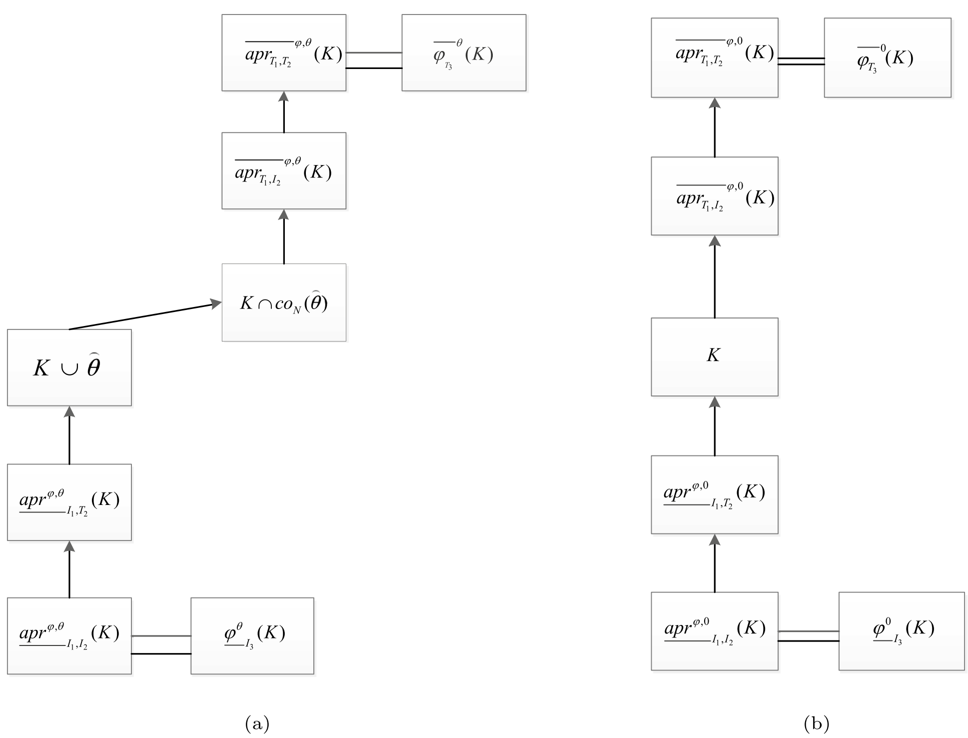

Fig. 7 The relationship among the 1-BGFVPRS, 2-BGFVPRS and 3-BGFVPRS models

Thus, $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K) = co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K)))$ can be obtained.

(2) For each $a \in \mathcal{S}$ and $\theta \in [0, 1)$,

$$\begin{aligned}
&co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(co_{\mathcal{N}}(K)))(a)\\
&= \mathcal{N}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(co_{\mathcal{N}}(K)(a)))\\
&= \mathcal{N}(\bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), \theta \vee \mathcal{N}(K(c)))))\\
&= \mathcal{N}(\bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} \mathcal{N}(T_1(\varphi(b,c), \mathcal{N}(\mathcal{N}(K(c)) \wedge \mathcal{N}(\theta))))))\\
&= \mathcal{N}(\bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} \mathcal{N}(T_1(\varphi(b,c), K(c) \wedge \mathcal{N}(\theta)))))\\
&= \mathcal{N}(\bigvee_{b\in\mathcal{S}} T_2(\varphi(b,a), \mathcal{N}(\bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), K(c) \wedge \mathcal{N}(\theta)))))\\
&= \bigwedge_{b\in\mathcal{S}} \mathcal{N}(T_2(\varphi(b,a), \mathcal{N}(\bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), K(c) \wedge \mathcal{N}(\theta)))))\\
&= \bigwedge_{b\in\mathcal{S}} (I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), K(c) \wedge \mathcal{N}(\theta))))\\
&= \overline{apr}^{\varphi,\theta}_{T_1,I_2}(K).
\end{aligned}$$

Thus, $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K) = co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(co_{\mathcal{N}}(K)))$ is proved. Similarly, the assertions (3) and (4) can be proved and thus we omit them here □

Proposition 12 *Assume that I_1, I_2 are two R-implicators based on continuous t-norms T_1, T_2, respectively. If $\mathcal{N}$ is a negator induced by implicator I, for each $k_1, k_2 \in [0, 1]$, $\mathcal{N}(\mathcal{N}(k_1)) \geq k_1$, and $\mathcal{N}(I(k_1, k_2)) \geq T(k_1, \mathcal{N}(k_2)), \forall K \in \mathcal{F}(\mathcal{S})$ and $\theta \in [0, 1)$, we have the following statements.*

(1) $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K)) \subseteq co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K))$.

(2) $\underline{apr}^{\varphi,\theta}_{I_1,T_2}(co_{\mathcal{N}}(K)) \subseteq co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K))$.

(3) $\overline{apr}^{\varphi,\theta}_{T_1,T_2}(co_{\mathcal{N}}(K)) \subseteq co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K))$.

(4) $\underline{apr}^{\varphi,\theta}_{I_1,I_2}(co_{\mathcal{N}}(K)) \subseteq co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K))$.

Proof Here, we only prove the first assertion in this proposition, others can be verified similarly. Under the condition given in this proposition, for each $a \in \mathcal{S}$,

$$\begin{aligned}
&\overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K))(a)\\
&= \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} T_1(\varphi(b,c), \mathcal{N}(\theta) \wedge \mathcal{N}(K(c))))\\
&\leq \bigwedge_{b\in\mathcal{S}} I_2(\varphi(b,a), \bigvee_{c\in\mathcal{S}} \mathcal{N}(I_1(\varphi(b,c), \theta \vee K(c))))\\
&\leq \bigwedge_{b\in\mathcal{S}} \mathcal{N}(T_2(\varphi(b,a), \mathcal{N}(\bigvee_{c\in\mathcal{S}} \mathcal{N}(I_1(\varphi(b,c), \theta \vee K(c))))))\\
&= \bigwedge_{b\in\mathcal{S}} \mathcal{N}(T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} \mathcal{N}(\mathcal{N}(I_1(\varphi(b,c), \theta \vee K(c))))))\\
&\leq \bigwedge_{b\in\mathcal{S}} \mathcal{N}(T_2(\varphi(b,a), \bigwedge_{c\in\mathcal{S}} I_1(\varphi(b,c), \theta \vee K(c))))\\
&= \mathcal{N}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)(a)).
\end{aligned}$$

Hence, $\overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K)) \subseteq co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K))$ holds. □

Proposition 13 *Let I_1, I_2 be two R-implicators based on continuous t-norms T_1, T_2, respectively. For each $K \in \mathcal{F}(\mathcal{S})$ and $k_1, k_2 \in [0, 1]$, if $\mathcal{N}(\mathcal{N}(k_1)) = k_1$ and $\mathcal{N}(I(k_1, k_2)) \leq T(k_1, \mathcal{N}(k_2))$, the following statements hold.*

(1) $co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K))$.

(2) $co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,T_2}(co_{\mathcal{N}}(K))$.

(3) $co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)) \subseteq \overline{apr}^{\varphi,\theta}_{T_1,T_2}(co_{\mathcal{N}}(K))$.

(4) $co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)) \subseteq \underline{apr}^{\varphi,\theta}_{I_1,I_2}(co_{\mathcal{N}}(K))$.

Proposition 14 *Let I_1, I_2 be two R-implicators based on continuous t-norms T_1, T_2, respectively. If $\mathcal{N}(\mathcal{N}(k_1)) = k_1$ and $\mathcal{N}(I(k_1, k_2)) = T(k_1, \mathcal{N}(k_2))$ for each $k_1, k_2 \in [0, 1]$, then the following results are obtained.*

(1) $co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,T_2}(K)) = \overline{apr}^{\varphi,\theta}_{T_1,I_2}(co_{\mathcal{N}}(K))$.

(2) $co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,I_2}(K)) = \underline{apr}^{\varphi,\theta}_{I_1,T_2}(co_{\mathcal{N}}(K))$.

(3) $co_{\mathcal{N}}(\underline{apr}^{\varphi,\theta}_{I_1,I_2}(K)) = \overline{apr}^{\varphi,\theta}_{T_1,T_2}(co_{\mathcal{N}}(K))$.

(4) $co_{\mathcal{N}}(\overline{apr}^{\varphi,\theta}_{T_1,T_2}(K)) = \underline{apr}^{\varphi,\theta}_{I_1,I_2}(co_{\mathcal{N}}(K))$.

4 The uncertainty measure and reduction

The uncertainty measure and reduction are two important concepts in classical RS model. In this section, the uncertainty measure of BGFVPRS models and reduction of fuzzy bisimulations are introduced. Furthermore, the related properties of them are discussed. In the following, we firstly investigate the uncertainty measure of BGFVPRS models.

4.1 The uncertainty measure

The inaccuracy of a set is caused by the existence of the boundary domain. To express this point more accurately, we introduce the concept of accuracy measures of BGFVPRS models of the fuzzy set K. Obviously that 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS are three fuzzy sets on $\mathcal{S}$. Then $\forall K_1, K_2 \in \mathcal{F}(\mathcal{S})$, the distance function of fuzzy sets is used to define the closeness measure between K_1 and K_2. As we all know that the Euclid distance function satisfies the conditions of distance. In general, it is used as a similarity. In the paper, by using the Euclid distance, we can compute the accuracy measure between fuzzy sets K_1 and K_2.

Suppose that $K_1 = \frac{K_1(a_1)}{a_1} + \frac{K_1(a_2)}{a_2} + \cdots + \frac{K_1(a_n)}{a_n}$, $K_2 = \frac{K_2(a_1)}{a_2} + \frac{K_2(a_2)}{a_2} + \cdots + \frac{K_2(a_n)}{a_n}$, $\mathcal{S} = \{a_1, a_2, \ldots, a_n\}, | \mathcal{S} |= n$. Assume that $K_t(a_i)$ is the membership degree that element a_i belongs to fuzzy set K_t where $t \in \{1, 2\}$ and $i \in \{1, 2, \ldots, n\}$. The Euclid distance function between fuzzy sets K_1 and K_2 is defined as:

$$d_E(K_1, K_2) = 1 - \frac{1}{\sqrt{n}}(\sum_{i=1}^{n}(K_1(a_i) - K_2(a_i))^2)^{\frac{1}{2}}.$$

Based on the Euclid distance function, a concept of the accuracy measures of the BGFVPRS models is given in the following.

Definition 13 Let $(\mathcal{S}, \varphi)$ be a fuzzy relational structure where $\theta \in [0, 1), \mathcal{S}_1 = \{a_1, a_2, \ldots, a_n\} \subseteq \mathcal{S}$. Then for each $K \in \mathcal{F}(\mathcal{S})$, the θ-accuracy measures of upper and lower approximation operators of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS for fuzzy set K are defined as follows:

$$d_{E,1}^{\varphi,\theta}(K) = 1 - \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}(\overline{apr}_{T_1,I_2}^{\varphi,\theta}(K)(a_i) - \underline{apr}_{I_1,T_2}^{\varphi,\theta}(K)(a_i))^2 \right)^{\frac{1}{2}}, \quad (21)$$

$$d_{E,2}^{\varphi,\theta}(K) = 1 - \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}(\overline{apr}_{T_1,T_2}^{\varphi,\theta}(K)(a_i) - \underline{apr}_{I_1,I_2}^{\varphi,\theta}(K)(a_i))^2 \right)^{\frac{1}{2}}, \quad (22)$$

$$d_{E,3}^{\varphi,\theta}(K) = 1 - \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}(\overline{\varphi}_{T_3}^{\theta}(K)(a_i) - \underline{\varphi}_{I_3}^{\theta}(K)(a_i))^2 \right)^{\frac{1}{2}}. \quad (23)$$

Meanwhile, another concept of the roughness measure of the BGFVPRS models is defined as follows.

Definition 14 Under the condition in Definition 13, for each $K \in \mathcal{F}(\mathcal{S})$, the θ-roughness measure of upper and lower approximation operators of λ-BGFVPRS for the fuzzy set K is defined as $\rho_{\lambda}^{\varphi,\theta}(K) = 1 - d_{E,\lambda}^{\varphi,\theta}(K)$, where $\lambda = 1, 2, 3$.

For better understand the above two definitions, an example is given in the following.

Example 7 Assume that in the condition of Example 4, if $I_1 = I_2 = I_3 = I_L$, $T_1 = T_2 = T_3 = T_L$, through Definitions 13 and 14, we have that $d_{E,1}^{\varphi,0.58}(K) = d_{E,2}^{\varphi,0.58}(K) = d_{E,3}^{\varphi,0.58}(K) = 0.7085$, $\rho_1^{\varphi,0.58}(K) = \rho_2^{\varphi,0.58}(K) = \rho_3^{\varphi,0.58}(K) = 0.2915$. In this case, the 0.58-accuracy measures of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS are identical meanwhile the 0.58-roughness measures of them are the same.

In the following, the relationships of the θ-accuracy measures (or the θ-roughness measures) of the BGFVPRS models are discussed.

Proposition 15 *If* $K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta})$, φ *is reflexive and* I_1, I_2 *are R-implicators, for each* $K \in \mathcal{F}(\mathcal{S})$ *and* $\theta \in [0, 1)$, *we have* $d_{E,1}^{\varphi,\theta}(K) \geq d_{E,2}^{\theta}(K), \rho_1^{\varphi,\theta}(K) \leq \rho_2^{\varphi,\theta}(K)$.

Proposition 16 *Suppose that* $K \cup \hat{\theta} \subseteq K \cap co_{\mathcal{N}}(\hat{\theta})$, $I_1 = I_2 = I_3 = I$ *and* $T_1 = T_2 = T_3 = T$ *where I is an R-implicator based on a continuous t-norm T. For each* $K \in \mathcal{F}(\mathcal{S})$ *and* $\theta \in [0, 1)$, *we have the following statements.*

(1) *If* φ *is symmetric and T-transitive fuzzy bisimulation, then* $d_{E,3}^{\varphi,\theta}(K) \leq d_{E,2}^{\varphi,\theta}(K), \rho_3^{\varphi,\theta}(K) \geq \rho_2^{\varphi,\theta}(K)$.
(2) *If* φ *is a fuzzy T-similarity bisimulation relation, then* $d_{E,3}^{\varphi,\theta}(K) \leq d_{E,2}^{\varphi,\theta}(K) \leq d_{E,1}^{\varphi,\theta}(K), \rho_3^{\varphi,\theta}(K) \geq \rho_2^{\varphi,\theta}(K) \geq \rho_1^{\varphi,\theta}(K)$.
(3) *If* φ *is a fuzzy equivalence bisimulation relation, then* $d_{E,3}^{\varphi,\theta}(K) \leq d_{E,2}^{\varphi,\theta}(K) \leq d_{E,1}^{\varphi,\theta}(K), \rho_3^{\varphi,\theta}(K) \geq \rho_2^{\varphi,\theta}(K) \geq \rho_1^{\varphi,\theta}(K)$.
(4) *If* φ *is a largest fuzzy bisimulation relation, then* $d_{E,3}^{\varphi,\theta}(K) \leq d_{E,2}^{\varphi,\theta}(K) \leq d_{E,1}^{\varphi,\theta}(K), \rho_3^{\varphi,\theta}(K) \geq \rho_2^{\varphi,\theta}(K) \geq \rho_1^{\varphi,\theta}(K)$.
(5) *Under the condition* (2) (*or* (3), *or* (4)), *if* $\theta = 0$, *then* $d_{E,3}^{\varphi,\theta}(K) \leq d_{E,2}^{\varphi,\theta}(K) \leq d_{E,1}^{\varphi,\theta}(K) \leq 1, \rho_3^{\varphi,\theta}(K) \geq \rho_2^{\varphi,\theta}(K) \geq \rho_1^{\varphi,\theta}(K) \geq 0$.
(6) *For each* $\theta_1, \theta_2 \in [0, 1)$, *if* $\theta_1 \leq \theta_2$, *then* $d_{E,i}^{\varphi,\theta_1}(K) \leq d_{E,i}^{\varphi,\theta_2}(K)$ *where* $(i = 1, 2, 3)$.

The above Propositions show that we can study RS models from a numerical point of view. Through Propositions 15 and 16, when describing a fuzzy set at the same time, the accuracy (or roughness) measures of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS may be different. The 1-BGFVPRS is the most accurate, 2-BGFVPRS is the second, and 3-BGFVPRS is the worst. Conversely, 3-BGFVPRS is the roughest, 2-BGFVPRS is the next, and 1-BGFVPRS is the least rough.

4.2 The reduction

In two models (or the model and itself), the largest fuzzy bisimulation is the union of all fuzzy bisimulations. If there are some redundant or unnecessary fuzzy bisimulations, when we compute the largest fuzzy bisimulation, we will spend a lot of unnecessary manpower and material resources. Especially when dealing with big data, these redundant data even make our work impossible because of the limited equipment and manpower.

To deal with these redundant data, some related concepts are described below.

Definition 15 Let $(\mathcal{S}, (R_i)_{i\in\Lambda})$ be a fuzzy relational structure. Assume that $H = \{\varphi_1, \varphi_2, \dots, \varphi_k\}$ which is the set of some fuzzy bisimulations in this structure. Then $\varphi_t \in H$ is called dispensable to φ^b if $\varphi^b - \varphi_t = \varphi^b$; otherwise, it is called indispensable to φ^b. The set of all indispensible fuzzy bisimulations to φ^b in H is named the reduction of φ^b, which is written as φ^b_r.

Note that fuzzy transition system in [7] is a special case of fuzzy relational structure with labels. In general, the labels are regarded as attributes. The following example is given to demonstrate the feasibility of the above definition.

Example 8 Assume $\mathfrak{T} = \{T = \{t_1, t_2, t_3\}, A = \{m\}, \delta, t_1\}$ is a labeled fuzzy transition system where T is a set of states, A is a set of labels, δ is the fuzzy transition relation and t_1 is the initial state. The fuzzy transition system is shown as Fig. 8 where $\delta(t_1, m, t_2) = \delta(t_1, m, t_3) = 0.8$ and δ takes values 0 for all other cases.

Suppose that

$$R_1 = \frac{1}{(t_1, t_1)} + \frac{1}{(t_2, t_2)} + \frac{1}{(t_3, t_3)},$$
$$R_2 = \frac{1}{(t_1, t_1)} + \frac{1}{(t_2, t_3)} + \frac{1}{(t_3, t_2)},$$
$$R_3 = \frac{1}{(t_1, t_1)} + \frac{1}{(t_2, t_2)} + \frac{1}{(t_3, t_3)} + \frac{1}{(t_2, t_3)} + \frac{1}{(t_3, t_2)}.$$

It is easy to prove that R_1, R_2, R_3 are three fuzzy bisimulations.

In this case, since $\varphi^b = \bigcup_{i=1}^3 R_i = R_3, \varphi^b - R_3 \neq \varphi^b$, then R_3 is an indispensable fuzzy bisimulation to φ^b. Due to $\varphi^b - R_1 = \varphi^b - R_2 = \varphi^b$, we have R_1, R_2 are two dispensable fuzzy bisimulations to φ^b. Hence, R_3 is the reduction to φ^b.

Proposition 17 *In a fuzzy relational structure* $(\mathcal{S}, (R_i)_{i\in\Lambda})$, *if* $H = \{\varphi_1, \varphi_2, \dots, \varphi^b_r\}$, $\psi = \bigcup_{t=1}^m \varphi^b_{rt}$, *and* $\varphi^b_r = \{\varphi^b_{r1}, \varphi^b_{r2}, \dots, \varphi^b_{rm}\}$ *which is a reduction of* φ^b, *then for each* $K \in \mathcal{F}(\mathcal{S})$ *and* $\theta \in [0, 1)$ *the following results are true.*

(1) $\underline{apr}^{\varphi^b,\theta}_{I_1,T_2}(K) = \underline{apr}^{\psi,\theta}_{I_1,T_2}(K), \overline{apr}^{\varphi^b,\theta}_{T_1,I_2}(K) = \overline{apr}^{\psi,\theta}_{T_1,I_2}(K)$.

(2) $\underline{apr}^{\varphi^b,\theta}_{I_1,I_2}(K) = \underline{apr}^{\psi,\theta}_{I_1,I_2}(K), \overline{apr}^{\varphi^b,\theta}_{T_1,T_2}(K) = \overline{apr}^{\psi,\theta}_{T_1,T_2}(K)$.

(3) $\underline{(\varphi^b)}^{\theta}_{I_3}(K) = \underline{\psi}^{\theta}_{I_3}(K), \overline{(\varphi^b)}^{\theta}_{T_3}(K) = \overline{\psi}^{\theta}_{T_3}(K)$.

(4) $d^{\varphi^b,\theta}_{E,1}(K) = d^{\psi,\theta}_{E,1}(K)$, $d^{\varphi^b,\theta}_{E,2}(K) = d^{\psi,\theta}_{E,2}(K)$, $d^{\varphi^b,\theta}_{E,3}(K) = d^{\psi,\theta}_{E,3}(K)$.

(5) $\rho^{\varphi^b,\theta}_1(K) = \rho^{\psi,\theta}_1(K), \rho^{\varphi^b,\theta}_2(K) = \rho^{\psi,\theta}_2(K), \rho^{\varphi^b,\theta}_3(K) = \rho^{\psi,\theta}_3(K)$.

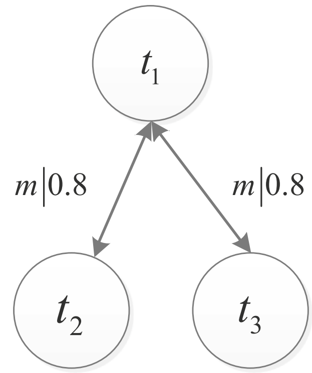

Fig. 8 A labeled fuzzy transition system

Of course, for the general fuzzy bisimulation, the reduction is very important. In order to maintain some properties, we can also propose some reduction methods of fuzzy bisimulation from other angles, such as keeping the positive region unchanged.

The paper from keeping the distance between the lower and the upper approximation operator invariable provides a method to depict the reduction of the general fuzzy bisimulation. Using our method, in the era of big data, human and material resources can be greatly saved. The reduction forms of the three models are similar. Next, only the reduction of 1-BGFVPRS model is discussed.

Definition 16 Let $(\mathcal{S}, (R_i)_{i\in\Lambda})$ be a finite fuzzy relational structure. Assume that $H = \{\varphi_1, \varphi_2, \dots, \varphi_k\}$ which is the set of all fuzzy bisimulations. If $d^{\varphi-\varphi_t,\theta}_{E,1}(K) \neq d^{\varphi,\theta}_{E,1}(K)$, then φ_t is relative indispensable to φ; otherwise it is relative dispensable to φ. The relative reduction of H is all relative indispensable fuzzy bisimulations for H.

To help us understand the above definition well, the following example is given.

Example 9 Under the condition of Example 8, let $I_1 = I_2 = I_L$, $T_1 = T_2 = T_L$, $\mathcal{N} = \mathcal{N}_s$ and $\theta = 0.58$. Assume that $K = \frac{0.8}{t_1} + \frac{0.5}{t_2} + \frac{0.7}{t_3}$. By Definition 8 and formula (3.21), we have the following results:

$$d^{\varphi^b-R_1,0.58}_{E,1}(K) = d^{\varphi^b,0.58}_{E,1}(K) = 0.74;$$
$$d^{\varphi^b-R_2,0.58}_{E,1}(K) = d^{\varphi^b,0.58}_{E,1}(K) = 0.74;$$
$$0 = d^{\varphi^b-R_3,0.58}_{E,1}(K) \neq d^{\varphi^b,0.58}_{E,1}(K) = 0.74.$$

Through Definition 16, we obtain that R_3 is relative indispensable while R_1, R_2 are relative dispensable to φ^b. Thus, R_3 is a relative reduction to φ^b.

In the following, a method to compute the relative reduction for a general fuzzy bisimulation is given. Through this method, the reduction to the largest fuzzy bisimulation φ^b can be easily obtained.

Let $H = \{\varphi_1, \varphi_2, \dots, \varphi_k\}$ be a set of k fuzzy bisimulations in the finite fuzzy relational structure $(\mathcal{S}, (R_i)_{i\in\Lambda})$. The fuzzy bisimulation ϖ is a relative reduction to the fuzzy bisimulation φ if it satisfies the following conditions:

(1) $d_{E,1}^{\varphi,\theta}(K) = d_{E,1}^{\varpi,\theta}(K)$,
(2) $\forall\, \varphi_t \in H, \varphi_t \subseteq \varpi, d_{E,1}^{\varphi,\theta}(K) \neq d_{E,1}^{\varphi_t,\theta}(K)$.

In the following, an example is given to illustrate the above method.

Example 10 From Example 9, through Definition 8 and formula (3.21), we have that $d_{E,1}^{\varphi^b,0.58}(K) = 0.74, d_{E,1}^{R_1,0.58}(K) = 0.71, d_{E,1}^{R_2,0.58}(K) = 0.15, d_{E,1}^{R_3,0.58}(K) = 0.74$. Based on this, we find that $d_{E,1}^{\varphi^b,0.58}(K) = d_{E,1}^{R_3,0.58}(K), d_{E,1}^{\varphi^b,0.58}(K) \neq d_{E,1}^{R_1,0.58}(K), d_{E,1}^{\varphi^b,0.58}(K) \neq d_{E,1}^{R_2,0.58}(K)$.

Since $R_1, R_2 \subset R_3$, according to the above method, R_3 is the relative reduction to φ^b while R_1 and R_2 are not the relative reduction.

The relationship between the reduction and relative reduction is discussed as follows.

Remark 5 Through the analysis of the above definitions and propositions, we have the following statements.

(1) If ψ is the relative reduction of φ^b, it may not the reduction for φ^b.
(2) If ψ is the reduction of φ^b, it is the relative reduction for φ^b.

5 The PROMETHEE II method based on BGFVPRS models

The classical PROMETHEE methods in [5, 6] and some MADM methods in [20, 21, 62–64] based on the generalized RS models have achieved many great results in applications. However, they seem powerless for the multiple relationship structure problems such as community promotion problems and social network analysis problems. In view of this, a novel MADM method is needed. In this section, based on the concept of fuzzy bisimulation and the thought of PROMETHEE II method, we want to provide an effective decision-making method based on the BGFVPRS models.

The BGFVPRS models are based on the fuzzy bisimulations. With them, the theoretical research of the classical rough set model has been enriched. These relations can excavate "multi-step" information for underlying relations in a fuzzy environment. By means of the fuzzy bisimulations, objects can be well distinguished. Based on this, we can approximate the uncertain concept well and make a better result.

From Fig. 7 in Section 3 and Propositions 15 and 16 in Section 4, 1-BGFVPRS is most accurate than the 2-BGFVPRS and 3-BGFVPRS. That is to say, the results by using 1-BGFVPRS are closer to the uncertainty concept K than the other BGFVPRS models provided in the paper. In order to make the decision results closer to the actual needs of DMs, the 1-BGFVPRS is selected to be applied. Of course, the other two types of BGFVPRS models can also be applied in this decision-making method, and then some different results may be obtained.

To make the decision-making method more general and be applied in widely areas, the general fuzzy logic operators are introduced. A threshold parameter θ where $\theta \in [0, 1)$, is given to control the influence of environmental noise data on our decision results. In decision-making, DMs can give the appropriate threshold θ according to the actual situation of the environment, which will make the decision-making result better. Because the BGFVPRS models are not easily disturbed by noise data, the decision-making method in the paper is robust.

The subjective weight is important in some decision-making methods. However, if the experts have different opinions, then the subjective weights are difficult to obtain. Or, if the DMs are lack of experience, the subjective weights obtained may not be right. Compared with it, the objective weight can avoid errors caused by human factors. In this section, by means of the concept of θ-accuracy measure which is uncertainty measure, an objective weight formula is proposed.

In classical rough set theory, the attribute reduction can help remove redundant attributes and improve the decision-making efficiency of the algorithm. In view of this, the reduction defined in Section 4 is applied in our decision-making method. When making a decision, we don't need to compute the decision-making result of every fuzzy bisimulation. Then, we will save a lot of time and human resources. In other words, it will improve the decision-making efficiency of our proposed algorithm.

Above, our method is robust, flexible, and effective.

5.1 The PROMETHEE II method based on 1-BGFVPRS model

Recently, MADM methods have drawn many scholars' attention and their contribution is enormous. Some scholars relied on RS models and presented many decision-making methods. However, some MADM methods in [5, 6, 20, 21, 62–64] have some defects in relational structure decision-making problems such as graph mining or social network analysis. Based on this, we hope to look for an effective approach to solve these problems.

On the basis of the 1-BGFVPRS model, combing the principle of the PROMETHEE II method, a novel decision-making method is provided. Furthermore, an example for selecting the optimal alternative by mass organizations in the Zachary karate club network is given to illustrate this method. Through the example, the flexibility and effectiveness are illustrated from two aspects: comparative analysis and sensitivity analysis.

5.2 Problem statement

An organization or a community wants to choose talent leaders. It is necessary to consider not only the belief ability, planning ability, and goal ability of alternatives, but also their influence ability, interpersonal ability, and communication ability. If the managers not only have political integrity and ability, but also have high leadership ability, the organization will make brilliant achievements under their leadership.

Let $\mathcal{S} = \{t_0, t_1, \dots, t_{n-1}\}$ be the universe of n alternatives in an organization or a community, $\mathcal{M} = \{m_0, m_1, \dots, m_{q-1}\}$ be the set of q labels, $\Upsilon = \{R_0, R_1, \dots, R_{y-1}\}$ be a set of fuzzy relations, and $\Psi = \{\varphi_0, \varphi_1, \dots, \varphi_{x-1}\}$ be a set of binary fuzzy relations. Here, $R_h(t_i, m_j, t_k) = h_{ijk}$ denotes the degree value of alternative t_i to alternative t_k with respect to the relation $R_h \in \mathcal{R}$ for label m_j where $t_i, t_k \in \mathcal{S}$, $m_j \in \mathcal{M}$, and $h_{ijk} \in [0, 1]$. Here, the n, q, y, x are natural numbers.

The information of $R_h(t_i, m_j, t_k)$ is provided by a lot of experts and shown as Table 2. The value of $\varphi_f(t_i, t_k) = f_{ik}$ shows that the membership degree for alternative t_i to alternative t_k with respect to binary fuzzy relation φ_f. Table 3 presents the information of the fuzzy set φ_f. Note that the labels can represent a certain identity, a certain characteristic, or a certain aspect, etc. Generally speaking, labels can be considered as attributes. Therefore, this kind of problem is an MADM with multiple relationships.

To select the optimal member, a decision-making method is given as follows.

5.3 Decision-making methodology

First of all, through the experience of experts, professional evaluation, and statistical analysis, we can get real data from a community or a social network. Through normalization, we get the fuzzy data sets Υ and Ψ.

Secondly, according to Definition 7 and the reduction of fuzzy bisimulation, some fuzzy bisimulations are selected. These fuzzy bisimulations constitute a new family of relation sets $\Xi = \{\psi_0, \psi_1, \cdots, \psi_{z-1}\} \subseteq \Psi$.

Table 2 The information of $R_h(t_i, m_j, t_k)$

R_p	t_0	t_1	$\cdots$	t_{n-1}
t_0	h_{0j0}	h_{0j1}	$\cdots$	$h_{0j(n-1)}$
t_1	h_{1j0}	h_{1j1}	$\cdots$	$h_{1j(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
t_{n-1}	$h_{(n-1)j0}$	$h_{(n-1)j1}$	$\cdots$	$h_{(n-1)j(n-1)}$

Table 3 The information of $\varphi_f(t_i, t_k)$

φ_f	t_0	t_1	$\cdots$	t_{n-1}
t_0	f_{00}	f_{01}	$\cdots$	$f_{0(n-1)}$
t_1	f_{10}	f_{11}	$\cdots$	$f_{1(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
t_{n-1}	$f_{(n-1)0}$	$f_{(n-1)1}$	$\cdots$	$f_{(n-1)(n-1)}$

Thirdly, based on the principle of PROMETHEE II, through Definition 8, the preference function is given as follows:

$$P_f(t_i, t_j) = D_f(t_i, t_j), \tag{24}$$

where $D_f(t_i, t_j) = [\underline{apr}^{\psi_f,\theta}_{I_1,T_2}(K)(t_i) + \overline{apr}^{\psi_f,\theta}_{T_1,I_2}(K)(t_i)] - [\underline{apr}^{\psi_f,\theta}_{I_1,T_2}(K)(t_j) + \overline{apr}^{\psi_f,\theta}_{T_1,I_2}(K)(t_j)]$ and $\psi_f \in \Xi$.

Fourthly, by the following formula, the overall preference index π is obtained:

$$\pi(t_i, t_j) = \sum_{f=0}^{z-1} P_f(t_i, t_j) \cdot \omega_f, \tag{25}$$

where $\omega_f = \frac{d^{\varphi_f,\theta}_{E,1}(K)}{\sum_{v=0}^{z-1} d^{\varphi_v,\theta}_{E,1}(K)}$. Especially, when $\sum_{v=0}^{z-1} d^{\varphi_v,\theta}_{E,1}(K) = 0$ and $w_f = 0$.

Furthermore, the leaving flow, the entering flow, and net flow are defined as :

$$\varrho^+(t_i) = \sum_{j=1}^{n-1} \pi(t_i, t_j), \tag{26}$$

$$\varrho^-(t_i) = \sum_{j=1}^{n-1} \pi(t_j, t_i), \tag{27}$$

$$\varrho(t_i) = \varrho^+(t_i) - \varrho^-(t_i). \tag{28}$$

Finally, the alternative t_i is ranked by the value of $\varrho(t_i)$. The best alternative is selected according to the ranking order.

In the following, an algorithm of our proposed method is given as follows.

5.4 Procedure for decision-making method

Algorithm 1 The algorithm of the PROMETHEE II method based on 1-BGFVPRS method in fuzzy relational structure

htbp

Input: A fuzzy relational structure $(\mathcal{S}, \Upsilon \cup \Psi)$

Output: The ranking of all members and the optimal member

1: **begin for** $\Xi = \emptyset, R_h \in \Upsilon, \varphi_f \in \Psi, h \in \{0, 1, \cdots, y-1\}, f \in \{0, 1; \cdots, x-1\}$ **do**
  if $(\varphi_f)^- \circ R_h \subseteq R_h \circ (\varphi_f)^-$ and $\varphi_f \circ R_h \subseteq R_h \circ \varphi_f$ **then** $\Xi = \varphi_f \cup \Xi$ **end**

2: Given a parameter θ and a fuzzy set K
  for $t_i \in \mathcal{S}, \varphi_f \in \Xi$
  compute: $\underline{apr}^{\varphi_f,\theta}_{I_1,T_2}(K)(t_i) = \bigvee_{b\in\mathcal{S}} T_2(\varphi_f(b, t_i), \bigwedge_{c\in\mathcal{S}} I_1(\varphi_f(b, c), \theta \vee K(c)))$
  compute: $\overline{apr}^{\varphi_f,\theta}_{T_1,I_2}(K)(t_i) = \bigwedge_{b\in\mathcal{S}} I_2(\varphi_f(b, t_i), \bigvee_{c\in\mathcal{S}} T_1(\varphi_f(b, c), \mathcal{N}(\theta) \wedge K(c)))$
  end

3: **for** $t_i, t_j \in \mathcal{S}$ **do**
  compute:
  $D(t_i, t_j) = [\underline{apr}^{\varphi_f,\theta}_{I_1,T_2}(K)(t_i) + \overline{apr}^{\varphi_f,\theta}_{T_1,I_2}(K)(t_i)] - [\underline{apr}^{\varphi_f,\theta}_{I_1,T_2}(K)(t_j) + \overline{apr}^{\varphi_f,\theta}_{T_1,I_2}(K)(t_j)]$
  end

4: **for** $t_i, t_j \in \mathcal{S}$
  compute: $P_f(t_i, t_j) = D_f(t_i, t_j)$
  end

5: **for** $\varphi_f \in \Xi$ **do**
  if $\sum_{v=0}^{z-1} d^{\varphi_v,\theta}_{E,1}(K) = 0$, **then** $\omega_f = 0$
  else $\omega_f = \frac{d^{\varphi_f,\theta}_{E,1}(K)}{\sum_{v=0}^{z-1} d^{\varphi_v,\theta}_{E,1}(K)}$
  end

6: **for** $t_i, t_j \in \mathcal{S}$ **do**
  compute: $\pi(t_i, t_j) = \sum_{f=0}^{z-1} P_f(t_i, t_j) \cdot \omega_f$
  end

7: **for** $t_i \in \mathcal{S}$ **do**
  compute: $\varrho^+(t_i) = \sum_{j=0}^{n-1} \pi(t_i, t_j)$
  compute: $\varrho^-(t_i) = \sum_{j=0}^{n-1} \pi(t_j, t_i)$
  compute: $\varrho(t_i) = \varrho^+(t_i) - \varrho^-(t_i)$
  end

8: **for** $t_i, t_j \in \mathcal{S}$
  if $\varrho(t_i) > \varrho(t_j)$, **then** $t_i > t_j$
  else $t_i \leq t_j$
  end

9: **return:** The ranking of all members and the optimal member.
  end

In this subsection, Algorithm 1 is the algorithm of our proposed method. The complexity of Algorithm 1 is $O(z * n^2 + xy)$. In this algorithm, its computational complexity is analyzed as follows. Step 1 is to obtain fuzzy bisimulations by Definition 7 and the computational complexity is $O(xy)$. Step 2 is to get the approximation operators of the fuzzy set K and the computational complexity is $O(n^2)$. Steps 3 and 4 are to calculate the preference between alternatives and the computational complexity is $O(2n^2)$. Steps 5 and 6 are to obtain the overall preference between alternatives and the computational complexity is $O(zn^2 + z)$. Step 7 is to get the net flow and the computational complexity is $O(n^2)$.

Step 8 is to rank the alternatives and the computational complexity is $O(n^2)$. In a word, the computational complexity of Algorithm 1 is $O(xy + zn^2)$.

In the following, an illustrative example is given for the above algorithm.

5.5 An illustrative example

Example 11 From reference [60], we observe that the karate club network shown as Fig. 9, was divided into two small subgroups by the conflict between the club president, John A., and Mr. Hi about the price of karate lessons. Mr. Hi wanted to raise the prices and asked the mechanism to give him the right to set his lesson fees due to that he was the instructor. However, John A, the chief executive of the club, insisted on stabilizing prices and claimed that he had the right to set the tuition.

As time went by, this contradiction split the original community into two small groups. However, these factions are only ideological groups, which have never materialized in the organization. It is worth noting that there are no political differences, and the club members do not name their names or even deny the existence of these factions. No one have tried to arrange or guide the political tactics of these groups.

To make the club better and stronger, the top management of the company came forward to make John A and Mr. Hi reconcile. For increasing the cohesion and competitiveness of the club, the board of directors unanimously decided to select an outstanding member from 34 members as CEO. Through a series of tests, the experts investigated the abilities of the 34 members in various aspects, such as the belief ability, planning ability, and target ability of alternative programs. Then, the comprehensive evaluation ability value of 34 members constitutes a fuzzy set K. For the selected member to better lead everyone, it is also necessary to examine the degree of communication among members.

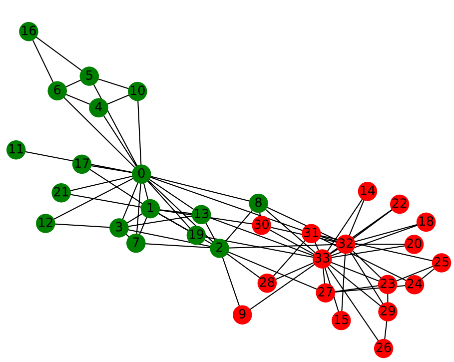

Fig. 9 Zachary karate club network

Figure 9 shows the social relationship of 34 members in the karate club. In this figure, the red ball 33 represents John A. and the other red numbered balls represent the supporters of his. John A. with his supporters constitute one community. Meanwhile, the green No.0 ball represents Mr. Hi, and other green numbered balls represent his supporters, forming another community. A line between two colored numbered balls represents consistently communicated in settings outside those of workouts, club meetings, and karate classes. Because in workouts, club meetings, and karate classes, they interact with each other almost identically. The labels considered here for all members are the same, which can be considered as having no labels. In other words, this is an unlabeled approximation space. The line can be regarded as an edge. The edge in Fig. 9 is bidirectional which makes that the fuzzy relation R is symmetrical.

Anthropologists have investigated and counted the frequency of interaction between club members, the depth of conversation, the influence of the conversation, and the mutual trust after communication. The degree of communication between members is shown in Tables 4 and 5. In the paper, $R(t_i, t_j) \neq 0$ denotes that there is one line between the colored numbered balls i and j, which shows that the degree of communication between the members t_i and t_j is not 0, vice versa. For the sake of simplicity, some unnecessary data representations are omitted. For example $R(t_{14}, t_j) = 0$ ($j \in \{0, 1, \ldots, 8\}$) is omitted.

Case 1: Let $I_1 = I_2 = I_L, T_1 = T_2 = T_L, \mathcal{N} = \mathcal{N}_s$, and $\otimes = \wedge$. The information of K is listed as Table 6. According to the actual demands and environmental factors, experts set $\theta = 0.45$. Since $R^{-} \circ R = R \circ R^{-}, R \circ R = R \circ R$, through Definition 7, R is the only fuzzy bisimulation relation. By Algorithm 1, some results are obtained as Tables 7, 8, 9, 10.

Through the value of $\varrho(t_i)$ in Table 10, the 34 members in Zachary karate club network are ranked as:

$$\begin{aligned} & t_{33} > t_0 > t_1 > t_{32} > t_{11} > t_{17} = t_{19} > t_4 > t_{23} \\ & = t_{24} > t_{26} > t_{27} > t_{25} > t_{12} > t_5 > \\ & t_{29} > t_{16} > t_{15} > t_{30} > t_7 > t_2 > t_{14} \\ & = t_{18} = t_{20} > t_8 > t_{21} > t_{22} > t_3 > t_{13} \\ & > t_{10} > t_{31} > t_9 = t_{28} > t_6. \end{aligned}$$

According to the ranking order, the member t_{33} should be selected to lead this club.

The result of this algorithm accords with real life, because the member t_{33} is the president of the club, and his technical and management ability is very strong. If he leads the team, the karate club will develop better.

Table 4 The information of fuzzy relation R

$R(t_i, t_j)$	t_0	t_1	t_2	t_3	t_4
t_0	0	0.5015	0.1381	0.6123	0.7508
t_1	0.5014	0	0.4762	0.4228	0
t_2	0.1381	0.4762	0	0.21	0
t_3	0.6123	0.4228	0.21	0	0
t_4	0.7508	0	0	0	0
t_5	0.1661	0	0	0	0
t_6	0.3146	0	0	0	0.4867
t_7	0.4855	0.4447	0.2177	0.5053	0
t_8	0.7258	0	0.4558	0	0
t_9	0	0	0.4868	0	0
t_{10}	0.5546	0	0	0	0.2545
t_{11}	0.7263	0	0	0	0
t_{12}	0.513	0	0	0.4034	0
t_{13}	0.5766	0.7454	0.5876	0.8032	0
t_{16}	0	0	0	0	0
t_{17}	0.8889	0.8348	0	0	0
t_{19}	0.7245	0.1776	0	0	0
t_{21}	0.1618	0.4564	0	0	0
t_{27}	0	0	0.3637	0	0
t_{28}	0	0	0.6358	0	0
t_{30}	0	0.6322	0	0	0
t_{31}	0.3578	0	0	0	0
t_{32}	0	0	0.2584	0	0
t_{33}	0	0	0	0	0

$R(t_i, t_j)$	t_5	t_6	t_7	t_8
t_0	0.1662	0.3146	0.4855	0.7258
t_1	0	0	0.4447	0
t_2	0	0	0.2178	0.4558
t_3	0	0	0.5053	0
t_4	0	0.4867	0	0
t_5	0	0.4867	0	0
t_6	0.6181	0	0	0
t_{10}	0.4849	0	0	0
t_{16}	0.3639	0.5237	0	0
t_{30}	0	0	0	0.3515
t_{32}	0	0	0	0.3697
t_{33}	0	0	0	0.2043

$R(t_i, t_j)$	t_9	t_{10}	t_{11}	t_{12}	t_{13}
t_0	0	0.5546	0.7263	0.513	0.5766
t_1	0	0	0	0	0.7454
t_2	0.4868	0	0	0	0.5876
t_3	0	0	0	0.4034	0.8032
t_4	0	0.2545	0	0	0
t_5	0	0.4849	0	0	0
t_{33}	0.2592	0	0	0	0.7413

$R(t_i, t_j)$	t_{14}	t_{15}	t_{16}	t_{17}
t_0	0	0	0	0.8889
t_1	0	0	0	0.8348

Table 4 (continued)

$R(t_i, t_j)$	t_{14}	t_{15}	t_{16}	t_{17}	
t_5	0	0	0.3639	0	
t_6	0	0	0.5237	0	
t_{32}	0.2446	0.7147	0	0	
t_{33}	0.6341	0.1818	0	0	
$R(t_i, t_j)$	t_{18}	t_{19}	t_{20}	t_{21}	t_{22}
t_0	0	0.7245	0	0.1618	0
t_1	0	0.1776	0	0.4564	0
t_2	0.4868	0	0	0	0.5876
t_{32}	0.2025	0	0.7056	0	0.4584
t_{33}	0.2754	0.5802	0.9985	0	0.5212
$R(t_i, t_j)$	t_{23}	t_{24}	t_{25}	t_{26}	
t_{23}	0	0	0.6547	0	
t_{24}	0	0	0.7537	0	
t_{25}	0.6547	0.7537	0	0	
t_{27}	0.6619	0.6736	0	0	
t_{29}	0.328	0	0	0.5642	
t_{31}	0	0.7191	0.3349	0	
t_{32}	0.568	0	0	0	
t_{33}	0.4903	0	0	0.6108	

Table 5 The information of fuzzy relation R

$R(t_i, t_j)$	t_{27}	t_{28}	t_{29}	t_{30}	t_{31}	t_{32}	t_{33}
t_0	0	0.7245	0	0.1618	0.3578	0	0
t_1	0	0	0	0.6322	0	0	0
t_2	0.3637	0.6358	0	0	0	0	0.2584
t_8	0	0	0	0.3515	0	0.3697	0.2043
t_9	0	0	0	0	0	0	0.2592
t_{13}	0	0	0	0	0	0	0.7413
t_{14}	0	0	0	0	0	0.2446	0.6341
t_{15}	0	0	0	0	0	0.7147	0.1818
t_{18}	0	0	0	0	0	0.2025	0.2754
t_{19}	0	0	0	0	0	0	0.5802
t_{20}	0	0	0	0	0	0.7056	0.9985
t_{22}	0	0	0	0	0	0.4584	0.5212
t_{23}	0.6619	0	0.328	0	0	0.568	0.4903
t_{24}	0.6736	0	0	0	0.7191	0	0
t_{25}	0	0	0	0	0.3349	0	0
t_{26}	0	0	0.5642	0	0	0	0.6108
t_{27}	0	0	0	0	0	0	0.7917
t_{28}	0	0	0	0	0.7589	0	0.112
t_{29}	0	0	0	0	0	0.5475	0.7715
t_{30}	0	0	0	0	0	0.8532	0.1807
t_{31}	0	0.7589	0	0	0	0.7231	0.5093
t_{32}	0	0	0.5475	0.8532	0.7231	0	0.9985
t_{33}	0.7917	0.112	0.7715	0.1807	0.5093	0.9985	0

Table 6 The information of fuzzy set K

t_i	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
$K(t_i)$	0.98	0.81	0.90	0.12	0.91	0.63	0.09	0.27	0.54	0.05
t_i	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}	t_{19}
$K(t_i)$	0.15	0.95	0.48	0.45	0.14	0.42	0.91	0.79	0.10	0.65
t_i	t_{20}	t_{21}	t_{22}	t_{23}	t_{24}	t_{25}	t_{26}	t_{27}	t_{28}	t_{29}
$K(t_i)$	0.03	0.84	0.93	0.58	0.75	0.74	0.39	0.65	0.17	0.70
t_i	t_{30}	t_{31}	t_{32}	t_{33}						
$K(t_i)$	0.03	0.27	0.75	0.94						

Table 7 The results of $\underline{apr}^{R,0.45}_{I_L,T_L}(K)(t_i)$

t_i	t_0	t_1	t_2	t_3	t_4	t_5	t_6
$\underline{apr}^{R,0.45}_{I_L,T_L}(K)(t_i))$	0.8641	0.81	0.4868	0.45	0.565	0.6181	0.45
t_i	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}
$\underline{apr}^{R,0.45}_{I_L,T_L}(K)(t_i)$	0.2997	0.54	0.301	0.3688	0.5405	0.3271	0.45
t_i	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}	t_{19}	t_{20}
$\underline{apr}^{R,0.45}_{I_L,T_L}(K)(t_i)$	0.0856	0.3115	0.5237	0.703	0.0	0.5387	0.45
t_i	t_{21}	t_{22}	t_{23}	t_{24}	t_{25}	t_{26}	t_{27}
$\underline{apr}^{R,0.45}_{I_L,T_L}(K)(t_i)$	0.1611	0.0552	0.58	0.679	0.6428	0.45	0.65
t_i	t_{28}	t_{29}	t_{30}	t_{31}	t_{32}	t_{33}	
$\underline{apr}^{R,0.45}_{I_L,T_L}(K)(t_i)$	0.45	0.5642	0.45	0.45	0.75	0.94	

Table 8 The results of $\overline{apr}^{R,0.45}_{T_L,I_L}(K)(t_i)$

t_i	t_0	t_1	t_2	t_3	t_4	t_5	t_6
$\overline{apr}^{R,0.45}_{T_L,I_L}(K)(t_i)$	0.55	0.55	0.55	0.4922	0.6814	0.55	0.3819
t_i	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}
$\overline{apr}^{R,0.45}_{T_L,I_L}(K)(t_i)$	0.7479	0.5818	0.5509	0.5151	0.7126	0.8498	0.45
t_i	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}	t_{19}	t_{20}
$\overline{apr}^{R,0.45}_{T_L,I_L}(K)(t_i)$	0.9144	0.8339	0.6361	0.55	1.0	0.7143	0.55
t_i	t_{21}	t_{22}	t_{23}	t_{24}	t_{25}	t_{26}	t_{27}
$\overline{apr}^{R,0.45}_{T_L,I_L}(K)(t_i)$	0.9283	1.0	0.649	0.55	0.55	0.7572	0.55
t_i	t_{28}	t_{29}	t_{30}	t_{31}	t_{32}	t_{33}	
$\overline{apr}^{R,0.45}_{T_L,I_L}(K)(t_i)$	0.4019	0.5965	0.6954	0.4268	0.55	0.55	

5.6 Comparative analysis

In this section, some comparative analyses are given. Firstly, a comparison of ranking results about our proposed method for different logical operators is demonstrated in the following.

5.6.1 A comparative analysis for different logical operators

Since S-implicator, R-implicator and QL-implicator are three kinds of popular and important implicators, by using these kinds of implicators and some usually used continuous t-norms in [41], we have 198 types of 1-BGFVPRS models, 198 types of 2-BGFVPRS models and 18 types of 3-BGFVPRS models. In the following, we random extract 15 types of 1-BGFVPRS models to apply our method for solving the problems. Table 11 gives the information of 15 cases and the optimal alternatives of each case. It should be noted that $I_\bullet$ and $T_\circ$ ($\bullet = L, G, \Delta, KD, \star, Z; \circ = M, L, P$) [41] are an implicator and a continuous t-norm, respectively.

In order to compare the correlation of ranking results, we introduce Spearman's rank correlation coefficient (SRCC). Suppose that the two random variables are A and B (can also be regarded as two sets), and the number of elements is k. The ith ($1 \leq i \leq k$) value of the two random variables is represented by X_i and Y_i respectively. Sorting A and B (ascending or descending at the same time) to obtain two element ranking sets X and Y where elements X_i and Y_i are

Table 9 The results of the overall preference $\pi(t_i, t_j)$

$\pi(t_i, t_j)$	t_0	t_1	t_2	⋯	t_{31}	t_{32}	t_{33}
t_0	0	0.0541	0.3773	⋯	0.5373	0.1141	− 0.0759
t_1	− 0.0541	0	0.3232	⋯	0.4832	0.06	− 0.13
t_2	− 0.3773	− 0.3232	0	⋯	0.16	− 0.2632	− 0.4532
t_3	− 0.4719	− 0.4178	− 0.0946	⋯	0.0654	− 0.3578	− 0.5478
t_4	− 0.1677	− 0.1136	0.2096	⋯	0.3696	− 0.0536	− 0.2436
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
t_{16}	− 0.2543	− 0.2002	0.123	⋯	0.283	− 0.1402	− 0.3302
t_{17}	− 0.1611	− 0.1070	0.2162	⋯	0.3762	− 0.047	− 0.237
t_{18}	− 0.4141	− 0.36	− 0.0368	⋯	0.1232	− 0.3	− 0.49
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
t_{30}	− 0.2687	− 0.2146	0.1086	⋯	0.2686	− 0.1546	− 0.3446
t_{31}	− 0.5373	− 0.4832	− 0.16	⋯	0	− 0.4232	− 0.6132
t_{32}	− 0.1141	− 0.06	0.2632	⋯	0.4232	0	− 0.19
t_{33}	0.0759	0.13	0.4532	⋯	0.6132	0.19	0

Table 10 The results of $\varrho^+(t_i)$, $\varrho^-(t_i)$ and $\varrho(t_i)$

t_i	t_0	t_1	t_2	t_3	t_4	t_5	t_6
$\varrho^+(t_i)$	10.0061	8.1667	− 2.8221	− 6.0385	4.3043	1.6421	− 9.7887
$\varrho^-(t_i)$	− 10.0061	− 8.1667	2.8221	6.0385	− 4.3043	− 1.6421	9.7887
$\varrho(t_i)$	20.0122	16.3334	− 5.6442	− 12.077	8.6086	3.2842	− 19.5774
t_i	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}
$\varrho^+(t_i)$	− 2.4549	0.0679	− 9.1087	− 8.0207	4.5321	1.9413	− 7.4733
$\varrho^-(t_i)$	2.4549	− 0.0679	9.1087	8.0207	− 4.5321	− 1.9413	7.4733
$\varrho(t_i)$	− 4.9098	0.1358	− 18.2174	− 16.0414	9.0642	3.8826	− 14.9466
t_i	t_{21}	t_{22}	t_{23}	t_{24}	t_{25}	t_{26}	t_{27}
$\varrho^+(t_i)$	− 1.0337	− 2.1965	3.7127	3.7127	2.4819	2.9715	2.7267
$\varrho^-(t_i)$	1.0337	2.1965	− 3.7127	− 3.7127	− 2.4819	− 2.9715	− 2.7267
$\varrho(t_i)$	− 2.0674	− 4.393	7.4254	7.4254	4.9638	5.943	5.4534
t_i	t_{28}	t_{29}	t_{30}	t_{31}	t_{32}	t_{33}	
$\varrho^+(t_i)$	− 9.1087	1.3905	0.8703	− 8.262	6.1267	12.5867	
$\varrho^-(t_i)$	9.1087	− 1.3905	− 0.8703	8.2621	− 6.1267	− 12.5867	
$\varrho(t_i)$	− 18.2174	2.781	1.7406	− 16.5242	12.2534	25.1734	

the ith values in the ranking of X and Y, respectively. The elements in sets X and Y are correspondingly subtracted to obtain a ranking difference set $d = \{d_i \mid i = 1, 2, \ldots, k\}$ where $d_i = X_i - Y_i$. The SRCC (ρ) between random variables A and B can be calculated as follows:

$$\rho = 1 - \frac{6\sum_{i=1}^{k} d_i^2}{k(k^2 - 1)}.$$

From Table 11, we find that in most cases, the best member is t_{33}, while in Case 2, the best member is t_0. The reason is that the selection of different logical operators has a certain influence on sorting order. Although the optimal results of Case 15 and Case 2 are inconsistent, the SRCC of the two ranking results is 0.7406, which is highly correlated.

5.6.2 A comparative analysis for different variable precision values

In this section, we investigate how θ affects the ranking orders of all considered members. When θ changes its value, we make some comparisons among several cases as figures 10 and 11. The value range of θ is [0, 1), and the step length is 0.1. From Figs. 10 and 11, the following conclusions can be drawn.

Table 11 The information of 15 cases and the optimal alternative of each case

Case	I_1	I_2	T_1	T_2	$\mathcal{N}$	θ	The optimal member
Case 1	I_L	I_L	T_L	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 2	I_{KD}	I_L	T_M	T_L	$\mathcal{N}_s$	0.45	t_0
Case 3	I_L	I_{KD}	T_L	T_M	$\mathcal{N}_s$	0.45	t_{33}
Case 4	I_L	I_{KD}	T_L	T_P	$\mathcal{N}_s$	0.45	t_{33}
Case 5	I_L	I_*	T_L	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 6	I_L	I_*	T_L	T_M	$\mathcal{N}_s$	0.45	t_{33}
Case 7	I_L	I_Z	T_L	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 8	I_*	I_Z	T_L	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 9	I_L	I_G	T_L	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 10	I_*	I_Z	T_P	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 11	I_Δ	I_L	T_L	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 12	I_Δ	I_*	T_P	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 13	I_Δ	I_Z	T_P	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 14	I_Δ	I_{KD}	T_P	T_L	$\mathcal{N}_s$	0.45	t_{33}
Case 15	I_Δ	I_G	T_P	T_L	$\mathcal{N}_s$	0.45	t_{33}

- With the change of variable precision value θ, the ranking of each case will change, but the optimal members of them are consistent. For example, in Fig. 10, except for Case 2, when taking the same θ, the optimal members of other five cases are highly consistent in most of the time.
- In Fig. 10, for Case 2, when the variable precision is greater than a certain value, its sensitivity is not high. In particular, when $\theta = 0.9$, using Case 2 cannot help DMs to select the best member. Therefore, the DMs may not consider Case 2 when making decisions.
- Through Fig. 11a, when $\theta = 0.95$, the best optimal member cannot be obtained in Cases 1–3. But, the optimal member can be selected in the other cases. In Cases 4 and 5, more than one optimal member is got while the best one can be obtained in Cases 6 and 15.
- Figures 10j and 11a show that when $0.9 < \theta < 1$, the optimal member cannot be selected in some cases. This phenomenon is caused by human subjective factors interfere so strongly that many objective data sets are invalid. In fact, in these conditions the results obtained are unfair for the members under the test. Generally speaking, the variable precision value should not be too large or infinitely close to 1. If an irrational decision-maker has to make a decision in this situation, we can still help him make a decision and choose the best member, such as Case 15.
- By Figs. 10 and 11a, the optimal member for Case 6 changes while the optimal member for Case 15 still stays the same i.e., t_{33}.

Above, the DMs can select different logical operators and variable precision values according to their actual needs. This shows the flexibility of the method in the paper.

5.6.3 A comparative analysis for different MADM methods

In this section, from a comparison among the classical PROMETHEE method (including PROMETHEE I and PROMETHEE II) in [5, 6], some popular MADM methods in [20, 21, 62–64] and our proposed method, the effectiveness and feasibility of our presented method are demonstrated. Through the analysis of the design principle and application environment of these methods, their differences are shown as Table 12. In this table, for simplicity, using FISs denotes functional information systems [42] while using RISs represents relational information systems [42].

By Table 12, some conclusions are made as follows:

- A data table in RS consists of a set of objects and a set of attributes. In the data table, every attribute is considered as a function from a set of objects to the attributes' domain of values. In [42], FISs are data tables that represent attribute data. In the FISs, the methods proposed in [5, 6, 20, 21, 62–64] only consider the information about individual objects but do not consider the relations between objects. Sometimes, it may be necessary to take into account the relations between objects. For example, in social network analysis, if you only consider the attribute data, you will lose much useful information since its main data types are attribute data [42] and relational data [42]. In this case, using the methods in [5, 6, 20, 21, 62–64] may be unavailable. The RISs [15] comprise a set of objects, a set of attributes and a set of relations between objects. In the RISs, using our proposed method can deal with the problems which the above mentioned

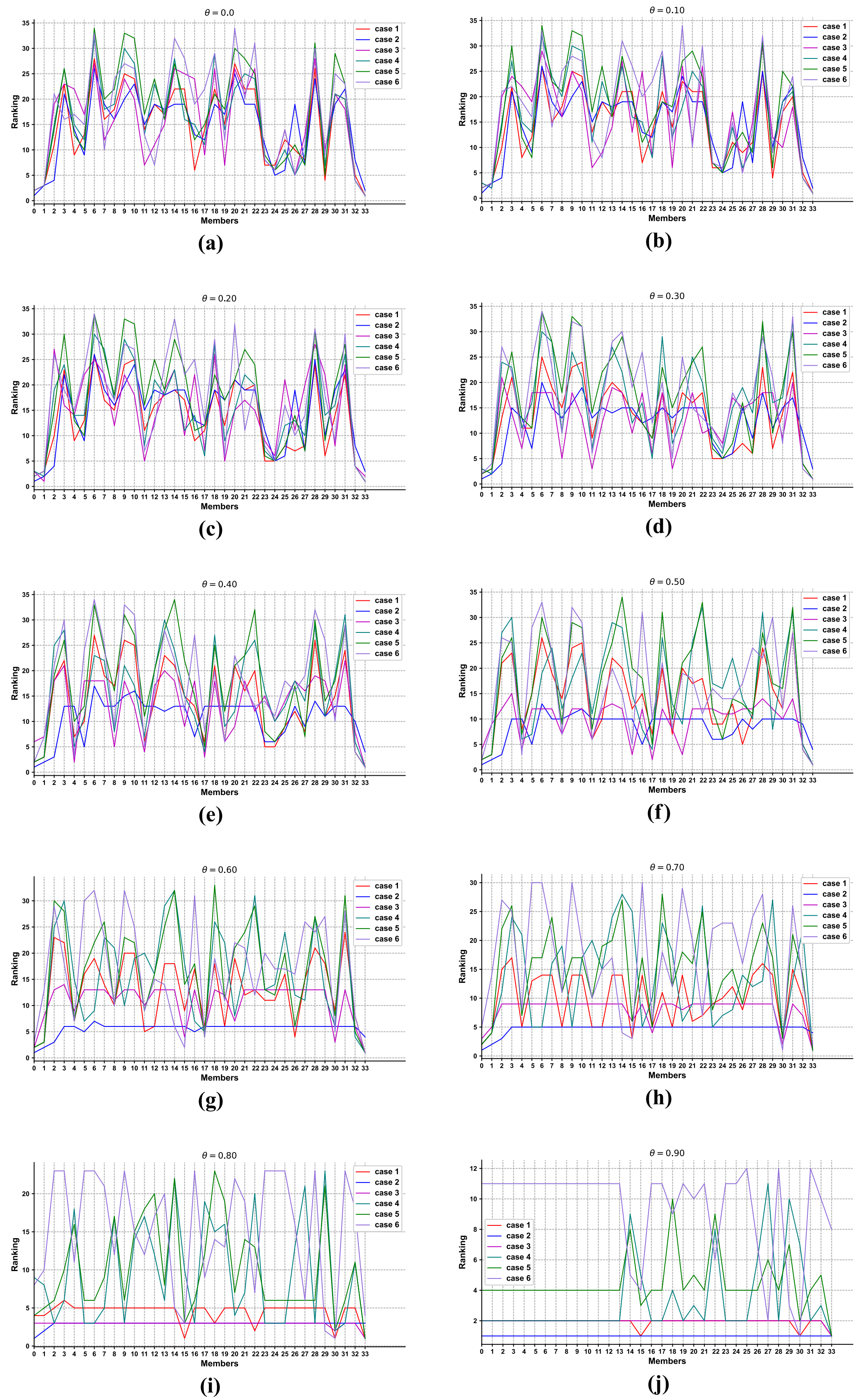

Fig. 10 The comparison of the ranking results for 6 cases with different θ

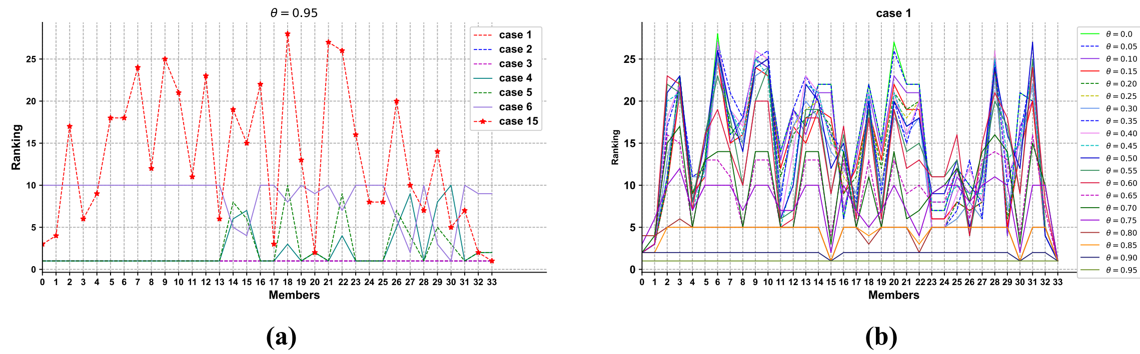

Fig. 11 A comparison for 7 cases when $\theta = 0.95$ and another comparison for Case 1 when $\theta \in [0, 1)$.

Table 12 The differences among some methods in [5, 6, 20, 21, 62–64] and our proposed method

Methods	Information systems	Weights	Relations
Brans and Vincke's method [5]	FISs	Objective	None
Brans et al.'s method [6]	FISs	Subjective	None
Jiang et al.'s method [20]	FISs	Subjective	Fuzzy β-neighborhoods operators
Jiang et al.'s method [21]	FISs	Subjective	Fuzzy neighborhood operators
Zhan et al.'s method [62]	FISs	Subjective	Fuzzy β-neighborhood operators
Zhang et al.'s [63]	FISs	Subjective	Fuzzy neighborhood operators
Zhang et al.'s [64]	FISs	Subjective	Fuzzy β-neighborhood operators
Our proposed method	RISs	Objective	Fuzzy bisimulations

methods cannot solve and helps DMs make a clear decision.

- From Table 12, we find that the weights in [6, 20, 21, 62–64] are given by experts or DMs while the weights in [5] and our proposed method are calculated from the raw data by a formula. In general, the weights given by experts or DMs are called subjective weights while the weights computed by a formula are named objective weights. For the methods in [6, 20, 21, 62–64], if the experts have different opinions, then the weights are difficult to obtain. Or, if the DMs are lack of experience, the weights obtained may not be suitable. That is, using these methods may be infeasible in these cases. But using objective weight formulas can effectively avoid these situations.
- From the above table, the two methods in [5, 6] are not based on relations. The methods in [20, 62, 64] are based on the fuzzy β-neighborhood operators while other methods in [21, 63] are based on the fuzzy neighborhood operators. Different from the aforementioned methods, our proposed method is on the basis of fuzzy bisimulations. It is noted that the fuzzy β-neighborhood operators and fuzzy neighborhood operators are binary fuzzy relations that only obtain "one step" information of the potential relations. However, for some complex problems, "one step" information may not be enough to discern objects. The fuzzy bisimulations which are natural generations of bisimulations can help us to characterize indiscernibility by exploiting "multi-step" information. With fuzzy bisimulations, the application scope of the classical PROMETHEE II has been expanded. For example, our proposed decision-making method can solve many MADM problems involving relational structure data where some popular MADM methods in [5, 6, 20, 21, 62–64] cannot solve.

Table 13 A comparison among some methods in Example 11

Methods	Ranking order	Optimal member
Brans and Vincke's method [5]	×	×
Brans et al.'s method [6]	×	×
Jiang et al.'s method [20]	×	×
Jiang et al.'s method [21]	×	×
Zhan et al.'s [62]	×	×
Zhang et al.'s [63]	×	×
Zhang et al.'s [64]	×	×
Our proposed method	✓	✓

Furthermore, we apply some methods in [5, 6, 20, 21, 62–64] and our proposed method in Example 11. A comparison among them is given as Table 13. From this table, we have the following statements:

- Using the methods in [5, 6, 20, 21, 62–64] cannot get a ranking order for the considered members while using our proposed method can obtain a ranking order for them.
- By means of the methods in [5, 6, 20, 21, 62–64], the optimal member cannot be obtained while using our proposed method can get an optimal member.

Above all, using the methods in [5, 6, 20, 21, 62–64] cannot help DMs make a clear decision while using our proposed method can do it. This demonstrates the feasibility and effectiveness of our proposed method in the paper.

5.6.4 A comparative analysis for different BGFVPRS models

Different models may affect the results of decision-making results. The reason is that the different construction principles of different models will affect the decision-making results in a certain extent. In the following, an example is given to demonstrate this.

Example 12 Let $I_1 = I_3 = I_{\triangle}, I_2 = I_G, T_1 = T_3 = T_P, T_2 = T_L, \bigotimes = \wedge$ and $\theta = 0.45$. Furthermore, 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS models are applied in Algorithm 1, respectively. The information of K is given as Table 6. Under the same condition in Case 1, through computation, a comparison of the results for these three type of models is shown as Table 14. From this table, we find that the optimal member of 3-BGFVPRS is t_{20} while the optimal members of 1-BGFVPRS and 2-BGFVPRS are t_{33}. The optimal member of 3-BGFVPRS is different from the other two types of BGFVPRS models. The ranking orders of three models are different. Above all, applying different models in Algorithm 1 may obtain different ranking orders of members and different optimal members. When making a decision, 1-BGFVPFRS model plays a better performance than the other two models. The reason is that the accuracy measure of 1-BGFVPRS model is bigger than other two models. As the accuracy measure of the BGFVPRS model increases, the decision-making results get more accurate and closer to the actual needs. In order to make the decision results closer to the DMs' actual needs, the 1-BGFVPRS is a better option than other two models.

5.7 Sensitivity analysis

The variable precision θ acts a critical role in BGFVPRS models, which is verified in Section 3. In subsection 5.6.2, the change of sorting results is studied when different variable precision values are taken in Case 1. Here, the value range of variable precision θ is [0,1), where the step size is 0.05. The comparison among 20 ranking results in Case 1 when taking different variable precision values is given in Fig. 11b. From the figure, the optimal member is t_{33} when $\theta \in [0, 0.8)$ while the optimal members are t_{15}, t_{30}, t_{33} with $\theta \in [0.8, 0.95)$. Especially, if $\theta \in [0.95, 1)$, the optimal member cannot be selected.

Remark 6 Through comparative analysis and sensitivity analysis, the following conclusions can be drawn.

(1) Our approach is highly flexible when solving complex problems by selecting different logical operators and changing the value of θ.
(2) Although the ranking order will fluctuate after selecting different logic operators, the optimal results are consistent, which shows the effectiveness of the decision-making method on the other hand.

Table 14 A comparison of the results for different BGFVPRS models

Models	Ranking orders of members	Optimal member
1-BGFVPRS	$t_{33} > t_0 > t_1 > t_{32} > t_{27} > t_{24} > t_{25} > t_{17} > t_5 > t_{29} > t_{23} > t_{16} > t_2 > t_4 > t_{11} > t_8 > t_{19} > t_{20} > t_6 > t_3 > t_{30} > t_{13} > t_{26}$ $t_{28} > t_{31} > t_{10} > t_{12} > t_{15} > t_7 > t_9 > t_{14} > t_{21} > t_{18} > t_{22}$	t_{33}
2-BGFVPRS	$t_{33} > t_0 > t_1 > t_2 > t_3 > t_4 > t_5 > t_6 > t_7 > t_8 > t_9 > t_{10} > t_{11} > t_{12} > t_{13} > t_{14} > t_{15} > t_{16} > t_{17} > t_{18} > t_{19} > t_{20} > t_{21} > t_{22} > t_{23} > t_{24} > t_{25} > t_{26} > t_{27} > t_{28} > t_{29} > t_{30} > t_{31} > t_{32}$	t_{33}
3-BGFVPRS	$t_{20} > t_{17} > t_{11} > t_8 > t_{19} > t_{15} > t_{14} > t_{30} > t_{23} > t_6 > t_4 > t_{26} > t_{27} > t_{10} > t_{25} > t_{22} > t_{12} > t_9 > t_{21} > t_0 > t_{29} > t_7 > t_{18} > t_{32} > t_1 > t_{16} > t_{24} > t_{33} > t_{31} > t_2 > t_{13} > t_{28} > t_5 > t_3$	t_{20}

(3) Compared with some popular MADM methods in [5, 6, 20, 21, 62–64], our provided method can solve the complex problems which consist of attribute data and relational data. That is to say, this method can solve the MADM problems involving relational structure data.

6 Conclusion

Aiming at the complex MADM problems including attribute data and relational data, the paper has proposed a decision-making method. This method is based on BGFVPRS models which are the natural generalizations of the classical RS model. The BGFVPRS models are based on the fuzzy bisimulations. With these relations, the theoretical research and the application of the classical RS model has been enriched and expanded, respectively. The BGFVPRS models have many important properties such as duality. Using the concept of reduction, the redundant data can be removed quickly. According to the θ-accuracy measures (or θ-roughness measures) of BGFVPRS models defined in the paper, the compactness of them can be compared very well. Compared with many MADM methods, the method presented in the paper is effective to solve the problems including relational data. The comparative analysis and sensitivity analysis have shown that the flexibility and effectiveness of our decision-making method in the Zachary karate club network. For example, although the ranking order will fluctuate after selecting different logic operators and the values of θ, the optimal results are consistent.

The future research directions are shown as follows.

- Because the data in real life does not always exist in the form of numbers, it should be considered to extend our BGFVPRS models to solve linguistic problems [37, 38] and design an effective decision-making method for dealing with these problems by building a tool to minimize information loss.
- Motivated by the concept of weak bisimulations [17], the BGFVPRS models can be generalized and some related important properties about the models may be got.
- Learning from many popular three-way decision-making methods [11, 19, 31, 52], we will further modify our decision-making method and apply it to more complex environments such as the IF [2] environment.
- The granular computing is an important concept. Many scholars applied it in three-way decision-making methods [30, 47]. Motivated by this, integrating the concepts of fuzzy bisimulations, three-way decision-making methods and granular computing will provide a novel decision-making method.

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