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ORIGINAL ARTICLE

On generalized fuzzy ideals of ordered $\mathcal{AG}$-groupoids

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Abstract We introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right, bi-) ideals of an ordered Abel Grassman's groupoids (AG-groupoid) and characterized intra-regular ordered AG-groupoids in terms of these generalized fuzzy ideals.

Keywords Ordered $\mathcal{AG}$-groupoids · Left invertive law · Medial law · Paramedial law · $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals

1 Introduction

The fundamental concept of a fuzzy set was introduced by Zadeh in his classic paper [29], which provides a natural framework for generalizing some of the basic notions of algebra. Kuroki [10] introduced the notion of fuzzy bi-ideals in semigroups. A new type of fuzzy subgroup, that is (α, β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [3] by using the notions of "belongingness and quasi-coincidence" of fuzzy points and fuzzy sets. The concepts of an $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroups [19]. It is now natural to investigate similar type of generalizations of existing fuzzy sub-systems of other algebraic structures. The concept of an $(\in, \in \vee q)$-fuzzy sub-near rings of a near ring introduced by Davvaz in [6]. Kazanci and Yamak [11] studied $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. Shabir et al. [20] characterized regular semigroups by the properties of $(\in, \in \vee q)$-fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals. Kazanci and Yamak [11] defined $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals in semigroups. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra. Generalizing the concept of $x_t qf$, Shabir and Jun [21], defined $x_t q_k f$ as $f(x) + t + k > 1$, where $k \in [0, 1)$. In [21], semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

Faisal and Khan [15] introduced the concept of an ordered $\mathcal{AG}$-groupoid and provided the basic theory for an ordered $\mathcal{AG}$-groupoid in terms of fuzzy subsets. The generalization of an ordered $\mathcal{AG}$-groupoid was also given by Faisal et al. [27] and they introduced the notion of an ordered Γ-$\mathcal{AG}^{**}$-groupoid.

The concept of a left almost semigroup ($\mathcal{LA}$-semigroup) was first introduced by Kazim and Naseeruddin [12] in 1972. In [7], the same structure was called a left invertive groupoid. Protic and Stevanovic [18] called it an Abel-Grassmann's groupoid ($\mathcal{AG}$-groupoid). An $\mathcal{AG}$-groupoid is a groupoid $\mathcal{S}$ whose elements satisfy the left invertive law $(ab)c = (cb)a, \forall a, b, c \in \mathcal{S}$. In an $\mathcal{AG}$-groupoid, the medial law [12] $(ab)(cd) = (ac)(bd), \forall a, b, c, d \in \mathcal{S}$ holds. An $\mathcal{AG}$-groupoid may or may not contains a left identity. The left identity of an $\mathcal{AG}$-groupoid allow us to introduce the inverses of elements in an $\mathcal{AG}$-groupoid. If an $\mathcal{AG}$-groupoid contains a left identity, then it is unique [16]. In an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the paramedial law $(ab)(cd) = (dc)(ba), \forall a, b, c, d \in \mathcal{S}$ holds. If an $\mathcal{AG}$-

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groupoid contains a left identity, then by using medial law, we get $a(bc) = b(ac), \forall a, b, c \in \mathcal{S}$. If an $\mathcal{AG}$-groupoid $\mathcal{S}$ satisfies $a(bc) = b(ac), \forall a, b, c \in \mathcal{S}$ without left identity, then $\mathcal{S}$ is called an $\mathcal{AG}^{**}$-groupoid. Several examples and interesting properties of $\mathcal{AG}$-groupoids can be found in [28], [16] and [22].

Motivated by the study of Khan et al. [14], Yin and Zhan [25] and Yin et al. [26] on generalized fuzzy ideals in ordered semigroups, we study the theory of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sets in ordered $\mathcal{AG}$-groupoids. $\mathcal{AG}$-groupoids are the generalization of the concept of semigroups and it was difficult to handle the results on $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sets in ordered $\mathcal{AG}$-groupoids. In this paper we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in an ordered $\mathcal{AG}$-groupoid and introduce some new results which are infect the generalization of the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in an ordered semigroup. We characterize an intra-regular ordered $\mathcal{AG}$-groupoid by the properties of its $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

2 Preliminaries and examples

In this section, we will present some basic definitions needed for next section.

Definition 1 An ordered $\mathcal{AG}$-groupoid ($\mathfrak{po}$-$\mathcal{AG}$-groupoid) is a structure $(G, ., \leq)$ in which the following conditions hold [15]:

(i) $(G, .)$ is an $\mathcal{AG}$-groupoid.
(ii) $(G, \leq)$ is a poset.
(iii) $\forall a, b, x \in G, a \leq b \Rightarrow ax \leq bx (xa \leq xb)$.

Example 1 Define a new binary operation "$\circ_e$" (e-sandwich operation) on an ordered $\mathcal{AG}$-groupoid $(\mathcal{S}, ., \leq)$ with left identity e as follows:

$$a \circ_e b = (ae)b \, \forall \, a, b \in \mathcal{S}.$$

Then $(\mathcal{S}, \circ_e, \leq)$ becomes an ordered semigroup.

An ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup because if an ordered $\mathcal{AG}$-groupoid has a right identity then it becomes an ordered semigroup.

Let A be a non-empty subset an ordered $\mathcal{AG}$-groupoid G, then

$$(A] = \{t \in S \mid t \leq a, \text{ for some a} \in \text{A}\}.$$

For $A = \{a\}$, we usually written as $(a]$.

Definition 2 Let G be an ordered $\mathcal{AG}$-groupoid. By a left (right) ideal of G, we mean a non-empty subset A of G such that $(GA] \subseteq A((AG] \subseteq A)$. By two-sided ideal or simply ideal, we mean a non-empty subset A of G which is both a left and a right ideal of G.

Definition 3 An $\mathcal{AG}$-subgroupoid A of G is called a bi-ideal of G if $((AG)A] \subseteq A$.

Definition 4 A non-empty subset A of G is called a generalized bi-ideal of G if $((AG)A] \subseteq A$.

A fuzzy subset f of a given set G is described as an arbitrary function $f : G \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of $G, f \subseteq g$ means that, $f(x) \leq g(x), \forall x \in G$. Let f and g be any fuzzy subsets of an ordered $\mathcal{AG}$-groupoid G, then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee_{a \leq bc} \{f(b) \wedge g(c)\}, & \text{if there exist b, c} \in \text{G, such that a} \leq \text{bc} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 5 A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy ordered $\mathcal{AG}$-subgroupoid of G if $f(xy) \geq f(x) \wedge f(y), \forall x, y \in G$.

Definition 6 A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy left (right) ideal of G if $f(xy) \geq f(y)(f(xy) \geq f(x)), \forall x, y \in G$.

Definition 7 A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy ideal of G if it is both fuzzy left and fuzzy right ideal of G.

Definition 8 A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called a fuzzy generalized bi-ideal of G if $f((xy)z) \geq f(x) \wedge f(z), \forall x, y$ and $z \in G$.

Let $\mathcal{F}(G)$ denotes the collection of all fuzzy subsets of an ordered $\mathcal{AG}$-groupoid G, then $(\mathcal{F}(G), \circ)$ becomes an ordered $\mathcal{AG}$-groupoid [15].

The characteristic function χ_A for a non-empty subset A of an ordered $\mathcal{AG}$-groupoid G is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if x} \in \text{A,} \\ 0, & \text{if x} \notin \text{A.} \end{cases}$$

A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G of the form

$$f(y) = \begin{cases} r(\neq 0), & if\ y \leq x \\ 0, & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r, where $r \in (0, 1]$.

In what follows let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, we define $X^\delta_{\gamma B}$ be the fuzzy subset of X by $X^\delta_{\gamma B}(x) \geq \delta$ and $X^\delta_{\gamma B}(x) \leq \gamma, \forall x \in B$. Otherwise, clearly $X^\delta_{\gamma B}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

Definition 9 For a fuzzy point x_r and a fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G, we say that:

(i) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.

(ii) $x_r q_\delta f$ if $f(x) + r > 2\delta$.
(iii) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now we introduce a new relation on $\mathcal{F}(G)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.

For any $f, g \in \mathcal{F}(G)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$, we mean that $x_r \in_\gamma f \Longrightarrow x_r \in_\gamma \vee q_\delta g, \forall x \in G$ and $r \in (\gamma, 1]$.

Moreover f and g are said to be (γ,δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

Example 2 Let $G = \{1, 2, 3\}$ be an ordered $\mathcal{AG}$-groupoid with the multiplication table and order below:

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	1	1

$\leq := \{(1,1), (2,2), (3,3), (1,3), (1,2)\}$

Define a fuzzy subset $f : G \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.9, & \text{for x} = 1 \\ 0.6, & \text{for x} = 2\,. \\ 0.7, & \text{for x} = 3 \end{cases}$$

Then by routine calculation it is easy to observe the following:

(i) f is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy ideal of G.
(ii) f is not an $(\in, \in \vee q_{0.3})$-fuzzy ideal of G, because $f(12) < f(2) \wedge \frac{1-0.3}{2}$.

Example 3 Let $S = \{0, 1, 2, 3\}$ be an ordered $\mathcal{AG}$-groupoid with the multiplication table and order below:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$\leq := \{(0,0), (1,1), (2,2), (3,3), (0,1)\}$

Define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.75 & \text{for x} = 0 \\ 0.65 & \text{for x} = 1 \\ 0.7 & \text{for x} = 2 \\ 0.5 & \text{for x} = 3 \end{cases}$$

Then clearly f is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy left ideal of S. If

$\leq := \{(a,b), (a,c), (a,d)\}$

Again define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.9 & \text{for x} = 0 \\ 0.7 & \text{for x} = 1 \\ 0.6 & \text{for x} = 2 \\ 0.5 & \text{for x} = 3 \end{cases}$$

Then f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.5})$-fuzzy bi-ideal.

Lemma 1 *Let $f, g, h \subseteq \mathcal{F}(G)$ and $\gamma, \delta \in [0,1]$, then*

(i) $f \subseteq \vee q_{(\gamma,\delta)} g (f \supseteq \vee q_{(\gamma,\delta)} g) \Leftrightarrow \max\{f(x), \gamma\} \leq \min\{g(x), \delta\} (\max\{f(x), \gamma\} \geq \min\{g(x), \delta\}), \forall x \in G$.
(ii) *If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.*

Proof The proof is straightforward. □

Corollary 1 $= \vee q_{(\gamma,\delta)}$ *is an equivalence relation on* $\mathcal{F}(G)$.

Definition 10 By Lemma 1, it is also notified that $f = \vee q_{(\gamma,\delta)} g \Leftrightarrow \max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}, \forall x \in G$, where $\gamma, \delta \in [0,1]$.

Lemma 2 *Let A and B be any subsets of an ordered $\mathcal{AG}$-groupoid G, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0,1]$, then*

(1) $A \subseteq B \Leftrightarrow \chi^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \chi^\delta_{\gamma B}$.
(2) $\chi^\delta_{\gamma A} \cap \chi^\delta_{\gamma B} =_{(\gamma,\delta)} \chi^\delta_{\gamma(A \cap B)}$.
(3) $\chi^\delta_{\gamma A} \circ \chi^\delta_{\gamma B} =_{(\gamma,\delta)} \chi^\delta_{\gamma(AB]}$.

Proof (1) : Assume that A and B are any subset of an ordered $\mathcal{AG}$-groupoid G. Let for any $x \in G$ such that $x \in A \subseteq B$, then by definition, we can write $\chi^\delta_{\gamma B} \geq \delta \to (i)$. Let $x_r \in_\gamma \chi^\delta_{\gamma A}$, then $\chi^\delta_{\gamma A}(x) \geq r > \gamma$. Now two cases arises for Eq. (i).

Case(a): if $\delta \geq r$, then $(i) \Rightarrow \chi^\delta_{\gamma B} \geq r$, therefore $x_r \in_\gamma \chi^\delta_{\gamma B}$.

Case(b): if $\delta < r$ then $(i) \Rightarrow \chi^\delta_{\gamma B} + r > 2\delta$, therefore $x_r q_\delta \chi^\delta_{\gamma B}$.

Hence $\chi^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \chi^\delta_{\gamma B}$.

Conversely, let $\chi^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \chi^\delta_{\gamma B}$. Let $x \in A$, then by definition we can write $\chi^\delta_{\gamma A} \geq \delta$. Let $x_r \in_\gamma \chi^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \chi^\delta_{\gamma B}$, where $\chi^\delta_{\gamma A}$ and $\chi^\delta_{\gamma B}$ are any fuzzy subsets of G. Thus $x_r \in_\gamma \chi^\delta_{\gamma A}$, $x_r \in_\gamma \chi^\delta_{\gamma B}$ or $x_r q_\delta \chi^\delta_{\gamma B}$. As $x_r \in_\gamma \chi^\delta_{\gamma A}$, then $\chi^\delta_{\gamma A}(x) \geq r > \gamma$ and $\chi^\delta_{\gamma B}(x) \geq r > \gamma$ or $\chi^\delta_{\gamma B}(x) + \delta > 2\delta \to (ii)$. Now here arises two cases for (ii).

Case(a): if $r<\delta$, then

$$(ii)\Rightarrow \chi^{\delta}_{\gamma B}(x)\geq 2\delta - r > \delta \Longrightarrow \chi^{\delta}_{\gamma B}(x) > \delta \Longrightarrow x_r \in_{\gamma} \chi^{\delta}_{\gamma B}.$$

Case(b): if $r\geq\delta$, then

$$(ii)\Rightarrow \chi^{\delta}_{\gamma B}(x)\geq r\geq\delta \Longrightarrow \chi^{\delta}_{\gamma B}(x)\geq\delta \Longrightarrow x\in B.$$

Hence $A\subseteq B$.

(2) : Assume that $\chi^{\delta}_{\gamma A}$ and $\chi^{\delta}_{\gamma B}$ are any fuzzy subsets of an ordered $\mathcal{AG}$-groupoid G. Let $x_r \in_{\gamma} \chi^{\delta}_{\gamma A}\cap\chi^{\delta}_{\gamma B}$, then $x_r \in_{\gamma} \chi^{\delta}_{\gamma A}$ and $x_r \in_{\gamma} \chi^{\delta}_{\gamma B} \Longrightarrow \chi^{\delta}_{\gamma A}(x)\geq r>\gamma$ and $\chi^{\delta}_{\gamma B}(x)\geq r>\gamma$. Let $x\in A\cap B$, then by definition $\chi^{\delta}_{\gamma(A\cap B)}(x)\geq\delta\rightarrow(iii)$. Here arises two cases for (iii).

Case(a): Let $r\leq\delta$, then

$$(iii)\Rightarrow \chi^{\delta}_{\gamma(A\cap B)}(x)\geq\delta\geq r\Rightarrow \chi^{\delta}_{\gamma(A\cap B)}(x)\geq r\Rightarrow x_r\in_{\gamma}\chi^{\delta}_{\gamma A\cap B}.$$

Case (b): Let $r>\delta$ then

$$(iii)\Rightarrow \chi^{\delta}_{\gamma A}(x)+r\geq\delta+\delta=2\delta\Rightarrow \chi^{\delta}_{\gamma A}(x)+r>2\delta\Longrightarrow x_r q_{\delta}\chi^{\delta}_{\gamma A\cap B}.$$

Hence $\chi^{\delta}_{\gamma A}\cap\chi^{\delta}_{\gamma B}\subseteq q_{(\gamma,\delta)}\chi^{\delta}_{\gamma(A\cap B)}\rightarrow(iv)$.

Assume that for any $x\in G$, there exists a fuzzy subset $\chi^{\delta}_{\gamma(A\cap B)}$ of G such that $x_r\in_{\gamma}\chi^{\delta}_{\gamma(A\cap B)}$, then $\chi^{\delta}_{\gamma(A\cap B)}(x)\geq r>\gamma$. Let $x\in A\cap B$, then by definition, we can write $\chi^{\delta}_{\gamma(A)}(x)\geq\delta$ and $\chi^{\delta}_{\gamma(B)}(x)\geq\delta$. Thus

$$\chi^{\delta}_{\gamma(A)}\cap\chi^{\delta}_{\gamma(B)}\geq\delta\cap\delta=\delta\Rightarrow\chi^{\delta}_{\gamma(A)}\cap\chi^{\delta}_{\gamma(B)}\geq\delta\rightarrow(v).$$

Here arises two cases for (v).

Case(a): if $r\leq\delta$, then

$$(v)\Rightarrow\chi^{\delta}_{\gamma(A)}\cap\chi^{\delta}_{\gamma(B)}\geq\delta\geq r\Longrightarrow\chi^{\delta}_{\gamma(A)}\cap\chi^{\delta}_{\gamma(B)}\geq r$$
$$\Longrightarrow x_r\in_{\gamma}\chi^{\delta}_{\gamma(A)}\cap\chi^{\delta}_{\gamma(B)}.$$

Case(b): if $r>\delta$, then

$$(v)\Rightarrow\chi^{\delta}_{\gamma(A)}\cap\chi^{\delta}_{\gamma(B)}+r\geq\delta+\delta>2\delta\Longrightarrow\chi^{\delta}_{\gamma(A)}\cap\chi^{\delta}_{\gamma(B)}+r>2\delta.$$

Hence $\chi^{\delta}_{\gamma(A\cap B)}\subseteq\vee q_{(\gamma,\delta)}\chi^{\delta}_{\gamma A}\cap\chi^{\delta}_{\gamma B}\rightarrow(vi)$. From (iv) and (vi), we get $\chi^{\delta}_{\gamma A}\cap\chi^{\delta}_{\gamma B}=_{(\gamma,\delta)}\chi^{\delta}_{\gamma(A\cap B)}$.

(3) : Assume that $\chi^{\delta}_{\gamma A}$ and $\chi^{\delta}_{\gamma B}$ are any fuzzy subsets of G. Let $x\in G$, then

$$x_r\in_{\gamma}\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}\Longrightarrow\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}\geq r>\gamma.$$

Now we have to show that $x_r\in_{\gamma}\chi^{\delta}_{\gamma(AB]}$ or $q_{\delta}\chi^{\delta}_{\gamma(AB]}$. Let $x\in(AB]$, then $\chi^{\delta}_{\gamma(AB]}(x)\geq\delta\rightarrow(vii)$. Now here arises two cases for (vii).

Case(a): if $r\leq\delta$, then

$$(vii)\Rightarrow\chi^{\delta}_{\gamma(AB]}(x)\geq\delta\geq r\Longrightarrow\chi^{\delta}_{\gamma(AB]}(x)\geq r\Rightarrow x_r\in_{\gamma}\chi^{\delta}_{\gamma(AB]}.$$

Case(b): if $r>\delta$, then

$$(vii)\Rightarrow\chi^{\delta}_{\gamma(AB]}(x)+r\geq\delta+\delta>2\delta\Rightarrow\chi^{\delta}_{\gamma(AB]}(x)+r>2\delta$$
$$\Rightarrow x_r q_{\delta}\chi^{\delta}_{\gamma(AB]}.$$

Hence $\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}\subseteq q_{(\gamma,\delta)}\chi^{\delta}_{\gamma(AB]}\rightarrow(viii)$.

Assume that for any $x\in G$ there exists $r\in(\gamma,1]$ and $\gamma,\delta\in[0,1]$ such that $x_r\in_{\gamma}\chi^{\delta}_{\gamma(AB]}\Longrightarrow\chi^{\delta}_{\gamma(AB]}(x)\geq r>\gamma$. Let $x\in(AB]$, then $x\leq ab$ for $a\in A$, $b\in B$, and if $a\in A$, then by definition, we can write $\chi^{\delta}_{\gamma A}(x)\geq\delta$ and similarly for $b\in B\Longrightarrow\chi^{\delta}_{\gamma B}(x)\geq\delta$. Thus we can write

$$\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}(x)=\vee_{x\leq ab}\{\chi^{\delta}_{\gamma A}(a)\cap\chi^{\delta}_{\gamma B}(b)\}$$
$$\geq\vee_{x\leq ab}\{\delta\cap\delta\}=\delta.$$

Thus $\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}(x)\geq\delta\rightarrow(ix)$. Here arises two cases for (ix).

Case(a): if $r\leq\delta$, then

$$(ix)\Rightarrow\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}(a)\geq\delta\geq r\Longrightarrow\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}(a)\geq r$$
$$\Rightarrow x_r\in_{\gamma}\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}.$$

Case(b): if $r>\delta$, then

$$(ix)\Rightarrow\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}(a)+r\geq\delta+\delta>2\delta$$
$$\Longrightarrow\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}(a)+r>2\delta. So\vee q_{(\gamma,\delta)}\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}.$$

Hence $\chi^{\delta}_{\gamma(AB]}\subseteq\vee q_{(\gamma,\delta)}\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}\rightarrow(x)$. From $(viii)$ and (x), we get $\chi^{\delta}_{\gamma A}\circ\chi^{\delta}_{\gamma B}=_{(\gamma,\delta)}\chi^{\delta}_{\gamma(AB]}$. □

Corollary 2 *Let G be an ordered $\mathcal{AG}$-groupoid and γ, γ_1, δ, $\delta_1\in[0,1]$ such that $\gamma<\delta$, $\gamma_1<\delta_1$, $\gamma<\gamma_1$ and $\delta_1<\delta$. Then any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy left ideal of G is an $(\in_{\gamma_1},\in_{\gamma_1}\vee q_{\delta_1})$-fuzzy left ideal over G.*

Definition 11 A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy $\mathcal{AG}$-subgroupoid of G if for all $a,b\in G$ and $s,t\in(\gamma,1]$, the following conditions hold:

(i) If $a\leq b$ and $b_t\in_{\gamma}f\Longrightarrow a_t\in_{\gamma}\vee q_{\delta}f$.
(ii) If $a_t\in_{\gamma}f$ and $b_t\in_{\gamma}f\Longrightarrow(ab)_{\min\{t,s\}}\in_{\gamma}\vee q_{\delta}f$.

Theorem 1 *A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy $\mathcal{AG}$-subgroupoid if for all $a,b\in G$ and $t\in(\gamma,1]$, the following conditions hold:*

(1) $\max\{f(a),\gamma\}\geq\min\{f(b),\delta\}$ *with* $a\leq b$.
(2) $\max\{f(ab),\gamma\}\geq\min\{f(a),f(b),\delta\}$.

Proof $(i)\Rightarrow(1)$: Assume that $a,b\in S$ and $t\in(\gamma,1]$ such that $\max\{f(a),\gamma\}<t\leq\min\{f(b),\delta\}$, then

$$f(a)<t\leq\gamma\Longrightarrow f(a)+t\leq 2\delta,$$

that is $a_t\overline{\in_{\gamma}\vee q_{\delta}}f$ and $f(b)\geq t>\gamma$ and therefore $b_t\in_{\gamma}f$, but $a_t\overline{\in_{\gamma}\vee q_{\delta}}f$ is a contradiction. Hence $\max\{f(a),\gamma\}\geq\min\{f(b),\delta\}$.

$(1) \Rightarrow (i)$: Let $a, b \in S, \gamma, \delta \in [0,1]$ and $b_t \in_\gamma f$, then by definition $f(b) \geq t > \gamma$. Now by (1)

$$\max\{f(a), \gamma\} \geq \min\{f(b), \delta\} \geq \min\{t, \delta\}.$$

We have to consider two cases here:
Case(a): If $t \leq \delta$, then

$$f(a) \geq t > \gamma \Longrightarrow a_t \in_\gamma f.$$

Case(b): If $t > \delta$, then

$$f(a) + t > 2\delta \Longrightarrow a_t q_\delta f.$$

From both cases we can write $a_t \in_\gamma \vee q_\delta f$.

$(ii) \Rightarrow (2)$: Let f be a fuzzy subset of an $\mathcal{AG}$-groupoid G which is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy $\mathcal{AG}$-subgroupoid of G. Assume that there exists $a, b \in G$ and $t \in (\gamma, 1]$ such that

$$\max\{f(ab), \gamma\} < t \leq \min\{f(a), f(b), \delta\},$$

Thus

$$\max\{f(ab), \gamma\} < t \Longrightarrow f(ab) < t \leq \gamma$$
$$\Longrightarrow (ab)_t \overline{\in_\gamma \vee q_\delta} f \text{ and } \min\{\mathrm{f(a), f(b)}, \delta\} \geq \mathrm{t}.$$

Therefore

$$\min\{f(a), f(b)\} \geq t \Longrightarrow f(a) \geq t > \gamma \text{ and } f(b) \geq t > \gamma$$
$$\Longrightarrow a_t \in_\gamma f, b_t \in_\gamma f,$$

but $(ab)_t \overline{\in_\gamma \vee q_\delta} f$ is a contradiction to the definition. Hence

$$\max\{f(ab), \gamma\} \geq \min\{f(a), f(b), \delta\}, \quad \forall \mathrm{a, b} \in \mathrm{G}.$$

$(2) \Rightarrow (ii)$: Assume that there exist $a, b \in G$ and $t, s \in (\gamma, 1]$ such that $a_t \in_\gamma f$ and $b_s \in_\gamma f$, then by definition

$$f(a) \geq t > \gamma \text{ and } f(b) \geq s > \gamma,$$

therefore

$$\max\{f(ab), \delta\} \geq \min\{f(a), f(b), \delta\} \Longrightarrow f(ab) \geq \min\{t, s, \delta\}.$$

We have to consider two cases here:
Case(a): If $\{t, s\} \leq \delta$, then

$$f(ab) \geq \min\{t, s\} > \gamma \Longrightarrow (ab)_{\min\{t,s\}} \in_\gamma f.$$

Case(b): If $\{t, s\} > \delta$, then

$$f(ab) + \min\{t, s\} > 2\delta \Longrightarrow (ab)_{\min\{t,s\}} q_\delta f.$$

From both cases, we can write $(ab)_{\min\{t,s\}} \in_\gamma \vee q_\delta f, \forall a, b \in G$. □

Definition 12 A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of G if for all $a, b \in G$ and $t \in (\gamma, 1]$, the following conditions hold:

(i) If $a \leq b$ and $b_t \in_\gamma f \Longrightarrow a_t \in_\gamma \vee q_\delta f$.

(ii) If $b_t \in_\gamma f \Longrightarrow (ab)_t \in_\gamma \vee q_\delta f$ $(a_t \in_\gamma f \Longrightarrow (ab)_t \in_\gamma \vee q_\delta f)$.

Theorem 2 *A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of G if for all $a, b \in G$ and $\gamma, \delta \in [0,1]$, the following conditions hold:*

(1) $\max\{f(a), \gamma\} \geq \min\{f(b), \delta\}$ *with* $a \leq b$.
(2) $\max\{f(ab), \gamma\} \geq \min\{f(b), \delta\}$.

Proof $(i) \Leftrightarrow (1)$: It is same as in Theorem 1.

$(ii) \Rightarrow (2)$: Assume that $a, b \in G$ and $t, s \in (\gamma, 1]$ such that

$$\max\{f(ab), \gamma\} < t \leq \min\{f(b), \delta\}.$$

Then

$$\max\{f(ab), \gamma\} < t \Longrightarrow f(ab) < t \leq \gamma \Longrightarrow (ab)_t \overline{\in}_\gamma f$$
$$\Longrightarrow (ab)_t \overline{\in_\gamma \vee q_\delta} f,$$

Also

$$\min\{f(b), \delta\} \geq t > \gamma \Longrightarrow f(b) \geq t > \gamma \Longrightarrow b_t \in_\gamma f.$$

But $(ab)_t \overline{\in_\gamma \vee q_\delta} f$ is a contradiction. Hence

$$\max\{f(ab), \gamma\} \geq \min\{f(b), \delta\}.$$

$(2) \Rightarrow (ii)$: Assume that $a, b \in G$ and $t, s \in (\gamma, 1]$ such that $b_t \in_\gamma f$, then by definition we can write $f(b) \geq t > \gamma$, therefore

$$\max\{f(ab), \delta\} \geq \min\{f(b), \delta\} \geq \min\{t, \delta\}.$$

We have to consider two cases here:
Case(a): If $t \leq \delta$, then

$$f(ab) \geq t > \gamma \Longrightarrow (ab)_t \in_\gamma f.$$

Case(b): If $t > \delta$, then

$$f(ab) + t > 2\delta \Longrightarrow (ab)_t q_\delta f.$$

From both cases, we have $(ab)_t \in_\gamma \vee q_\delta f, \forall a, b \in G$. □

Lemma 3 *Let f be a fuzzy subset of an ordered $\mathcal{AG}$-groupoid G and $\gamma, \delta \in [0,1]$, then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of G if and only if f satisfies the following conditions.*

(i) $x \leq y \Rightarrow \max\{f(x), \gamma\} \geq \min\{g(x), \delta\}, \forall x, y \in G$.
(ii) $S \circ f \subseteq \vee q_{(\gamma,\delta)} f$ *and* $f \circ S \subseteq \vee q_{(\gamma,\delta)} f$ $(S \circ f \subseteq \vee q_{(\gamma,\delta)} f$ *and* $f \circ S \subseteq \vee q_{(\gamma,\delta)} f)$.

Proof The proof is straightforward. □

Definition 13 A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G if for all $x, y, z \in G$ and $s, t \in (\gamma, 1]$, the following conditions hold:

(i) If $a \leq b$ and $b_t \in_\gamma f \Longrightarrow a_t \in_\gamma \vee q_\delta f$.
(ii) if $x_t \in_\gamma f$ and $y_s \in_\gamma f \Longrightarrow (xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.
(iii) if $x_t \in_\gamma f$ and $z_s \in_\gamma f \Longrightarrow ((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

Theorem 3 *A fuzzy subset f of an ordered $\mathcal{AG}$-groupoid G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G if for all $x, y, z \in G$, $s, t \in (\gamma, 1]$ and $\gamma, \delta \in [0, 1]$, the following conditions hold:*

(1) $\max\{f(a), \gamma\} \geq \min\{f(b), \delta\}$ *with* $a \leq b$.
(2) $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.
(3) $\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}$.

Proof $(i) \Leftrightarrow (1)$: It is same as in Theorem 1.

$(ii) \Rightarrow (2)$: Assume that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of an ordered $\mathcal{AG}$-groupoid G. Let $x, y \in G$ and $s, t \in (\gamma, 1]$ such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}$$

Now

$$f(xy) < t \leq \gamma \Rightarrow (xy)_t \bar{\in}_\gamma f \Rightarrow (xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f.$$

As

$$\min\{f(x), f(y), \delta\} \geq t\{f(x), f(y)\} \geq t \Longrightarrow f(x) \geq t > \gamma, f(y) \geq t > \gamma \Longrightarrow x_t \in_\gamma f, y_s \in_\gamma f,$$

But $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$, which is a contradiction. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}.$$

$(2) \Rightarrow (ii)$: Assume that $x, y \in G$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$, $y_s \in_\gamma f$, but $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$, then $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$, $f(xy) < \min\{f(x), f(y), \delta\}$ and $f(xy) + \min\{t, s\} \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Which is a contradiction. Hence

$$x_t \in_\gamma f, y_s \in_\gamma f \Longrightarrow (xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f, \forall \mathrm{x}, \mathrm{y} \in \mathrm{G}.$$

$(iii) \Rightarrow (3)$: Assume that $x, y, z \in G$ and $t, s \in (\gamma, 1]$ such that

$$\max\{f((xy)z), \gamma\} < t \leq \min\{f(x), f(z), \delta\}.$$

Now

$$\max\{f((xy)z), \gamma\} < t \Longrightarrow f((xy)z) < t \Rightarrow ((xy)z)_t \bar{\in}_\gamma f \Rightarrow ((xy)z)_t \overline{\in_\gamma \vee q_\delta} f,$$

and

$$\min\{f(x), f(z), \delta\} \geq t > \gamma \Longrightarrow f(x) \geq t > \gamma, f(z) \geq t > \gamma \Longrightarrow x_t \in_\gamma f, z_t \in_\gamma f,$$

but $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$, which is a contradiction. Hence

$$\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}.$$

$(3) \Rightarrow (iii)$: Assume that $x, y, z \in G$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$, $z_s \in_\gamma f$, but $((xy)z)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$,

then $f(x) \geq t > \gamma$, $f(z) \geq s > \gamma$, $f((xy)z) < \min\{f(x), f(y), \delta\}$ and $f((xy)z) + \min\{t, s\} \leq 2\delta$. It follows that $f((xy)z) < \delta$ and so $\max\{f((xy)z), \gamma\} < \min\{f(x), f(y), \delta\}$ is a contradiction. Hence

$$x_t \in_\gamma f, z_s \in_\gamma f \Longrightarrow ((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f, \forall \mathrm{x}, \mathrm{y} \in \mathrm{G}.$$

□

Lemma 4 *A non-empty subset B of an ordered $\mathcal{AG}$-groupoid G is a bi-ideal of $G \Leftrightarrow \chi^\delta_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G, where $\gamma, \delta \in [0, 1]$.*

Proof Let B be a bi-ideal of G and assume that $x, y \in B$, then for any $a \in G$, we have $(xa)y \in B$, thus $\chi^\delta_{\gamma B}((xa)y) \geq \delta > \gamma$ and therefore $\chi^\delta_{\gamma B}(x) \geq \delta, \chi^\delta_{\gamma B}(y) \geq \delta$ which shows that $\chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \geq \delta$. Thus

$$\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \chi^\delta_{\gamma B}((xa)y) \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B} \wedge \delta = \delta.$$

Hence $\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta$.

Let $x \in B, y \notin B$, then

$$(xa)y \notin B, \forall \mathrm{a} \in \mathrm{G} \Longrightarrow \chi^\delta_{\gamma B}((xa)y) \leq \gamma < \delta, \chi^\delta_{\gamma \mathrm{B}}(\mathrm{x}) \geq \delta > \gamma \text{ and } \chi^\delta_{\gamma \mathrm{B}}(\mathrm{y}) < \gamma < \delta.$$

Therefore

$$\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \gamma \text{ and } \chi^\delta_{\gamma \mathrm{B}}(\mathrm{x}) \wedge \chi^\delta_{\gamma \mathrm{B}}(\mathrm{y}) \wedge \delta = \chi^\delta_{\gamma \mathrm{B}}(\mathrm{y}).$$

Hence $\chi^\delta_{\gamma B}((xa)y) \vee \gamma \geq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta$.

Let $x \notin B, y \in B$, then

$$(xa)y \notin B, \forall \mathrm{a} \in \mathrm{G} \Longrightarrow \chi^\delta_{\gamma \mathrm{B}}((\mathrm{xa})\mathrm{y}) \vee \gamma \geq \delta > \gamma, \chi^\delta_{\gamma B}(x) < \delta, \chi^\delta_{\gamma B}(y) \geq \delta \text{ and } \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta = \chi^\delta_{\gamma B}(x).$$

Therefore

$$\chi^\delta_{\gamma B}((xa)y) \vee \delta \geq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta.$$

Let $x, y \notin B$, then

$$(xa)y \notin B, \forall \mathrm{a} \in \mathrm{G} \Longrightarrow \chi^\delta_{\gamma \mathrm{B}}(\mathrm{x}) \wedge \chi^\delta_{\gamma \mathrm{B}}(\mathrm{y}) \leq \gamma \text{ and } \chi^\delta_{\gamma \mathrm{B}}((\mathrm{xa})\mathrm{y}) \leq \gamma.$$

Thus

$$\chi^\delta_{\gamma B}((xa)y) \vee \gamma = \gamma \text{ and } \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \wedge \delta \leq \chi^\delta_{\gamma B}(x) \wedge \chi^\delta_{\gamma B}(y) \leq \gamma.$$

Hence $\chi^{\delta}_{\gamma B}((xa)y) \vee \gamma \geq \chi^{\delta}_{\gamma B}(x) \wedge \chi^{\delta}_{\gamma B}(y) \wedge \delta$. Converse is simple. □

Lemma 5 *Let A be a non-empty set of an ordered $\mathcal{AG}$-groupoid G, then A is a left (right , two-sided) ideal of G $\Leftrightarrow \chi^{\delta}_{\gamma A}$ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left (right , two-sided) ideal of G, where $\gamma, \delta \in [0, 1]$.*

Proof The proof is straightforward. □

Lemma 6 ([9]) *In an ordered $\mathcal{AG}$-groupoid G, the following are true.*

(i) $A \subseteq (A], \forall A \subseteq G$.
(ii) $A \subseteq B \subseteq G \Longrightarrow (A] \subseteq (B], \forall A, B \subseteq G$.
(iii) $(A](B] \subseteq (AB], \forall A, B \subseteq G$.
(iv) $(A] = ((A]], \forall A \subseteq G$.
(vi) $((A](B]] = (AB], \forall A, B \subseteq G$.

3 Main results

This section contains the main results of the paper.

Definition 14 An element a of an ordered $\mathcal{AG}$-groupoid G is called an intra-regular element of G if there exists $x \in G$ such that $a \leq (xa^2)y$ and G is called an intra-regular, if every element of G is an intra-regular or equivalently, $A \subseteq ((GA^2)G], \forall A \subseteq G$ [15].

From now onward, G will denote an ordered $\mathcal{AG}$-groupoid unless otherwise specified.

Theorem 4 *For G with left identity, the following conditions are equivalent.*

(i) *G is an intra-regular.*
(ii) *For every left ideal L and bi-ideal B of G, $L \cap B \subseteq ((LB]B]$.*
(iii) *For an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal f and an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal g of $G, f \cap g \subseteq \vee_{q_{(\gamma,\delta)}} (f \circ g) \circ g$.*

Proof $(i) \Rightarrow (iii)$: Let f be an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal and g be an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of an intra-regular G with left identity, then for any $a \in G$, there exist some $x, y \in G$ such that

$$\begin{aligned} a &\leq (xa^2)y = (a(xa))y = (y(xa))a \leq (y(x((xa^2)y)))a \\ &= (x(y((a(xa))y)))a = (x(a(xa))y^2)a = ((a(xa))(xy^2))a \\ &= (((xy^2)(xa))a)a = (((ay^2)x^2)a)a \\ &= (((x^2y^2)a)a)a = (ba)a, \text{ where } \mathrm{b} = (\mathrm{x}^2\mathrm{y}^2)\mathrm{a}. \end{aligned}$$

Thus $(ba, a) \in A_a$. Therefore

$$\begin{aligned} \max\{((f\circ g)\circ g)(a),\gamma\} &= \max\left[\bigvee_{(ba,a)\in A_a} (f\circ g)(ba)\wedge g(a),\gamma\right] \\ &\geq \max[\min\{(f\circ g)(ba),g(a)\},\gamma] \\ &= \min[\max\{(f\circ g)(ba),\gamma\},\max\{g(a),\gamma\}] \\ &= \min\left[\begin{array}{l}\max\left\{\bigvee_{(b,a)\in A_a}\{f((x^2)y^2)a)\wedge g(a),\gamma\}\right\} \\ ,\max\{g(a),\gamma\}\end{array}\right] \\ &\geq \min[\max\{f((x^2)y^2)a)\wedge g(a),\gamma\},\max\{g(a),\gamma\}] \\ &= \min[\max\{\min\{f((x^2)y^2)a),g(a)\},\gamma\},\max\{g(a),\gamma\}] \\ &= \min\left[\begin{array}{l}\max\{\max\{f((x^2)y^2)a),\gamma\},\max\{g(a),\gamma\}\} \\ ,\max\{g(a),\gamma\}\end{array}\right] \\ &\geq \min[\min\{f(a)\wedge g(a),\delta\},\min\{g(a),\delta\}] \\ &= \min\{(f\cap g)(a),\delta\}, \end{aligned}$$

which shows that $f \cap g \subseteq \vee_{q_{(\gamma,\delta)}} ((f \circ g) \circ g)$.

$(iii) \Rightarrow (ii)$: Let L be a left ideal and B be a bi-ideal of G with left identity, then by Lemma 5, $\chi^{\delta}_{\gamma L}$ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal of G and by Lemma 4, $\chi^{\delta}_{\gamma B}$ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G. Therefore by using Lemma 2, we get

$$\begin{aligned} \chi^{\delta}_{\gamma L\cap B} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma L} \cap \chi^{\delta}_{\gamma B} &\subseteq \vee_{q_{(\gamma,\delta)}} (\chi^{\delta}_{\gamma L} \circ \chi^{\delta}_{\gamma B}) \circ \chi^{\delta}_{\gamma B} \\ &\subseteq \vee_{q_{(\gamma,\delta)}} (C_{(LB]}) \circ \chi^{\delta}_{\gamma B} = C_{((LB]B]}. \end{aligned}$$

Therefore $L \cap B \subseteq ((LB]B]$.

$(ii) \Rightarrow (i)$: Clearly $(Ga]$ is both left and bi-ideal of G with left identity, then

$$\begin{aligned} a \in (Ga] \cap (Ga] &= (((Ga](Ga]](Ga]] = (((Ga](Ga])(Ga)] \\ &= (((Ga)(Ga))(Ga)] = (((GG)(aa))(Ga)] \subseteq ((Ga^2)G]. \end{aligned}$$

Therefore G is an intra-regular. □

Theorem 5 *For G with left identity, the following conditions are equivalent.*

(i) *G is an intra-regular.*
(ii) *$f \cap g \cap h \subseteq \vee_{q_{(\gamma,\delta)}} (f \circ g) \circ (f \circ h)$, where f is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal, h is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy right ideal and g is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G.*

Proof $(i) \Rightarrow (ii)$: Assume that f is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal, h is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy right ideal and g is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of an intra-regular G with left identity. Then for any $a \in G$ there exist $x, y \in G$ such that

$$\begin{aligned} a &\leq (xa^2)y = (a(xa))y = y(xa) \leq y(x((xa^2)y) \\ &= y(x(a(xa))y) = y((a(xa))(xy)) \\ &= (a(xa))(y(xy)) = ((y(xy))(xa))a \\ &= ba, \text{ where } \mathrm{b} = (\mathrm{y}(\mathrm{xy}))(\mathrm{xa}) \end{aligned}$$

and

$$a \le (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = ca.$$

Thus $(c, a) \in A_a$ and $(y(xa), a) \in A_a$. Since $A_a = \emptyset$, therefore

$$\begin{aligned}\max\{(f \circ g)(a), \gamma\} &= \max\left[\bigvee_{(c,a)\in A_a} f(c) \wedge g(a), \gamma\right] \\ &\ge \max[\min\{f(y(xa)), g(a), \gamma] \\ &= \min[\max\{f(y(xa)), \gamma\}, \max\{g(a), \gamma\}] \\ &\ge \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}] \\ &= \min\{(f \cap g)(a), \delta\},\end{aligned}$$

and similarly we can show that $\max\{(f \circ h)(a), \gamma\} = \min\{(f \cap h)(a), \delta\}$, therefore $f \cap g \cap h \subseteq \vee_{q_{(\gamma,\delta)}} (f \circ g) \circ (f \circ h)$.

$(ii) \Rightarrow (i)$: Since $\chi^{\delta}_{\gamma G}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of G, therefore

$$f \cap g = f \cap g \cap C_G \subseteq \vee_{q_{(\gamma,\delta)}} (f \circ g) \circ (g \circ C_G) \subseteq \vee_{q_{(\gamma,\delta)}} (f \circ g) \circ g,$$

which implies that $f \cap g \subseteq \vee_{q_{(\gamma,\delta)}} (f \circ g) \circ g$. Hence by Theorem 4, G is an intra-regular. □

Lemma 7 *For G with left identity, the following conditions are equivalent.*

(i) *G is an intra-regular.*
(ii) *$f \circ g = f \cap g$, for an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of G such that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.*

Proof The proof is straightforward. □

Note that every intra-regular $\mathcal{AG}$-groupoid G with left identity is regular but the converse is not true in general [28].

Theorem 6 *For G with left identity, the following conditions are equivalent.*

(i) *G is an intra-regular.*
(ii) *$h \cap f \cap g \subseteq \vee_{q_{(\gamma,\delta)}} (h \circ f) \circ g$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal h, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of G.*
(ii) *$h \cap f \cap g \subseteq \vee_{q_{(\gamma,\delta)}} (h \circ f) \circ g$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal h, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of G.*

Proof $(i) \Longrightarrow (iii)$: Assume that G is an intra-regular with left identity. Let h, f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G respectively. Now for $a \in G$, there exist $x, y, u \in G$ such that, $a \le (au)a$ and $a \le (xa^2)y$. Now

$$\begin{aligned}au &\le \left((xa^2)y\right)u = (uy)\left(xa^2\right) = (ux)\left(ya^2\right) = \left(a^2x\right)(yu) \\ &= ((yu)x)(aa) = ((yu)a)(xa) = (ax)(a(yu)) \\ &= (aa)(x(yu)) = ((x(yu))a)a = ((x(yu))(ea))a \\ &= ((xe)((yu)a))a = ((ae)((yu)x))a.\end{aligned}$$

Thus $(((ae)((yu)x)), a) \in A_{au}$. Therefore

$$\begin{aligned}\max\{((h\circ f)\circ g)(a),\gamma\} &= \max\left[\bigvee_{(au,a)\in A_a} (h\circ f)(au)\wedge g(a),\gamma\right] \\ &\ge \max[\min(h\circ f)(au), g(a),\gamma] \\ &= \min[\max\{(h\circ f)(au),\gamma\},\max\{g(a),\gamma\}] \\ &= \min\left[\begin{matrix}\max\left\{\bigvee_{(((ae)((yu)x)),a)\in A_{au}} h((ae)((yu)x))\wedge f(a),\gamma\right\} \\ ,\max\{g(a),\gamma\}\end{matrix}\right] \\ &\ge \min[\max\{h((ae)((yu)x))\wedge f(a),\gamma\},\max\{g(a),\gamma\}] \\ &= \min[\max\{\min\{h((ae)((yu)x)), f(a)\},\gamma\},\max\{g(a),\gamma\}] \\ &= \min\left[\begin{matrix}\max\{\max\{h((ae)((yu)x)),\gamma\},\max\{f(a),\gamma\}\} \\ ,\max\{g(a),\gamma\}\end{matrix}\right] \\ &\ge \min[\min\{h(a),\delta\},\min\{f(a),\delta\},\min\{g(a),,\delta\}] \\ &= \min\{h\cap f\cap g)(a),\delta\},\end{aligned}$$

therefore $h \cap f \cap g \subseteq \vee_{q_{(\gamma,\delta)}} (h \circ f) \circ g$.

$(iii) \Longrightarrow (ii)$: is straightforward.

$(ii) \Longrightarrow (i)$: By using the given assumption, it is easy to show that $h \circ g \subseteq \vee_{q_{(\gamma,\delta)}} h \cap g$ and therefore by using Lemma 7, G is an intra-regular. □

Theorem 7 *For G with left identity, the following conditions are equivalent.*

(i) *G is an intra-regular.*
(ii) *$(f \circ g) \cap (g \circ f) \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of G.*
(iii) *$(f \circ g) \cap (g \circ f) \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal ideal g of G.*
(iv) *$(f \circ g) \cap (g \circ f) \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal ideal g of G.*
(v) *$(f \circ g) \cap (g \circ f) \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$, for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals f and g of G.*
(vi) *$(f \circ g) \cap (g \circ f) \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$, for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals f and g of G.*

Proof $(i) \Longrightarrow (vi)$: Let f and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals of an intra-regular G with left identity. Now for $a \in G$, there exist $x, y, u \in G$, such that $a \le (xa^2)y$ and $a \le (au)a$.

$$a \le \left(xa^2\right)y = (a(xa))y = (y(xa))a,$$

and

$$\begin{aligned} y(xa) &\leq y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(y(xy)) = (xa^2)(xy^2) \\ &= (xx)(a^2y^2) = a^2(x^2y^2) = (aa)(x^2y^2) = ((x^2y^2)a)a \\ &= ((x^2y^2)((xa^2)y))a = ((xa^2)((x^2y^2)y))a = ((x(x^2y^2))(a^2y))a \\ &= ((y(x^2y^2))(a^2x))a = (a^2((y(x^2y^2))x))a = ((x(y(x^2y^2)))a^2)a \\ &= (a(x(y(x^2y^2)))a)a = (av)a, \text{ where } v = (x(y(x^2y^2)))a. \end{aligned}$$

Therefore

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left[\bigvee_{((av)a,a)\in Aa} f((av)a) \wedge g(a), \gamma\right] \\ &\geq \max[\min\{f((av)a), g(a), \gamma] \\ &= \min[\max\{f((av)a), \gamma\}, \max\{g(a), \gamma\}] \\ &\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}] \\ &= \min\{(f \cap g)(a), \delta\}, \end{aligned}$$

which shows that $f \circ g \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$ and similarly we can show that $g \circ f \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$. Therefore $(f \circ g) \cap (g \circ f) \supseteq \vee_{q_{(\gamma,\delta)}} f \cap g$.

$(vi) \Longrightarrow (v) \Longrightarrow (iv) \Rightarrow (iii) \Longrightarrow (ii)$ are obvious cases.

$(ii) \Longrightarrow (i)$: By using the given assumption, it is easy to show that $f \circ g \subseteq \vee_{q_{(\gamma,\delta)}} f \cap g$ and therefore by using Lemma 7, G is an intra-regular. □

4 Conclusion

Fuzzy set theory introduced by Zadeh is a generalization of classical set theory. Fuzzy set theory has been advanced to a powerful mathematical theory. In more than 30,000 publications, it has been applied to many mathematical areas, such as algebra, analysis, clustering, control theory, graph theory, measure theory, optimization, operations research, topology, artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, pattern recognition, robotics and so on. In the present paper we applied the more general form of fuzzy set theory to the theory of AG-groupoids to discuss the fuzzy ideals in AG-groupoids. We introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right, bi-) ideals of an ordered Abel Grassman's groupoids and characterized intra-regular ordered $\mathcal{AG}$-groupoids in terms of these generalized fuzzy ideals.

In our future study, may be the following topics should be considered:

1. Characterization of completely regular ordered $\mathcal{AG}$-groupoids in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.
2. On $(\in, \in \vee q_k)$-fuzzy soft ideals of ordered $\mathcal{AG}$-groupoids.
3. A study of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals in ordered $\mathcal{AG}$-groupoids.

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