Spin quantum entanglement in non-commutative curved space–time

A Mohadi^{1*} , N Mebarki¹, H Aissaoui¹ and M Boussahel^{1,2}

¹Laboratoire de Physique Mathématique et Subatomique, Mentouri University, Constantine, Algeria

²Laboratoire de Physique et Chimie des Matériaux, Mohamed Boudiaf University, Msila, Algeria

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Abstract: The elaboration of a general formalism on quantum spin entanglement in curved space–time is presented by a system of two particles described by wave packets moving in a gravitational field (GF). This formulation allows us to study different models in curved space–time. In this work, the non-commutative Reissner–Nordström model is considered. The spin entanglement of a system of two spin 1/2 particles is discussed. With particularity that contains multiple and various physical parameters, allowing for a detailed study of this purely quantum phenomenon in different frames of space and geometry or both at the same time.

Keywords: Non-commutative space; Quantum information; Curved space; Spin Entanglement

1. Introduction

During the last decade, great interest has been devoted to quantum entanglement and information theory [1–5]. The spin quantum entanglement of a bipartite system plays an important role in most physical systems, such as condensed matter. Recently, the effect of relativistic motion on the entanglement correlation of quantum spin states has been the focus of many physicists, where the spin entanglement of massive particles can change under Lorentz transformations. The entangled momentum of rotation in a flat space–time is discussed by Peres, Scudo and Terno [6], in the same year Gingrich and Adami [7] showed that the entanglement between the spins is affected by the Wigner rotation. This latter in special relativity is known as the product of two Lorentz boost in different directions. Furthermore, this study is extended to a curved space–time [8–13], where Terashima and Ueda [8, 9] studied the EPR (Einstein–Podolsky–Rosen) correlation and Bell’s inequality in the Schwarzschild space–time. By considering accelerated particles in the gravitational field (GF), they showed that the acceleration and the gravity deteriorate the perfect anti-correlation of a pair of EPR spins in the same direction. On the other hand, in [9] they showed when the spin entropy of a spin-1/2 particle moving in the

gravitational field can be generated. Considering that if the spin state of the particle is pure at one point in space–time, it becomes mixed at another point. Because the local inertial frames of reference at different points are different in general. Moreover, they showed that the spin entropy of particles in a circular motion is quickly incremented close to the event horizon of the Schwarzschild black hole. Also, the spin entanglement can be more powerful against changes brought about by motion in the singlet state than in the triplet state [10].

The very early quantum space–time model based on non-commutative (NC) algebra was suggested by Snyder in 1949 [14] to ameliorate short-distance singularities in quantum field theory. This idea was the motivation behind studying non-commutative space with cosmological models [15–17], where NC Seiberg Witten space–time has played an important role in studying many phenomena in particle physics and cosmology [18–24], where some authors [25, 26] have suggested some non-commutative models in classical cosmology to explain the accelerated expansion of our universe, and NC opened the door for a new explanation of dark matter and dark energy as well as the cosmic microwave background (C.M.B) and its anisotropies [27–32].

Emerging of the entanglement entropy concept and its application to black hole entropy issues [33, 34], another exciting area has attracted many physicists: the relationship between the structure of space–time and entanglement. Where it was considered, the non-commutativity can

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*Corresponding author, E-mail: aicha.mohadi@univ-msila.dz

induce entanglement [35–37]. Abhishek Muhuri and others [38] showed that even in non-commutative space, the entanglement is generated only if the harmonic oscillator is anisotropic.

The model we present here is one that tries to understand quantum entanglement behavior, which can be a better alternative to experiment or to verify the effects of the NC space on quantum entanglement, as was done in studies [39, 40] the effects of the passing gravitational wave on the quantum states of a system of N spin-1/2 particles have been investigated by Ye Yeo et al.

Based on previous work, this article discusses the effect of the gravitational field (near or far from the black hole) on the quantum spin entanglement (QSE) of a bipartite system. The system is described by packets of centroid waves as a momentum representation [11]. Using the idea of local inertial frames, both the increasing speed of the centroid and the shape of the gravitational field cause a Wigner rotation that influences the wave packet. As a result of this fact, we try to extend our study to a metric or to different metrics in general. In order to be able to study the effects of both the GF shape and various parameters of the black hole, either in a commutative framework of geometry or even non-commutative. In Sect. 2, we present a general mathematical formalism. In Sect. 3, the non-commutative Reissner–Nordström space–time is considered. In Sect. 4, we compare the behavior of entanglement in singlet and triplet state, and in Sect. 5, we draw our conclusion by focusing on the SE of the centroid packet and how it is affected by various parameters like the acceleration of the centroid, the distance from a massive body, and the NC of space.

2. Mathematical formalism

In order to study the spin of a particle in curved space–time, one has to use an inertial local frame at each point. This can be done at the tangent at a point of curved space–time using the vierbein (or tetrad) e_a^μ (μ (resp.) is a curved (resp. flat) index) defined by:

$$g_{\mu\nu} e_a^\mu e_b^\nu = \eta_{ab} \quad (1)$$

Where $g_{\mu\nu}$ and η_{ab} ($\eta_{ab} = \text{diag}(-1, 1, 1, 1)$) are the curved and Minkowski space–time metric, respectively. Let us introduce one fermionic particle $|P, \sigma\rangle$ with a 4-momentum P^μ and spin $\sigma (= \uparrow, \downarrow)$ at some point of the space–time. If we move from one point to another, this state becomes (in a local frame) [10, 11]:

$$\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, P)) |\Lambda P, \sigma'\rangle \quad (2)$$

Where Λ is the Lorentz transformation matrix and $W(\Lambda, P)$ is the Wigner rotation operator corresponding to Λ . $D_{\sigma'\sigma}$ denotes the two-dimensional representation of the Wigner rotation operator [41].

Let us consider a system of two non-interacting spin 1/2 particles, where its center of mass system can be described by an initial wave packet $|\psi^i\rangle$ given in a local frame by [11] [9]:

$$|\psi^i\rangle = \sum_{\sigma_1 \sigma_2} \int d^3 p_1 d^3 p_2 \psi_{\sigma_1 \sigma_2}(p_1, p_2) |P_1, \sigma_1; P_2, \sigma_2\rangle \quad (3)$$

With the normalization condition:

$$\sum_{\sigma_1 \sigma_2} \int d^3 p_1 d^3 p_2 |\psi_{\sigma_1 \sigma_2}(p_1, p_2)|^2 = 1 \quad (4)$$

Here, P_1 and P_2 are 4-momentum of the particles 1 and 2, respectively. $\psi_{\sigma_1 \sigma_2}(p_1, p_2)$ are wave functions determining momentum and spin distribution. It can be used to express momentum entanglement, spin entanglement, and even entanglement between spins and momenta. Now, it is easy to show that when the system reaches another point of the inertial local frame, the wave packet becomes $|\psi^f\rangle$ like this:

$$\begin{aligned} |\psi^f\rangle &= U(\Lambda_1(x_f, x_i)) \otimes U(\Lambda_2(x_f, x_i)) |\psi^i\rangle \\ &= \sum_{\sigma_1 \sigma_2 \sigma'_1 \sigma'_2} \int d^3 p_1 d^3 p_2 \sqrt{\frac{(\Lambda_1 P_1)^0 (\Lambda_2 P_2)^0}{P_1^0 P_2^0}} \psi_{\sigma_1 \sigma_2}(p_1, p_2) \\ &\quad \times D_{\sigma'_1 \sigma_1}(W(\Lambda_1, P_1)) D_{\sigma'_2 \sigma_2}(W(\Lambda_2, P_2)) |\Lambda_1 P_1, \sigma'_1; \Lambda_2 P_2, \sigma'_2\rangle \end{aligned} \quad (5)$$

Where $U(\Lambda_1(x_f, x_i))$ is a unitary operator, x_f, x_i are the centroid location at a final and initial point, respectively. The Wigner rotation operator can have the following formula [9]

$$W(\Lambda_1, P_1) = T \exp \left[\int_{x_i}^{x_f} w(x(\tau)) d\tau \right] \quad (6)$$

T here is the time-ordering operator, τ proper time and w is a matrix whose elements are given by

$$w_k^i = \lambda_k^i + \frac{\lambda_{k0}^i p_k - \lambda_{k0}^0 p^i}{p^0 + mc^2} \quad (7)$$

Where $i, k = (1, 2, 3)$, with m being the mass of the particle. Where the infinitesimal Lorentz transformation matrix elements $\lambda_b^a(x)$ have the form:

$$\lambda_b^a(x) = -\frac{1}{mc^2} [a^a(x) q_b(x) - q^a(x) a_b(x)] + \chi_b^a(x) \quad (8)$$

With:

$$\chi_b^a(x) = -u^\mu(x)\omega_{\mu b}^a \quad (9)$$

And:

$$\omega_{\mu b}^a = -e_b^\nu(x)\nabla_\mu e_\nu^a(x) \quad (10)$$

Here, $\omega_{\mu b}^a$ is a spin connection, where $\chi_b^a(x)$ represents its change along the direction of the 4-vector velocity of the centroid, $u^\mu(x)$ is the four-velocity of the centroid, ∇_μ stands for the covariant derivative and $a^a(x)$ the 4-vector acceleration produced by a classical force as measured in the local frame which is given by

$$a^a(x) = e_\mu^a(x)(u^\nu(x)\nabla_\nu u^\mu(x)) \quad (11)$$

To mention where q comes from, let us consider a system of two non-interacting spin 1/2 particles (wave packet) whose center of mass is described by an equatorial plane with $\theta = \pi/2$. The motion has a radius with constant speed v . After obtaining a central force motion, the components of the centroid 4-momentum in the local inertial frame are given by [12]

$$q^0 = \gamma mc \quad q^1 = q^2 = 0 \quad q^3 = \gamma mv \quad (12)$$

Where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

Now, in order to measure entanglement between 2 particles in a gravitational field, let us consider the following space-time where the metric ds^2 has the form

$$ds^2 = F(r)dt^2 + G(r)dr^2 + H(r, \theta)d\theta^2 + I(r, \theta)d\varphi^2 \quad (13)$$

$F(r), G(r), H(r, \theta), I(r, \theta)$ are arbitrary functions that have a linear relation with the coordinates (r) or (r, θ) , let us make a diagonal choice of the tetrad

$$e_0^t = \frac{1}{\sqrt{F(r)}}, e_1^r = \frac{1}{\sqrt{G(r)}}, e_2^\theta = \frac{1}{\sqrt{H(r, \theta)}}, e_3^\varphi = \frac{1}{\sqrt{I(r, \theta)}} \quad (14)$$

Thus, the non-vanishing spin connection elements are

$$\begin{aligned} \omega_{t1}^0 &= \frac{1}{2} \frac{F'}{\sqrt{GF}}, \omega_{\varphi 3}^0 = \frac{\dot{I}}{2\sqrt{IF}}, \omega_{\theta 2}^1 = -\frac{1}{2\sqrt{GH}}H', \omega_{\varphi 3}^1 \\ &= -\frac{1}{2} \frac{I'}{\sqrt{GI}}, \omega_{\varphi 3}^2 = -\frac{1}{2} \frac{1}{\sqrt{HI}}\partial_\theta I \end{aligned} \quad (15)$$

Where $\dot{I} = \frac{\partial I}{\partial t}$ and $I' = \frac{\partial I}{\partial r}$, $H' = \frac{\partial H}{\partial r}\partial_\theta = \frac{\partial}{\partial \theta}$. Furthermore, the non-vanishing components $u^\nu, \chi_b^a(x)$ and λ_a^b , for a circular motion and constant angular velocity $\frac{d\varphi}{dt}$ on the equatorial plane where $\theta = \pi/2$ are given by

$$u^t(x) = \frac{\gamma c}{\sqrt{F}} u^\varphi(x) = \frac{1}{\sqrt{I}} \gamma r \frac{d\varphi}{dt} \quad (16)$$

$$\begin{aligned} \chi_1^0(x) &= -u^t(x)\omega_{t1}^0, \chi_3^0 = -u^\varphi\omega_{\varphi 1}^0, \chi_2^1(x) = -u^\theta\omega_{\theta 2}^1, \\ \chi_3^1(x) &= -u^\varphi(x)\omega_{\varphi 3}^1, \chi_3^2(x) = -u^\varphi\omega_{\varphi 3}^2 \end{aligned} \quad (17)$$

And

$$\begin{aligned} \lambda_1^0 &= \frac{1}{mc^2}(p^0 a_1) + \chi_1^0, \lambda_3^1 = -\frac{1}{mc^2}(a^1 p_3) + \chi_3^1, \lambda_3^2 = \chi_3^2, \lambda_2^0 \\ &= \chi_2^0 \end{aligned} \quad (18)$$

It is important to mention that the two non-vanishing components of the 4-vector velocity u^t and u^φ can be rewritten as

$$u^t(x) = \frac{\cosh \xi}{\sqrt{F}} \quad \text{and} \quad u^\varphi(x) = \frac{c \sinh \xi}{\sqrt{I(r, \theta)}} \quad (19)$$

Where ξ is the rapidity in the local inertial frame such that $\frac{v}{c} = \tanh \xi$.

To quantify the spin entanglement of the two particles system, we use the Wootters concurrence [42–44] for the mixed state $|P_1, \uparrow, P_2, \downarrow$ defined by

$$C(\rho) = \max(0, \sqrt{\lambda_1}, -\sqrt{\lambda_2}, -\sqrt{\lambda_3}, -\sqrt{\lambda_4}) \quad (20)$$

Where $\sqrt{\lambda_i}$ are the square roots of the eigenvalues of the matrix $\rho \tilde{\rho}$ with: $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$, σ_y here is the Pauli matrix, and ρ is the state density matrix: $\rho = |\psi\rangle\langle\psi|$, where ψ take this following expression [11]

$$\psi_{\sigma_1 \sigma_2}(p_1, p_2) = \varepsilon_i f(p_1) f(p_2) \quad (21)$$

Where ε_i is one of the Bell states; this choice allows us to assume a maximum spin entanglement, $f(p)$ is a normalized function which is defined by

$$f(p) = \frac{\sqrt{\delta(p^1)}\sqrt{\delta(p^2)}}{\sqrt{\pi^2 b m c}} \exp\left(-\frac{(p^3 - q^3)^2}{2b^2 m^2 c^2}\right) \quad (22)$$

Where b is width. To get more simplification of calculations, let $p^1 = 0, p^2 = 0, b = 1$.

If λ_i are positive real numbers, the entanglement can be quantified by the spin entanglement $E(\rho)$ defined as [11]

$$E(\rho) = h\left(\frac{1 + \sqrt{1 - C^2(\rho^f)}}{2}\right) \quad (23)$$

Where:

$$h(x) = -x \log_2 x - (1-x) \log_2 (1-x) \quad (24)$$

Equation (20) can be shown to have the following expression [10], in the case of spin singlet state in curved space-time

$$C(\rho^f) = \langle \cos \Theta^2 \rangle + \langle \sin \Theta^2 \rangle \quad (25)$$

In the case of the spin triplet state in curved space-time

$$C(\rho^f) = \sqrt{\langle \cos 2\Theta \rangle^2 + \langle \sin 2\Theta \rangle^2} \quad (26)$$

With

$$\langle \cos x \rangle = \int dp |f(p)|^2 \cos x \quad (27)$$

Θ here is a shorthand notation for $\tau\Theta_3^1$ (Θ_3^1 is the only non-vanishing component of Θ_k^i where $w_k^i(x) = \Theta_k^i$, τ is propre time). The two-dimensional representation of the Wigner rotation matrix $D(\Theta)$ is

$$D(\Theta) = e^{-iJ_2\Theta} = \begin{pmatrix} \cos \frac{\Theta}{2} & -\sin \frac{\Theta}{2} \\ \sin \frac{\Theta}{2} & \cos \frac{\Theta}{2} \end{pmatrix} \quad (28)$$

Where J_2 is the 2-component of the angular momentum operator. By using (7), (8) and (25), Θ can be rewritten as

$$\begin{aligned} \Theta = \Theta_3^1 \tau = & -\frac{\alpha}{2G} \left\{ \left(AD^2 + B(D^2 - 1) - I' \sqrt{\frac{G}{HI}} \right) \right. \\ & \left. + \frac{D}{C} p \left(AD^2 + B(D^2 - 1) + A\sqrt{G} \right) \right\} \end{aligned} \quad (29)$$

$$\text{Where: } A = \frac{F'}{F}, \quad B = \frac{I'}{H}, \quad D = \sqrt{1 + q^2}, \quad C = 1 + \sqrt{1 + p^2}, \quad \alpha = \frac{\tau}{r_s} \dots$$

3. Spin entanglement in Reissner–Nordström non-commutative space-time

We consider the Reissner–Nordström metric for a charged non-rotating black hole in commutative space-time. It is given by [45]

$$ds^2 = -c^2 \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (30)$$

M and Q are mass and charge, respectively, and t is the time coordinate, r is the radial coordinate, (θ, φ) are the spherical angles, the Schwarzschild radius of the body given by $r_s = 2M$, where r_s does not represent a singularity, but in this case it is only a parameter. And the new singularities are

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}$$

There is an external event horizon at r_+ . The internal Cauchy horizon is the other horizon r_- .

The extremal case is defined as the limiting case where $Q = M$ and $r_+ = r_-$.

Following ref [46], the Seiberg Witten vierbein $\hat{e}_\mu^a$ in a non-commutative gauge gravity is given by

$$\hat{e}_\mu^a = e_\mu^a(x) - i\tilde{\eta}^{\nu\rho} e_{\mu\nu\rho}^a(x) + \tilde{\eta}^{\nu\rho} \tilde{\eta}^{\lambda\tau} e_{\mu\nu\rho\lambda\tau}^a(x) + O(\tilde{\eta}^3) \quad (31)$$

Where

$$e_{\mu\nu\rho}^a = \frac{1}{4} \left(\omega_\nu^{ac} \partial_\rho e_\mu^d + \left(\partial_\rho \omega_\mu^{ac} + R_{\rho\mu}^{ac} \right) e_\nu^d \right) \eta_{cd} \quad (32)$$

And

$$\begin{aligned} e_{\mu\nu\rho\lambda\tau}^a = & \left[\frac{1}{32} 2 \{ R_{\tau\nu}, R_{\mu\rho} \}^{ab} e_\lambda^c - \omega_\lambda^{ab} \left(D_\rho R_{\tau\mu}^{cd} + \partial_\rho R_{\tau\mu}^{cd} \right) e_\nu^m \eta_{dm} \right. \\ & - \{ \omega_\nu, (D_\rho R_{\tau\mu} + \partial_\rho R_{\tau\mu}) \}^{cd} e_\lambda^c - \partial_\tau \{ \omega_\nu, (\partial_\rho \omega_\mu + R_{\rho\mu}) \}^{ab} e_\lambda^c \\ & - \omega_\lambda^{ab} \partial_\tau \left(\omega_\nu^{cd} \partial_\rho e_\mu^m + \left(\partial_\rho \omega_\mu^{cd} + R_{\rho\mu}^{cd} \right) e_\nu^m \right) \eta_{dm} + 2 \partial_\nu \omega_\lambda^{ab} \partial_\rho \partial_\tau e_\mu^c \\ & - 2 \partial_\rho \left(\partial_\tau \omega_\mu^{ab} + R_{\tau\mu}^{ab} \right) \partial_\nu e_\lambda^c - \{ \omega_\nu, (\partial_\rho \omega_\lambda + R_{\rho\lambda}) \}^{ab} \partial_\tau e_\mu^c \\ & \left. - \left(\partial_\tau \omega_\mu^{ab} + R_{\tau\mu}^{ab} \right) \left(\omega_\nu^{cd} \partial_\rho e_\lambda^m + \left(\partial_\rho \omega_\lambda^{cd} + R_{\rho\lambda}^{cd} \right) e_\nu^m \right) \eta_{dm} \right] \eta_{bc} \end{aligned} \quad (33)$$

Where $\tilde{\eta}^{\nu\rho}$ is non-commutativity anti-symmetric matrix elements defined as

$$[\tilde{x}^\mu, \tilde{x}^\nu] = i\tilde{\eta}^{\mu\nu} \quad (34)$$

And $\tilde{x}^\mu$ are the non-commutative space-time coordinates operators. Here, ω_λ^{ab} (resp. D_ρ) is the commutative spin connection (resp. covariant derivative) and $R_{\mu\nu}^{ad} = e_\alpha^a e_\beta^b R_{\mu\nu}^{\alpha\beta}$, where $R_{\mu\nu}^{\alpha\beta}$ is the Riemann tensor. The commutative space-time vierbein and the Minkowski metric are denoted by e_μ^a and $\eta_{\mu\nu}$, respectively. The non-commutative metric is defined by

$$\hat{g}_{\mu\nu} = \frac{1}{2} \left(\hat{e}_\mu^a * \hat{e}_{\nu}^a + \hat{e}_\nu^a * \hat{e}_\mu^a \right) \quad (35)$$

Where “ $*$ ” is the Moyal star product [47], straightforward calculations using Maple 13 and setting $z = \frac{r}{r_s}$, $y = \frac{Q^2}{r_s^2}$, $\lambda = \frac{\tilde{\eta}^2}{r_s^2}$ (in the case $\theta = \pi/2$, with choosing the only non-vanishing components of the NC parameter $\tilde{\eta}^{01} = -\tilde{\eta}^{10} = \tilde{\eta}$) one has

$$\begin{aligned} F = & -\left(1 - \frac{1}{z} + \frac{y}{z^2} \right) - \frac{(2z^3 - 9yz^2 - \frac{11}{4}z^2 + 15zy - 14y^2)\lambda}{4z^2} \\ G = & \frac{1}{\left(1 - \frac{1}{z} + \frac{y}{z^2} \right)} + \frac{(-z^3 + \frac{3}{4}z^2 + 3yz^2 - 3yz + 2y^2)\lambda}{4z^2(z^2 - z + y)^2} \\ H = & (z^2 + \frac{(z^4 - \frac{17}{2}z^3 + \frac{17}{2}z^2 + 27yz^2 - \frac{75}{2}zy + 30y^2)\lambda}{16z^2(z^2 - z + y)}) \\ I = & (z^2 + \frac{(-2z^3 + 8yz^2 + z^2 - 8zy + 8y^2)\lambda}{16(z^2 - z + y)}) \end{aligned} \quad (36)$$

We have two singularities $z_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + y}$.

Figure 1 displays the variation of the entanglement $E(\varrho)$ as a function of the NC parameter $\tilde{\eta}^2$, for a non-charged ($Q = 0$) black hole and fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$. Notice that if $\tilde{\eta}^2$ increases, $E(\varrho)$ decreases. Thus, $\tilde{\eta}^2$ plays an important role in the value changing of entanglement. In fact, as it was pointed out in ref [46], the NC parameter $\tilde{\eta}$ can be considered as a magnetic field contributing to the matter density ρ and therefore affecting the curvature of the space–time through its contribution to GF. Consequently if $\tilde{\eta}^2$ increases, the GF increases and the information decreases. Including the contribution of NC of space–time, it generates additional terms proportional to $\tilde{\eta}^2$. In fact, the gravitational potential $\hat{g}_{00}$ will be of the form

$$\hat{g}_{00} = \hat{A} + \hat{B}Q^2 + \tilde{\eta}^2(\hat{D}Q^4 + \hat{C}Q^2 + \hat{F}) \quad (37)$$

Where

$$\begin{aligned} \hat{A} &= -1 + \frac{1}{z}, \hat{B} = -\frac{1}{zr_s^2}, \hat{D} = \frac{7}{z^6 r_s^6}, \hat{C} \\ &= (9z - 15) \frac{1}{4z^5 r_s^4}, \hat{F} = \left(-2z + \frac{11}{4}\right) \frac{1}{4z^4 r_s^4} \end{aligned} \quad (38)$$

The behavior of the entanglement $E(\varrho)$ depends strongly on the sign of $(\hat{D}Q^4 + \hat{C}Q^2 + \hat{F})$. Considering $\hat{A}$ and $\hat{B}$ negative:

- (1) If $Q \gg 1$, the term $\hat{D}Q^4$ dominates. Since $\hat{D} > 0$, and if $\tilde{\eta}^2$ increases the GF decreases leading to an increase in $E(\varrho)$ (as is the case in Fig. 2).
- (2) If $Q \ll 1$, then $\hat{F}$ dominates and its sign will determine the behavior of $E(\varrho)$ as a function of $\tilde{\eta}^2$. If $\hat{F} > 0$ GF increases and $E(\varrho)$ decreases then we return to the case in Fig. 1.

Figure 3 represents the variation of $E(\varrho)$ as a function of z for fixed $Q = 0, \tilde{\eta} = 0, \alpha = 1, q = 0.01$, (the case of commutative Schwarzschild space–time). Notice that we will reproduce the same behavior as in ref [10].

Figure 4 shows the variation of $E(\varrho)$ as a function of z for fixed $Q \neq 0, \tilde{\eta} = 0$ (the case of commutative Reissner–Nordstrom space–time). Notice that the same behavior as in ref [11] is obtained.

Figure 5 shows the variation of $E(\varrho)$ as a function of z and fixed $\lambda = 0.01, y = 0, \alpha = 1, q = 0.01$; this case is the Schwarzschild black hole in non-commutative space–time.

Figure 6 represents the variation of $E(\varrho)$ as a function of z for fixed $\lambda = 0.1, y = 2, \alpha = 1, q = 0.01$ it is the case of Reissner–Nordstrom Black Hole in non-commutative space–time. Notice that far from the oscillatory behavior region, when z (or r) increases, the GF $\hat{g}_{00}$ decreases until reaching a saturation value (~ 1) where $E(\varrho)$ is maximal. Notice that for smaller values of r ($\rightarrow 0$ near black hole

singularity) where the gravitational field is infinite, the entanglement is minimal. If we go far from the singularity (z increases), the gravitational field decreases and therefore the information increases and thus the entanglement. The oscillatory behavior disappears when we enter the stability region where $E(\varrho) \sim 0.67$. The number of picks and minima depends strongly on the values of the various parameters λ, y, α and q . Concerning the non-commutativity effect on the $E(\varrho)$, it is clear from Eq. (37) that for smaller values of z , as $\tilde{\eta}$ increases the gravitational field $\hat{g}_{00}$ becomes more important (increases) and therefore $E(\varrho)$ decreases. For larger values of z , the effect is almost negligible since the terms in order of $\frac{1}{z^4}, \frac{1}{z^5}, \frac{1}{z^6}$ decrease faster than the commutative terms in order of $\frac{1}{z}$. Notice also that y increases, the GF increases (the term $\tilde{\eta}^2(\hat{D}Q^4)$ dominates at larger value of Q). Thus, the NC effect on the $E(\varrho)$ becomes more important for charged black hole than neutral ones (if the charge Q increases, $E(\varrho)$ decreases).

Table 1 summarizes the effect of the black hole charge on the $E(\varrho)$. It is worth mentioning that in order to keep the perturbative expansion with respect to $\tilde{\eta}^2$ reliable, one must have

$$|\tilde{\eta}^2 A_1| < |A_0| \quad (39)$$

Where $A_0 = \hat{A} + \hat{B}Q^2$ and $A_1 = (\hat{D}Q^4 + \hat{C}Q^2 + \hat{F})$, this implies new constraints on the space parameters z, y .

4. Comparison between singlet and triplet state of entanglement

To gain a thorough understanding, we compare the entanglement behavior in the triplet and singlet states, by using concurrence.

Figure 7 shows how the concurrence varies as a function of z for fixed values of q, y, α and λ in singlet and triplet state, respectively. We found the same behavior with Fig. 6, where for smaller values of r ($r \rightarrow 0$ near the black hole horizon), the gravitational field is infinite, the entanglement is minimal ($C(\rho^f) \sim 0$). If we go far away from the singularity, (z increases) the gravitational field decreases, so the information increases until a saturated bound of the maximal entanglement ($C(\rho^f) \sim 1$). By monitoring both of the curves, we notice that in singlet state when z is at the value of 1.23, $C(\rho^f)$ takes the value of 0.7789. While in triplet state it gives $z = 1.23, C(\rho^f) = 0.5962$.

Figure 8 displays the variation of the concurrence as a function of λ by fixing $\alpha = 1, z = 1.5$ and $q = 0.01$ for both state singlet and triplet, the concurrence is a decreasing function. This is due to the fact that the gravitational potential g_{00} increases, as we mentioned in Fig. 2. Take

Fig. 1 $E(\varrho)$ as a function of λ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$

Fig. 2 $E(\varrho)$ as a function of λ for fixed $z = 4, y = 0.6, \alpha = 1, q = 0.01$

note of this for singlet state when $z = 0.1, C(\rho^f) = 0.9069$, and triplet state when $z = 0.1, C(\rho^f) = 0.8547$.

As it is displayed in Figs. 7 and 8, we can say that the information (entanglement) for the first is greater or equal to the second, and that the singlet state is more resistant to changes induced by motion than the triplet state, this is due to the fact that for the single state, there is a minimum number of parameters and as mentioned before, gravity decreases the information (the effect of gravity on the single state is less than that of the triplet state).

5. Conclusions

Throughout this paper, we have studied the spin entanglement of two particles system quantified by entanglement

Fig. 3 The variation of $E(\varrho)$ as a function of z for fixed $\lambda = 0, \alpha = 1, y = 0, q = 0.01$

Fig. 4 The variation of $E(\varrho)$ as a function of z for fixed $y = 0.1, \lambda = 0, y = 0, \alpha = 1, q = 0.01$

Fig. 5 $E(\varrho)$ as a function of z and fixed $\lambda = 0.01, y = 0, \alpha = 1, q = 0.01$

$E(\varrho)$. Regarding the non-commutative case (as shown in Figs. 1, 2, 3, 4, 5 and 6), the variation of the quantum entanglement as a function of z , the NC parameter λ and

Fig. 6 The variation of $E(\varrho)$ as a function of z for fixed $\lambda = 0.1, y = 2, \alpha = 1, q = 0.01$

Table 1 Illustrative values of $E(\varrho)$ as a function of z for $y = 0$ and $y = 10$

z	$E(\varrho)(y = 0)$	$E(\varrho)(y = 10)$
2	0.64694	0.6072
4	0.664	0.6567
5	0.6644	0.6626
6	0.6646	0.6641

Fig. 7 The variation of $C(\rho^f)$ as a function of z for fixed $\lambda = 0, y = 0, \alpha = 1, q = 0.01$

the black hole charge y is discussed. We have noticed that the NC effect on $E(\varrho)$ becomes more important in a charged black hole, so the behavior of $E(\varrho)$ depends on the black hole's characteristics and not only on the kind of particles (bosons or fermions) [48]. We found as NC

Fig. 8 The variation of $C(\rho^f)$ as a function of λ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$

parameter increases, $E(\varrho)$ decreases, as if NC parameter is playing the role of gravity. On the other hand, as we mentioned in the introduction, NC parameter was considered as having antigravity properties (quintessence, dark energy, etc.), so NC parameter can induce two terms with opposing signs and that was confirmed in [48, 49]

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