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ORIGINAL RESEARCH

# Exact Solutions for a Class of Variable Coefficients Fractional Differential Equations Using Mellin Transform and the Invariant Subspace Method

M. S. Abdel Latif[1,2] · D. Baleanu[3,4] · A. H. Abdel Kader[1,2]

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**Abstract**
In this paper, we propose a class of variable coefficients fractional ordinary differential equations (FODEs). Using Mellin transform (MT), we have transformed this class into a functional equation which can't be solved in general. So, we have selected many special cases of this functional equation that can be solved exactly. After solving these special cases of the functional equation and using the inverse MT, we obtained some exact solutions for the proposed class. The obtained solutions are given in the form of $H$-function and the Wright function. The results, as special cases, contain some special forms given in the literature. Also, the invariant subspace method (ISM) is utilized for solving a class of nonlinear fractional diffusion equations with variable coefficients. The solutions of this class of nonlinear fractional diffusion equations depend upon the solutions of the proposed class of FODEs.

**Keywords** Mellin transform · $H$-function · Caputo fractional derivative · Invariant subspace method

**Mathematics Subject Classification** 26A33 · 34A08 · 44A10

## Introduction

Recently, there has been a lot of interest in studying the qualitative and quantitative behavior of solutions of FODEs [1–8]. The qualitative behavior of solutions of FODEs contains the asymptotic behavior of solutions like oscillation, stability, periodicity, etc.

✉ M. S. Abdel Latif
Mohamed.soror@nmu.edu.eg

[1] Department of Mathematics, Faculty of Science, New Mansoura University, New Mansoura City, Egypt

[2] Mathematics and Engineering Physics Department, Engineering Faculty, Mansoura University, Mansoura, Egypt

[3] Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon

[4] Institute of Space Sciences, Magurele, P. O. Box MG-23, 76900 Bucharest, Romania

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The quantitative behavior of solutions of FODEs is to find the exact forms of solutions which is not an easy task especially for equations with variable coefficients. Integral transforms are considered as one of the most useful techniques used in solving such problems [9–13]. Some of these transforms are, Laplace transform, Fourier transform, Hankel transform and MT [10–13]. Recently, the MT is widely used for finding the exact solutions of variable coefficients FDEs [10, 11].

Our interest in studying this type of FDEs with variable coefficients because they are widely used in modeling many engineering and physical phenomena such as, heat transfer, anomalous diffusion, quantum mechanics and quantum optics, see [14–18] and the references cited therein.

In this paper, we are interested in seeking solutions of the following class of variable coefficients FDEs

$$D_t^{\beta}\left(t^{v}f'(t)\right)+c_1t^{\alpha}f(t)=0, t\geq 0, \tag{1}$$

where $\alpha, v$ and $c_1$ are constants and $\beta$ is a positive constant such that $0<\beta<1$ and $D_t^{\beta}u$ is the Caputo fractional derivative of order $\beta$ which is given by [19]

$$D_t^{\beta}u=\frac{1}{\Gamma(1-\beta)}\int_0^t(t-\tau)^{-\beta}u'(\tau)d\tau.$$

We note that when $\beta=1$, Eq. (1) becomes the second order ODE

$$\left(t^{v}f'(t)\right)'+c_1t^{\alpha}f(t)=0, t\geq 0, \tag{2}$$

which has been widely studied in the literature for special cases of the constant $\alpha$. For examples, when $v=0$, and $\alpha=-2$, Eq. (2) becomes the Cauchy-Euler ODE [20]

$$f''(t)+\frac{c_1}{t^2}f(t)=0, t\geq 0,$$

which has a solution of the form $f(t)=t^{\lambda}$, and the characteristic equation proves the existence of oscillatory solution when $c_1>\frac{1}{4}$.

When $v=1, c_1=-1$ and $\alpha=0$, Eq. (2) becomes the Bessel-type ODE [21]

$$\left(tf'(t)\right)'-f(t)=0, t\geq 0,$$

which is utilized for modeling some phenomena such as Laguerre-type population dynamics [22].

The exact solution for Eq. (1) has been obtained in [21] for the special case when $\alpha=v-1, c_1=-\beta$. Our aim in this paper is to apply the MT to find the exact solution of Eq. (1) for the general case. The technique based on the reduction of Eq. (1) into a functional equation and by using the inverse MT, we obtained some exact solutions for Eq. (1). Also, the ISM will be utilized with the help of solutions of Eq. (1) to get the exact solutions for a class of nonlinear diffusion equations with variable coefficients.

The rest of the paper is organized as follows: In Sect. 2, we give some basic definitions and properties of fractional derivatives and MT. In Sect. 3, we establish the forms of the exact solutions of Eq. (1). We will derive some special cases of these forms to cover the existing results in the literature. In Sect. 4, the ISM will be used to get some exact solutions for a class of nonlinear diffusion equations with variable coefficients.

## Some Basic Definitions

In this section, we give the basic definitions that are needed in the rest of this paper.

**Definition 1** [23] The MT of $f(t)$ is given by

$$\mathcal{M}\{f(t)\} = F(p) = \int_0^\infty t^{p-1} f(t)dt.$$

**Lemma 1** [23] *If the function* $f(t)$ *is a continuous function for* $t > 0$, *then*

1. The MT of the function $t^v f(t)$ is given by
$$\mathcal{M}\{t^v f(t)\} = F(p+v). \tag{3}$$
2. The MT of $f^{(n)}(t)$ is
$$\mathcal{M}\left\{f^{(n)}(t)\right\} = \frac{\Gamma(1-p+n)}{\Gamma(1-p)} F(p-n). \tag{4}$$
3. The MT of the Caputo fractional derivative $D_t^\beta f(t)$ is
$$\mathcal{M}\left\{D_t^\beta f(t)\right\} = \frac{\Gamma(1-p+\beta)}{\Gamma(1-p)} F(p-\beta).$$
4. The MT of $t^\beta D_t^\beta f(t)$ is given by
$$\mathcal{M}\left\{t^\beta D_t^\beta f(t)\right\} = \frac{\Gamma(1-p)}{\Gamma(1-p-\beta)} F(p). \tag{5}$$

**Definition 2** [23] The inverse MT of $f(t)$ is given by

$$\mathcal{M}^{-1}\{F(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-p} F(p)dp.$$

**Definition 3** [24] The H-Function is defined by

$$H_{P,q}^{M,N}\left( z \middle| \begin{matrix} (a_1,A_1)(a_2,A_2)\dots(a_p,A_P) \\ (b_1,B_1)(b_2,B_2)\dots(b_q,B_q) \end{matrix} \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{M} \Gamma(b_j + B_j p) \prod_{j=1}^{N} \Gamma(1 - a_j - A_j p)}{\prod_{j=M+1}^{q} \Gamma(1 - b_j - B_j p) \prod_{j=N+1}^{P} \Gamma(a_j + A_j p)} z^{-p} dp,$$

where $B_j, A_j, a_j, b_j$ are constants, $M, N, P, q$ are integer numbers, $0 \le N \le P, 1 \le M \le q, A_i, B_j \in R_+, a_i, b_j \in R$.

**Definition 4** [24] The Wright function is defined by

$$W_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(p)}{\Gamma(\beta - \alpha p)} (-z)^{-p} dp.$$

## Some Exact Solutions for Eq. (1)

In this section, we establish the closed form of the solution of Eq. (1).

Multiplying Eq. (1) by $t^\beta$, we obtain

$$t^\beta D_t^\beta\left(t^v f'(t)\right) + c_1 t^{\alpha+\beta} f(t) = 0. \tag{6}$$

Applying MT to Eq. (6) and using Eq. (5), we obtain

$$\frac{\Gamma(1-p)}{\Gamma(1-p-\beta)} M\left(t^v f'(t)\right) + c_1 F(p+\alpha+\beta) = 0. \tag{7}$$

Using Eqs. (3) and (4), Eq. (7) becomes

$$\frac{\Gamma(1-p)}{\Gamma(1-p-\beta)}(p+v-1)F(p+v-1) + c_1 F(p+\alpha+\beta) = 0. \tag{8}$$

Assume

$$p \to 1 - v + (1 - v + \alpha + \beta)p.$$

Hence, Eq. (8) becomes

$$\frac{(1-v+\alpha+\beta)p\Gamma(v-p(1-v+\alpha+\beta))}{\Gamma(v-\beta-p(1-v+\alpha+\beta))} - c_1\frac{F((1-v+\alpha+\beta)(p+1))}{F((1-v+\alpha+\beta)p)} = 0. \tag{9}$$

Let,

$$F((1-v+\alpha+\beta)p) = R(p), \tag{10}$$

In this case, Eq. (9) becomes

$$\frac{(1-v+\alpha+\beta)p\Gamma(v-p(1-v+\alpha+\beta))}{\Gamma(v-\beta-p(1-v+\alpha+\beta))} - c_1\frac{R(p+1)}{R(p)} = 0. \tag{11}$$

Equation (11) can't be solved in general. So, we consider some special cases as follows:

**Case 1.** Assume the solution of Eq. (11) takes the form

$$R(p) = \frac{A(p)}{\Gamma(1+r+\alpha-(d-r+\beta)p)},$$

$$v = 1 - d + r + \alpha, \tag{12}$$

where $r, d$ are integers, $A(p)$ is a function which will be determined later. By choosing different values for $r$ and $d$ we can obtain many subcases. Some e

xamples of these subcases are given as follows:

**Case 1.1.** When $r = d = 1$, we get

$$A(p) = c_2(1+\alpha-\beta p)\Gamma(p)\left(\frac{\beta}{c_1}\right)^p, v = 1+\alpha, \tag{13}$$

where $c_2$ is a constant. Substitute Eqs. (13) and (12) into Eq. (10), to get

$$F(p) = c_2 \frac{\Gamma\left(\frac{p}{\beta}\right)}{\Gamma(1-p+\alpha)} \left(\frac{\beta}{c_1}\right)^{\frac{p}{\beta}}. \tag{14}$$

Applying the inverse MT to Eq. (14), we get

$$\begin{aligned} f(t) =& \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-p} c_2 \frac{\Gamma\left(\frac{p}{\beta}\right)}{\Gamma(1-p+\alpha)} \left(\frac{\beta}{c_1}\right)^{\frac{p}{\beta}} dp \\ =& c_2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{p}{\beta}\right)}{\Gamma(1-p+\alpha)} \left(\left(\frac{c_1}{\beta}\right)^{\frac{1}{\beta}} t\right)^{-p} dp \\ =& c_2 H_{0,2}^{1,0}\left(\left(\frac{c_1}{\beta}\right)^{\frac{1}{\beta}} t \,\middle|\, \begin{matrix} - \\ \left(0, \frac{1}{\beta}\right), (-\alpha, 1) \end{matrix}\right), \end{aligned} \tag{15}$$

***Remark 1*** Equation (15) is equivalent to

$$\begin{aligned} f(t) =& c_2 \frac{\beta}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(u)}{\Gamma(1-\beta u+\alpha)} \left(\left(\frac{c_1}{\beta}\right)^{\frac{1}{\beta}} t^{\beta}\right)^{-\mathrm{u}} du \\ =& c_2 \beta W_{\beta,1+\alpha}\left(-\left(\frac{c_1}{\beta}\right)^{\frac{1}{\beta}} t^{\beta}\right). \end{aligned} \tag{16}$$

A special case of the solution (16) is obtained in [21], when $c_1 = -\beta$ and $\alpha = v - 1$.
**Case 1.2.** When $r = 0, d = 1$, we get

$$A(p) = c_3 \frac{\Gamma(p)}{\Gamma\left(p - \frac{\alpha}{\beta+1}\right)} \left(\frac{-1}{c_1}\right)^{\mathrm{p}}, v = \alpha, \tag{17}$$

where $c_3$ is a constant. Substitute Eqs. (17) and (12) into Eq. (10), to get

$$F(p) = c_3 \frac{\Gamma\left(\frac{p}{\beta+1}\right)}{\Gamma(-p+\alpha+1)\Gamma\left(\frac{p}{\beta+1} - \frac{\alpha}{\beta+1}\right)} \left(\frac{-1}{c_1}\right)^{\frac{p}{\beta+1}}. \tag{18}$$

Applying the inverse MT to Eq. (18), we get

$$\begin{aligned} f(t) =& \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{p}{\beta+1}\right)}{\Gamma(-p+\alpha+1)\Gamma\left(\frac{p}{\beta+1} - \frac{\alpha}{\beta+1}\right)} \left(\left(-\mathrm{c}_1\right)^{\frac{1}{\beta+1}} t\right)^{-p} dp \\ =& c_3 H_{1,2}^{1,0}\left( \left(c_1\right)^{\frac{1}{\beta+1}} t \,\middle|\, \begin{matrix} \left(\frac{-\alpha}{\beta+1}, \frac{1}{\beta+1}\right) \\ \left(0, \frac{1}{\beta+1}\right), (-\alpha, 1) \end{matrix}\right). \end{aligned} \tag{19}$$

**Case 1.3.** When $r = -2, d = 1$, we get

$$A(p) = c_4 \frac{\Gamma(p)\left(\frac{-1}{(\beta+3)^2 c_1}\right)^p}{\Gamma\left(p + \frac{2-\alpha}{\beta+3}\right)\Gamma\left(p + \frac{-\alpha+\beta+3}{\beta+3}\right)\Gamma\left(p + \frac{-\alpha+\beta+4}{\beta+3}\right)}, v = \alpha - 2, \tag{20}$$

where $c_4$ is a constant. Substitute Eqs. (20) and (12) into Eq. (10), to get

$$F(p) = c_4 \frac{c_8\Gamma\left(\frac{p}{\beta+3}\right)}{\Gamma(-p+\alpha-1)\Gamma\left(\frac{p-\alpha+2}{\beta+3}\right)\Gamma\left(\frac{p-\alpha+\beta+3}{\beta+3}\right)\Gamma\left(\frac{p-\alpha+\beta+4}{\beta+3}\right)}\left(\frac{-1}{(\beta+3)^2 c_1}\right)^{\frac{p}{\beta+3}}. \tag{21}$$

Applying the inverse MT to Eq. (21), we get

$$\begin{aligned} f(t) &= c_4 \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{p}{\beta+3}\right)\left(\left(-(\beta+3)^2 c_1\right)^{\frac{1}{\beta+3}} t\right)^{-p}}{\Gamma(-p+\alpha-1)\Gamma\left(\frac{p-\alpha+2}{\beta+3}\right)\Gamma\left(\frac{p-\alpha+\beta+3}{\beta+3}\right)\Gamma\left(\frac{p-\alpha+\beta+4}{\beta+3}\right)} dp \\ &= c_4 H_{3,2}^{1,0}\left( \left(-(\beta+3)^2 c_1\right)^{\frac{1}{\beta+3}} t \,\middle|\, \begin{matrix} \left(\frac{2-\alpha}{3+\beta}, \frac{1}{\beta+3}\right), \left(\frac{3-\alpha+\beta}{3+\beta}, \frac{1}{\beta+3}\right), \left(\frac{4-\alpha+\beta}{3+\beta}, \frac{1}{\beta+3}\right) \\ \left(0, \frac{1}{3+\beta}\right), (-\alpha+2, 1) \end{matrix} \right). \end{aligned} \tag{22}$$

**Case 1.4.** When $r = -2, d = 2$, we get

$$A(p) = c_5 \frac{\Gamma(p)\left(\frac{1}{(\beta+4)^3 c_1}\right)^{\mathrm{p}}}{\Gamma\left(\mathrm{p} + \frac{2-\alpha}{\beta+4}\right)\Gamma\left(\mathrm{p} + \frac{3-\alpha}{\beta+4}\right)\Gamma\left(\mathrm{p} + \frac{-\alpha+\beta+4}{\beta+4}\right)\Gamma\left(p + \frac{-\alpha+\beta+5}{\beta+4}\right)}, \tag{23}$$

where $c_5$ is a constant and $v = 5 + \alpha$. Substitute Eq. (23) and Eq. (12) into Eq. (10), to get

$$F(p) = \frac{c_5\Gamma\left(\frac{p}{\beta+4}\right)\left(\frac{1}{(\beta+4)^3 c_1}\right)^{\frac{p}{\beta+4}}}{\Gamma(-p+\alpha-1)\Gamma\left(\frac{p}{\beta+4} + \frac{2-\alpha}{\beta+4}\right)\Gamma\left(\frac{p}{\beta+4} + \frac{3-\alpha}{\beta+4}\right)\Gamma\left(\frac{p}{\beta+4} + \frac{-\alpha+\beta+4}{\beta+4}\right)\Gamma\left(\frac{p}{\beta+4} + \frac{-\alpha+\beta+5}{\beta+4}\right)}. \tag{24}$$

Apply the inverse MT to Eq. (24), to get

$$\begin{aligned} f(t) &= c_5 \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{p}{\beta+4}\right)\left(\left((\beta+4)^3 c_1\right)^{\frac{1}{\beta+4}} t\right)^{-p}}{\Gamma(-p+\alpha-1)\Gamma\left(\frac{p}{\beta+4} + \frac{2-\alpha}{\beta+4}\right)\Gamma\left(\frac{p}{\beta+4} + \frac{3-\alpha}{\beta+4}\right)\Gamma\left(\frac{p}{\beta+4} + \frac{-\alpha+\beta+4}{\beta+4}\right)\Gamma\left(\frac{p}{\beta+4} + \frac{-\alpha+\beta+5}{\beta+4}\right)} dp \\ &= c_5 H_{4,2}^{1,0}\left( \left((\beta+4)^2 c_1\right)^{\frac{1}{\beta+4}} t \,\middle|\, \begin{matrix} \left(\frac{2-\alpha}{4+\beta}, \frac{1}{\beta+4}\right), \left(\frac{3-\alpha}{4+\beta}, \frac{1}{\beta+4}\right), \left(\frac{4-\alpha+\beta}{4+\beta}, \frac{1}{\beta+4}\right), \left(\frac{5-\alpha+\beta}{4+\beta}, \frac{1}{4+\beta}\right) \\ \left(0, \frac{1}{4+\beta}\right), (-\alpha+2, 1) \end{matrix} \right), \end{aligned} \tag{25}$$

**Case 1.5.** When $r = -3, d = 2$, we get

$$A(p) = \frac{c_6\Gamma(p)\left(\frac{-1}{(\beta+5)^4 c_1}\right)^p}{\Gamma\left(p + \frac{3-\alpha}{\beta+5}\right)\Gamma\left(p + \frac{4-\alpha}{\beta+5}\right)\left(p + \frac{-\alpha+\beta+5}{\beta+5}\right)\Gamma\left(p + \frac{-\alpha+\beta+6}{\beta+5}\right)\Gamma\left(p + \frac{-\alpha+\beta+7}{\beta+5}\right)}, \tag{26}$$

where $c_5$ is a constant and $v = \alpha - 4$. Substitue Eq. (26) and Eq. (12) into Eq. (10), to get

$$F(p)=\frac{c_6\left(-(\beta+5)^4c_1\right)^{\frac{-p}{\beta+5}}\Gamma\left(\frac{p}{\beta+5}\right)}{\Gamma(-p+\alpha-2)\Gamma\left(\frac{p-\alpha+3}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+4}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+\beta+5}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+\beta+6}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+\beta+7}{\beta+5}\right)}. \tag{27}$$

Apply the inverse MT to Eq. (27), to get

$$\begin{aligned} f(t)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{c_6\left(\left(-(\beta+5)^4c_1\right)^{\frac{1}{\beta+5}}t\right)^{-p}\Gamma\left(\frac{p}{\beta+5}\right)}{\Gamma(-p+\alpha-2)\Gamma\left(\frac{p-\alpha+3}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+4}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+\beta+5}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+\beta+6}{\beta+5}\right)\Gamma\left(\frac{p-\alpha+\beta+7}{\beta+5}\right)}dp\\ &=c_6H^{1,0}_{5,2}\left(\left(-(\beta+5)^4c_1\right)^{\frac{1}{\beta+5}}t\middle|\begin{matrix}\left(\frac{3-\alpha}{5+\beta},\frac{1}{5+\beta}\right),\left(\frac{4-\alpha}{5+\beta},\frac{1}{5+\beta}\right),\left(\frac{5-\alpha+\beta}{5+\beta},\frac{1}{5+\beta}\right),\left(\frac{6-\alpha+\beta}{5+\beta},\frac{1}{5+\beta}\right),\left(\frac{7-\alpha+\beta}{6+\beta},\frac{1}{5+\beta}\right)\\ \left(0,\frac{1}{5+\beta}\right),(3-\alpha,1)\end{matrix}\right), \end{aligned} \tag{28}$$

**Case 2.** The solution of Eq. (11) can be written in the form

$$R(p)=A(p)\Gamma\left(1+r+\alpha-\frac{d-r+\beta}{m}p\right),$$

$$v=1-d+r+\alpha. \tag{29}$$

Also, by choosing different values for $r$ and $d$ we can obtain many subcases. Some examples of these subcases are given as follows:

**Case 2.1.** When $r=\beta, d=1-\beta$, we get

$$A(p)=c_7\Gamma(p)\left(\frac{(\beta-1)^2}{-c_1}\right)^p\Gamma\left(p+\frac{\alpha+\beta}{\beta-1}\right), v=\alpha+2\beta, \tag{30}$$

where $c_7$ is a constant. Substitute Eqs. (30) and (29) into Eq. (10), to get

$$F(p)=c_7\left(\frac{(\beta-1)^2}{-c_1}\right)^{-\frac{p}{\beta-1}}\Gamma\left(\frac{p}{1-\beta}\right)\Gamma\left(\frac{-p+\alpha+\beta}{\beta-1}\right)\Gamma(-p+\alpha+\beta+1). \tag{31}$$

Apply the inverse MT to Eq. (31), to get

$$\begin{aligned} f(t)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}c_7\left(\left(\frac{(\beta-1)^2}{-c_1}\right)^{\frac{1}{\beta-1}}t\right)^{-p}\Gamma\left(\frac{p}{1-\beta}\right)\Gamma\left(\frac{p-\alpha-\beta}{1-\beta}\right)\Gamma(-p+\alpha+\beta+1)dp\\ &=c_7H^{2,1}_{1,2}\left(\left(\frac{(\beta-1)^2}{-c_1}\right)^{\frac{1}{\beta-1}}t\middle|\begin{matrix}(-\alpha-\beta,1)\\ \left(0,\frac{1}{1-\beta}\right),\left(\frac{\alpha+\beta}{-1+\beta},\frac{1}{1-\beta}\right)\end{matrix}\right). \end{aligned} \tag{32}$$

**Case 2.2.** When $r=\beta, d=2-\beta$, we get

$$A(p)=c_8\Gamma(p)\Gamma\left(p+\frac{\alpha+2\beta-3}{\beta-2}-1\right)\Gamma\left(p+\frac{\alpha+2\beta-2}{\beta-2}-1\right)\left(\frac{(\beta-2)^3}{c_1}\right)^p, \tag{33}$$

where $c_8$ is a constant and $v=\alpha+2\beta-1$. Substite Eq. (33) and Eq. (29) into Eq. (10), to get

$$F(p) = c_8\left(\frac{(\beta-2)^3}{-c_1}\right)^{-\frac{p}{\beta-2}}\Gamma\left(\frac{p}{2-\beta}\right)\Gamma\left(\frac{p-\alpha-\beta+1}{2-\beta}\right)\Gamma\left(\frac{p-\alpha-\beta}{2-\beta}\right)\Gamma(-p+\alpha+\beta+1). \tag{34}$$

Apply the inverse MT to Eq. (34), to get

$$f(t) = \frac{c_8}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\left(\frac{(\beta-2)^3}{-c_1}\right)^{\frac{1}{\beta-2}}t\right)^{-p}\Gamma\left(\frac{p}{2-\beta}\right)\Gamma\left(\frac{p-\alpha-\beta+1}{2-\beta}\right)\Gamma\left(\frac{p-\alpha-\beta}{2-\beta}\right)\Gamma(-p+\alpha+\beta+1)dp$$
$$= c_8 H^{3,1}_{1,3}\left(\left(\frac{(\beta-2)^3}{-c_1}\right)^{\frac{1}{\beta-2}}t\left|\begin{matrix}(-\alpha-\beta,1)\\ \left(0,\frac{1}{2-\beta}\right),\left(\frac{-1+\alpha+\beta}{-2+\beta},\frac{1}{2-\beta}\right),\left(\frac{\alpha+\beta}{-2+\beta},\frac{1}{2-\beta}\right)\end{matrix}\right.\right) \tag{35}$$

## Exact Solutions of Nonlinear Fractional Diffusion Equations with Variable Coefficients Using the ISM

In this section, the ISM will be utilized for seeking some exact solutions for a class of nonlinear diffusion equations with variable coefficients. The details of the ISM can be found in [14, 26].

Consider the nonlinear variable coefficients fractional diffusion equation

$$t^{-\alpha}D_t^{\beta}\left(t^{v}u_t\right) = bu_x^2 + \delta uu_{xx} + \gamma uu_x, t \geq 0, \tag{36}$$

where $b, \delta$ and $\gamma$ are constants.

The ISM can be used for obtaining the solutions of Eq. (36) as follows:

**Step 1.** Equation (36) can be rewritten as follows:

$$t^{-\alpha}D_t^{\beta}\left(t^{v}u_t\right) = F[u] = bu_x^2 + \delta uu_{xx} + \gamma uu_x.$$

**Step 2.** The solution of Eq. (36) can be written as

$$u = A_1(t)B_1(x) + A_2(t)B_2(x), \tag{37}$$

where $A_1, A_2, B_1, B_2$ are some functions which will be determined later.

**Step 3.** Compute the functions $B_1(x), B_2(x)$ as follows:

- Determine the values of the two constants $a_0$ and $a_1$ from the following equation

$$\left(\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0\right)F\left[y(x)\right] = 0,$$

which can be written in the form

$$\left(\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0\right)\left(b\left(y'\right)^2 + \delta yy'' + \gamma yy'\right) = 0. \tag{38}$$

Equation (38) after cancelling the independence using the second order ODE $y'' + a_1y' + a_0y = 0$, can be written as

$$\mathrm{a}_0^2(2b+\delta)y^2 + \mathrm{a}_0\left(4\mathrm{a}_1 b - 3\gamma + 3\mathrm{a}_1\delta\right)yy' - \left(2\mathrm{a}_1\gamma - 2\mathrm{a}_1^2(\beta+\delta) + \mathrm{a}_0(\beta+2\delta)\right)y'^2 = 0.$$

Equating the coefficients of $y^2, yy', y'^2$ to zero and solving the resulting algebraic equations, we obtain

$$a_0 = 0, a_1 = \frac{\gamma}{b+\delta}. \tag{39}$$

- Using Eq. (39), we solve the following ODE

$$\left(\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0\right)y(x) = 0,$$

to get

$$y(x) = h_1 + h_2 \exp\left(-\frac{\gamma}{b+\delta}x\right),$$

where $h_1$ and $h_2$ are arbitrary constants. Hence, the basis of the invariant subspace is $\left\{1, \exp\left(-\frac{\gamma}{b+\delta}x\right)\right\}$ and we can assume that

$$B_1(x) = 1, B_2(x) = \exp\left(-\frac{\gamma}{b+\delta}x\right), \tag{40}$$

**Step 4.** Substituting Eq. (40) into Eq. (37), we obtain

$$u = A_1(t) + A_2(t)\exp\left(-\frac{\gamma}{b+\delta}x\right). \tag{41}$$

Substituting Eq. (41) into Eq. (36), we obtain

$$t^{-\alpha}D_t^{\beta}\left(t^{\nu}A_1'\right) + t^{-\alpha}D_t^{\beta}\left(t^{\nu}A_2'\right)\exp\left(-\frac{\gamma}{b+\delta}x\right) = -\frac{b\gamma^2}{(b+\delta)^2}A_1A_2\exp\left(-\frac{\gamma}{b+\delta}x\right). \tag{42}$$

Comparing the two sides of Eq. (42), we obtain

$$t^{-\alpha}D_t^{\beta}\left(t^{\nu}A_1'\right) = 0, \tag{43}$$

$$t^{-\alpha}D_t^{\beta}\left(t^{\nu}A_2'\right) = -\frac{b\gamma^2}{(b+\delta)^2}A_1A_2. \tag{44}$$

Equation (43) has the solution

$$A_1 = C_0, \tag{45}$$

where $C_0$ is a constant. Substituting Eq. (45) into Eq. (44), we obtain

$$t^{-\alpha}D_t^{\beta}\left(t^{\nu}A_2'\right) = -\frac{b\gamma^2 C_0}{(b+\delta)^2}A_2. \tag{46}$$

Equation (46) is similar to Eq. (1) with $c_1 = \frac{b\gamma^2 C_0}{(b+\delta)^2}$. Using the obtained solutions of Eq. (1), we can get some solutions of Eq. (36) as follows:

**Case 1:** For $\nu = 1 + \alpha$, using Eq. (16), we obtain

$$A_2(t) = c_2 \beta W_{\beta,1+\alpha}\left( -\left( \frac{1}{\beta} \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta}} t^{\beta} \right). \tag{47}$$

Substituting Eqs. (45) and (47) into Eq. (41), we get the following solution for Eq. (36)

$$u = C_0 + c_2 \beta W_{\beta,1+\alpha}\left( -\left( \frac{1}{\beta} \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta}} t^{\beta} \right) \exp\left( -\frac{\gamma}{b+\delta} x \right).$$

**Case 2:** For $v = \alpha$, using Eq. (19), we obtain

$$A_2(t) = c_3 H^{1,0}_{1,2}\left( \left( \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+1}} t \,\middle|\, \begin{matrix} \left( \frac{-\alpha}{\beta+1}, \frac{1}{\beta+1} \right) \\ \left( 0, \frac{1}{\beta+1} \right), (-\alpha, 1) \end{matrix} \right). \tag{48}$$

Substituting Eqs. (45) and (48) into Eq. (41), we get

$$u = C_0 + c_3 H^{1,0}_{1,2}\left( \left( \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+1}} t \,\middle|\, \begin{matrix} \left( \frac{-\alpha}{\beta+1}, \frac{1}{\beta+1} \right) \\ \left( 0, \frac{1}{\beta+1} \right), (-\alpha, 1) \end{matrix} \right) \exp\left( -\frac{\gamma}{b+\delta} x \right).$$

**Case 3:** For $v = \alpha - 2$, using Eq. (22), we get

$$A_2(t) = c_4 H^{1,0}_{3,2}\left( \left( -(\beta+3)^2 \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+3}} t \,\middle|\, \begin{matrix} \left( \frac{2-\alpha}{3+\beta}, \frac{1}{\beta+3} \right), \left( \frac{3-\alpha+\beta}{3+\beta}, \frac{1}{\beta+3} \right), \left( \frac{4-\alpha+\beta}{3+\beta}, \frac{1}{\beta+3} \right) \\ \left( 0, \frac{1}{3+\beta} \right), (-\alpha+2, 1) \end{matrix} \right). \tag{49}$$

Substituting Eqs. (45) and (49) into Eq. (41), we get

$$u = C_0 + c_4 H^{1,0}_{3,2}\left( \left( \frac{-(\beta+3)^2 b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+3}} t \,\middle|\, \begin{matrix} \left( \frac{2-\alpha}{3+\beta}, \frac{1}{\beta+3} \right), \left( \frac{3-\alpha+\beta}{3+\beta}, \frac{1}{\beta+3} \right), \left( \frac{4-\alpha+\beta}{3+\beta}, \frac{1}{\beta+3} \right) \\ \left( 0, \frac{1}{3+\beta} \right), (-\alpha+2, 1) \end{matrix} \right) \exp\left( -\frac{\gamma}{b+\delta} x \right).$$

**Case 4:** For $v = \alpha + 5$, using Eq. (25), we obtain

$$A_2(t) = c_5 H^{1,0}_{4,2}\left( \left( (\beta+4)^2 \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+4}} t \,\middle|\, \begin{matrix} \left( \frac{2-\alpha}{4+\beta}, \frac{1}{\beta+4} \right), \left( \frac{3-\alpha}{4+\beta}, \frac{1}{\beta+4} \right), \left( \frac{4-\alpha+\beta}{4+\beta}, \frac{1}{\beta+4} \right), \left( \frac{5-\alpha+\beta}{4+\beta}, \frac{1}{4+\beta} \right) \\ \left( 0, \frac{1}{4+\beta} \right), (-\alpha+2, 1) \end{matrix} \right). \tag{50}$$

Substituting Eqs. (45) and (50) into Eq. (41), we get

$$u = C_0 + c_5 H^{1,0}_{4,2}\left( \left( (\beta+4)^2 \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+4}} t \,\middle|\, \begin{matrix} \left( \frac{2-\alpha}{4+\beta}, \frac{1}{\beta+4} \right), \left( \frac{3-\alpha}{4+\beta}, \frac{1}{\beta+4} \right), \left( \frac{4-\alpha+\beta}{4+\beta}, \frac{1}{\beta+4} \right), \left( \frac{5-\alpha+\beta}{4+\beta}, \frac{1}{4+\beta} \right) \\ \left( 0, \frac{1}{4+\beta} \right), (-\alpha+2, 1) \end{matrix} \right) \exp\left( -\frac{\gamma}{b+\delta} x \right)$$

**Case 5:** For $v = \alpha - 4$, using Eq. (28), we obtain

$$A_2(t) = c_6 H^{1,0}_{5,2}\left( \left( -(\beta+5)^4 \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+5}} t \,\middle|\, \begin{matrix} \left( \frac{3-\alpha}{5+\beta}, \frac{1}{5+\beta} \right), \left( \frac{4-\alpha}{5+\beta}, \frac{1}{5+\beta} \right), \left( \frac{5-\alpha+\beta}{5+\beta}, \frac{1}{5+\beta} \right), \left( \frac{6-\alpha+\beta}{5+\beta}, \frac{1}{5+\beta} \right), \left( \frac{7-\alpha+\beta}{6+\beta}, \frac{1}{5+\beta} \right) \\ \left( 0, \frac{1}{5+\beta} \right), (3-\alpha, 1) \end{matrix} \right). \tag{51}$$

Substituting Eqs. (45) and (51) into Eq. (41), we obtain

$$u = C_0 + c_6 H^{1,0}_{5,2} \left( \left( -(\beta+5)^4 \frac{b\gamma^2 C_0}{(b+\delta)^2} \right)^{\frac{1}{\beta+5}} t \middle| \left(\frac{3-\alpha}{5+\beta}, \frac{1}{5+\beta}\right), \left(\frac{4-\alpha}{5+\beta}, \frac{1}{5+\beta}\right), \left(\frac{5-\alpha+\beta}{5+\beta}, \frac{1}{5+\beta}\right), \left(\frac{6-\alpha+\beta}{5+\beta}, \frac{1}{5+\beta}\right), \left(\frac{7-\alpha+\beta}{6+\beta}, \frac{1}{5+\beta}\right) \left(0, \frac{1}{5+\beta}\right), (3-\alpha, 1) \right) \exp\left(-\frac{\gamma}{b+\delta}x\right)$$

**Case 6:** For $v = \alpha + 2\beta$, using Eq. (32), we obtain

$$A_2(t) = c_7 H^{2,1}_{1,2}\left( \left( \frac{(\beta-1)^2}{-\frac{b\gamma^2 C_0}{(b+\delta)^2}} \right)^{\frac{1}{\beta-1}} t \middle| \begin{matrix} (-\alpha-\beta, 1) \\ \left(0, \frac{1}{1-\beta}\right), \left(\frac{\alpha+\beta}{-1+\beta}, \frac{1}{1-\beta}\right) \end{matrix} \right). \tag{52}$$

Substituting Eqs. (45) and (52) into Eq. (41), we obtain

$$u = C_0 + c_7 H^{2,1}_{1,2}\left( \left( \frac{(\beta-1)^2}{-\frac{b\gamma^2 C_0}{(b+\delta)^2}} \right)^{\frac{1}{\beta-1}} t \middle| \begin{matrix} (-\alpha-\beta, 1) \\ \left(0, \frac{1}{1-\beta}\right), \left(\frac{\alpha+\beta}{-1+\beta}, \frac{1}{1-\beta}\right) \end{matrix} \right) \exp\left(-\frac{\gamma}{b+\delta}x\right).$$

**Case 7:** For $v = \alpha + 2\beta - 1$, using Eq. (35), we obtain

$$A_2(t) = c_8 H^{3,1}_{1,3}\left( \left( \frac{(\beta-2)^3}{-\frac{b\gamma^2 C_0}{(b+\delta)^2}} \right)^{\frac{1}{\beta-2}} t \middle| \begin{matrix} (-\alpha-\beta, 1) \\ \left(0, \frac{1}{2-\beta}\right), \left(\frac{-1+\alpha+\beta}{-2+\beta}, \frac{1}{2-\beta}\right), \left(\frac{\alpha+\beta}{-2+\beta}, \frac{1}{2-\beta}\right) \end{matrix} \right). \tag{53}$$

Substituting Eqs. (45) and (53) into Eq. (41), we obtain

$$u = C_0 + c_8 H^{3,1}_{1,3}\left( \left( \frac{(\beta-2)^3}{-\frac{b\gamma^2 C_0}{(b+\delta)^2}} \right)^{\frac{1}{\beta-2}} t \middle| \begin{matrix} (-\alpha-\beta, 1) \\ \left(0, \frac{1}{2-\beta}\right), \left(\frac{-1+\alpha+\beta}{-2+\beta}, \frac{1}{2-\beta}\right), \left(\frac{\alpha+\beta}{-2+\beta}, \frac{1}{2-\beta}\right) \end{matrix} \right) \exp\left(-\frac{\gamma}{b+\delta}x\right).$$

## Conclusion

In this paper, established new forms of exact solutions of FODE (1) which generalized the work that has been obtained in [21] for Eq. (1) when $c_1 = -\beta$ and $\alpha = v - 1$. We obtained our results by using the MT which reduce the equation to a functional equation which is solved when $v = 1 - d + r + \alpha$. By choosing different values of $r$ and $d$, we obtained exact solutions for Eq. (1) in the form of the $H$-function. Finally, the effectiveness of the ISM for getting exact solutions for FDEs with variable coefficients is illustrated when solving Eq. (36).

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.



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