Weighted Choquard Equation Perturbed with Weighted Nonlocal Term

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Abstract

We investigate the following problem

$$-\operatorname{div}(v(x)|\nabla u|^{m-2}\nabla u) + V(x)|u|^{m-2}u = \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha}\right) \frac{|u|^{b-2}}{|x|^\alpha}u + \lambda \left(|x|^{-\gamma} * \frac{|u|^c}{|x|^\beta}\right) \frac{|u|^{c-2}}{|x|^\beta}u \quad \text{in } \mathbb{R}^N,$$

where $b, c, \alpha, \beta > 0$, $\theta, \gamma \in (0, N)$, $N \geq 3$, $2 \leq m < \infty$ and $\lambda \in \mathbb{R}$. Here, we are concerned with the existence of groundstate solutions and least energy sign-changing solutions and that will be done by using the minimization techniques on the associated Nehari manifold and the Nehari nodal set respectively.

Keywords Choquard Equation · Weighted m -Laplacian · Weighted nonlocal perturbation · Groundstate solution · Least energy sign-changing solutions

Mathematics Subject Classification 35A15 · 35B20 · 35Q40 · 35Q75

Introduction

In this paper, we study the problem

$$-\operatorname{div}(v(x)|\nabla u|^{m-2}\nabla u) + V(x)|u|^{m-2}u = \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha}\right) \frac{|u|^{b-2}}{|x|^\alpha}u + \lambda \left(|x|^{-\gamma} * \frac{|u|^c}{|x|^\beta}\right) \frac{|u|^{c-2}}{|x|^\beta}u \quad \text{in } \mathbb{R}^N, \quad (1)$$

where $b, c, \alpha, \beta > 0$, $\theta, \gamma \in (0, N)$, $2 \leq m < \infty$, $N \geq 3$, $\lambda \in \mathbb{R}$ and $\operatorname{div}(v(x)|\nabla u|^{m-2}\nabla u)$ is the weighted m -Laplacian. Here v is a Muckenhoupt weight and $|x|^{-\xi}$ is the Riesz potential of order $\xi \in (0, N)$. The function $V \in C(\mathbb{R}^N)$ must satisfy either one or both of the following conditions:

(A1) $\inf_{\mathbb{R}^N} V(x) \geq A_0 > 0$;

(A2) For all $B > 0$ the set $\{x \in \mathbb{R}^N : V(x) \leq B\}$ has finite Lebesgue measure.

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By taking $\lambda = 0$, the Eq. (1) becomes the weighted Choquard equation driven by weighted m -Laplacian and is given by

$$-\operatorname{div}(v(x)|\nabla u|^{m-2}\nabla u) + V(x)|u|^{m-2}u = \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha}\right) \frac{|u|^{b-2}}{|x|^\alpha}u \quad \text{in } \mathbb{R}^N. \quad (2)$$

The case of $v(x) = V(x) \equiv 1$, $m = 2$, $\theta = b = 2$ and $\alpha = 0$ in (2) refers to the Choquard or nonlinear Schrödinger–Newton equation, that is,

$$-\Delta u + u = (|x|^{-2} * u^2)u \quad \text{in } \mathbb{R}^N, \quad (3)$$

and it was first studied by Pekar [32] in 1954 for $N = 3$. The Eq. (3) had been used by Penrose in 1996 as a model in self-gravitating matter (see [33, 34]). Also, if $v(x) \equiv 1$, $m = 2$ and $\alpha = \lambda = 0$, then (2) becomes stationary Choquard equation

$$-\Delta u + V(x)u = (|x|^{-\theta} * |u|^b)|u|^{b-2}u \quad \text{in } \mathbb{R}^N,$$

which arises in quantum theory and in the theory of Bose–Einstein condensation. The Choquard equation has received a considerable attention in the last few decades and has been appeared in many different contexts and settings (see [1, 3, 23, 29, 31]). In [17], Du, Gao and Yang studied the following nonlinear weighted Choquard equation,

$$-\Delta u = \frac{1}{|x|^a} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*}}{|x-y|^b|y|^c} dy \right) |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \quad (4)$$

where $N \geq 3$, $b \in (0, N)$, $a \geq 0$, $2a + b \leq N$ and $2^* = \frac{2N-2a-b}{N-2}$ is the critical exponent. The authors proved the existence of positive groundstate solutions by using the Schwarz symmetrization in subcritical case and by a nonlocal version of concentration–compactness principle in the critical case.

For constant weight function v , singular problems of the type

$$\begin{cases} -\operatorname{div}(v(x)|\nabla u|^{m-2}\nabla u) = f(x)u^{-\delta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and $\delta > 0$, has been considered widely in the last few decades, see [4–6, 8]. The case $v \not\equiv \text{constant}$ and $m = 2$ has also received a considerable attention and was considered by Hadji and Yazidi in [24] for existence and nonexistence results, see also [21, 25]. In [9], Boccardo–Orsina studied the following singular problem

$$\begin{cases} \operatorname{div}(v(x)\nabla u) = f(x)u^{-\delta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (6)$$

where $\delta > 0$ is arbitrary, $v(x)$ is a weight function satisfying $v(x)\eta \geq L|\eta|^2$, $|v(x)| \leq M$ for some positive constants L , M and $\eta \in \mathbb{R}^N$. The authors were able to prove the existence of weak solution $u \in H_0^1(\Omega)$ for $0 < \delta < 1$ and $u \in H_{loc}^1(\Omega)$ in case of $\delta > 1$ such that $u^{\frac{\eta+1}{2}} \in H_0^1(\Omega)$. In [7], Benhamida and Yazidi investigated the critical Sobolev problem

$$\begin{cases} -\operatorname{div}(v(x)|\nabla u|^{m-2}\nabla u) = |u|^{b^*-2}u + \lambda|u|^{c-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (7)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N > b \geq 2$, $b \leq c < b^*$ and $b^* = \frac{Nb}{N-b}$ is called the critical Sobolev exponent. They investigated the existence of positive solutions which depends on the weight $v(x)$. In [11], Brezis and Nirenberg studied the problem (7) for $v(x) \equiv 1$ and $m = 2$ and it has stimulated a several work. In this article, we intent to choose the weight function v which belongs to the class of Muckenhoupt weight A_p , see [16, 19, 20] and this class of weights were first introduced by Muckenhoupt [30], where the author had proved that these are the only class of weights such that the Hardy–Littlewood maximal operator is bounded from the weighted Lebesgue space into itself and plays a very important role in harmonic analysis.

In the recent past, researchers are very interested in studying the problems on degenerate elliptic operators with Muckenhoupt weights. Results related to weighted Poincaré and Sobolev inequalities were obtained by Chanillo and Wheeden, see [14]. In [18], De Cicco–Vivaldi, proved a Liouville theorem for the weight $w(x) = |x|^r$ where $r > -N$ and $N > 2$. In [26], Kawohl et al studied the related degenerate eigenvalue problem.

In this paper, we are interested in the groundstate solutions and least energy sign-changing solutions to (1) and one could easily see that (1) has a variational structure. To this aim, in the subsection below we provide variational framework and main results.

Variational Framework and Main Results

Definition 1 (*Muckenhoupt Weight*) Let $v \in \mathbb{R}^N$ be a locally integrable function such that $0 < v < \infty$ a.e. in $\mathbb{R}^N$. Then $v \in A_m$, that is, the Muckenhoupt class if there exists a positive constant $C_{m,v}$ depending on m and v such that for all balls $B \in \mathbb{R}^N$, we have

$$\left(\frac{1}{|B|} \int_B v dx \right) \left(\frac{1}{|B|} \int_B v^{-\frac{1}{m-1}} dx \right)^{m-1} \leq C_{m,v}.$$

Definition 2 (*Weighted Sobolev Space*) For any $v \in \mathbb{R}^N$, we denote the weighted Sobolev space by $W^{1,m}(\mathbb{R}^N, v)$ and is defined as

$$W^{1,m}(\mathbb{R}^N, v) = \{u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : \|u\|_{1,m,v} < \infty\},$$

with respect to the norm

$$\|u\|_{1,m,v} = \left(\int_{\mathbb{R}^N} |u(x)|^m v(x) dx + \int_{\mathbb{R}^N} |\nabla u|^m v(x) dx \right)^{\frac{1}{m}}. \quad (8)$$

And the space $X = W_0^{1,m}(\mathbb{R}^N, v)$ is the closure of $(C_c^\infty(\mathbb{R}^N), \|\cdot\|_{1,m,v})$ with respect to the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |\nabla u|^m v(x) dx \right)^{\frac{1}{m}}. \quad (9)$$

Definition 3 (*Subclass of A_m*) Let us denote the subclass of A_m by A_p and define A_p as

$$A_p = \left\{ v \in A_m : v^{-p} \in L^1(\mathbb{R}^N) \text{ for some } p \in \left[\frac{1}{m-1}, \infty \right) \cap \left(\frac{N}{m}, \infty \right) \right\}.$$

Definition 4 (*Weighted Morrey space*) Assume $1 < m < \infty$, $r > 0$ and $v \in A_m$. Then $u \in L^{m,r}(\mathbb{R}^N, v)$ - the weighted Morrey space, if $u \in L^m(\mathbb{R}^N, v)$, where

$$L^m(\mathbb{R}^N, v) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} v(x)|u|^m dx < \infty \right\},$$

and

$$\|u\|_{L^{m,r}(\mathbb{R}^N, v)} = \sup_{x \in \mathbb{R}^N, R > 0} \left(L \int_{B(x, R)} v(y)|u(y)|^m dy \right)^{\frac{1}{m}} < \infty,$$

where $L = \frac{R^r}{\int_{B(x, R)} v(x) dx}$ and $B(x, R)$ is the ball centered at x and radius R .

Next, let us define the functional space

$$X_v(\mathbb{R}^N) = \left\{ u \in X : \int_{\mathbb{R}^N} V(x)|u|^m < \infty \right\},$$

endowed with the norm

$$\|u\|_{X_v} = \left[\int_{\mathbb{R}^N} v(x)|\nabla u|^m + \int_{\mathbb{R}^N} V(x)|u|^m \right]^{\frac{1}{m}}.$$

Also, we assume that b satisfies

$$\frac{mp(2N - 2\alpha - \theta)}{2N(p + 1)} < b < \frac{mp(2N - 2\alpha - \theta)}{2N + 2p(N - m)}, \quad (10)$$

or

$$\frac{2N - 2\alpha - \theta}{2N} < b < \infty, \quad (11)$$

and c satisfies

$$\frac{mp(2N - 2\beta - \gamma)}{2N(p + 1)} < c < \frac{mp(2N - 2\beta - \gamma)}{2N + 2p(N - m)}, \quad (12)$$

or

$$\frac{2N - 2\beta - \gamma}{2N} < c < \infty. \quad (13)$$

We also need the following double weighted Hardy-Littlewood-Sobolev inequality by Stein and Weiss(see [35])

$$\left| \int_{\mathbb{R}^N} \left(|x|^{-\delta} * \frac{u}{|x|^\mu} \right) \frac{v}{|x|^\mu} \right| \leq C \|u\|_p \|v\|_q, \quad (14)$$

for $\delta \in (0, N)$, $\mu \geq 0$, $u \in L^p(\mathbb{R}^N)$ and $v \in L^q(\mathbb{R}^N)$ such that

$$1 - \frac{1}{q} - \frac{\delta}{N} < \frac{\mu}{N} < 1 - \frac{1}{q} \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} + \frac{\delta + 2\mu}{N} = 2.$$

Define the energy functional $\mathcal{L}_\lambda : X_v(\mathbb{R}^N) \rightarrow \mathbb{R}$ by

$$\mathcal{L}_\lambda(u) = \frac{1}{m} \|u\|_{X_v}^m - \frac{1}{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha} - \frac{\lambda}{2c} \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|u|^c}{|x|^\beta} \right) \frac{|u|^c}{|x|^\beta}, \quad (15)$$

which is well defined by using (10) to (13) together with the double weighted Hardy–Littlewood–Sobolev inequality (14) and also $\mathcal{L}_\lambda \in C^1(X_v)$. Any solution of (1) is a critical point of the energy functional $\mathcal{L}_\lambda$. Firstly, we deal with the existence of groundstate solutions for Eq. (1). We shall be using a minimization method on the associated Nehari manifold, which is defined as

$$\mathcal{N}_\lambda = \{u \in X_v(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{L}'_\lambda(u), u \rangle = 0\}, \quad (16)$$

and the groundstate solutions will be obtained as minimizers of

$$d_\lambda = \inf_{u \in \mathcal{N}_\lambda} \mathcal{L}_\lambda(u).$$

Now, we present our main result regarding the existence of groundstate solutions.

Theorem 1.1 *Let $N > m \geq 2$, $b > c > \frac{m}{2}$, $\lambda > 0$, $\theta + 2\alpha < N, \gamma + 2\beta < N$. If b, c satisfies (10) and (12) or if b, c satisfy (11) and (13) and V satisfies (A1), then Eq. (1) has a groundstate solution $u \in X_v(\mathbb{R}^N)$.*

Next, we study the least energy sign-changing solutions of (1). Now, we use the minimization method on the Nehari nodal set defined as

$$\overline{\mathcal{N}}_\lambda = \{u \in X_v(\mathbb{R}^N) : u^\pm \neq 0 \text{ and } \langle \mathcal{L}'_\lambda(u), u^\pm \rangle = 0\},$$

and solutions will be obtained as minimizers for

$$\overline{d}_\lambda = \inf_{u \in \overline{\mathcal{N}}_\lambda} \mathcal{L}_\lambda(u).$$

Here, we have

$$\begin{aligned} \langle \mathcal{L}'_\lambda(u), u^\pm \rangle &= \|u^\pm\|_{X_v}^m - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u^\pm)^b}{|x|^\alpha} \right) \frac{(u^\pm)^b}{|x|^\alpha} - \lambda \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u^\pm)^c}{|x|^\beta} \right) \frac{(u^\pm)^c}{|x|^\beta} \\ &\quad - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u^\pm)^b}{|x|^\alpha} \right) \frac{(u^\mp)^b}{|x|^\alpha} - \lambda \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u^\pm)^c}{|x|^\beta} \right) \frac{(u^\mp)^c}{|x|^\beta}. \end{aligned}$$

We now state our second main result in reference to the least energy sign-changing solutions.

Theorem 1.2 *Let $N > m \geq 2$, $b > c > m$, $\lambda \in \mathbb{R}$, $\theta + 2\alpha < m, \gamma + 2\beta < m$. If b, c satisfies (10) and (12) or if b, c satisfy (11) and (13) and V satisfies both (A1) and (A2), then Eq. (1) has a least energy sign-changing solution $u \in X_v(\mathbb{R}^N)$.*

Rest of the paper is organized as follows. In Sect. 2 we collect some preliminary results. Sects. 3 and 4 consists of the proofs of our main results.

Preliminary Results

Lemma 2.1 ([2, 19, 22]) *For any $v \in A_p$, the inclusion map*

$$X_v \hookrightarrow W_0^{1,m_p}(\mathbb{R}^N) \hookrightarrow \begin{cases} L^s(\mathbb{R}^N), & \text{for } m_p \leq s \leq m_p^*, \\ L^s(\mathbb{R}^N), & \text{for } 1 \leq s < \infty, \end{cases} \quad \begin{array}{l} \text{when } 1 \leq m_p < N, \\ \text{when } m_p = N, \end{array}$$

is continuous, where $m_p = \frac{mp}{p+1}$ and $m_p^ = \frac{Nm_p}{N-m_p}$. Here, m_p^* is called the critical Sobolev exponent. Moreover, the embeddings are compact except when $s = m_p^*$ in case of $1 \leq m_p < N$.*

Lemma 2.2 ([27, Lemma 1.1], [28, Lemma 2.3]) *There exists a constant $C_0 > 0$ such that for any $u \in X_v(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} |u|^r \leq C_0 \|u\| \left(\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u|^r \right)^{1-\frac{2}{r}},$$

where $r \in [m_p, m_p^]$.*

Lemma 2.3 ([10, Proposition 4.7.12]) *Let (z_n) be a bounded sequence in $L^r(\mathbb{R}^N)$ for some $r \in (1, \infty)$ and let (z_n) converges to z almost everywhere. Then, we have that $z_n \rightharpoonup z$ weakly in $L^r(\mathbb{R}^N)$.*

Lemma 2.4 (Local Brezis–Lieb lemma) *Let (z_n) be a bounded sequence in $L^r(\mathbb{R}^N)$ for some $r \in (1, \infty)$ such that (z_n) converges to z almost everywhere. Then,*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |z_n|^q - |z_n - z|^q - |z|^q = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |z_n|^{q-1} z_n - |z_n - z|^{q-1} (z_n - z) - |z|^{q-1} z = 0,$$

for every $q \in [1, r]$.

Proof Let $\varepsilon > 0$ be fixed, then there exists a constant $C(\varepsilon) > 0$ such that

$$|g + h|^q - |g|^q - |h|^q \leq \varepsilon |g|^r + C(\varepsilon) |h|^r, \quad (17)$$

for all $g, h \in \mathbb{R}$. By Eq. (17), we have

$$|f_{n,\varepsilon}| = \left(||z_n|^q - |z_n - z|^q - |z|^q| \right)^{\frac{r}{q}} - \varepsilon |z_n - z|^r \Big) + \\ \leq (1 + C(\varepsilon))|z|^r.$$

Next, by Lebesgue Dominated Convergence theorem, we get

$$\int_{\mathbb{R}^N} f_{n,\varepsilon} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (18)$$

Hence, we deduce that

$$||z_n|^q - |z_n - z|^q - |z|^q| \leq f_{n,\varepsilon} + \varepsilon |z_n - z|^r,$$

and this further gives

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} ||z_n|^q - |z_n - z|^q - |z|^q| \leq c\varepsilon,$$

where $c = \sup_n |z_n - z|^r < \infty$. In order to conclude our proof, we let $\varepsilon \rightarrow 0$. $\square$

Lemma 2.5 (Weighted Nonlocal Brezis–Lieb lemma ([28, Lemma 2.4]) *Let $N \geq 3$, $\alpha \geq 0$, $\theta \in (0, N)$, $\theta + 2\alpha < N$ and $b \in [1, \frac{2N}{2N-2\alpha-\theta})$. Let (u_n) be a bounded sequence in $L^{\frac{2N}{2N-2\alpha-\theta}}(\mathbb{R}^N)$ such that $u_n \rightarrow u$ almost everywhere in $\mathbb{R}^N$. Then*

$$\int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n|^b}{|x|^\alpha} \right) \frac{|u_n|^b}{|x|^\alpha} dx - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n - u|^b}{|x|^\alpha} \right) \frac{|u_n - u|^b}{|x|^\alpha} dx \rightarrow \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha} dx.$$

Proof It could be easily seen that

$$\int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n|^b}{|x|^\alpha} \right) \frac{|u_n|^b}{|x|^\alpha} dx - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n - u|^b}{|x|^\alpha} \right) \frac{|u_n - u|^b}{|x|^\alpha} dx \\ = \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |u_n - u|^b \right) \right] \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |u_n - u|^b \right) dx \quad (19) \\ + 2 \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |u_n - u|^b \right) \right] \frac{1}{|x|^\alpha} |u_n - u|^b dx.$$

Next, by taking $q = b$, $r = \frac{2Nb}{2N-2\alpha-\theta}$ in Lemma 2.4, we get $|u_{n_N} - u|^b - |u_n|^b \rightarrow |u|^b$ strongly in $L^{\frac{2N}{2N-2\alpha-\theta}}(\mathbb{R}^N)$. Also, we have $|u_n - u|^b \rightarrow 0$ weakly in $L^{\frac{2N}{2N-2\alpha-\theta}}(\mathbb{R}^N)$ by Lemma 2.3. Further, using the double weighted Hardy–Littlewood–Sobolev inequality (14) we get

$$|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n - u|^b - \frac{1}{|x|^\alpha} |u_n|^b \right) \rightarrow |x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \quad \text{in } L^{\frac{2N}{\theta+2\alpha}}(\mathbb{R}^N).$$

Hence, passing to the limit in (19) together with the above arguments, we get the desired result. $\square$

Lemma 2.6 $\frac{2Nb}{2N-2\alpha-\theta}$ *Let $N \geq 3$, $\alpha \geq 0$, $\theta \in (0, N)$, $\theta + 2\alpha < N$ and $b \in [1, \frac{2N}{2N-2\alpha-\theta})$. Assume $(u_n) \in L^{\frac{2N}{2N-2\alpha-\theta}}(\mathbb{R}^N)$ be a bounded sequence such that $u_n \rightarrow u$ almost everywhere in $\mathbb{R}^N$. Then*

$$\int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n|^b}{|x|^\alpha} \right) \frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h \, dx \rightarrow \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{1}{|x|^\alpha} |u|^{b-2} u h \, dx,$$

for any $h \in L^{\frac{2N\theta}{2N-2\alpha-\theta}}(\mathbb{R}^N)$.

Proof Let $h = h^+ - h^-$ and $v_n = u_n - u$. Here, it will be sufficient to prove the lemma for $h \geq 0$. One could easily notice that

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n|^b}{|x|^\alpha} \right) \frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h &= \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |v_n|^b \right) \right] \\ &\quad \left(\frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h - \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \right) \\ &\quad + \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |v_n|^b \right) \right] \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \\ &\quad + \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h - \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \right) \right] \frac{|v_n|^b}{|x|^\alpha} \\ &\quad + \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|v_n|^b}{|x|^\alpha} \right) \frac{1}{|x|^\alpha} |v_n|^{p-2} v_n h. \end{aligned} \quad (20)$$

Now, by taking $q = b$ and $r = \frac{2Nb}{2N-2\alpha-\theta}$ in Lemma 2.4 and by letting $(z_n, z) = (u_n, u)$ and then $(z_n, z) = (u_n h^{1/b}, u h^{1/b})$ respectively, we get

$$\begin{cases} \frac{|u_n|^b}{|x|^\alpha} - \frac{|v_n|^b}{|x|^\alpha} \rightarrow \frac{|u|^b}{|x|^\alpha} \\ \frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h - \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \rightarrow \frac{1}{|x|^\alpha} |u|^{b-2} u h \end{cases} \quad \text{strongly in } L^{\frac{2N}{2N-2\alpha-\theta}}(\mathbb{R}^N).$$

Further, using the double weighted Hardy-Littlewood-Sobolev inequality we obtain

$$\begin{cases} |x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |v_n|^b \right) \rightarrow |x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \\ |x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h - \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \right) \rightarrow |x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u|^{b-2} u h \right) \end{cases} \quad \text{strongly in } L^{\frac{2N}{\theta+2\alpha}}(\mathbb{R}^N). \quad (21)$$

Using Lemma 2.3 we get

$$\begin{cases} \frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h \rightarrow \frac{1}{|x|^\alpha} |u|^{b-2} u h \\ \frac{1}{|x|^\alpha} |v_n|^b \rightarrow 0 \\ \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \rightarrow 0 \end{cases} \quad \text{weakly in } L^{\frac{2N}{\theta+2\alpha}}(\mathbb{R}^N) \quad (22)$$

Next, by (21) and (22) we have

$$\begin{aligned}
 & \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |v_n|^b \right) \right] \left(\frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h - \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \right) \\
 & \rightarrow \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{1}{|x|^\alpha} |u|^{b-2} u h, \\
 & \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^b - \frac{1}{|x|^\alpha} |v_n|^b \right) \right] \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \rightarrow 0, \\
 & \int_{\mathbb{R}^N} \left[|x|^{-\theta} * \left(\frac{1}{|x|^\alpha} |u_n|^{b-2} u_n h - \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \right) \right] \frac{|v_n|^b}{|x|^\alpha} \rightarrow 0.
 \end{aligned} \tag{23}$$

Using the double weighted Hardy-Littlewood-Sobolev inequality and Hölder's inequality, we find

$$\begin{aligned}
 \left| \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|v_n|^b}{|x|^\alpha} \right) \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h \right| & \leq \|v_n\|_{\frac{2N}{2N-2\alpha-\theta}}^b \| |v_n|^{b-1} h \|_{\frac{2N}{2N-2\alpha-\theta}} \\
 & \leq C \| |v_n|^{b-1} h \|_{\frac{2N}{2N-2\alpha-\theta}}.
 \end{aligned} \tag{24}$$

Also, $v_n^{\frac{2N(b-1)}{2N-2\alpha-\theta}} \rightharpoonup 0$ weakly in $L^{\frac{b}{b-1}}(\mathbb{R}^N)$ by Lemma 2.3. Hence,

$$\| |v_n|^{b-1} h \|_{\frac{2N}{2N-2\alpha-\theta}} = \left(\int_{\mathbb{R}^N} |v_n|^{\frac{2N(b-1)}{2N-2\alpha-\theta}} |h|^{\frac{2N}{2N-2\alpha-\theta}} \right)^{\frac{2N-2\alpha-\theta}{2N}} \rightarrow 0.$$

Therefore, by (24) we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|v_n|^b}{|x|^\alpha} \right) \frac{1}{|x|^\alpha} |v_n|^{b-2} v_n h = 0, \tag{25}$$

and then by passing to the limit in (20) and using (23) and (25) we conclude our proof. $\square$

In the next section, we investigate the groundstate solutions to (1).

Proof of Theorem 1.1

Proof of Theorem 1.1 depends on the analysis of the Palais–Smale sequences for $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$. In this section, we prove that any Palais–Smale sequence of $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$ is either converging strongly to its weak limit or differs from it by a finite number of sequences, which then will be the translated solutions of (2) by following the ideas from [12, 13]. Here, our approach will be depending on several weighted nonlocal Brezis–Lieb results which we have presented in Section 2. Assume $\lambda > 0$. For $u, \phi \in X_\nu(\mathbb{R}^N)$ we have

$$\begin{aligned}\langle \mathcal{L}'_{\lambda}(u), \phi \rangle &= \int_{\mathbb{R}^N} v(x) |\nabla u|^{m-2} \nabla u \nabla \phi + \int_{\mathbb{R}^N} V(x) |u|^{m-2} u \phi - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^{\alpha}} \right) \frac{|u|^{b-1}}{|x|^{\alpha}} \phi \\ &\quad - \lambda \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|u|^c}{|x|^{\beta}} \right) \frac{|u|^{c-1}}{|x|^{\beta}} \phi.\end{aligned}$$

Also, we have

$$\langle \mathcal{L}'_{\lambda}(tu), tu \rangle = t^m \|u\|_{X_v}^m - t^{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^{\alpha}} \right) \frac{|u|^b}{|x|^{\alpha}} - \lambda t^{2c} \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|u|^c}{|x|^{\beta}} \right) \frac{|u|^c}{|x|^{\beta}},$$

for some $t > 0$.

As $b > c > \frac{m}{2}$, so the equation $\langle \mathcal{L}'_{\lambda}(tu), tu \rangle = 0$ has a unique positive solution $t = t(u)$, which is called the *projection of u on $\mathcal{N}_{\lambda}$* . Next, we present the main properties of the Nehari manifold $\mathcal{N}_{\lambda}$ which we have used in this paper by the following lemmas:

Lemma 3.1 $\mathcal{L}_{\lambda}|_{\mathcal{N}_{\lambda}}$ is coercive and bounded from below by a positive constant.

Proof First we show that $\mathcal{L}_{\lambda}|_{\mathcal{N}_{\lambda}}$ is coercive. Note that

$$\begin{aligned}\mathcal{L}_{\lambda}(u) &= \mathcal{L}_{\lambda}(u) - \frac{1}{2c} \langle \mathcal{L}'_{\lambda}(u), u \rangle \\ &= \left(\frac{1}{m} - \frac{1}{2c} \right) \|u\|_{X_v}^m + \left(\frac{1}{2c} - \frac{1}{2b} \right) \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^{\alpha}} \right) \frac{|u|^b}{|x|^{\alpha}} \\ &\geq \left(\frac{1}{m} - \frac{1}{2c} \right) \|u\|_{X_v}^m.\end{aligned}$$

Next, using the double weighted Hardy–Littlewood–Sobolev inequality together with the continuous embeddings $X_v(\mathbb{R}^N) \hookrightarrow L^{\frac{2Nb}{2N-2a-\theta}}(\mathbb{R}^N)$ and $X_v(\mathbb{R}^N) \hookrightarrow L^{\frac{2Nc}{2N-2\beta-\gamma}}(\mathbb{R}^N)$, for any $u \in \mathcal{N}_{\lambda}$ we have

$$\begin{aligned}0 = \langle \mathcal{L}'_{\lambda}(u), u \rangle &= \|u\|_{X_v}^m - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^{\alpha}} \right) \frac{|u|^b}{|x|^{\alpha}} - \lambda \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|u|^c}{|x|^{\beta}} \right) \frac{|u|^c}{|x|^{\beta}} \\ &\geq \|u\|_{X_v}^m - C \|u\|_{X_v}^{2b} - C_{\lambda} \|u\|_{X_v}^{2c}.\end{aligned}$$

Therefore, there exists $C_0 > 0$ such that

$$\|u\|_{X_v} \geq C_0 > 0 \quad \text{for all } u \in \mathcal{N}_{\lambda}. \quad (26)$$

Hence, using coercivity of $\mathcal{L}_{\lambda}|_{\mathcal{N}_{\lambda}}$ and (26), we get

$$\mathcal{L}_{\lambda}(u) \geq \left(\frac{1}{m} - \frac{1}{2c} \right) C_0^m > 0.$$

□

Lemma 3.2 Let u be any critical point of $\mathcal{L}_{\lambda}|_{\mathcal{N}_{\lambda}}$. Then, it is a free critical point.

Proof Let us assume $\mathcal{K}(u) = \langle \mathcal{L}'_{\lambda}(u), u \rangle$ for any $u \in X_v(\mathbb{R}^N)$. Using (26), for any $u \in \mathcal{N}_{\lambda}$ we get

$$\begin{aligned}
 \langle \mathcal{K}'(u), u \rangle &= m \|u\|^m - 2b \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha} - 2c\lambda \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|u|^c}{|x|^\beta} \right) \frac{|u|^c}{|x|^\beta} \\
 &= (m - 2c) \|u\|_{X_v}^m - 2(b - c) \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha} \\
 &\leq -(2c - m) \|u\|_{X_v}^m \\
 &< -(2c - m) C_0.
 \end{aligned} \tag{27}$$

Now, say $u \in \mathcal{N}_\lambda$ is a critical point of $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$. Then, by the Lagrange multiplier theorem, there exists $v \in \mathbb{R}$ such that $\mathcal{L}'_\lambda(u) = v\mathcal{K}'(u)$. Therefore, we have $\langle \mathcal{L}'_\lambda(u), u \rangle = v\langle \mathcal{K}'(u), u \rangle$. Since $\langle \mathcal{K}'(u), u \rangle < 0$, which gives us that $v = 0$. Hence, $\mathcal{L}'_\lambda(u) = 0$. $\square$

Lemma 3.3 Any sequence (u_n) which is a (PS) sequence for $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$ is a (PS) sequence for $\mathcal{L}_\lambda$.

Proof Assume that $(u_n) \subset \mathcal{N}_\lambda$ is a (PS) sequence for $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$. As,

$$\mathcal{L}_\lambda(u_n) \geq \left(\frac{1}{m} - \frac{1}{2c} \right) \|u_n\|_{X_v}^m,$$

this gives us that (u_n) is bounded in X_v . Next, we show that $\mathcal{L}'_\lambda(u_n) \rightarrow 0$. Since,

$$\mathcal{L}'_\lambda(u_n) - v_n \mathcal{K}'(u_n) = \mathcal{L}'_\lambda|_{\mathcal{N}_\lambda}(u_n) = o(1),$$

for some $v_n \in \mathbb{R}$, we get

$$v_n \langle \mathcal{K}'(u_n), u_n \rangle = \langle \mathcal{L}'_\lambda(u_n), u_n \rangle + o(1) = o(1).$$

Using (27), we have that $v_n \rightarrow 0$ which implies that $\mathcal{L}'_\lambda(u_n) \rightarrow 0$. $\square$

Compactness Result

Define the energy functional $\mathcal{I} : X_v(\mathbb{R}^N) \rightarrow \mathbb{R}$ by

$$\mathcal{I}(u) = \frac{1}{m} \|u\|^m - \frac{1}{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha},$$

and the associated Nehari manifold for $\mathcal{I}$ by

$$\mathcal{N}_\mathcal{I} = \{u \in X_v(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{I}'(u), u \rangle = 0\},$$

and let

$$d_\mathcal{I} = \inf_{u \in \mathcal{N}_\mathcal{I}} \mathcal{I}(u).$$

Also, for all $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\langle \mathcal{I}'(u), \phi \rangle = \int_{\mathbb{R}^N} v(x) |\nabla u|^{m-2} \nabla u \nabla \phi + \int_{\mathbb{R}^N} V(x) |u|^{m-2} u \phi - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^{b-1}}{|x|^\alpha} \phi.$$

and

$$\langle \mathcal{I}(u), u \rangle = \|u\|_{X_v}^m - \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha}.$$

Lemma 3.4 *Let us assume that $(u_n) \subset \mathcal{N}_\mathcal{I}$ is a (PS) sequence of $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$, that is,*

- (a) $(\mathcal{L}_\lambda(u_n))$ is bounded;
- (b) $\mathcal{L}'_\lambda|_{\mathcal{N}_\lambda}(u_n) \rightarrow 0$ strongly in $X_v^{-1}(\mathbb{R}^N)$.

Then there exists a solution $u \in X_v(\mathbb{R}^N)$ of (1) such that, if we replace the sequence (u_n) with a subsequence, then one of the following alternative holds:

(A₁) either $u_n \rightarrow u$ strongly in $X_v(\mathbb{R}^N)$;

or

(A₂) $u_n \rightarrow u$ weakly in $X_v(\mathbb{R}^N)$ and there exists a positive integer $k \geq 1$ and k functions $u_1, u_2, \dots, u_k \in X_v(\mathbb{R}^N)$ which are nontrivial weak solutions to (2) and k sequences of points $(w_{n,1}), (w_{n,2}), \dots, (w_{n,k}) \subset \mathbb{R}^N$ such that the following conditions hold:

- (i) $|w_{n,j}| \rightarrow \infty$ and $|w_{n,j} - w_{n,i}| \rightarrow \infty$ if $i \neq j, n \rightarrow \infty$;
- (ii) $u_n - \sum_{j=1}^k u_j(\cdot + w_{n,j}) \rightarrow u$ in $X_v(\mathbb{R}^N)$;
- (iii) $\mathcal{L}_\lambda(u_n) \rightarrow \mathcal{L}_\lambda(u) + \sum_{j=1}^k \mathcal{I}(u_j)$.

Proof As $(u_n) \in X_v(\mathbb{R}^N)$ is a bounded sequence, so there exists $u \in X_v(\mathbb{R}^N)$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } X_v(\mathbb{R}^N), \\ u_n \rightharpoonup u & \text{weakly in } L^s(\mathbb{R}^N), \ m_p \leq s \leq m_p^*, \\ u_n \rightarrow u & \text{a.e. in } \mathbb{R}^N. \end{cases} \quad (28)$$

Using (28) together with Lemma 2.6, we get

$$\mathcal{L}'_\lambda(u) = 0.$$

Hence, $u \in X_v(\mathbb{R}^N)$ is a solution of (1). Now, if $u_n \rightarrow u$ strongly in $X_v(\mathbb{R}^N)$ then (A₁) holds and we are done.

Next, let us assume that $(u_n) \in X_v(\mathbb{R}^N)$ does not converge strongly to u and define $y_{n,1} = u_n - u$. Then $(y_{n,1})$ converges weakly (not strongly) to zero in $X_v(\mathbb{R}^N)$ and

$$\|u_n\|_{X_v}^m = \|u\|_{X_v}^m + \|y_{n,1}\|_{X_v}^m + o(1). \quad (29)$$

Using Lemma 2.5 we get

$$\int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n|^b}{|x|^\alpha} \right) \frac{|u_n|^b}{|x|^\alpha} = \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha} + \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|y_{n,1}|^b}{|x|^\alpha} \right) \frac{|y_{n,1}|^b}{|x|^\alpha} + o(1). \quad (30)$$

By (29) and (30) we have

$$\mathcal{L}_\lambda(u_n) = \mathcal{L}_\lambda(u) + \mathcal{I}(y_{n,1}) + o(1). \quad (31)$$

Now, by Lemma 2.6, for any $h \in X_v(\mathbb{R}^N)$, we have

$$\langle \mathcal{I}'(y_{n,1}), h \rangle = o(1). \quad (32)$$

Further, using Lemma 2.5 we get

$$\begin{aligned} 0 = \langle \mathcal{L}'_\lambda(u_n), u_n \rangle &= \langle \mathcal{L}'_\lambda(u), u \rangle + \langle \mathcal{I}'(y_{n,1}), y_{n,1} \rangle + o(1) \\ &= \langle \mathcal{I}'(y_{n,1}), y_{n,1} \rangle + o(1), \end{aligned}$$

which yields

$$\langle \mathcal{I}'(y_{n,1}), y_{n,1} \rangle = o(1). \quad (33)$$

Next, we claim that

$$\Delta := \limsup_{n \rightarrow \infty} \left(\sup_{w \in \mathbb{R}^N} \int_{B_1(w)} |y_{n,1}|^{\frac{2Nb}{2N-2\alpha-\theta}} \right) > 0.$$

Let us assume that $\Delta = 0$. Using Lemma 2.2 we have $y_{n,1} \rightarrow 0$ strongly in $L^{\frac{2Nb}{2N-2\alpha-\theta}}(\mathbb{R}^N)$. By double weighted Hardy–Littlewood–Sobolev inequality we deduce that

$$\int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|y_{n,1}|^b}{|x|^\alpha} \right) \frac{|y_{n,1}|^b}{|x|^\alpha} = o(1).$$

Combining this together with (33), we have $y_{n,1} \rightarrow 0$ strongly in $X_v(\mathbb{R}^N)$, which gives us a contradiction and therefore, we get $\Delta > 0$.

As $\Delta > 0$, one could find $w_{n,1} \in \mathbb{R}^N$ such that

$$\int_{B_1(w_{n,1})} |y_{n,1}|^{\frac{2Nb}{2N-2\alpha-\theta}} > \frac{\Delta}{2}. \quad (34)$$

For the sequence $(y_{n,1}(\cdot + w_{n,1}))$, there exists $u_1 \in X_v(\mathbb{R}^N)$ such that, up to a subsequence, we have

$$\begin{aligned} y_{n,1}(\cdot + w_{n,1}) &\rightharpoonup u_1 \quad \text{weakly in } X_v(\mathbb{R}^N), \\ y_{n,1}(\cdot + w_{n,1}) &\rightarrow u_1 \quad \text{strongly in } L^{\frac{2Nb}{2N-2\alpha-\theta}}_{loc}(\mathbb{R}^N), \\ y_{n,1}(\cdot + w_{n,1}) &\rightarrow u_1 \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

By passing to the limit in (34), we have

$$\int_{B_1(0)} |u_1|^{\frac{2Nb}{2N-2\alpha-\theta}} \geq \frac{\Delta}{2},$$

hence, $u_1 \not\equiv 0$. As $(y_{n,1})$ converges weakly to zero in $X_v(\mathbb{R}^N)$, we get that $(w_{n,1})$ is unbounded. Therefore, passing to a subsequence, one could assume that $|w_{n,1}| \rightarrow \infty$. Using (33), we have $\mathcal{I}'(u_1) = 0$, which further implies that u_1 is a nontrivial solution of (2). Now, we define

$$y_{n,2}(x) = y_{n,1}(x) - u_1(x - w_{n,1}).$$

Then, following the same procedure as before, we get

$$\|y_{n,2}\|^m = \|u_1\|^m + \|y_{n,2}\|^m + o(1).$$

By Lemma 2.5 we have

$$\int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|y_{n,1}|^b}{|x|^\alpha} \right) \frac{|y_{n,1}|^b}{|x|^\alpha} = \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_1|^b}{|x|^\alpha} \right) \frac{|u_1|^b}{|x|^\alpha} + \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|y_{n,2}|^b}{|x|^\alpha} \right) \frac{|y_{n,2}|^b}{|x|^\alpha} + o(1).$$

Therefore,

$$\mathcal{I}(y_{n,1}) = \mathcal{I}(u_1) + \mathcal{I}(y_{n,2}) + o(1).$$

Using (31), we have

$$\mathcal{L}_\lambda(u_n) = \mathcal{L}_\lambda(u) + \mathcal{I}(u_1) + \mathcal{I}(y_{n,2}) + o(1).$$

Using the same approach as above, we get

$$\langle \mathcal{I}'(y_{n,2}), h \rangle = o(1) \quad \text{for any } h \in X_v(\mathbb{R}^N)$$

and

$$\langle \mathcal{I}'(y_{n,2}), y_{n,2} \rangle = o(1).$$

Further, if $(y_{n,2}) \rightarrow 0$ strongly, then we are done by taking $k = 1$ in the Lemma 3.4. Assume $y_{n,2} \rightarrow 0$ weakly (not strongly) in $X_v(\mathbb{R}^N)$, then we could iterate the whole process and in k number of steps we find a set of sequences $(w_{n,j}) \subset \mathbb{R}^N$, $1 \leq j \leq k$ with

$$|w_{n,j}| \rightarrow \infty \quad \text{and} \quad |w_{n,i} - w_{n,j}| \rightarrow \infty \quad \text{as } n \rightarrow \infty, i \neq j$$

and k nontrivial solutions $u_1, u_2, \dots, u_k \in X_v(\mathbb{R}^N)$ of (2) such that, by denoting

$$y_{n,j}(x) := y_{n,j-1}(x) - u_{j-1}(x - w_{n,j-1}), \quad 2 \leq j \leq k,$$

we get

$$y_{n,j}(x + w_{n,j}) \rightharpoonup u_j \quad \text{weakly in } X_v(\mathbb{R}^N)$$

and

$$\mathcal{L}_\lambda(u_n) = \mathcal{L}_\lambda(u) + \sum_{j=1}^k \mathcal{I}(u_j) + \mathcal{I}(y_{n,k}) + o(1).$$

Now, as $\mathcal{L}_\lambda(u_n)$ is bounded and $\mathcal{I}(u_j) \geq d_{\mathcal{I}}$, one could iterate the process only a finite number of times and with this, we conclude our proof. $\square$

Corollary 1 Any $(PS)_c$ sequence of $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$ is relatively compact for any $c \in (0, d_{\mathcal{I}})$.

Proof Let us assume that (u_n) is a $(PS)_c$ sequence of $\mathcal{L}_\lambda|_{\mathcal{N}_\lambda}$. Then, by Lemma 3.4 we get $\mathcal{I}(u_j) \geq d_{\mathcal{I}}$ and upto a subsequence $u_n \rightarrow u$ strongly in $X_v(\mathbb{R}^N)$ and hence, u is a solution of (1). $\square$

Completion of the Proof of Theorem 1.1

We need the following result in order to complete the proof of Theorem 1.1.

$$d_\lambda < d_{\mathcal{I}}.$$

Lemma 3.5

Proof Let us assume that $P \in X_v(\mathbb{R}^N)$ is a groundstate solution of (2) (see [8, 15]). Let us denote by tP , the projection of P on $\mathcal{N}_\lambda$, that is, $t = t(P) > 0$ is the unique real number such that $tP \in \mathcal{N}_\lambda$. Since, $P \in \mathcal{N}_{\mathcal{I}}$ and $tP \in \mathcal{N}_\lambda$, we have

$$\|P\|^m = \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|P|^b}{|x|^\alpha} \right) \frac{|P|^b}{|x|^\alpha} \quad (35)$$

and

$$t^m \|P\|^m = t^{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|P|^b}{|x|^\alpha} \right) \frac{|P|^b}{|x|^\alpha} + \lambda t^{2c} \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|P|^c}{|x|^\beta} \right) \frac{|P|^c}{|x|^\beta}.$$

Therefore, we get $t < 1$. Now,

$$\begin{aligned} d_\lambda &\leq \mathcal{L}_\lambda(tP) = \frac{1}{m} t^m \|P\|^m - \frac{1}{2b} t^{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|P|^b}{|x|^\alpha} \right) \frac{|P|^b}{|x|^\alpha} - \frac{\lambda}{2c} t^{2c} \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|P|^c}{|x|^\beta} \right) \frac{|P|^c}{|x|^\beta} \\ &= \left(\frac{t^m}{m} - \frac{t^{2b}}{2b} \right) \|P\|^m - \frac{1}{2c} \left(t^m \|P\|^m - t^{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|P|^b}{|x|^\alpha} \right) \frac{|P|^b}{|x|^\alpha} \right) \\ &= t^m \left(\frac{1}{m} - \frac{1}{2c} \right) \|P\|^m + t^{2b} \left(\frac{1}{2c} - \frac{1}{2b} \right) \|P\|^m \\ &< \left(\frac{1}{m} - \frac{1}{2c} \right) \|P\|^m + \left(\frac{1}{2c} - \frac{1}{2b} \right) \|P\|^m \\ &< \left(\frac{1}{m} - \frac{1}{2b} \right) \|P\|^m = \mathcal{I}(P) = d_{\mathcal{I}}. \end{aligned}$$

Hence, we are done. $\square$

Next, we use the Ekeland variational principle, that is, for any $n \geq 1$ there exists $(u_n) \in \mathcal{N}_\lambda$ such that

$$\begin{aligned} \mathcal{L}_\lambda(u_n) &\leq d_\lambda + \frac{1}{n} && \text{for all } n \geq 1, \\ \mathcal{L}_\lambda(u_n) &\leq \mathcal{L}_\lambda(\tilde{u}) + \frac{1}{n} \|\tilde{u} - u_n\| && \text{for all } \tilde{u} \in \mathcal{N}_\lambda, n \geq 1. \end{aligned}$$

Further, one could easily find that $(u_n) \in \mathcal{N}_\lambda$ is a $(PS)_{d_\lambda}$ sequence for $\mathcal{L}_\lambda$ on $\mathcal{N}_\lambda$. Then, by Lemma 3.5 and Corollary 1 we have that up to a subsequence $u_n \rightarrow u$ strongly in $X_v(\mathbb{R}^N)$ which is a groundstate solution of the $\mathcal{L}_\lambda$.

Proof of Theorem 1.2

In this section, we are concerned the existence of a least energy sign-changing solution of (1).

Proof of Theorem

Lemma 4.1 *Let $N > m \geq 2$, $b > c > m$ and $\lambda \in \mathbb{R}$. There exists a unique pair $(\tau_0, \delta_0) \in (0, \infty) \times (0, \infty)$ such that, for any $u \in X_v(\mathbb{R}^N)$ and $u^\pm \neq 0$, we have $\tau_0 u^+ + \delta_0 u^- \in \bar{\mathcal{N}}_\lambda$. Also, if $u \in \bar{\mathcal{N}}_\lambda$ then for all $\tau, \delta \geq 0$ we have $\mathcal{L}_\lambda(u) \geq \mathcal{L}_\lambda(\tau u^+ + \delta u^-)$.*

Proof In order to prove this lemma, we follow the idea of [36]. Define the function $\varphi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(\tau, \delta) &= \mathcal{L}_\lambda(\tau^{\frac{1}{2b}} u^+ + \delta^{\frac{1}{2b}} u^-) \\ &= \frac{\tau^{\frac{m}{2b}}}{m} \|u^+\|_{X_v}^m + \frac{\delta^{\frac{m}{2b}}}{m} \|u^-\|_{X_v}^m - \lambda \frac{\tau^{\frac{c}{2b}}}{2c} \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u^+)^c}{|x|^\beta} \right) \frac{(u^+)^c}{|x|^\beta} - \lambda \frac{\delta^{\frac{c}{2b}}}{2c} \\ &\quad \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u^-)^c}{|x|^\beta} \right) \frac{(u^-)^c}{|x|^\beta} \\ &\quad - \lambda \frac{\tau^{\frac{c}{2b}} \delta^{\frac{c}{2b}}}{2c} \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u^+)^c}{|x|^\beta} \right) \frac{(u^-)^c}{|x|^\beta} - \frac{\tau}{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u^+)^b}{|x|^\alpha} \right) \frac{(u^+)^b}{|x|^\alpha} \\ &\quad - \frac{\delta}{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u^-)^b}{|x|^\alpha} \right) \frac{(u^-)^b}{|x|^\alpha} - \frac{\tau^{\frac{1}{2}} \delta^{\frac{1}{2}}}{2b} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u^+)^b}{|x|^\alpha} \right) \frac{(u^-)^b}{|x|^\alpha}. \end{aligned}$$

One could observe that φ is strictly concave. Hence, φ has at most one maximum point. On the other hand we have

$$\lim_{\tau \rightarrow \infty} \varphi(\tau, \delta) = -\infty \text{ for all } \delta \geq 0 \quad \text{and} \quad \lim_{\delta \rightarrow \infty} \varphi(\tau, \delta) = -\infty \text{ for all } \tau \geq 0, \quad (36)$$

and it could be easily seen that

$$\lim_{\tau \searrow 0} \frac{\partial \varphi}{\partial \tau}(\tau, \delta) = \infty \text{ for all } \delta > 0 \quad \text{and} \quad \lim_{\delta \searrow 0} \frac{\partial \varphi}{\partial \delta}(\tau, \delta) = \infty \text{ for all } \tau > 0. \quad (37)$$

Therefore, by (36) and (37) maximum cannot be achieved at the boundary. Hence, φ has exactly one maximum point $(\tau_0, \delta_0) \in (0, \infty) \times (0, \infty)$. $\square$

Next, we divide our proof into two steps.

Step 1. *The energy level $\bar{d}_\lambda \geq 0$ is achieved by some $\sigma \in \bar{\mathcal{N}}_\lambda$.*

Let us assume that $(u_n) \subset \bar{\mathcal{N}}_\lambda$ be a minimizing sequence for $\bar{d}_\lambda$. Observe that

$$\begin{aligned}
 \mathcal{L}_\lambda(u_n) &= \mathcal{L}_\lambda(u_n) - \frac{1}{2c} \langle \mathcal{L}'_\lambda(u_n), u_n \rangle \\
 &= \left(\frac{1}{m} - \frac{1}{2c} \right) \|u_n\|_{X_v}^m + \left(\frac{1}{2c} - \frac{1}{2b} \right) \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u|^b}{|x|^\alpha} \right) \frac{|u|^b}{|x|^\alpha} \\
 &\geq \left(\frac{1}{m} - \frac{1}{2c} \right) \|u_n\|_{X_v}^m \\
 &\geq C \|u_n\|_{X_v}^m,
 \end{aligned}$$

for some positive constant $C_1 > 0$. Hence, for $C_2 > 0$ we have

$$\|u_n\|_{X_v}^m \leq C_2 \mathcal{L}_\lambda(u_n) \leq M,$$

that is, (u_n) is bounded in $X_v(\mathbb{R}^N)$. This further implies that (u_n^+) and (u_n^-) are also bounded in $X_v(\mathbb{R}^N)$. Therefore, passing to a subsequence, there exists $u^+, u^- \in X_v(\mathbb{R}^N)$ such that

$$u_n^+ \rightharpoonup u^+ \text{ and } u_n^- \rightharpoonup u^- \quad \text{weakly in } X_v(\mathbb{R}^N).$$

As $b, c > m \geq 2$ satisfy (10) and (12) or (11) and (13), we have that the embeddings $X_v(\mathbb{R}^N) \hookrightarrow L^{\frac{2Nb}{2N-2\alpha-\theta}}(\mathbb{R}^N)$ and $X_v(\mathbb{R}^N) \hookrightarrow L^{\frac{2Nc}{2N-2\beta-\gamma}}(\mathbb{R}^N)$ are compact. Hence,

$$u_n^\pm \rightarrow u^\pm \quad \text{strongly in } L^{\frac{2Nb}{2N-2\alpha-\theta}}(\mathbb{R}^N) \cap L^{\frac{2Nc}{2N-2\beta-\gamma}}(\mathbb{R}^N). \quad (38)$$

Using the double weighted Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}
 C \left(\|u_n^\pm\|_{L^{\frac{2Nb}{2N-2\alpha-\theta}}}^m + \|u_n^\pm\|_{L^{\frac{2Nc}{2N-2\beta-\gamma}}}^m \right) &\leq \|u_n^\pm\|_{X_v}^m \\
 &= \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{|u_n|^b}{|x|^\alpha} \right) \frac{|u_n|^b}{|x|^\alpha} \\
 &\quad + |\lambda| \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{|u_n|^c}{|x|^\beta} \right) \frac{|u_n|^c}{|x|^\beta} \\
 &\leq C \left(\|u_n^\pm\|_{L^{\frac{2Nb}{2N-2\alpha-\theta}}}^b + \|u_n^\pm\|_{L^{\frac{2Nc}{2N-2\beta-\gamma}}}^c \right) \\
 &\leq C \left(\|u_n^\pm\|_{L^{\frac{2Nb}{2N-2\alpha-\theta}}}^m + \|u_n^\pm\|_{L^{\frac{2Nc}{2N-2\beta-\gamma}}}^m \right) \\
 &\quad \left(\|u_n^\pm\|_{L^{\frac{2Nb}{2N-2\alpha-\theta}}}^{b-m} + \|u_n^\pm\|_{L^{\frac{2Nc}{2N-2\beta-\gamma}}}^{c-m} \right).
 \end{aligned}$$

As $u_n^\pm \neq 0$, we get

$$\|u_n^\pm\|_{L^{\frac{2Nb}{2N-2\alpha-\theta}}}^{b-m} + \|u_n^\pm\|_{L^{\frac{2Nc}{2N-2\beta-\gamma}}}^{c-m} \geq C > 0 \quad \text{for all } n \geq 1. \quad (39)$$

Therefore, using (38) and (39) one could have that $u^\pm \neq 0$. Next, using (38) together with double weighted Hardy–Littlewood–Sobolev inequality, we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u_n^\pm)^b}{|x|^\alpha} \right) \frac{(u_n^\pm)^b}{|x|^\alpha} &\rightarrow \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u^\pm)^b}{|x|^\alpha} \right) \frac{(u^\pm)^b}{|x|^\alpha}, \\ \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u_n^+)^b}{|x|^\alpha} \right) \frac{(u_n^-)^b}{|x|^\alpha} &\rightarrow \int_{\mathbb{R}^N} \left(|x|^{-\theta} * \frac{(u^+)^b}{|x|^\alpha} \right) \frac{(u^-)^b}{|x|^\alpha}, \\ \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u_n^\pm)^c}{|x|^\beta} \right) \frac{(u_n^\pm)^c}{|x|^\beta} &\rightarrow \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u^\pm)^c}{|x|^\beta} \right) \frac{(u^\pm)^c}{|x|^\beta}, \end{aligned}$$

and

$$\int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u_n^+)^c}{|x|^\beta} \right) \frac{(u_n^-)^c}{|x|^\beta} \rightarrow \int_{\mathbb{R}^N} \left(|x|^{-\gamma} * \frac{(u^+)^c}{|x|^\beta} \right) \frac{(u^-)^c}{|x|^\beta}.$$

Next, by using Lemma 4.1, we get that there exists a unique pair (τ_0, δ_0) such that $\tau_0 u^+ + \delta_0 u^- \in \bar{\mathcal{N}}_\lambda$. Further, using the fact that the norm $\|\cdot\|_{X_v}$ is weakly lower semi-continuous, we get

$$\begin{aligned} \bar{d}_\lambda &\leq \mathcal{L}_\lambda(\tau_0 u^+ + \delta_0 u^-) \leq \liminf_{n \rightarrow \infty} \mathcal{L}_\lambda(\tau_0 u^+ + \delta_0 u^-) \\ &\leq \limsup_{n \rightarrow \infty} \mathcal{L}_\lambda(\tau_0 u^+ + \delta_0 u^-) \\ &\leq \lim_{n \rightarrow \infty} \mathcal{L}_\lambda(u_n) \\ &= \bar{d}_\lambda. \end{aligned}$$

We conclude by taking $\sigma = \tau_0 u^+ + \delta_0 u^- \in \bar{\mathcal{N}}_\lambda$.

Step 2. $\mathcal{L}'_\lambda(\sigma) = 0$, that is, $\sigma \in \bar{\mathcal{N}}_\lambda$ is the critical point of $\mathcal{L}_\lambda : X_v(\mathbb{R}^N) \rightarrow \mathbb{R}$.

Say σ is not a critical point of $\mathcal{L}_\lambda$, then there exists $\kappa \in C_c^\infty(\mathbb{R}^N)$ such that $\langle \mathcal{L}'_\lambda(\sigma), \kappa \rangle = -2$. As $\mathcal{L}_\lambda$ is continuous and differentiable, so there exists $\zeta > 0$ small such that

$$\langle \mathcal{L}'_\lambda(\tau u^+ + \delta u^- + \omega \bar{\sigma}), \bar{\sigma} \rangle \leq -1 \quad \text{if } (\tau - \tau_0)^2 + (\delta - \delta_0)^2 \leq \zeta^2 \text{ and } 0 \leq \omega \leq \zeta. \quad (40)$$

Next, let us assume that $D \subset \mathbb{R}^2$ is an open disc of radius $\zeta > 0$ centered at (τ_0, δ_0) and define a continuous function $\Phi : D \rightarrow [0, 1]$ by

$$\Phi(\tau, \delta) = \begin{cases} 1 & \text{if } (\tau - \tau_0)^2 + (\delta - \delta_0)^2 \leq \frac{\zeta^2}{16}, \\ 0 & \text{if } (\tau - \tau_0)^2 + (\delta - \delta_0)^2 \geq \frac{\zeta^2}{4}. \end{cases}$$

Also, let us define a continuous map $T : D \rightarrow X_v(\mathbb{R}^N)$ as

$$T(\tau, \delta) = \tau u^+ + \delta u^- + \zeta \Phi(\tau, \delta) \bar{\sigma} \quad \text{for all } (\tau, \delta) \in D$$

and $Q : D \rightarrow \mathbb{R}^2$ as

$$Q(\tau, \delta) = (\langle \mathcal{L}'_\lambda(T(\tau, \delta)), T(\tau, \delta)^+ \rangle, \langle \mathcal{L}'_\lambda(T(\tau, \delta)), T(\tau, \delta)^- \rangle) \quad \text{for all } (\tau, \delta) \in D.$$

As the mapping $u \mapsto u^+$ is continuous in $X_v(\mathbb{R}^N)$, we get that Q is also continuous. Furthermore, if we are on the boundary of D , that is, $(\tau - \tau_0)^2 + (\delta - \delta_0)^2 = \zeta^2$, then $\Phi = 0$ according to the definition. Therefore, we get $T(\tau, \delta) = \tau u^+ + \delta u^-$ and by Lemma 4.1, we deduce

$$Q(\tau, \delta) \neq 0 \quad \text{on } \partial D.$$

Hence, the Brouwer degree is well defined and $\deg(Q, \text{int}(D), (0, 0)) = 1$ and there exists $(\tau_1, \delta_1) \in \text{int}(D)$ such that $Q(\tau_1, \delta_1) = (0, 0)$. Therefore, we get that $T(\tau_1, \delta_1) \in \bar{N}_\lambda$ and by the definition of $\bar{d}_\lambda$ we deduce that

$$\mathcal{L}_\lambda(T(\tau_1, \delta_1)) \geq \bar{d}_\lambda. \quad (41)$$

Next, by equation (40), we have

$$\begin{aligned} \mathcal{L}_\lambda(T(\tau_1, \delta_1)) &= \mathcal{L}_\lambda(\tau_1 u^+ + \delta_1 u^-) + \int_0^1 \frac{d}{dt} \mathcal{L}_\lambda(\tau_1 u^+ + \delta_1 u^- + \zeta t \Phi(\tau_1, \delta_1) \bar{\sigma}) dt \\ &= \mathcal{L}_\lambda(\tau_1 u^+ + \delta_1 u^-) - \zeta \Phi(\tau_1, \delta_1). \end{aligned} \quad (42)$$

Now, by definition of Φ we have $\Phi(\tau_1, \delta_1) = 1$ when $(\tau_1, \delta_1) = (\tau_0, \delta_0)$. Hence, we deduce that

$$\mathcal{L}_\lambda(T(\tau_1, \delta_1)) \leq \mathcal{L}_\lambda(\tau_1 u^+ + \delta_1 u^-) - \zeta \leq \bar{d}_\lambda - \zeta < \bar{d}_\lambda.$$

The case when $(\tau_1, \delta_1) \neq (\tau_0, \delta_0)$, then by Lemma 4.1 we have

$$\mathcal{L}_\lambda(\tau_1 u^+ + \delta_1 u^-) < \mathcal{L}_\lambda(\tau_0 u^+ + \delta_0 u^-) = \bar{d}_\lambda,$$

which further gives

$$\mathcal{L}_\lambda(T(\tau_1, \delta_1)) \leq \mathcal{L}_\lambda(\tau_1 u^+ + \delta_1 u^-) < \bar{d}_\lambda.$$

This contradicts Eq. (41) and with this we conclude our proof.

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