On the Maximal Output Admissible Set for a Class of Bilinear Discrete-time Systems

Youssef Benfatah, Amine El Bhih, Mostafa Rachik, and Abdessamad Tridane*

Abstract: Given a discrete-time controlled bilinear systems with initial state x_0 and output function y_i , we investigate the maximal output set $\Theta(\Omega) = \{x_0 \in \mathbb{R}^n, y_i \in \Omega, \forall i \geq 0\}$ where Ω is a given constraint set and is a subset of $\mathbb{R}^p$. Using some stability hypothesis, we show that $\Theta(\Omega)$ can be determined via a finite number of inequations. Also, we give an algorithmic process to generate the set $\Theta(\Omega)$. To illustrate our theoretical approach, we present some examples and numerical simulations. Moreover, to demonstrate the effectiveness of our approach in real-life problems, we provide an application to the SI epidemic model and the SIR model.

Keywords: Asymptotic stability, bilinear systems, constraint set, discrete-time systems, output admissible set.

1. INTRODUCTION

Output admissible sets arise in many essential practical applications such as stability analysis and control of constrained systems (see [1–6]). This concept has been considerably investigated (see [7,8] and the references therein), but not in the case of the bilinear systems. Consequently, its applicability is limited.

Bilinear systems were introduced into control theory in the 1960s. They were a distinct type of nonlinear systems, in which nonlinear terms are developed by way of multiplication of control vector and state vector. These type of systems has attracted a large community of researchers in almost half a century (see [9]). Such systems' significance lies in the fact that many necessary engineering processes [10] can be modeled through bilinear systems [11].

As we know, a nonlinear discrete-time finite-dimensional system can be represented by the following dynamic equation:

$$x_{i+1} = f(x_i, u_i), \quad (1)$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^q$, for $i \geq 0$, are respectively the state and the control.

If we assume that f is linear with respect to the state or the control, we obtain, respectively.

$$x_{i+1} = h_1(u_i)x_i + f_1(u_i), \quad (2)$$

and

$$x_{i+1} = h_2(x_i)u_i + f_2(x_i), \quad (3)$$

we obtain an another form of bilinear systems given as follows:

$$x_{i+1} = Ax_i + \tilde{B}x_iu_i + Bu_i, \quad (4)$$

where for each $i \geq 0$, A is a $n \times n$ matrix, B is a $n \times q$ matrix. $\tilde{B}x_iu_i$ is a bilinear form in the x_i, u_i variables that can be expressed as

$$\tilde{B}x_iu_i = \sum_{j=1}^q u_i^j B_j x_i, \quad (5)$$

where $u_i = \{u_i^j\}_{j=1}^q$ and B_j is a $n \times n$ matrix.

The use of bilinear systems has two significant advantages: First, they provide better modeling of a nonlinear phenomenon than a linear system [12]). Second, many real-life problems can be model using bilinear systems.

1.1. Related work

The maximal output admissible (MOA) sets are an important concept for analyzing controlled systems with constraints. The MOA set has been well studied particularly for linear systems with state and control constraints [13,14]. This concept provides an understanding of the analysis of constrained control systems. Besides, it is widely used in control system design methods [15].

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The MOA set for a class of nonlinear systems has recently been studied by Rachik *et al.* in [16]. Later on, Hirata and Ohta [17] considered a special class of nonlinear systems, the so-called polynomial systems, and discussed the finite determinability of the MOA set.

Some other resources that provide information about the study of the concept MOA sets are given in the references [18–24]. In [18], Ossareh considered a Lyapunov-stable periodic system and gave the development of the MOA set theory. He also investigated some of its geometric and algebraic properties. In [19], Yamamoto presented MOA's computational procedure for a trajectory tracking control of biped robots.

Various algorithms have been added in the literature for determining the maximal state constraint sets [13,25]. In [25] the authors considered autonomous linear discrete-time systems subject to linear constraints, and proposed an effective procedure to determine the maximal set of admissible initial states. Other models from population dynamics and optimal control problems can be found in [26,27].

1.2. Problem statement

This work aims to present a new approach to studying the MOA set for a class of bilinear discrete-time systems. To the best of our knowledge, the MOA sets for this type of system have not been investigated before. More specifically, we characterize the initial states of a controlled bilinear system whose resulting trajectory verify a point-wise constraint. Moreover, our approach is applicable to epidemiological, sociological, and physiological models. In these examples, the MOA set represents the set of initial data that guarantee for these systems do not violate certain constraints.

This paper considers discrete-time controlled bilinear systems. More precisely, the system has the following form:

$$\begin{cases} x_{i+1} = Ax_i + \sum_{j=1}^q u_i^j B_j x_i + Bv_i, & i \geq 0, \\ x_0 \in \mathbb{R}^n, \end{cases} \quad (6)$$

the corresponding output is

$$y_i = Cx_i, \quad i \in \mathbb{N}, \quad (7)$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is the matrix of the state, $\mathbb{R}$, is the set of real numbers, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, is the set of real matrices of order $n \times n$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is the matrix of the input, $C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ is the matrix of the output, $x_i \in \mathbb{R}^n$ is the state variable, $B_j \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ for all $j \in \{1, 2, \dots, q\}$, $\mathbb{N}$, is the set of nonnegative integers, n, m, p and q are the nonnegative integer and u_i^j, v_i are the feedback controls defined by

$$u_i^j = K^j y_i \quad \text{and} \quad v_i = Ly_i \quad (8)$$

with $L \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^m)$ and $K^j \in \mathcal{L}(\mathbb{R}^p, \mathbb{R})$, $j \in \{1, \dots, q\}$ are matrices with single row, and are given by

$$K^j = \begin{bmatrix} k_1^j & k_2^j & k_3^j & \dots & k_{p-1}^j & k_p^j \end{bmatrix}. \quad (9)$$

Our purpose right here is to determine all vectors x_0 (the initial states) such that the corresponding signal output $(y_i)_i$ verify the condition $y_i \in \Omega$, $\forall i \in \mathbb{N}$, where $\Omega \subset \mathbb{R}^p$ is a given constraint set. The set of all such vectors is the maximal output admissible set $\Theta(\Omega)$.

For technical consideration we restrict our own selves in the current work to the study of $\Theta^\lambda(\Omega)$, the maximal output λ -admissible set, which defined by

$$\Theta^\lambda(\Omega) = \Theta(\Omega) \cap B(0, \lambda), \quad (10)$$

where $B(0, \lambda)$ is the ball of center 0 and radius $\lambda > 0$, this means we study the set of all vectors $x_0 \in \mathbb{R}^n$ such that $\|x_0\| \leq \lambda$ and $y_i \in \Omega$, $\forall i \in \mathbb{N}$, where λ is any positive number.

Moreover, we give some useful properties of $\Theta^\lambda(\Omega)$, and we investigate its determination by way of a finite number of functional inequalities.

The rest of this paper is structured as follows: In Section 2, we present some preliminary results. In Section 3, we characterize the maximal output λ -admissible set. Moreover, in Section 4 we give some sufficient conditions to ensure the finite determination of the maximal output λ -admissible set. In the following section, we propose an algorithm for determining the output admissibility index and, consequently the set $\Theta^\lambda(\Omega)$. Various numerical examples to illustrate our results are given in Section 6. An application to a controlled SI epidemic model and SIR model is given, to demonstrate the effectiveness of our approach in Section 7. The last section includes conclusion.

In the next section, we will give some necessary mathematical preliminaries that will be used in this paper.

2. PRELIMINARY RESULTS

In this section we consider the discrete-time controlled bilinear systems described by (6)-(7).

We replace u_i^j and v_i by its values in system (6) we obtain

$$\begin{cases} x_{i+1} = (A + BLC)x_i + \sum_{j=1}^q K^j Cx_i B_j x_i, & i \geq 0, \\ x_0 \in \mathbb{R}^n. \end{cases} \quad (11)$$

Notations: In what follows we will denote the matrix $A + BLC$ by $\tilde{A}$, $K^j C$ by $\tilde{K}^j$ and $\{i, i+1, \dots, k\}$, $i \leq k$, the finite subset of $\mathbb{N}$ by σ_i^k .

Remark 1: In case $q = 1$ the system (11) becomes

$$\begin{cases} x_{i+1} = \tilde{A}x_i + \delta x_i B_1 x_i, & i \geq 0 \\ x_0 \in \mathbb{R}^n \end{cases} \quad (12)$$

where $\delta = K^1 C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

Hence and by taking into account the previous notations, we consider in this paper the class of discrete systems defined by

$$\begin{cases} x_{i+1} = \tilde{A}x_i + \sum_{j=1}^q \tilde{K}^j x_i B_j x_i, & i \geq 0, \\ x_0 \in \mathbb{R}^n, \end{cases} \quad (13)$$

increased by the output

$$y_i = Cx_i, \quad \forall i \geq 0, \quad (14)$$

and we are interested in the theoretical and numerical characterization of the set

$$\Theta^\lambda(\Omega) = \{x_0 \in B(0, \lambda) \cap \mathbb{R}^n \mid y_i \in \Omega, \forall i \in \mathbb{N}\}. \quad (15)$$

Indeed, let ψ be the function defined as follows:

$$\begin{aligned} \psi: \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto \psi(x) = \sum_{j=1}^q \tilde{K}^j x B_j x. \end{aligned} \quad (16)$$

Let K be the matrix defined by

$$K = \begin{pmatrix} k_1^1 & k_2^1 & \cdots & k_p^1 \\ k_1^2 & k_2^2 & \cdots & k_p^2 \\ \vdots & \vdots & \ddots & \vdots \\ k_1^q & k_2^q & \cdots & k_p^q \end{pmatrix}. \quad (17)$$

Definition 1: Let $\Omega \subset \mathbb{R}^p$ and $x_0 \in \mathbb{R}^n$. The initial state x_0 is said to be Ω -output admissible if

$$y_i \in \Omega, \quad \forall i \in \mathbb{N}.$$

The set of all such initial states is defined by

$$\Theta(\Omega) = \{x_0 \in \mathbb{R}^n \mid y_i \in \Omega, \forall i \in \mathbb{N}\}.$$

The general solution of system (6) is given by

$$x_i = \Phi^i(x_0), \quad \forall i \in \mathbb{N}, \quad (18)$$

where the function Φ is defined by

$$\begin{aligned} \Phi: \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto \Phi(x) = \tilde{A}x + \psi(x), \end{aligned}$$

and

$$\Phi^i = \overbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}^{i\text{-times}}.$$

Using (18) the set $\Theta(\Omega)$ could be rewritten as follows:

$$\Theta(\Omega) = \{x_0 \in \mathbb{R}^n \mid C\Phi^i(x_0) \in \Omega, \forall i \in \mathbb{N}\}.$$

In this paper we restrict our-selves to the study of the set $\Theta^\lambda(\Omega)$ defined by

$$\begin{aligned} \Theta^\lambda(\Omega) &= \Theta(\Omega) \cap B(0, \lambda) \\ &= \{x_0 \in B(0, \lambda) \cap \mathbb{R}^n \mid C\Phi^i(x_0) \in \Omega, \forall i \in \mathbb{N}\}. \end{aligned} \quad (19)$$

The restriction to the set $\Theta^\lambda(\Omega)$ will not reduce the value of the work and this for the following reason. Having an initial state $x_0 \in \mathbb{R}^n$, we would like to know if x_0 is Ω -output admissible or not. In order to give an answer to such a question, we first identify the set $\Theta^\lambda(\Omega) = \Theta(\Omega) \cap B(0, \lambda)$ where λ is a real that verifies $\|x_0\| \leq \lambda$ and we check if $x_0 \in \Theta^\lambda(\Omega)$ or not.

Remark 2: 1) The general solution of system (6) is given by

$$x_i = \prod_{k=0}^{i-1} \left(\tilde{A} + \sum_{j=1}^q u_k^j B_j \right) x_0, \quad \forall i \geq 1. \quad (20)$$

2) $\forall x, y \in \mathbb{R}^n$,

$$\begin{aligned} &\|\psi(x) - \psi(y)\| \\ &\leq \left\{ \sum_{j=1}^q \|(\tilde{K}^j)^\top\| \|B_j\| (\|x\| + \|y\|) \right\} \times \|x - y\|, \end{aligned} \quad (21)$$

where $\top$ denotes the transpose.

3) For $x_0 \in B(0, \lambda)$, we have

$$\|x_i\| \leq F^i(\lambda), \quad \forall i \in \mathbb{N} \quad (22)$$

where F is defined by

$$F(t) = \alpha t + C_K t^2, \quad \forall t \in \mathbb{R}, \quad (23)$$

with

$$\alpha = \|\tilde{A}\| \quad \text{and} \quad C_K = \sum_{j=1}^q \|(\tilde{K}^j)^\top\| \|B_j\|.$$

Now, we give some conditions which are sufficient to ensure, for convenient initial states x_0 , the asymptotic stability of system (6). Our main result in this direction is the following.

Proposition 1: Suppose the following hypothesis to hold.

- 1) There exists $\tau \geq 1$ and $\theta \in]0, 1[$ such that $\|\tilde{A}\|^k \leq \tau \theta^k, \forall k \in \mathbb{N}$ (the matrix L can be chosen such that the hypothesis (1) holds),
- 2) $\theta + C_K \lambda \tau^2 < 1$, where

$$C_K = \sum_{j=1}^q \|(\tilde{K}^j)^\top\| \|B_j\|. \quad (24)$$

Then the system (6) is asymptotically stable in the ball of center 0 and radius λ , i.e.,

$$\lim_{i \rightarrow \infty} \|x_i\| = \lim_{i \rightarrow \infty} \|\Phi^i(x_0)\| = 0, \quad \forall x_0 \in B(0, \lambda). \quad (25)$$

Proof: Let $x_0 \in B(0, \lambda)$. Then the general solution of system (6) can be written as

$$x_i = \tilde{A}^i x_0 + \sum_{k=0}^{i-1} \tilde{A}^{i-k-1} \psi(x_k), \quad \forall i \geq 1 \quad (26)$$

$$\Rightarrow \|x_i\| \leq \|\tilde{A}^i\| \|x_0\| + \sum_{k=0}^{i-1} \|\tilde{A}^{i-k-1}\| \|\psi(x_k)\|$$

$$\Rightarrow \|x_i\| \leq \tau \theta^i \|x_0\| + \sum_{k=0}^{i-1} \tau \theta^{i-k-1} \|\psi(x_k)\|.$$

On the other hand, we have

$$\|\psi(x_k)\| = \left\| \sum_{j=1}^q \tilde{K}^j x_k B_j x_k \right\| \quad (27)$$

$$\leq \sum_{j=1}^q \|(\tilde{K}^j)^\top\| \|x_k\| \|B_j\| \|x_k\| \quad (28)$$

$$= \left(\sum_{j=1}^q \|(\tilde{K}^j)^\top\| \|B_j\| \right) \|x_k\|^2 \quad (29)$$

$$= C_K \|x_k\|^2, \quad (30)$$

we have used Cauchy-Schwarz inequality. This leads to

$$\begin{aligned} \|x_i\| &\leq \tau \theta^i \|x_0\| \\ &\quad + C_K \tau \theta^{i-1} \sum_{k=0}^{i-1} \theta^{-k} \|x_k\|^2, \quad \forall i \geq 1. \end{aligned}$$

Let $z_i = \frac{x_i}{\lambda}$. Then

$$\begin{aligned} \|z_i\| &\leq \tau \theta^i \|z_0\| \\ &\quad + \lambda C_K \tau \theta^{i-1} \sum_{k=0}^{i-1} \theta^{-k} \|z_k\| \|z_k\|, \quad \forall i \geq 1. \end{aligned} \quad (31)$$

Since $\|z_0\| \leq \tau$, we prove that

$$\|z_i\| \leq \tau, \quad \forall i \in \mathbb{N}. \quad (32)$$

Indeed, if we assume that

$$\|z_i\| \leq \tau, \quad \forall i \in \sigma_0^N, \quad (33)$$

then using (31) it follows

$$\begin{aligned} \|z_i\| &\leq \tau \theta^i \|z_0\| \\ &\quad + \lambda C_K \tau^2 \theta^{i-1} \sum_{k=0}^{i-1} \theta^{-k} \|z_k\|, \quad \forall i \in \sigma_1^{N+1}. \end{aligned} \quad (34)$$

Take

$$y_i = \theta^{-i} \|z_i\| \quad \text{and} \quad \Gamma_i = \alpha + \beta \sum_{k=0}^{i-1} y_k, \quad (35)$$

where

$$\alpha = \tau \|z_0\| \quad \text{and} \quad \beta = \lambda C_K \tau^2 \theta^{-1}. \quad (36)$$

From (34) we conclude

$$y_i \leq \Gamma_i, \quad \forall i \in \sigma_1^{N+1}. \quad (37)$$

We have $y_i = \frac{\Gamma_{i+1} - \Gamma_i}{\beta}$. Then

$$\frac{\Gamma_{i+1} - \Gamma_i}{\beta} \leq \Gamma_i, \quad \forall i \in \sigma_1^{N+1},$$

which implies

$$y_i \leq \Gamma_i \leq (1 + \beta)^{i-1} \Gamma_1, \quad \forall i \in \sigma_1^{N+1}. \quad (38)$$

If one replaces y_i and Γ_1 by their values, we deduce from (38) that $\forall i \in \sigma_1^{N+1}$,

$$\|z_i\| \leq \frac{(\theta + C_K \lambda \tau^2)^i (\tau + C_K \lambda \tau^2 \theta^{-1})}{1 + C_K \lambda \tau^2 \theta^{-1}} \|z_0\|. \quad (39)$$

Because, $\theta + C_K \lambda \tau^2 < 1$ and $\|z_0\| \leq 1$ it follows

$$\|z_i\| \leq \tau, \quad \forall i \in \sigma_1^{N+1},$$

and

$$\|z_i\| \leq \tau, \quad \forall i \in \mathbb{N}. \quad (40)$$

Using (39) for all $i \geq 0$, we deduce that

$$\lim_{i \rightarrow \infty} \|z_i\| = 0 \quad \text{since} \quad \theta + C_K \lambda \tau^2 \leq 1.$$

Therefore

$$\lim_{i \rightarrow \infty} \|\Phi^i(x_0)\| = 0. \quad (41)$$

This completes the proof. $\square$

The following result assumes that $0 \in \overset{\circ}{\Omega}$ ($\overset{\circ}{\Omega}$ denoted the interior of Ω), this assumption is satisfied in any reasonable application and has nice consequences (see [13]). In practice, the set Ω has the form

$$\Omega = \{y \in \mathbb{R}^p \mid f_1(y) \leq 0, f_2(y) \leq 0, \dots, f_s(y) \leq 0\}, \quad (42)$$

where $s \in \mathbb{N}$ and $f_i, i \in \sigma_1^s$ are continuous functions such that $f_i(0) \leq 0, \forall i \in \sigma_1^s$, such a sets have many importance in a practical view.

Remark 3: The condition taken above $f_i(0) \leq 0, \forall i \in \sigma_1^s$ implies $0 \in \Theta^\lambda(\Omega)$ and then the set $\Theta^\lambda(\Omega)$ nonempty. Indeed, the set $\Theta^\lambda(\Omega)$ is given by

$$\begin{aligned} \Theta^\lambda(\Omega) \\ = \{x_0 \in B(0, \lambda) \cap \mathbb{R}^n \mid C\Phi^i(x_0) \in \Omega, \forall i \in \mathbb{N}\} \end{aligned}$$

$$= \{x_0 \in B(0, \lambda) \mid f_j(C\Phi^i(x_0)) \leq 0, \forall j \in \sigma_1^s, \forall i \in \mathbb{N}\},$$

and we have

$$f_j(C\Phi^i(0)) = f_j(0) \leq 0, \forall j \in \sigma_1^s, \forall i \in \mathbb{N}, \quad (43)$$

since $\Phi(0) = 0$.

Hence $0 \in \Theta^\lambda(\Omega)$ because $0 \in B(0, \lambda)$.

Imposing special conditions on $\tilde{A}$, $\tilde{K}^j$, B_j , $j \in \{1, \dots, q\}$ and Ω which imposes corresponding conditions on $\Theta^\lambda(\Omega)$.

Proposition 2: 1) If Ω is closed, then the set $\Theta^\lambda(\Omega)$ is also closed.

2) If we assume the following assumptions to hold:

(a) there exists $\tau \geq 1$ and $\theta \in]0, 1[$ such that $\|\tilde{A}\|^k \leq \tau\theta^k$, $\forall k \in \mathbb{N}$,

(b) $\theta + C_K \lambda \tau^2 \leq 1$, $C_K = \sum_{j=1}^q \|(\tilde{K}^j)^\top\| \|B_j\|$,

(c) $0_{\mathbb{R}^p} \in \overset{\circ}{\Omega}$,

then $0_{\mathbb{R}^n} \in \overset{\circ}{\Theta^\lambda(\Omega)}$.

Proof: 1) Let $i \in \mathbb{N}$. Then we define the functions

$$\Psi_i : B(0, \lambda) \cap \mathbb{R}^n \rightarrow \mathbb{R}^p, \quad x_0 \mapsto C\Phi^i(x_0). \quad (44)$$

Using these functions, the set $\Theta^\lambda(\Omega)$ can be written as

$$\Theta^\lambda(\Omega) = \bigcap_{i \in \mathbb{N}} \Psi_i^{-1}(\Omega). \quad (45)$$

Since Ω is closed and Ψ_i , $i \geq 0$ are continuous functions (since $\psi(x) = \sum_{j=1}^q K^j x B_j x$ is continuous), $\Psi_i^{-1}(\Omega)$ is closed and then $\Theta^\lambda(\Omega)$ is closed.

2) By hypothesis (a), (b), and (c), and from Proposition 1 we have

$$\forall z_0 \in B(0, \lambda), \quad \lim_{i \rightarrow \infty} \|\Phi^i(z_0)\| = 0. \quad (46)$$

This leads to

$$\begin{aligned} \forall z_0 \in B(0, \lambda), \quad \forall \varepsilon > 0, \quad \exists i_0 \in \mathbb{N} : \\ \|\Phi^i(z_0)\| \leq \varepsilon, \quad \forall i \geq i_0. \end{aligned}$$

i.e.,

$$\begin{aligned} \forall z_0 \in B(0, \lambda), \quad \forall \varepsilon > 0, \quad \exists i_0 \in \mathbb{N} : \\ \Phi^i(z_0) \in B(0, \varepsilon), \quad \forall i \geq i_0. \end{aligned}$$

On the other hand

$$0 \in \overset{\circ}{\Omega} \implies \exists \eta > 0 : B(0, \eta) \subset \Omega. \quad (47)$$

For $\varepsilon = \frac{\eta}{\|C\|}$, we have

$$\forall z_0 \in B(0, \lambda), \quad \exists i_0 \in \mathbb{N}$$

such that

$$C\Phi^i(z_0) \in B(0, \eta) \subset \Omega, \quad \forall i \geq i_0. \quad (48)$$

It remains to show that

$$C\Phi^i(z_0) \in \Omega, \quad \forall i \in \sigma_0^{i_0-1}. \quad (49)$$

Φ^{i_0-1} continuous in 0 implies

$$\begin{aligned} \exists \eta_{i_0-1} > 0, \quad \forall z_0 \in B(0, \eta_{i_0-1}), \\ \|\Phi^{i_0-1}(z_0) - \Phi^{i_0-1}(0)\| \leq \frac{\eta}{\|C\|} \\ \implies \exists \eta_{i_0-1} > 0, \quad \forall z_0 \in B(0, \eta_{i_0-1}), \\ C\Phi^{i_0-1}(z_0) \in \Omega. \end{aligned}$$

Φ^{i_0-2} continuous in 0 implies

$$\exists \eta_{i_0-2} > 0, \quad \forall z_0 \in B(0, \eta_{i_0-2}), \quad C\Phi^{i_0-2}(z_0) \in \Omega$$

by a similar way, using the continuity of Φ^j , $j \in \sigma_1^{i_0-3}$ in 0, we obtain

$$\begin{aligned} \forall j \in \sigma_1^{i_0-3}, \\ \exists \eta_j > 0, \quad \forall z_0 \in B(0, \eta_j), \quad C\Phi^j(z_0) \in \Omega. \end{aligned} \quad (50)$$

We have also

$$\forall z_0 \in B\left(0, \frac{\eta}{\|C\|}\right), \quad C z_0 \in \Omega. \quad (51)$$

If we take $\varsigma = \inf(\eta_{i_0-1}, \dots, \eta_1, \frac{\eta}{\|C\|})$, we get

$$\forall z_0 \in B(0, \varsigma), \quad C\Phi^i(z_0) \in \Omega, \quad \forall i \in \sigma_0^{i_0-1}. \quad (52)$$

If we choose this time $\zeta = \inf(\lambda, \varsigma)$, it follows

$$\forall z_0 \in B(0, \zeta), \quad C\Phi^i(z_0) \in \Omega, \quad \forall i \in \mathbb{N}. \quad (53)$$

Thus $B(0, \zeta) \subset \Theta^\lambda(\Omega)$, and consequently $0 \in \overset{\circ}{\Theta^\lambda(\Omega)}$. This completes the proof. $\square$

3. CHARACTERIZATION OF THE MAXIMAL OUTPUT λ -ADMISSIBLE SET

It is more difficult to characterize the set $\Theta^\lambda(\Omega)$ defined in (19), for that reason we define for each $k \in \mathbb{N}$ the set

$$\Theta_k^\lambda(\Omega) = \{x_0 \in B(0, \lambda) \cap \mathbb{R}^n : C\Phi^i(x_0) \in \Omega, \forall i \in \sigma_0^k\}, \quad (54)$$

and we introduce the following definition.

Definition 2: The set $\Theta^\lambda(\Omega)$ is said to be finitely determined if there exists an integer k_0 such that $\Theta^\lambda(\Omega) = \Theta_{k_0}^\lambda(\Omega)$.

Remark 4: 1) For $r, s \in \mathbb{N}$ such that $r \leq s$, we have

$$\Theta^\lambda(\Omega) \subset \Theta_s^\lambda(\Omega) \subset \Theta_r^\lambda(\Omega). \quad (55)$$

- 2) If $\Theta^\lambda(\Omega)$ is finitely determined, and k_0 the smallest k such that $\Theta_k^\lambda(\Omega) = \Theta_{k+1}^\lambda(\Omega)$, then we have

$$\Theta^\lambda(\Omega) = \Theta_{k_0}^\lambda(\Omega) = \Theta_k^\lambda(\Omega), \forall k \geq k_0. \quad (56)$$

Conditions that demonstrate finite determinability, are examined in the following proposition.

- Proposition 3:** 1) If $\Theta^\lambda(\Omega)$ is finitely determined then there exists $k_0 \in \mathbb{N}$ such that $\Theta_{k_0}^\lambda(\Omega) = \Theta_{k_0+1}^\lambda(\Omega)$.
2) If $\Phi(B(0, \lambda)) \subset B(0, \lambda)$ and $\Theta_{k_0}^\lambda(\Omega) = \Theta_{k_0+1}^\lambda(\Omega)$ for some $k_0 \in \mathbb{N}$ then $\Theta^\lambda(\Omega)$ is finitely determined.

Proof: 1) Assume that the set $\Theta^\lambda(\Omega)$ is finitely determined. Then by Definition 2 it follows

$$\exists k_0 \in \mathbb{N} \text{ such that } \Theta^\lambda(\Omega) = \Theta_{k_0}^\lambda(\Omega). \quad (57)$$

Using Remark 4 we have $\Theta_{k_0+1}^\lambda(\Omega) \subset \Theta_{k_0}^\lambda(\Omega)$ since $k_0 + 1 \geq k_0$. On the other hand,

$$\Theta^\lambda(\Omega) \subset \Theta_{k_0+1}^\lambda(\Omega). \quad (58)$$

We deduce from (57) and (58) that

$$\Theta_{k_0}^\lambda(\Omega) \subset \Theta_{k_0+1}^\lambda(\Omega). \quad (59)$$

Therefore

$$\Theta_{k_0}^\lambda(\Omega) = \Theta_{k_0+1}^\lambda(\Omega) \text{ for some } k_0 \in \mathbb{N}. \quad (60)$$

- 2) Let $x_0 \in \Theta_{k_0}^\lambda(\Omega)$, then $x_0 \in \Theta_{k_0+1}^\lambda(\Omega)$.

This leads

$$x_0 \in B(0, \lambda), \quad C\Phi^i(x_0) \in \Omega, \quad \forall i \in \sigma_0^{k_0+1}. \quad (61)$$

it follows

$$C\Phi^i(\Phi(x_0)) \in \Omega, \quad \forall i \in \sigma_0^{k_0}. \quad (62)$$

Thus $\Phi(x_0) \in \Theta_{k_0}^\lambda(\Omega)$ since

$$\Phi(x_0) \in \Phi(B(0, \lambda)) \subset B(0, \lambda). \quad (63)$$

By iteration

$$\begin{aligned} x_0 \in \Theta_{k_0}^\lambda(\Omega) &\implies \Phi^j(x_0) \in \Theta_{k_0}^\lambda(\Omega), \forall j \in \mathbb{N} \\ &\implies C\Phi^i(\Phi^j(x_0)) \in \Omega, \quad \forall i \in \sigma_0^{k_0} \\ &\implies C\Phi^i(x_0) \in \Omega, \quad \forall i \in \mathbb{N} \\ &\implies x_0 \in \Theta^\lambda(\Omega) \text{ since } x_0 \in B(0, \lambda). \end{aligned}$$

Hence

$$\Theta_{k_0}^\lambda(\Omega) \subset \Theta^\lambda(\Omega). \quad (64)$$

Using Remark 4, we conclude

$$\Theta_{k_0}^\lambda(\Omega) = \Theta^\lambda(\Omega) \text{ for some } k_0 \in \mathbb{N}, \quad (65)$$

which completes the proof. $\square$

4. SUFFICIENT CONDITIONS FOR FINITE DETERMINATION OF $\Theta^\lambda(\Omega)$

The following two theorems are the main results that give simple conditions to guarantee the finite determination of the set $\Theta^\lambda(\Omega)$.

Theorem 1: Assume the following hypothesis to hold

- (i) There exists $\tau \geq 1$ and $\theta \in]0, 1[$ such that $\|\tilde{A}\|^k \leq \tau\theta^k, \forall k \in \mathbb{N}$,
(ii) $\theta + C_K \lambda \tau^2 \leq 1, C_K = \sum_{j=1}^q \|(\tilde{K}^j)^\top\| \|B_j\|$,

$$(iii) \quad 0 \in \overset{\circ}{\Omega},$$

$$(iv) \quad \Phi(B(0, \lambda)) \subset B(0, \lambda).$$

Then, $\Theta^\lambda(\Omega)$ is finitely determined.

Theorem 2: If we assume that

- (i) $\|\Phi(z)\| \leq \nu \|z\|, \forall z \in \mathbb{R}^n$ and $\nu \in]0, 1[$.
(ii) $0 \in \overset{\circ}{\Omega}$.

Then, $\Theta^\lambda(\Omega)$ is finitely determined.

5. ALGORITHMIC DETERMINATION

In this section we note k_0^* the smallest integer such that

$$\Theta^\lambda(\Omega) = \Theta_{k_0^*}^\lambda(\Omega), \quad (66)$$

we call k_0^* the admissibility index. We shall provide a useful algorithm (see [13]) for identifying this admissibility index k_0^* and therefore the characterization of the set $\Theta^\lambda(\Omega)$.

Remark 5: 1) Algorithm 1 produce k_0^* and $\Theta^\lambda(\Omega)$ if and only if $\Theta^\lambda(\Omega)$ is finitely determined.

- 2) If Algorithm 1 does not converge then the set $\Theta^\lambda(\Omega)$ is not finitely determined.

Algorithm 1 is not practical due to the fact it does not describe how the test $\Theta_k^\lambda(\Omega) = \Theta_{k+1}^\lambda(\Omega)$ is implemented. To deal with this difficulty we will consider the following approach.

Let Ω be defined as

$$\Omega = \{y \in \mathbb{R}^p / f_1(y) \leq 0, f_2(y) \leq 0, \dots, f_s(y) \leq 0\}. \quad (67)$$

Algorithm 1: Determination of k_0^* .

Require: $\lambda > 0, n, p, q \in \mathbb{N}^*, \tilde{A}, C, \psi, \Omega \subset \mathbb{R}^p$

$k \leftarrow 0$

if $\Theta_k^\lambda(\Omega) = \Theta_{k+1}^\lambda(\Omega)$ **then**

 let $k_0^* \leftarrow k$ and stop

else

 let $k \leftarrow k + 1$ and continue (return to previous test)

end if

Algorithm 2 Determination of k_0^*

Require: $\lambda > 0, n, p, q, s \in \mathbb{N}^*, \tilde{A}, C, \psi, f_i, i = 1, \dots, s$
 $k \leftarrow 0$
Repeat
for $i=1, \dots, s$ **do**
Maximize $J_i(x) = f_i(C\Phi^{k+1}(x))$
Subject to the constraints

$$\begin{cases} f_j(C\Phi^l(x)) \leq 0, & x \in B(0, \lambda), \\ \forall j \in \{1, \dots, s\}, \forall l \in \{0, \dots, k\}. \end{cases}$$

end for
 $J_i^* \leftarrow \max\{J_i(x)\}$
if $J_i^* \leq 0, \forall i = 1, 2, \dots, s$ **then**
 $k_0^* \leftarrow k$
break
else
 $k \leftarrow k + 1$
end if

Thus, for all $k \in \mathbb{N}$, the set $\Theta_k^\lambda(\Omega)$ can be written as

$$\Theta_k^\lambda(\Omega) = \{x \in B(0, \lambda) \mid f_j(C\Phi^i(x)) \leq 0, j = 1, \dots, s, i = 0, \dots, k\}. \quad (68)$$

On the other hand,

$$\Theta_{k+1}^\lambda(\Omega) = \{x \in \Theta_k^\lambda(\Omega, K) \mid f_j(C\Phi^{k+1}(x)) \leq 0, j = 1, \dots, s\}. \quad (69)$$

Now, since $\Theta_{k+1}^\lambda(\Omega) \subset \Theta_k^\lambda(\Omega)$ for all $k \in \mathbb{N}$, we deduce

$$\begin{aligned} \Theta_{k+1}^\lambda(\Omega) &= \Theta_k^\lambda(\Omega, K) \\ \iff \Theta_k^\lambda(\Omega) &\subset \Theta_{k+1}^\lambda(\Omega) \\ \iff \forall x \in \Theta_k^\lambda(\Omega), & f_j(C\Phi^{k+1}(x)) \leq 0, \forall j \in \{1, \dots, s\} \\ \iff \sup_{\substack{x \in B(0, \lambda), f_i(C\Phi^l(x)) \leq 0 \\ \forall i \in \{1, \dots, s\}, \forall l \in \{0, \dots, k\}}} & f_j(C\Phi^{k+1}(x)) \leq 0, \forall j \in \{1, \dots, s\}. \end{aligned}$$

Therefore, the test $\Theta_k^\lambda(\Omega) = \Theta_{k+1}^\lambda(\Omega)$ leads to a set of mathematical programming problems. We will suggest another version of Algorithm 1, this new algorithm is given by Algorithm 2.

Remark 6: If the matrix $\tilde{A}$ is Lyapunov stable then the supremum in Algorithm 2 is defined for all $x \in \mathbb{R}^n$. Indeed, using the notation defined in Proposition 1 it follows

$$\|C\Phi(x)\| = \left\| C\tilde{A}x + C \sum_{j=1}^q \tilde{K}^j x Bx \right\| \quad (70)$$

$$\leq \|C\tilde{A}x\| + \|C\| C_K \|x\|^2. \quad (71)$$

On the other hand, the assumption of Lyapunov stability of $\tilde{A}$ implies $\|C\tilde{A}x\| \leq \gamma \|x\|$ for some positive γ . This leads

$$\|C\Phi(x)\| \leq (\gamma + \|C\| C_K \|x\|) \|x\|. \quad (72)$$

Let $k \in \mathbb{N}$ and define the function F by

$$F(\alpha) = (\gamma + \|C\| C_K \alpha) \alpha, \forall \alpha \in \mathbb{R}. \quad (73)$$

Then

$$\|C\Phi^{k+1}(x)\| \leq F^{k+1}(\|x\|), \quad (74)$$

and

$$C\Phi^{k+1}(x) \in B(0, F^{k+1}(\|x\|)), \quad (75)$$

using the compactness of B and the continuity of the functions f_i , the sequence $f_i(C\Phi^{k+1}(x))$ is bounded from above.

Remark 7: 1) This algorithm can not be effectively used if there is not a feasible way to find global optima. When Ω is a polyhedron (i.e., the functions f_j are affine for all j), the difficulty becomes less.

2) Assumptions of our two previous results (Theorem 1 and 2 of Section 4) are sufficient but not necessary. If these assumptions are not established, then the convergence of Algorithm 2 is not guaranteed.

To illustrate our results, we should give several examples, which will be given in the upcoming section.

5.1. Algorithm 2 steps

In case $k = 0$ we have

for $i = 1$

$$\begin{aligned} &\max f_1(C\Phi(x)) \\ s.c \quad &\begin{cases} f_1(Cx) \leq 0, \dots, f_s(Cx) \leq 0, \\ |x_1| \leq \lambda, \dots, |x_n| \leq \lambda, \end{cases} \end{aligned}$$

for $i = 2$

$$\begin{aligned} &\max f_2(C\Phi(x)) \\ s.c \quad &\begin{cases} f_1(Cx) \leq 0, \dots, f_s(Cx) \leq 0, \\ |x_1| \leq \lambda, \dots, |x_n| \leq \lambda, \end{cases} \end{aligned}$$

we continue like this until $i = s$

$$\begin{aligned} &\max f_s(C\Phi(x)) \\ s.c \quad &\begin{cases} f_1(Cx) \leq 0, \dots, f_s(Cx) \leq 0, \\ |x_1| \leq \lambda, \dots, |x_n| \leq \lambda. \end{cases} \end{aligned}$$

If $\max f_i(C\Phi(x)) \leq 0, \forall i \in \{1, \dots, s\}$ then we stop and $k_0^* = 0$ else we continue.

In case $k = 1$ we have

for $i = 1$

$$\begin{aligned} &\max f_1(C\Phi^2(x)) \\ s.c \quad &\begin{cases} f_1(Cx) \leq 0, \dots, f_s(Cx) \leq 0, \\ f_1(C\Phi(x)) \leq 0, \dots, f_s(C\Phi(x)) \leq 0, \\ |x_1| \leq \lambda, \dots, |x_n| \leq \lambda, \end{cases} \end{aligned}$$

for $i = 2$

$$\max f_2(C\Phi^2(x))$$

$$s.c \begin{cases} f_1(Cx) \leq 0, \dots, f_s(Cx) \leq 0, \\ f_1(C\Phi(x)) \leq 0, \dots, f_s(C\Phi(x)) \leq 0, \\ |x_1| \leq \lambda, \dots, |x_n| \leq \lambda, \end{cases}$$

we continue until $i = s$

$$\max f_s(C\Phi^2(x))$$

$$s.c \begin{cases} f_1(Cx) \leq 0, \dots, f_s(Cx) \leq 0, \\ f_1(C\Phi(x)) \leq 0, \dots, f_s(C\Phi(x)) \leq 0, \\ |x_1| \leq \lambda, \dots, |x_n| \leq \lambda. \end{cases}$$

If $\max f_i(C\Phi^2(x)) \leq 0, \forall i \in \{1, \dots, s\}$ then we stop and $k_0^* = 1$ else we continue.

For example in step $k = k^*$ we have for $i \in \{1, \dots, s\}$

$$\max f_i(C\Phi^{k^*+1}(x))$$

$$s.c \begin{cases} f_1(Cx) \leq 0, \dots, f_s(Cx) \leq 0, \\ f_1(C\Phi(x)) \leq 0, \dots, f_s(C\Phi(x)) \leq 0, \\ \vdots \\ f_1(C\Phi^{k^*}(x)) \leq 0, \dots, f_s(C\Phi^{k^*}(x)) \leq 0, \\ |x_1| \leq \lambda, \dots, |x_n| \leq \lambda. \end{cases}$$

If $\max f_i(C\Phi^{k^*+1}(x)) \leq 0, \forall i \in \{1, \dots, s\}$ then we stop and $k_0^* = k^*$ else we continue. We can use a predefined output function to be called at each iteration.

Remark 8: Let us suppose that the cost of function of maximization is $Cost_{\max}$, the cost of repeat loop is $Cost_{Repeat}$ and the cost associated to the computation of $C\Phi^i, i \in \{0, 1, \dots, Cost_{Repeat}\}$ is $Cost_{fct}$. Then the computational complexity of Algorithm 2 is given by

$$Cost_{algo} = (s \times Cost_{\max} + 2s) \times Cost_{Repeat} + Cost_{fct}. \quad (76)$$

6. NUMERICAL EXAMPLES

To illustrate our results, we present some examples in two and three dimensional cases ($n = 2$ and $n = 3$). For this reason, we construct the matrices $K^j, j \in \sigma_1^q$ (in the case when $p = 1$) by the following way

$$K^j = \frac{1}{Nb \times \|B_j\| \times \|C\| \times q \times \lambda \times \tau^2},$$

where the choice of the number $Nb > 1$ depend on the value of θ which satisfies the first hypothesis of Proposition 1. This choice of K^j guarantee the verification of the second hypothesis of Proposition 1. The matrix $\tilde{A}$ is chosen such that $\|\tilde{A}\| < 1$. Even if $\|A\| > 1$, this choice is possible by selecting some matrix L since $\tilde{A} = A + BLC$.

Using these matrices, we guarantee the asymptotic stability of the considered systems. Assumption $\Phi(B(0, \lambda)) \subset B(0, \lambda)$ is also taken into consideration. In all examples the dotted region will denote the set $\Theta^\lambda(\Omega)$. For Example 4, we have $k_0^* = \infty$, which means that Algorithm 2 does not converge. Various selections of matrices determining the systems are considered.

In Examples 1-7, we construct Ω as follows:

$$\Omega = \{y \in \mathbb{R}^p \mid f_i(y) \leq 0, i = 1, \dots, 2p\},$$

where $f_i : \mathbb{R}^p \rightarrow \mathbb{R}$ are defined for all $x = (x_1, \dots, x_p) \in \mathbb{R}^p$ by

$$\begin{cases} f_{2m-1}(x) = x_m - \varepsilon, & m \in \{1, 2, \dots, p\}, \\ f_{2m}(x) = -x_m - \varepsilon, & m \in \{1, 2, \dots, p\}, \end{cases} \quad \varepsilon > 0,$$

clearly $0 \in \overset{\circ}{\Omega}$, $f_i, i \in \sigma_1^s$ are continuous functions such that $f_i(0) \leq 0, \forall i \in \sigma_1^s$.

Example 1: Let λ, q and n defined by

$$\lambda = \frac{1}{2}, \quad q = 1, \quad n = 2.$$

Let the matrix $\tilde{A}, B_1, C$ and $\tilde{K}^1$ defined by

$$\tilde{A} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{8} & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\tilde{K}^1 = \begin{pmatrix} \frac{1}{64} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then the functions $\Phi(x), C\Phi(x)$ and $C\Phi^2(x)$ are defined by

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{64}x^2 + \frac{1}{6}x + \frac{1}{6}y \\ \frac{1}{8}x \end{pmatrix},$$

$$C\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{64}x^2 + \frac{1}{6}x + \frac{1}{6}y,$$

$$C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{36}x + \frac{1}{36}y + \frac{1}{64} \left(\frac{1}{64}x^2 + \frac{1}{6}x + \frac{1}{6}y \right)^2 + \frac{1}{384}x^2.$$

Using Algorithm 2 with $\varepsilon = 0.2$ we get $k_0^* = 0$

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}, |x| \leq \varepsilon \right\},$$

and using the same algorithm with $\varepsilon = 0.1$ we get $k_0^* = 1$

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{l} |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}, \\ \left| \frac{1}{64}x^2 + \frac{1}{6}x + \frac{1}{6}y \right| \leq \varepsilon \end{array} \right\}.$$

Fig. 1. The set $\Theta^\lambda(\Omega)$ corresponding to Example 1 ($\varepsilon = 0.2$).

Fig. 2. The set $\Theta^\lambda(\Omega)$ corresponding to Example 1 ($\varepsilon = 0.1$).

Example 2: Let λ, q, n and ε be defined as

$$\lambda = 1, \quad q = 1, \quad n = 2, \quad \varepsilon = 0.3.$$

Let the matrix $\tilde{A}, C, \tilde{K}^1$ and B_1 defined as

$$\tilde{A} = \begin{pmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{8} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \\ \tilde{K}^1 = \begin{pmatrix} \frac{1}{8} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then the functions $\Phi \begin{pmatrix} x \\ y \end{pmatrix}$, $C\Phi \begin{pmatrix} x \\ y \end{pmatrix}$ and $C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix}$ are defined by

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{8}x^2 + \frac{1}{6}x + \frac{1}{4}y \\ \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{8}y \end{pmatrix}, \\ C\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{8}x^2 + \frac{1}{6}x + \frac{1}{4}y,$$

Fig. 3. The set $\Theta^\lambda(\Omega)$ corresponding to Example 2.

$$C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{512}x^4 + \frac{1}{192}x^3 + \frac{1}{128}x^2y + \frac{25}{288}x^2 \\ + \frac{1}{96}xy + \frac{11}{72}x + \frac{1}{128}y^2 + \frac{7}{96}y.$$

Using Algorithm 2 we obtain $k_0^* = 1$, and then the set $\Theta^\lambda(\Omega)$

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{l} |x| \leq 1, |y| \leq 1, |x| \leq \varepsilon, \\ \left| \frac{1}{8}x^2 + \frac{1}{6}x + \frac{1}{4}y \right| \leq \varepsilon \end{array} \right\}.$$

Example 3: Let λ, q, n and ε be defined as

$$\lambda = 2, \quad q = 2, \quad n = 2 \text{ and } \varepsilon = 0.1.$$

Let the matrix $\tilde{A}, C, \tilde{K}^1, \tilde{K}^2, B_1$ and B_2 be defined as

$$\tilde{A} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, \\ \tilde{K}^1 = \begin{pmatrix} \frac{1}{32} & 0 \end{pmatrix}, \quad \tilde{K}^2 = \begin{pmatrix} \frac{1}{64} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x - \frac{1}{2}y \\ \frac{2}{3}x + \frac{1}{3}y \end{pmatrix}, \quad C\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}x - \frac{1}{2}y, \\ \Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}x - \frac{5}{12}y \\ \frac{5}{9}x - \frac{2}{9}y \end{pmatrix}, \quad C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{12}x - \frac{5}{12}y, \\ \Phi^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{23}{72}x - \frac{7}{72}y \\ \frac{7}{54}x - \frac{19}{54}y \end{pmatrix}, \quad C\Phi^3 \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{23}{72}x - \frac{7}{72}y.$$

Using Algorithm 2 we obtain $k_0^* = 2$, and then the set $\Theta^\lambda(\Omega)$

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{l} |x| \leq \lambda, |y| \leq \lambda, |x| \leq \varepsilon, \\ \left| \frac{1}{2}x - \frac{1}{2}y \right| \leq \varepsilon, \\ \left| -\frac{1}{12}x - \frac{5}{12}y \right| \leq \varepsilon \end{array} \right\}.$$

Fig. 4. The set $\Theta^\lambda(\Omega)$ corresponding to Example 3.

Example 4: Let λ, q, n and ε defined by

$$\lambda = 2, \quad q = 2, \quad n = 2, \quad \varepsilon = 0.3.$$

Let the matrix $\tilde{A}, B_1, B_2, C, \tilde{K}^1$ and $\tilde{K}^2$ defined as

$$\tilde{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, \\ C = (1 \ 0), \quad \tilde{K}^1 = \begin{pmatrix} \frac{1}{32} & 0 \end{pmatrix}, \quad \tilde{K}^2 = \begin{pmatrix} \frac{1}{64} & 0 \end{pmatrix}.$$

Then

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{3}y \\ \frac{1}{3}x + \frac{2}{3}y \end{pmatrix}, \quad C\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}x + \frac{1}{2}y, \\ C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{5}{12}x + \frac{7}{12}y, \quad C\Phi^3 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{29}{72}x + \frac{43}{72}y, \\ C\Phi^4 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{173}{432}x + \frac{259}{432}y, \\ C\Phi^5 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1037}{2592}x + \frac{1555}{2592}y, \\ C\Phi^6 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{6221}{15552}x + \frac{9331}{15552}y, \\ C\Phi^7 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{37325}{93312}x + \frac{55987}{93312}y.$$

Using Algorithm 2 we get $k_0^* = \infty$.

Example 5: Let λ, q, n and ε defined by

$$\lambda = 2, \quad q = 2, \quad n = 2, \quad \varepsilon = 0.1.$$

Let the matrix $\tilde{A}, B_1, B_2, C, \tilde{K}^1$ and $\tilde{K}^2$ defined as

$$\tilde{A} = \begin{pmatrix} \frac{1}{8} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{24} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}, \\ C = (1 \ 0), \quad \tilde{K}^1 = \begin{pmatrix} \frac{1}{64} & 0 \end{pmatrix}, \quad \tilde{K}^2 = \begin{pmatrix} \frac{1}{128} & 0 \end{pmatrix}.$$

Fig. 5. The set $\Theta^\lambda(\Omega)$ corresponding to Example 5.

Then, the functions $\Phi \begin{pmatrix} x \\ y \end{pmatrix}, C\Phi \begin{pmatrix} x \\ y \end{pmatrix}$ and $C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix}$ are defined by

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{16}x^2 + \frac{1}{8}x + \frac{1}{4}y \\ \frac{5}{128}x^2 + \frac{1}{3}x + \frac{1}{24}y \end{pmatrix}, \\ C\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{16}x^2 + \frac{1}{8}x + \frac{1}{4}y, \\ C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4096}x^4 + \frac{1}{1024}x^3 + \frac{1}{512}x^2y + \frac{19}{1024}x^2 \\ + \frac{1}{256}xy + \frac{19}{192}x + \frac{1}{256}y^2 + \frac{1}{24}y.$$

Using Algorithm 2 we get $k_0^* = 1$

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{l} |x| \leq 2, |y| \leq 2, |x| \leq \varepsilon, \\ \left| \frac{1}{16}x^2 + \frac{1}{8}x + \frac{1}{4}y \right| \leq \varepsilon \end{array} \right\}.$$

Example 6: Let $\tilde{A}, B_1, \tilde{K}^1, C, q, \varepsilon$ and λ defined as

$$C = (1 \ 0 \ 0), \quad \lambda = 2, \quad q = 1, \quad \varepsilon = 0.05, \\ B_1 = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \frac{1}{6} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{8} & \frac{1}{7} \\ 0 & \frac{1}{10} & 0 \end{pmatrix}, \quad \tilde{K}^1 = \begin{pmatrix} \frac{1}{40} & 0 & 0 \end{pmatrix}.$$

Then, we have

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{8} & \frac{1}{7} \\ 0 & \frac{1}{10} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ + \frac{1}{40}x \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ = \begin{pmatrix} \frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \\ \frac{1}{3}x + \frac{1}{8}y + \frac{1}{7}z + \frac{3}{40}xy + \frac{1}{20}x^2 \\ \frac{1}{40}x^2 + \frac{1}{40}zx + \frac{1}{10}y \end{pmatrix},$$

$$\begin{aligned}
C\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2, \\
\Phi^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} \frac{1}{9}x + \frac{223}{2400}y + \frac{29}{420}z + \\ \frac{1}{40} \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right)^2 + \\ \frac{1}{10} \left(\frac{1}{40}x^2 + \frac{1}{40}xz + \frac{1}{10}y \right) \times \\ \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right) + \\ \frac{3}{160}xy + \frac{13}{600}xz + \frac{13}{600}x^2 \\ \frac{7}{72}x + \frac{761}{6720}y + \frac{71}{840}z + \\ \frac{1}{20} \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right)^2 + \\ \frac{3}{40} \left(\frac{1}{40}x^2 + \frac{1}{40}xz + \frac{1}{10}y \right) \times \\ \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right) + \\ \frac{3}{320}xy + \frac{31}{840}xz + \frac{61}{3360}x^2 \\ \frac{1}{30}x + \frac{1}{80}y + \frac{1}{70}z + \\ \frac{1}{40} \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right)^2 + \\ \frac{1}{40} \left(\frac{1}{40}x^2 + \frac{1}{40}xz + \frac{1}{10}y \right) \times \\ \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right) + \\ \frac{3}{400}xy + \frac{1}{200}x^2 \end{pmatrix}, \\
C\Phi^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{9}x + \frac{223}{2400}y + \frac{29}{420}z \\
&+ \frac{1}{40} \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right)^2 \\
&+ \frac{1}{10} \left(\frac{1}{40}x^2 + \frac{1}{40}xz + \frac{1}{10}y \right) \\
&\times \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right) \\
&+ \frac{3}{160}xy + \frac{13}{600}xz + \frac{13}{600}x^2.
\end{aligned}$$

Using Algorithm 2 we obtain $k_0^* = 2$, and then

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid \begin{aligned} &|x| \leq \lambda, |y| \leq \lambda, \\ &|y| \leq \lambda, |x| \leq \varepsilon, \\ &\left| \frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right| \leq \varepsilon, \\ &\left| \frac{1}{9}x + \frac{223}{2400}y + \frac{29}{420}z + \frac{1}{40} \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right)^2 + \right. \\ &\quad \left. + \frac{1}{10} \left(\frac{1}{40}x^2 + \frac{1}{40}xz + \frac{1}{10}y \right) \times \left(\frac{1}{6}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{10}xz + \frac{1}{40}x^2 \right) + \right. \\ &\quad \left. + \frac{3}{160}xy + \frac{13}{600}xz + \frac{13}{600}x^2 \right| \leq \varepsilon \end{aligned} \right\}.$$

Example 7: To show the efficiency of the stability we take the following example.

Let $\lambda, q, n, \varepsilon$ and τ defined as

$$\lambda = 1, \quad q = 1, \quad n = 2, \quad \varepsilon = 0.1, \quad \tau = 1.$$

Fig. 6. The set $\Theta^\lambda(\Omega)$ corresponding to Example 8.

Fig. 7. The set $\Theta^\lambda(\Omega)$ corresponding to Example 9.

Let $B_1, \tilde{A}, \tilde{K}^1, C, C_K$ and θ defined by

$$\begin{aligned}
B_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0.1 + \gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \end{pmatrix}, \\
\tilde{K}^1 &= \begin{pmatrix} \frac{1}{10} & 0 \end{pmatrix}, \quad C_K = \frac{1}{10}, \quad \theta = \|\tilde{A}\|.
\end{aligned}$$

Now, we use Algorithm 2 we obtain

γ	$\theta + C_K \lambda \tau^2$	k_0^*
0	0.2	0
0.5	0.7	1
0.6	0.8	1
0.9	1.1	3
1	1.2	∞ .

6.1. Comment

In Examples 1-7, we have determined the index of admissibility k_0^* using the predefined function **fmincon** in

Matlab [28], which allow us to solve the maximization problems in Algorithm 2. In Figs. 1-5, we have plotted all the constraints forming the sets $\Theta^\lambda(\Omega)$, in order to visualize such sets of Examples 1, 2, 3, and 5.

Example 8: In this example we take Ω in the form

$$\Omega = \{y \in \mathbb{R} \mid f(y) = y^2 + y - 2 \leq 0\}.$$

It is easy to see that $0 \in \overset{\circ}{\Omega}$, f is continuous function such that $f(0) \leq 0$ and that Ω is closed since $\Omega = f^{-1}([-\infty, 0])$.

Now, we use the results of Example 3 and Algorithm 2 it follows

$$\begin{aligned} \text{Maximize } f(C\Phi(x, y)) &= \left(\frac{1}{2}x - \frac{1}{2}y\right)^2 + \left(\frac{1}{2}x - \frac{1}{2}y\right) - 2 \\ \text{s.c } \begin{cases} f\left(C\begin{pmatrix} x \\ y \end{pmatrix}\right) &= x^2 + x - 2 \leq 0, \\ |x| \leq 2, \quad |y| \leq 2, \end{cases} \end{aligned}$$

the solution of this problem gives 1.7500.

Maximize

$$\begin{aligned} f(C\Phi^2(x, y)) &= \left(-\frac{1}{12}x - \frac{5}{12}y\right)^2 + \left(-\frac{1}{12}x - \frac{5}{12}y\right) - 2 \\ \text{s.c } \begin{cases} f\left(C\begin{pmatrix} x \\ y \end{pmatrix}\right) &= x^2 + x - 2 \leq 0, \\ f\left(C\Phi\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left(\frac{1}{2}x - \frac{1}{2}y\right)^2 + \left(\frac{1}{2}x - \frac{1}{2}y\right) - 2 \leq 0, \\ |x| \leq 2, \quad |y| \leq 2, \end{cases} \end{aligned}$$

the solution of this problem gives: $-1.2000e^{-06}$.

Therefore, the value of index of admissibility is $k_0^* = 1$. Hence

$$\begin{aligned} \Theta^\lambda(\Omega) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{l} |x| \leq 2, \quad |y| \leq 2, \quad x^2 + x - 2 \leq 0, \\ \left(\frac{x}{2} - \frac{y}{2}\right)^2 + \left(\frac{x}{2} - \frac{y}{2}\right) - 2 \leq 0 \end{array} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| \leq 2, \quad |y| \leq 2, \quad x^2 + x - 2 \leq 0 \right\} \\ &\quad \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \left(\frac{x}{2} - \frac{y}{2}\right)^2 + \left(\frac{x}{2} - \frac{y}{2}\right) - 2 \leq 0 \right\} \\ &= [-2, 1] \times [-2, 2] \\ &\quad \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x - 2 \leq y \leq x + 4 \right\}. \end{aligned}$$

Example 9: In this example the set Ω is represented by

$$\Omega = \left\{ y \in \mathbb{R} \mid \begin{array}{l} f_1(y) = y^3 + y - 3 \leq 0, \\ f_2(y) = y^2 + 2y - 4 \leq 0 \end{array} \right\}.$$

It is easy to see that $f_i(0) \leq 0, \forall i \in \{1, 2\}$ and the functions $f_i, i = 1, 2$ are continuous.

Using results of Example 5 and Algorithm 2 we obtain $k_0^* = 0$, and then the maximal output λ -admissible set $\Theta^\lambda(\Omega)$ is given by

$$\begin{aligned} \Theta^\lambda(\Omega) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{l} |x| \leq 2, \quad |y| \leq 2, \\ x^3 + x - 3 \leq 0, \\ x^2 + 2x - 4 \leq 0 \end{array} \right\} \\ &= [-2, 2] \times [-2, 2] \cap [-2, \sqrt{5} - 1] \times [-2, 2] \\ &\quad \cap [-2, r] \times [-2, 2] \\ &= [-2, r] \times [-2, 2], \end{aligned}$$

$$\text{with } r = \sqrt[3]{\frac{3}{2} - \frac{1}{18}\sqrt{3}\sqrt{247}} + \sqrt[3]{\frac{1}{18}\sqrt{3}\sqrt{247} + \frac{3}{2}}.$$

6.2. Comment

In Examples 8 and 9, we have modified the set Ω which affects the mathematical programming problems that arise in Algorithm 2, we have plotted again the sets $\Theta^\lambda(\Omega)$ in Figs. 6 and 7.

In order to obtain the numerical computational (elapsed time in seconds (s)) of Examples 1, 2, 3, 5, and 6 using Algorithm 2 the predefined output function **fmincon** (see [28]) is applied with different values for the coefficient λ , ε , k_0^* , n to solve the maximization problems which arise in such algorithm. The obtained results are presented in Table 1.

Remark 9: In case $p \geq 2$ we have to reconstruct the matrices K^j such that the condition $\theta + \lambda C_K \tau^2 < 1$ of Proposition 1 is verified.

Table 1. Parameters values and elapsed time of Examples 1-3, 5, and 6.

Examples	$\tilde{A}$	λ	s	k_0^*	ε	Time (s)
Examples 1	$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{8} & 0 \end{pmatrix}$	$\frac{1}{2}$	2	0	0.2	1.660970
Examples 1	$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{8} & 0 \end{pmatrix}$	$\frac{1}{2}$	2	1	0.1	3.1711
Examples 2	$\begin{pmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{8} \end{pmatrix}$	1	2	1	0.3	3.2184
Examples 3	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$	2	2	2	0.1	3.0743
Examples 5	$\begin{pmatrix} \frac{1}{8} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{24} \end{pmatrix}$	2	2	1	0.1	3.5962
Examples 6	$\begin{pmatrix} \frac{1}{6} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{8} & \frac{1}{7} \\ 0 & \frac{1}{10} & 0 \end{pmatrix}$	2	2	1	0.05	8.5662

7. APPLICATIONS

This section aims to apply the approach developed in this paper to epidemiological and sociological models that includes control terms.

7.1. Application in epidemiological model (App 1)

We consider a deterministic model of susceptible- infected (SI), where S_i represents the number of individuals susceptible to infection of the disease but not yet infected at the time i . I_i denotes the number of individuals who have been infected with the disease, and they are infectious. These types of models are used to model many infectious diseases such as HIV. The model is based on the assumption that disease transmission happens as a result of an “effective” contact between a person who is susceptible and an infectious person. Because of the nonlinear contact dynamics within the population, the αSI incidence function is used to reflect successful disease transmission, where α is the rate of effective contact with infected individuals in compartment I .

The treatment and awareness campaigns are well known as efficient strategies to control and prevent the spread of such infectious diseases. Consequently, we introduce in the following system with two controls u_i and v_i .

u_i : refers to the effort to protect individuals who are susceptible to becoming infected. This effort to provide awareness.

v_i : refers to the effort to provide treatment.

Therefore, we use the following model:

$$\begin{cases} S_{i+1} = S_i - \alpha S_i I_i - u_i S_i, \\ I_{i+1} = I_i + \alpha S_i I_i - v_i I_i, \quad i \geq 0, \\ S(0) = S_0, I(0) = I_0. \end{cases} \quad (77)$$

The corresponding output is

$$y_i = \gamma S_i + \beta I_i = (\gamma \quad \beta) \begin{pmatrix} S_i \\ I_i \end{pmatrix}, \quad (78)$$

(y_i can be the cost of treatment).

Take $x_i = \begin{pmatrix} S_i \\ I_i \end{pmatrix}$ and u_i, v_i feedback controls defined by

$$u_i = m + ky_i, \quad v_i = \tilde{m} + \tilde{k}y_i. \quad (79)$$

We assume that our control depends on the level of endemicity in the population (given the observation times k and $\tilde{k}$) and the available resource to control the spread of the disease (m and $\tilde{m}$, respectively). In fact, the public health authorities are all-time facing limited resources to contain the infection. Therefore, our control is a function of the observation y_i , which gives the decision-maker an optimal way to treat and vaccinate the population.

Then, the system (77) can be rewritten as follows:

$$\begin{cases} S_{i+1} = (1-m)S_i - \alpha S_i I_i - k S_i S_i, \\ I_{i+1} = (1-\tilde{m})I_i + \alpha S_i I_i - \tilde{k} S_i I_i, \end{cases} \quad i \geq 0, \quad (80)$$

or equivalently

$$\begin{pmatrix} S_{i+1} \\ I_{i+1} \end{pmatrix} = \begin{pmatrix} 1-m & 0 \\ 0 & 1-\tilde{m} \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} + (\alpha \quad 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} + (\alpha \quad 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} + (k \quad 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} + (\tilde{k} \quad 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix}.$$

This model can be written as

$$\begin{cases} x_{i+1} = \tilde{A}x_i + \sum_{j=1}^q \tilde{K}^j x_i B_j x_i, \quad i \geq 0, \\ x_0 \in \mathbb{R}^2, \\ y_i = (\gamma \quad \beta)x_i = Cx_i, \end{cases} \quad (81)$$

where

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} 1-m & 0 \\ 0 & 1-\tilde{m} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \\ B_4 &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \tilde{K}^1 = (\alpha \quad 0), \\ \tilde{K}^2 &= (\alpha \quad 0), \quad \tilde{K}^3 = (k \quad 0), \\ \tilde{K}^4 &= (\tilde{k} \quad 0), \quad q = 4. \end{aligned}$$

Now we select $\tilde{K}^3$ and $\tilde{K}^4$ as follows:

$$\tilde{K}^3 = \frac{1}{4}, \quad \tilde{K}^4 = \frac{1}{3},$$

and the parameter α is taken as

$$\alpha = \frac{1}{2}.$$

The second hypothesis of Proposition 1 is verified. Indeed,

$$\theta + C_K \lambda \tau^2 = 0.3 + \frac{19}{12} \lambda \tau^2 = 0.99271 < 1,$$

with $\theta = \|\tilde{A}\| = 0.3$, $\lambda = \frac{42}{96}$, $\tau = 1$ ($m = \tilde{m} = 0.7$).

Therefore,

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} x \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned}
& + \frac{1}{2}x \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{4}x \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
& + \frac{1}{3}x \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
& = \begin{pmatrix} 0.3x - \frac{1}{2}xy - \frac{1}{4}x^2 \\ 0.3y + \frac{1}{6}xy \end{pmatrix}, \\
C\Phi \begin{pmatrix} x \\ y \end{pmatrix} & = \begin{pmatrix} \frac{9}{10} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0.3x - \frac{1}{2}xy - \frac{1}{4}x^2 \\ 0.3y + \frac{1}{6}xy \end{pmatrix} \\
& = 0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2, \\
\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} & = \begin{pmatrix} 0.09x + \frac{1}{2} \left(0.3y + \frac{1}{6}xy \right) \left(\frac{1}{2}xy - 0.3x + \frac{1}{4}x^2 \right) \\ -\frac{1}{4} \left(\frac{1}{2}xy - 0.3x + \frac{1}{4}x^2 \right)^2 - 0.15xy - 0.075x^2 \\ 0.09y - \frac{1}{6} \left(0.3y + \frac{1}{6}xy \right) \left(\frac{1}{2}xy - 0.3x + \frac{1}{4}x^2 \right) \\ + 0.05xy \end{pmatrix}, \\
C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} & = 0.081x + 0.045y \\
& + \frac{11}{30} \left(0.3y + \frac{1}{6}xy \right) \left(\frac{1}{2}xy - 0.3x + \frac{1}{4}x^2 \right) \\
& - \frac{9}{40} \left(\frac{1}{2}xy - 0.3x + \frac{1}{4}x^2 \right)^2 - 0.11xy \\
& - 0.0675x^2.
\end{aligned}$$

We have $\Phi(B(0, \frac{42}{96})) \subset B(0, \frac{42}{96})$ since

$$0.3 \frac{42}{96} + \frac{1}{2} \frac{42}{96} \frac{42}{96} + \frac{1}{4} \frac{42^2}{96} = 0.2748 < \frac{42}{96},$$

and

$$0.3 \frac{42}{96} + \frac{1}{6} \frac{42}{96} \frac{42}{96} = 0.16315 < \frac{42}{96}.$$

Algorithm 2

In case $k = 0$, we have

for $i = 1$

$$\max 0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2 - 0.03$$

$$s.c \begin{cases} \left| \frac{9}{10}x + \frac{1}{2}y \right| \leq 0, \\ |x| \leq \frac{42}{96}, |y| \leq \frac{42}{96}, \end{cases}$$

for $i = 2$

$$\max - \left(0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2 \right) - 0.03$$

$$s.c \begin{cases} \left| \frac{9}{10}x + \frac{1}{2}y \right| \leq 0, \\ |x| \leq \frac{42}{96}, |y| \leq \frac{42}{96}, \end{cases}$$

we have $J_1^*, J_2^* > 0$, we then go to the next step ($k = 1$).

For $i = 1$

$$\begin{aligned}
& \max C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} - 0.03 \\
& s.c \begin{cases} \left| 0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2 \right| \leq 0.03, \\ \left| \frac{9}{10}x + \frac{1}{2}y \right| \leq 0, \\ |x| \leq \frac{42}{96}, |y| \leq \frac{42}{96}, \end{cases}
\end{aligned}$$

for $i = 2$

$$\begin{aligned}
& \max - \left(C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix} \right) - 0.03 \\
& s.c \begin{cases} \left| 0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2 \right| \leq 0.03, \\ \left| \frac{9}{10}x + \frac{1}{2}y \right| \leq 0.03, \\ |x| \leq \frac{42}{96}, |y| \leq \frac{42}{96}, \end{cases}
\end{aligned}$$

since $J_1^*, J_2^* \leq 0$, we stop.

The computational complexity of applying this algorithm (see Remark 8) is described as follows:

$$Cost_{algo} = (s \times Cost_{max} + 2s) \times Cost_{Repeat} + Cost_{fct},$$

where $s = 2$, $Cost_{Repeat} = 2$, $Cost_{fct}$ is the number of operations to compute $C \begin{pmatrix} x \\ y \end{pmatrix}$, $C\Phi \begin{pmatrix} x \\ y \end{pmatrix}$ and $C\Phi^2 \begin{pmatrix} x \\ y \end{pmatrix}$ and $Cost_{max}$ is the number of operations ($\times$ and $+$) applying the predefined output function **fmincon** from the Matlab Optimization Toolbox [28] to solve the maximization problems which arise in each iteration.

Thus, $k_0^* = 1$ and the set $\Theta^\lambda(\Omega)$ is given by

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{l} 0 \leq x \leq \frac{42}{96}, 0 \leq y \leq \frac{42}{96}, \\ \left| \frac{9}{10}x + \frac{1}{2}y \right| \leq 0.03, \\ \left| 0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2 \right| \leq 0.03 \end{array} \right\}.$$

The plot of the set $\Theta^\lambda(\Omega)$ is given in Fig. 8.

7.2. Application in rumor spreading model (App 2)

Our next application is to consider a SIR type model of rumor [29], where the population is divided into three categories: ignorants (people who never heard the rumor), spreaders (people who have heard the rumor and transmitted it to others), and stiflers (having heard the rumor but

Fig. 8. The set $\Theta^\lambda(\Omega)$.

have not passed it to others). The variables S_i , I_i , and R_i to denote the population densities of ignorant users, spreading users, and stifter users, respectively. This model's dynamic is based on the following assumption: Rumor transmission happens as a result of adequate contact between a person who is ignorant and a person who is a spreader. The ignorant person accepts the rumor and becoming a spreading at a rate β . We assume that the propagation of a constant rumor in a population with a constant emigration rate μ .

Our model is described by

$$\begin{cases} S_{i+1} = S_i - \beta S_i I_i - \mu S_i - u_i S_i, \\ I_{i+1} = I_i + \beta S_i I_i - \mu I_i, \\ R_{i+1} = R_i - \mu R_i + u_i S_i, \end{cases} \quad i \geq 0, \quad (82)$$

where u_i represent the awareness and refers to the effort to protect individuals who are ignorants against becoming spreaders.

The input u_i is taken as feedback of y_i , i.e.,

$$u_i = ky_i, \quad (83)$$

where y_i is the corresponding output, i.e.,

$$y_i = I_i = (0 \ 1 \ 0) \begin{pmatrix} S_i \\ I_i \\ R_i \end{pmatrix} = Cx_i. \quad (84)$$

Using the same approach developed before it follows

$$\begin{cases} x_{i+1} = \tilde{A}x_i + \sum_{j=1}^q \tilde{K}^j x_i B_j x_i, \quad i \geq 0, \\ x_0 \in \mathbb{R}^3, \end{cases} \quad (85)$$

where

$$\tilde{A} = \begin{pmatrix} 1-\mu & 0 & 0 \\ 0 & 1-\mu & 0 \\ 0 & 0 & 1-\mu \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{K}^1 = (0 \ \beta \ 0),$$

$$\tilde{K}^2 = (0 \ k \ 0), \quad \tilde{K}^3 = (0 \ \beta \ 0),$$

$$\tilde{K}^4 = (0 \ k \ 0), \quad q = 4.$$

We choose λ , μ , β and k as

$$\lambda = 1, \quad \mu = 0.2, \quad \beta = k = \frac{1}{18}.$$

Therefore

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.8x - \frac{1}{9}xy \\ 0.8y + \frac{1}{18}xy \\ 0.8z + \frac{1}{18}xy \end{pmatrix},$$

$$C\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.8y + \frac{1}{18}xy.$$

Algorithm 2

In case $k = 0$, we have

for $i = 1$

$$\begin{aligned} & \max 0.8y + \frac{1}{18}xy - 0.9 \\ & s.c \begin{cases} |y| \leq 0.9, \\ |x| \leq 1, \quad |y| \leq 1, \end{cases} \end{aligned}$$

for $i = 2$

$$\begin{aligned} & \max - \left(0.8y + \frac{1}{18}xy \right) - 0.9 \\ & s.c \begin{cases} |y| \leq 0.9, \\ |x| \leq 1, \quad |y| \leq 1, \end{cases} \end{aligned}$$

since $J_1^*, J_2^* \leq 0$, we stop.

The computational complexity of applying this algorithm (see Remark 8) is described as follows:

$$Cost_{algo} = (s \times Cost_{max} + 2s) \times Cost_{Repeat} + Cost_{fct},$$

where $Cost_{Repeat} = 1$ and $s = 2$.

Therefore $k_0^* = 0$ and $\Theta^\lambda(\Omega)$ is given by

$$\Theta^\lambda(\Omega) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid \begin{matrix} 0 \leq x \leq 1, & 0 \leq y \leq 0.9, \\ 0 \leq z \leq 1 \end{matrix} \right\}.$$

The plot of the set $\Theta^\lambda(\Omega)$ is given in Fig. 9. In order to obtain the numerical computational (elapsed time in seconds (s)) of two applications (Apps) using Algorithm 2, the predefined output function **fmincon** (see [28]) is applied with different values for the coefficient λ , ε , k_0^* , n to solve the maximization problems which arise in such algorithm. The obtained results are presented in Table 2.

Table 2. Parameters values and elapsed time of two applications.

Apps	$\tilde{A}$	λ	s	k_0^*	ε	Time (s)
App 1	$\begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}$	$\frac{42}{96}$	2	1	0.03	5.444
App 2	$\begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{pmatrix}$	1	2	0	0.9	2.856

Fig. 9. The set $\Theta^\lambda(\Omega)$.

Remark 10: The computational complexity of Algorithm 2 depend on the following factors:

- 1) The maximization problems which occur in such algorithm,
- 2) the value of index of admissibility k_0^* ,
- 3) the dimension n ,
- 4) the parameters λ , ε and s .

In the following section, we will present a conclusion of our work.

8. CONCLUSION

This paper presented a new problem of maximal output admissible set for discrete-time bilinear controlled systems. First, we gave a new sufficient condition that assures the asymptotic stability of the system. Then, we characterized the maximal output λ -admissible set $\Theta^\lambda(\Omega)$, and we showed that, under certain specific conditions, the finite determinations of $\Theta^\lambda(\Omega)$. Moreover, we proposed an algorithmic approach to determine the admissibility index k_0^* and, consequently, the set $\Theta^\lambda(\Omega)$. Our approach was illustrated with several numerical examples with different dimensions and parameters. In these examples, we were

able to determine the maximal output and to sketch it in the 2 and 3 dimensions.

To give more applications to our approach, we applied our technique to an *SI* epidemic model. Our goal was to find the initial data that guarantees the control of the spread of the epidemic via awareness and treatment. In addition, we also used our approach of a rumor spread model. To contain the spread of a rumor, the control term aimed to increase the awareness of the population. In the future and inspired by the work of Larrache *et al.* [30], we investigate the problem of the maximal output set for discrete-time bilinear distributed systems.

APPENDIX A: PROOFS OF THEOREMS

A.1. Proof of Theorem 1

By hypothesis (i), (ii), (iii), we can show from Proposition 2 that

$$\begin{aligned} \forall z_0 \in B(0, \lambda), \exists i_0 \in \mathbb{N} : \\ C\Phi^i(z_0) \in B(0, \eta) \subset \Omega, \forall i \geq i_0. \end{aligned}$$

For $i = i_0$, we have

$$\forall z_0 \in B(0, \lambda), C\Phi^{i_0}(z_0) \in \Omega. \quad (\text{A.1})$$

Let $z_0 \in \Theta_{i_0-1}^\lambda(\Omega)$, then

$$z_0 \in B(0, \lambda) \text{ and } C\Phi^i(\Omega) \in \Omega, \forall i \in \sigma_0^{i_0-1}, \quad (\text{A.2})$$

using (A.1) we conclude that $z_0 \in \Theta_{i_0}^\lambda(\Omega)$. Thus $\Theta_{i_0-1}^\lambda(\Omega) = \Theta_{i_0}^\lambda(\Omega)$ and $\Theta^\lambda(\Omega)$ is finitely determined (using this time hypothesis (iv) and Proposition 3).

A.2. Proof of Theorem 2

It is easy to verify that

$$\|\Phi^i(z_0)\| \leq v^i \|z_0\|, \forall i \in \mathbb{N}. \quad (\text{A.3})$$

Using limit definition ($\lim_{i \rightarrow \infty} v^i = 0$) it follows

$$\exists i_0 \in \mathbb{N}, v^i \leq \frac{\eta}{\lambda \|C\|}, \forall i \geq i_0. \quad (\text{A.4})$$

For $z_0 \in B(0, \lambda)$ we have

$$\|C\Phi^i(z_0)\| \leq \eta, \forall i \geq i_0. \quad (\text{A.5})$$

Let $z_0 \in \Theta_{i_0-1}^\lambda(\Omega)$. Then

$$z_0 \in B(0, \lambda) \text{ and } C\Phi^i(z_0) \in \Omega, \forall i \in \sigma_0^{i_0-1}. \quad (\text{A.6})$$

From (A.5) we have $C\Phi^{i_0}(z_0) \in B(0, \eta) \subset \Omega$. Therefore $z_0 \in \Theta_{i_0}^\lambda(\Omega)$.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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