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# Existence of Solutions for $p(x)$-Laplacian Elliptic BVPs on a Variable Sobolev Space Via Fixed Point Theorems

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**Abstract**
In this paper, we prove some existence theorems for elliptic boundary value problems within the $p(x)$-Laplacian on a variable Sobolev space. For this purpose, the main problem is transformed into a fixed point problem and then fixed point arguments such as Schaefer's and Schauder's theorems are used. Our approach involves fewer stringent assumptions on the nonlinearity function than the prior findings. An interesting example is presented to examine the validity of the theoretical findings.

**Keywords** $p(x)$-Laplacian · Variable exponent Sobolev space · Schaefer fixed point · Schauder fixed point

**Mathematics Subject Classification** 46E35 · 35D30 · 35J25 · 47H10

## 1 Introduction

The study of differential equations and variational problems involving non-standard $p$-growth conditions has attracted special attention. Such problems often arise in the modeling of several phenomena such as electrorheological fluids, image restoration,

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magnetostatics problems, and many other problems in elastic mechanics [33, 36, 46]. Due to their widespread applications, the investigation of nonlinear elliptic boundary value problems involving the $p(x)$-Laplacian operators have been reported using different approaches [1, 5, 13, 16, 18, 25, 26, 39, 42–44]. Meanwhile, there have been many authors who focused their works on the study of the equations involving the $p(x)$-Laplacian operators and obtained some important results; the reader can consult the papers [4, 7, 15, 17, 20–22, 27, 29, 30, 34, 35].

On the other hand, the notion of variable exponent Lebesgue spaces introduced by Orlicz in the paper [31]. In [46], Zhikov presented a new direction of investigation, which created the relationship between spaces with variable exponent and variational integrals under nonstandard growth conditions. Some interesting results concerning the generalized Lebesgue spaces and the generalized Lebesgue–Sobolev spaces can be found in [8, 9], and references therein.

Recently in [40], Vetro investigated a nonlinear $p(x)$-Kirchhoff type problem with the Dirichlet boundary condition, in the case of a reaction term depending on the gradient (convection). Utilizing a topological attitude based on the Galerkin method, he studied the existence of two notions of solutions: strong generalized solutions and weak solutions. Further, the authors in [46] used minimax approaches in conjunction with the Trudinger-Moser inequality to investigate a particular kind of the weighted Kirchhoff problem and found some existence results of a solution in the subcritical exponential growth situation with positive energy. For more information in this area refer to [14, 17, 20, 41]. Yet, in the event that the Catheodory reaction is gradient dependent, some approximation methods such as the Galerkin method are regarded as essential resources for understanding the $p(x)$-Laplace equation since the problem loses its variational character in this scenario. In [38], Sousa studied a version of the abstract lower and upper-solution method for some operators. Indeed, he presented, the existence of a positive solution for a new class of fractional systems of the Kirchhoff type with $\psi$-Hilfer operators via the method of sub and supersolutions.

Motivated by the results of the above-mentioned work, we prove some existence results of solutions for elliptic boundary value problems with $p(x)$-Laplacian of the following form

$$\begin{cases} -\Delta_{p(x)} \jmath(x) = f(x, \jmath(x)), & x \in \Omega, \\ \jmath(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where

$$-\Delta_{p(x)}(\jmath) = div\left(\|\nabla \jmath(x)\|^{p(x)-2} \nabla \jmath(x)\right),$$

and $\Omega$ is a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, $p(x) \in C(\overline{\Omega})$ is log-Hölder continuous with values in $(1, +\infty)$, and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, satisfying some growth conditions [12]. Problem (1.1) naturally describes applications in fluid mechanics and thus it is of a great significance for engineers and scientists. Fixed point arguments based on Schaefer's and Schauder's fixed point theorems as well as the method of sub-supersolution are used for proving the main theorems. Indeed, the existence results of the weak solutions for the $p$-Laplacian BVPs have been established in [19, 23, 28], while the variational and topological methods were used in [6, 11, 32, 45], and the simple variational arguments based on the Mountain-Pass theorem was used in [28]. The structure of the nonlinearity $f$ in (1.1) suggests that the assumptions made about it are less restrictive than in previous discoveries. The paper discusses major findings within the variable exponent Sobolev space, as well as via utilizing fundamental properties of the $p(x)$-Laplacian and Nemytskii operators.

The article is structured as follows. Section 2 presents some fundamental concepts which are essential to prove the results in the next sections. The main results together with an interesting example are explicitly delivered in Sects. 2 and 3. A brief conclusion is given in Sect. 4.

## 2 Preliminary Results

Here, we present some essential results on the variable exponent Sobolev space and state some basic properties of the $p(x)$-Laplacian and Nemytskii operators. For more details, we orient the reader to [8, 9, 12, 19, 23, 28].

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open bounded subset with smooth boundary $\partial\Omega$ and set

$$C_+(\overline{\Omega}) := \{g : g \in C(\overline{\Omega}),\ g(x) > 1 \text{ for all } \ x \in \overline{\Omega}\},$$

and

$$D = \{\jmath : \Omega \to \mathbb{R};\ \text{is a measurable real–valued}\}\,.$$

For $p \in C_+(\overline{\Omega})$, we define

$$p^- := \min_{x\in\overline{\Omega}} p(x), \quad p^+ := \max_{x\in\overline{\Omega}} p(x).$$

Clearly, $1 < p^- \leq p^+ < \infty$, and

$$L^{p(x)}(\Omega) = \left\{\jmath \in D : \int_\Omega |\jmath(x)|^{p(x)} dx < \infty\right\}.$$

We define the Luxemburg norm by:

$$\|\jmath\|_{p(x)} = \inf \left\{\tau > 0 : \int_\Omega |\frac{\jmath(x)}{\tau}|^{p(x)} dx \leq 1\right\}.$$

Further, $\left(L^{p(x)}(\Omega), \|\jmath\|_{p(x)}\right)$ are Banach spaces which are reflexive if and only if $1 < p^- \leq p^+ < \infty$. Due to the inclusion of the Lebesgue spaces, we get: if $0 < |\Omega| < \infty$ and $p_1(\cdot)$, $p_2(\cdot)$ are variable exponents such that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$, then, there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Moreover, if $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$, then:

$$\Big|\int_\Omega \jmath v dx\Big| \leq \Big(\frac{1}{p^-}+\frac{1}{(p')^-}\Big)\|\jmath\|_{p(x)}\|v\|_{p'(x)} \leq 2\|\jmath\|_{p(x)}\|v\|_{p'(x)}, \ \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega). \tag{2.1}$$

In the sequel, we employ the modular and its properties, which is $\sigma_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, defined by

$$\sigma_{p(x)}(\jmath) := \int_\Omega |\jmath|^{p(x)} dx.$$

It should be noticed that the log–Hölder continuous and Carathéodory functions are defined as follows.

**Definition 2.1** The variable exponent $p(x)$ is log–Hölder continuous if there exists $C > 0$ with

$$|p(x) - p(y)| \leq \frac{C}{-log\|x-y\|},$$

for $x, y \in \mathbb{R}^N$, and $\|x - y\| \leq \frac{1}{2}$.

**Definition 2.2** A function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is Carathéodory function, if $f(., \eta)$ be measurable on $\mathbb{R}$ and $f(\xi, .)$ is continuous on $\mathbb{R}$.

**Proposition 2.3** [12] For all $\jmath, v \in L^{p(x)}(\Omega)$, we have

$$\begin{aligned}
&\|\jmath\|_{p(x)} < 1 \ (resp. = 1; > 1) \Leftrightarrow \sigma_{p(x)}(\jmath) < 1 \ (resp. = 1; > 1),\\
&\|\jmath\|_{p(x)} < 1 \Longrightarrow \|\jmath\|_{p(x)}^{p^+} \leq \sigma_{p(x)}(\jmath) \leq \|\jmath\|_{p(x)}^{p^-},\\
&\|\jmath\|_{p(x)} > 1 \Longrightarrow \|\jmath\|_{p(x)}^{p^-} \leq \sigma_{p(x)}(\jmath) \leq \|\jmath\|_{p(x)}^{p^+},\\
&\sigma_{p(x)}(\jmath - v) \to 0 \Leftrightarrow |\jmath - v|_{p(x)} \to 0.
\end{aligned}$$

It follows that

$$\|\jmath\|_{p(x)} \leq \sigma_{p(x)}(\jmath) + 1,$$

and

$$\sigma_{p(x)}(\jmath) \leq \|\jmath\|_{p(x)}^{p^+} + \|\jmath\|_{p(x)}^{p^-}.$$

**Definition 2.4** The variable Sobolev space $W^{1,p(x)}(\Omega)$ is defined as follows

$$W^{1,p(x)}(\Omega) = \Big\{\jmath \in L^{p(x)}(\Omega) : \|\nabla \jmath\| \in L^{p(x)}(\Omega)\Big\},$$

with the norm

$$\|\jmath\|_{1,p(x)} := \|\jmath\|_{p(x)} + \|\nabla \jmath\|_{p(x)}.$$

**Remark 2.5** The closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ is denoted by $W_0^{1,p(x)}(\Omega)$, and we define

$$p^*(x) := \begin{cases} \dfrac{Np(x)}{N-p(x)}, & \text{if } \ p(x) < N, \\ +\infty, & \text{if } \ p(x) \geq N. \end{cases}$$

**Definition 2.6** Suppose $\Gamma$ and $\Lambda$ be real Banach spaces, $S \subset \Gamma$ be nonempty and the symbols $\rightharpoonup$ and $\rightarrow$ denotes the weak and strong convergence, respectively. The mapping $\mathcal{L} : S \rightarrow \Lambda$ is called bounded if the projection of any bounded set in $\Gamma$ under $\mathcal{L}$ be a bounded set in $\Lambda$.

**Definition 2.7** Suppose $\Gamma$ be a real reflexive Banach space, $\Gamma^*$ be its dual and $S \subset \Gamma$ be nonempty. A mapping $\mathcal{L} : S \rightarrow \Gamma^*$ is said to be of class $(S_+)$, if for any $\{\gamma_n\}$ in $\Gamma$, where $\gamma_n \rightharpoonup \gamma$ and $\overline{lim}\ \langle \mathcal{L}(\gamma_n), \gamma_n - \gamma \rangle \leq 0$, we can conclude that $\gamma_n \rightarrow \gamma$.

**Proposition 2.8** *([8])*

*(i) The spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*
*(ii) If $\gamma \in C_+(\overline{\Omega})$ and $\gamma(x) < p^*(x)$ for $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{\gamma(x)}(\Omega)$ is continuous and compact.*
*(iii) There exists $\vartheta_0 > 0$, with*

$$\|\jmath\|_{p(x)} \leq \vartheta_0 \|\nabla \jmath\|_{p(x)}, \quad \jmath \in W_0^{1,p(x)}(\Omega), \tag{2.2}$$

*which implies that, $\|\nabla \jmath\|_{p(x)}$ and $\|\jmath\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.*

The Nemytskii operator will play an essential role in the subsequent discussion.

**Proposition 2.9** *([19]) Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function and satisfies the growth condition*

$$|f(x,s)| \leq a|s|^{\frac{p_1(x)}{p_2(x)}} + g(x), \ \forall x \in \Omega, s \in \mathbb{R},$$

*where $p_1(.), p_2(.) \in C_+(\overline{\Omega})$, $g \in L^{p_2(x)}(\Omega)$ and $a \geq 0$ is a constant. Then the Nemytskii operator $N_f$ satisfies:*

*(i) $N_f\left(L^{p_1(x)}(\Omega)\right) \subset L^{p_2(x)}(\Omega)$,*
*(ii) $N_f$ is continuous from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$,*
*(iii) $N_f$ maps bounded set into bounded set.*

Further, it is important to review the properties of $p(x)$-Laplacian operator. We consider the following functional $\varphi$ defined by

$$\varphi(\jmath) = \int_{\Omega} \frac{1}{p(x)} \|\nabla \jmath\|^{p(x)}\, dx, \qquad \jmath \in W_0^{1,p(x)}(\Omega).$$

Here $\varphi$ is continuously Fréchet differentiable and $\varphi'(\jmath) = -\Delta_{p(x)}\jmath$, for all $\jmath \in W_0^{1,p(x)}(\Omega)$ where

$$-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \longrightarrow \widetilde{W}_0^{1,p(x)}(\Omega),$$

with

$$\langle -\Delta_{p(x)}\jmath, v\rangle = \int_{\Omega} \frac{1}{p(x)} \|\nabla \jmath\|^{p(x)-2} \nabla \jmath \nabla v\, dx, \quad \jmath, v \in W_0^{1,p(x)}(\Omega),$$

where, $\widetilde{W}_0^{1,p(x)}(\Omega)$ is a dual space of $W_0^{1,p(x)}(\Omega)$.

**Proposition 2.10** [12] The mapping $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \longrightarrow \widetilde{W}_0^{1,p(x)}(\Omega)$ satisfies the following:

(i) $-\Delta_{p(x)}$ is a homeomorphism from $W_0^{1,p(x)}(\Omega)$ to $\widetilde{W}_0^{1,p(x)}(\Omega)$;
(ii) $-\Delta_{p(x)}$ is a continuous, bounded and monotone operator;
(iii) $-\Delta_{p(x)}$ is a mapping of type $S_+$;
(iv) The operator $-\Delta_{p(x)}$ has a continuous inverse mapping $\left(-\Delta_{p(x)}\right)^{-1} : \widetilde{W}_0^{1,p(x)}(\Omega) \longrightarrow W_0^{1,p(x)}(\Omega)$;
(v) $\left(-\Delta_{p(x)}\right)^{-1}$ is bounded and satisfies the $S_+$ condition.

***Proof*** The proofs of (i–iii) are given in literature. We only prove (iv) and (v).

Proof of (iv). Suppose $\beta_n, \beta \in \widetilde{W}_0^{1,p(x)}(\Omega)$, and $\beta_n \to \beta$. Let $\alpha_n = \left(-\Delta_{p(x)}\right)^{-1}(\beta_n)$ and $\alpha = \left(-\Delta_{p(x)}\right)^{-1}(\beta)$, then it's obvious that $-\Delta_{p(x)}(\alpha_n) = \beta_n$ and $-\Delta_{p(x)}(\alpha) = \beta$, thus $\{\alpha_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. If $\alpha_n \to \alpha^*$, then

$$lim_{n\to\infty} \left\langle -\Delta_{p(x)}(\alpha_n) - \left(-\Delta_{p(x)}\right)\left(\alpha^*\right), \alpha_n - \alpha^* \right\rangle = lim_{n\to\infty} \left\langle \beta_n, \alpha_n - \alpha^* \right\rangle = 0,$$

considering that $-\Delta_{p(x)}$ is of type $S_+$ and $\alpha_n \to \alpha^*$, we conclude that $\alpha_n \to \alpha$. Therefore $\left(-\Delta_{p(x)}\right)^{-1}$ is continuous.

Proof of (v). Due to the continuity of $\left(-\Delta_{p(x)}\right)^{-1}$, it is clear that it is bounded. Now we show that $\left(-\Delta_{p(x)}\right)^{-1}$ is of type $S_+$. Suppose $\beta_n \in \widetilde{W}_0^{1,p(x)}(\Omega)$ and $\beta_n \rightharpoonup \beta$, so there exist $\alpha_n, \alpha \in W_0^{1,p(x)}(\Omega)$ such that $\left(-\Delta_{p(x)}\right)^{-1}\beta_n = \alpha_n$ and $\left(-\Delta_{p(x)}\right)^{-1}\beta = \alpha$ and $\alpha_n \rightharpoonup \alpha$, that is, $\left(-\Delta_{p(x)}\right)\alpha_n = \beta_n$ and $\left(-\Delta_{p(x)}\right)\alpha = \beta$. Further, assume that $limsup_{n\to\infty} \left\langle \left(-\Delta_{p(x)}\right)^{-1}\beta_n, \beta_n - \beta \right\rangle \leq 0$, so we get

$$limsup_{n\to\infty} \left\langle \left(\alpha_n, -\Delta_{p(x)}(\alpha_n) - \left(-\Delta_{p(x)}\beta\right)\right)\right\rangle \leq 0.$$

Since $-\Delta_{p(x)}$ is of type $S_+$, we conclude that $\alpha_n \to \alpha$. By the continuity of $\left(-\Delta_{p(x)}\right)^{-1}$, we get $\alpha_n \to \alpha$. □

We close this section with recalling some important fixed point theorems.

**Theorem 2.11** (Classical Schaefer's fixed point theorem [37]) Let $T : X \to X$ a continuous function which is compact on each bounded subset of $X$. Then, either $\jmath = \lambda T \jmath$ has a solution for $\lambda = 1$, or the set of all such solutions $\jmath$, $(0 < \lambda < 1)$ are unbounded.

**Theorem 2.12** (Schauder's fixed point theorem [2, 3]) *Let $D$ be a closed and convex subset of a normed linear space $X$. Then every compact, continuous map $T : D \longrightarrow D$ has at least one fixed point.*

## 3 Main Results

Let $I_1$ be the embedding of $W_0^{1,p(x)}(\Omega)$ into $L^{\eta(x)}(\Omega)$ and $I_2$ the embedding of $L^{\delta(x)}(\Omega)$ into $\widetilde{W}_0^{1,p(x)}(\Omega)$ which is the dual space of $W_0^{1,p(x)}(\Omega)$. We recall that $\delta(x)$ is the conjugate of $\eta(x)$. Let $F : L^{\eta(x)}(\Omega) \longrightarrow L^{\delta(x)}(\Omega)$ be the Nemytskii operator defined by $F\jmath(x) = f(x, \jmath(x))$ and $\overline{F} : W_0^{1,p(x)}(\Omega) \longrightarrow \widetilde{W}_0^{1,p(x)}(\Omega)$, defined by

$$\langle \overline{F}\jmath, v\rangle = \int_\Omega f(x, \jmath) v \, dx, \quad \forall \jmath, v \in W_0^{1,p(x)}(\Omega),$$

where

$$\overline{F} : W_0^{1,p(x)}(\Omega) \xrightarrow{I_1} L^{\eta(x)}(\Omega) \xrightarrow{F} L^{\delta}(\Omega) \xrightarrow{I_2} \widetilde{W}_0^{1,p(x)}(\Omega).$$

Let

$$T : W_0^{1,p(x)}(\Omega) \longrightarrow W_0^{1,p(x)}(\Omega)$$

be the operator given by

$$T\jmath = (-\Delta_{p(x)})^{-1}\overline{F}\jmath.$$

We transfer problem (1.1) into a fixed point problem of some operators. One can easily see that $u$ is a solution of problem (1.1) if only if

$$\jmath = (-\Delta_{p(x)})^{-1}\overline{F}\jmath.$$

Let $\vartheta_0$ be such that $\|\jmath\|_{p(x)} \le \vartheta_0 \|\nabla \jmath\|_{p(x)}$, $\jmath \in W_0^{1,p(x)}(\Omega)$ and $\vartheta_1$ the constant of the embedding of $L^{p(x)}(\Omega)$ into $L^{\eta(x)}(\Omega)$. We make use of the following assumptions:

$A_1$. $|f(x,s)| \leq \zeta(x) + c|s|^{\eta(x)-1}$ where $c > 0$, $\eta(x) \in C_+(\overline{\Omega})$, $\eta(x) < p^*(x)$, $\forall (x,s) \in \Omega \times \mathbb{R}$, with $1 < \wp^- \leq \wp(x) \leq \wp^+ < p^-$, $\frac{1}{\wp(x)} + \frac{1}{\delta(x)} = 1$, $\zeta(x) \in \left(L^{\delta(x)}(\Omega) \cap L^{p'(x)}(\Omega)\right)$,

$A_2$. $0 < c < \dfrac{1}{2\kappa^{p^+-1}(\vartheta_0\vartheta_1)^{p^-}}$, and $\|\zeta\|_{p'(x)} < \dfrac{1}{2\vartheta_0\kappa^{p^+-1}} - c\vartheta_0^{p^--1}\vartheta_1^{p^-}$, for some $\kappa$ in $[0\ 1]$, where $p'(x)$ is the conjugate of $p(x)$.

**Theorem 3.1** *Under the assumptions $A_1$ and $A_2$, problem (1.1) admits at least a non-trivial solution in $W_0^{1,p(x)}(\Omega)$.*

***Proof*** To complete the proof, we split the process into three parts: the operator $T$ is well defined, $T$ is compact and the set $B = \left\{ \jmath \in W_0^{1,p(x)}(\Omega) : \ \jmath = \kappa T\jmath,\ \kappa \in [0\ 1] \right\}$ is bounded.

(i) $T$ is well defined. It is clear that the operator $T : W_0^{1,p(x)}(\Omega) \overset{\overline{F}}{\longrightarrow} \widetilde{W}_0^{1,p(x)}(\Omega) \overset{(-\Delta_{p(x)})^{-1}}{\longrightarrow} W_0^{1,p(x)}(\Omega)$ is well defined;

(ii) $T$ is compact. Let $(\jmath_n)$ be a bounded sequence in the reflexive space $W_0^{1,p(x)}(\Omega)$. Then, there exist $\jmath_0$ and a subsequence which we also denote $(\jmath_n)$ that $\jmath_n$ converge weakly to $\jmath_0$ in $W_0^{1,p(x)}(\Omega)$. By the strong continuity of $\overline{F}$ (Lemma 2.7, [28]), we have $\overline{F}(\jmath_n) \longrightarrow \overline{F}(\jmath_0)$ in $\widetilde{W}_0^{1,p(x)}(\Omega)$. By the continuity of the operator $(-\Delta_{p(x)})^{-1}$ (according to the fourth assertion in Proposition 2.10), we have $(-\Delta_{p(x)})^{-1}\overline{F}(\jmath_n) \longrightarrow (-\Delta_{p(x)})^{-1}\overline{F}(\jmath_0)$, that is, $T(\jmath_n) \longrightarrow T(\jmath_0)$;

(iii) The set $B$ is bounded. In fact, we prove that for $\jmath \in B$, there exists $R > 0$ with $\|\nabla \jmath\|_{p(x)} \leq R$. Considering $\jmath \in B$, we have two cases.

- If $\|\nabla \jmath\| \leq 1$, then $B$ is bounded.
- If $\|\nabla \jmath\| > 1$, then by using modular's properties, we proceed as follows: for $\jmath = \kappa T\jmath$, and assuming $\kappa \neq 0$, we have $\dfrac{\jmath}{\kappa} = T\jmath = (-\Delta_{p(x)})^{-1}\overline{F}\jmath$. Therefore,

$$-\Delta_{p(x)}\left(\frac{\jmath}{\kappa}\right) = \overline{F}\jmath,$$

and consequently,

$$\int_\Omega |\nabla\left(\frac{\jmath}{\kappa}\right)|^{p(x)-2}\nabla\left(\frac{\jmath}{\kappa}\right)\nabla v\,dx = \int_\Omega f(x,\jmath)v\,dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

Using Hölder–type inequality, propositions 2.3–2.8 and the embedding of $L^{p(x)}(\Omega)$ into $L^{\eta(x)}(\Omega)$, we have

$$\frac{1}{\kappa^{p^+}}\sigma_{p(x)}(\nabla\jmath) \leq \sigma_{p(x)}\nabla\left(\frac{\jmath}{\kappa}\right) \leq \frac{1}{\kappa}\int_\Omega f(x,\jmath(x))\jmath(x)\,dx.$$

Furthermore, we get

$$\begin{aligned}
\sigma_{p(x)}(\nabla \jmath) &\le \kappa^{p^+-1}\int_\Omega f(x,\jmath(x))\jmath(x)\,dx\\
&\le \kappa^{p^+-1}\Big[\int_\Omega |\zeta(x)\jmath(x)| + c\int_\Omega |\jmath(x)|^{\eta(x)}\,dx\Big],\\
&\le \kappa^{p^+-1}\Big[2|\zeta|_{p'(x)}\|\jmath\|_{p(x)} + c\sigma_{\eta(x)}(\jmath)\\
&\le \kappa^{p^+-1}\Big[2|\zeta|_{p'(x)}\vartheta_0\|\nabla\jmath\|_{p(x)} + c\sigma_{\eta(x)}(\jmath)\\
&\le \kappa^{p^+-1}\Big[2|\zeta|_{p'(x)}\vartheta_0\left(\sigma_{p(x)}(\nabla\jmath)+1\right) + c\sigma_{\eta(x)}(\jmath).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left(1-2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)}\right)\sigma_{p(x)}(\nabla\jmath) \le 2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)} + \kappa^{p^+-1}c\sigma_{\eta(x)}(\jmath),\\
&\left(1-2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)}\right)\sigma_{p(x)}(\nabla\jmath) \le 2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)} + \kappa^{p^+-1}c\left(\|\jmath\|_{\eta(x)}^{\eta^+} + \|\jmath\|_{\eta(x)}^{\eta^-}\right),\\
&\left(1-2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)}\right)\sigma_{p(x)}(\nabla\jmath) \le 2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)} + 2\kappa^{p^+-1}c(\vartheta_0\vartheta_1)^{p^-}\|\nabla\jmath\|_{p(x)}^{p^-},\\
&\left(1-2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)}\right)\|\nabla\jmath\|_{p(x)}^{p^-}\\
&\quad\le 2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)} + 2\kappa^{p^+-1}c(\vartheta_0\vartheta_1)^{p^-}\|\nabla\jmath\|_{p(x)}^{p^-},\\
&\left(1-2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)} - 2\kappa^{p^+-1}c(\vartheta_0\vartheta_1)^{p^-}\right)\|\nabla\jmath\|_{p(x)}^{p^-} \le 2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)}.
\end{aligned}$$

From assumption $A_2$; $\|\zeta\|_{p'(x)} < \frac{1}{2v_0\kappa^{p^+-1}} - cv_0{}^{p^--1}v_1{}^{p^-}$, hence by multiplying the both sides of this inequality in $2v_0\kappa^{p^+-1}$ we get

$$2v_0\kappa^{p^+-1}\|\zeta\|_{p'(x)} < 1 - 2v_0\kappa^{p^+-1}cv_0{}^{p^--1}v_1{}^{p^-},$$

Thus

$$1-2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)} - 2\kappa^{p^+-1}c(\vartheta_0\vartheta_1)^{p^-} > 0,$$

and

$$\|\nabla\jmath\|_{p(x)}^{p^-} \le \frac{2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)}}{1-2\kappa^{p^+-1}\vartheta_0|\zeta|_{p'(x)} - 2\kappa^{p^+-1}c(\vartheta_0\vartheta_1)^{p^-}}.$$

That is, $B$ is bounded.

By Schaefer's fixed point theorem, the operator $T$ has a fixed point $\jmath$ which is the solution of (1.1). □

For our purpose, we consider the following associated problem to (1.1)

$$\begin{cases} \displaystyle\int_\Omega \|\nabla \jmath\|^{p(x)-2}\nabla \jmath \nabla v\, dx = \int_\Omega f(x,\jmath) v\, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega), \\ \jmath \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{3.1}$$

Our approach is based on the principle of the sub-super-solution for which the reader can consult [10, 24]. By the sub-solution and the super-solution of (3.1), where, $\vartheta, \nu \in W_0^{1,p(x)}(\Omega)$ satisfying

$$\int_\Omega |\nabla \vartheta|^{p(x)-2}\nabla \vartheta \nabla v\, dx \le \int_\Omega f(x,\vartheta) v\, dx,$$

and

$$\int_\Omega |\nabla \jmath|^{p(x)-2}\nabla \nu \nabla v\, dx \ge \int_\Omega f(x,\nu) v\, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega), v \ge 0.$$

We assume the following conditions:

$A_3$. There exist a subsolution $\vartheta \in W_0^{1,p(x)}(\Omega)$ and a supersolution $\nu \in W_0^{1,p(x)}(\Omega)$ of (3.1) with $\vartheta \le \nu$.

$A_4$. For fixed $x \in \Omega$, $f(x,s)$ is a nondecreasing function of $s$ for $\vartheta(x) \le s \le \nu(x)$.

**Theorem 3.2** *Under the assumptions $A_1$, $A_3$ and $A_4$, (1.1) has at least one solution.*

***Proof*** Consider the subset $D \subset W_0^{1,p(x)}(\Omega)$ defined as follow

$$D = \left\{ \jmath \in W_0^{1,p(x)}(\Omega) : \vartheta(x) \le \jmath(x) \le \nu(x), \ \forall x \in \Omega \right\},$$

and

$$T : D \longrightarrow W_0^{1,p(x)}(\Omega), \quad T\jmath = (-\Delta_{p(x)})^{-1}\overline{F}\jmath.$$

The proof will be completed in several steps.

Step 1: $D$ is convex. For $\jmath, w \in D$ and $t \in [0,1]$, it is clear to see that $t\jmath + (1-t)w \in D$.

Step 2: $D$ is closed. If $(\jmath_n)$ be a sequence in $D$ which converge to $\jmath_0$ in $W_0^{1,p(x)}(\Omega)$. Then, $\jmath_n \longrightarrow \jmath_0$ in $L^{p(x)}(\Omega)$. Therefore, there exists a subsequence also denoted $(\jmath_n) \in L^{p(x)}(\Omega)$ and $\psi \in L^{p(x)}(\Omega)$ with $\jmath_n(x) \longrightarrow \jmath_0(x)$ and $|\jmath_n(x)| \le \psi(x)$, $a.e., x \in \Omega$. Since $\vartheta(x) \le \jmath_n(x) \le \nu(x)$, passing to the limit when $n \to \infty$, we get $\vartheta(x) \le \jmath(x) \le \nu(x)$, and hence $\jmath \in D$.

Step 3: $T : D \xrightarrow{\overline{F}} \widetilde{W}_0^{1,p(x)}(\Omega) \xrightarrow{(-\Delta_{p(x)})^{-1}} W_0^{1,p(x)}(\Omega)$ is strongly continuous. Let $(\jmath_n) \subset D$ be a sequence such that converges weakly to $\jmath_0$ in $D$. By the strong continuity of $\overline{F}$, it follows that $\overline{F}(\jmath_n) \longrightarrow \overline{F}(\jmath_0)$ and the strong convergence of $T(\jmath_n) \longrightarrow T(\jmath_0)$ is ensured by the continuity of $(-\Delta_{p(x)})^{-1}$.

Step 4: $T$ is compact.
Step 5: $D$ is bounded. For $\jmath \in D$, we have $\|\jmath\| \leq \|\vartheta\| + \|\nu\|$.
Step 6: $T(D) \subset D$. Because, if $\jmath \in D$, then $\vartheta(x) \leq \jmath(x) \leq \nu(x),\ x \in \Omega$. Using assumptions $A_3$ and $A_4$ we have $f(x, \vartheta(x)) \leq f(x, \jmath(x)) \leq f(x, \nu(x))$. Therefore, for any $v \in W_0^{1,p(x)}(\Omega),\ v \geq 0$ we have

$$\int_\Omega f(x, \vartheta(x))v(x)\,dx \leq \int_\Omega f(x, \jmath(x))v(x)\,dx \leq \int_\Omega f(x, \nu(x))v(x)\,dx.$$

By the definition of $\vartheta$ and $\nu$ we can write

$$\begin{aligned}
\int_\Omega |\nabla\vartheta|^{p(x)-2}\nabla\vartheta\nabla v\,dx &\leq \int_\Omega f(x, \vartheta(x))v(x)\,dx \\
&\leq \int_\Omega f(x, \nu(x))v(x)\,dx \\
&\leq \int_\Omega |\nabla\nu|^{p(x)-2}\nabla\nu\nabla v\,dx,
\end{aligned}$$

that is,

$$\langle -\Delta_{p(x)}\vartheta, v\rangle \leq \langle \overline{F}\jmath, v\rangle \leq \langle -\Delta_{p(x)}\nu, v\rangle.$$

According to the maximum principle and the monotonicity of $(-\Delta_{p(x)})^{-1}$, we have

$$\vartheta \leq T\jmath \leq \nu,$$

which implies

$$\vartheta(x) \leq T\jmath(x) \leq \nu(x), \qquad \forall x \in \Omega.$$

By the Schauder's fixed point Theorem 2.12, $T$ has a fixed point which is a solution for (1.1). □

**Remark 3.3** If in Theorems 3.1 and 3.2, we assume that there exists $x_0 \in \Omega$ with $f(x_0, 0) \neq 0$, then the solution of problem (1.1) is non trivial.

For the purpose of support, we provide an example that is completely consistent with the theoretical findings.

**Example 3.4** Let $X = \mathbb{R}^2$, $\Omega = \{(x_1, x_2) : \frac{1}{9} < {x_1}^2 + {x_2}^2 < \frac{1}{4}\}$. Define $p : \overline{\Omega} \to \mathbb{R}$, by

$$p(x = (x_1, x_2)) := 1 + \frac{{x_2}^2}{{x_1}^2}.$$

Therefore, we have

$$p'(x = (x_1, x_2)) := \begin{cases} \infty, & x \in \Omega_1^p, \\ 1, & x \in \Omega_\infty^p, \\ \frac{p(x)}{p(x)-1}, & x \in \Omega, \end{cases} \tag{3.2}$$

where $\Omega_1^p = \{x \in \Omega : p(x) = 1\}$ and $\Omega_\infty^p = \{x \in \Omega : p(x) = \infty\}$, we can obtain; $p'(x := (x_1, x_2)) = 1 + \frac{x_1{}^2}{x_2{}^2}$.

Consider the following $p(x)$–Laplacian Dirichlet problem:

$$\begin{cases} -\Delta_{p(x)} \jmath = f(x, \jmath) = f(x_1, x_2, \jmath((x_1, x_2)) = \jmath(x_1, x_2) + 2x_1x_2, \\ \quad x = (x_1, x_2) \in \Omega, \\ \jmath = 0 \qquad\qquad x \in \partial\Omega. \end{cases} \tag{3.3}$$

Setting $\vartheta_1 = 1$, $\kappa = 1$ and $\vartheta_0 \geq \frac{\|\jmath\|_{p(x)}}{\|\nabla \jmath\|_{p(x)}}$, we obtain

$$0 < c < \frac{1}{2\vartheta_0{}^{p^-}}.$$

Therefore, condition $A_2$ of Theorem 3.1 is satisfied. Moreover, $A_1$ of Theorem 3.1 holds. Hence, it is easy to see that $f$ is a Carathéodory function and the conditions of Theorem 3.1 are satisfied. Therefore, we conclude that problem (3.3) admits at least a non–trivial solution in $W_0^{1,p(x)}(\Omega)$.

## 4 Conclusion

Despite its significance, the investigation of the existence of solutions for elliptic BVPs with the $p(x)$-Laplacian equation has received limited attention from interested scholars who have elaborated on this topic using various methodologies.

In this paper, we present some different conditions that allow using of variational and topological methods in the case of $p(x)$-Laplacian to prove the existence of solutions for elliptic BVPs with the $p(x)$-Laplacian equation. Main theorems are proven using fixed point theorems and the sub-supersolution approach. The paper presents key discoveries on the variable exponent Sobolev space, as well as basic features of the $p(x)$-Laplacian and Nemytskii operators. When compared to earlier findings, the assumptions made concerning the nonlinearity $f$ in (1.1) appear to be less restrictive given its structure.

It should be emphasised that the findings of this study are significant since they provide a novel approach for establishing the primary results, which can be used to examine solutions for Kirchhoff's non-local version of the $p$-Laplace equations in the future.

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## Declaration

**Conflict of interest** The authors declare that they have no Conflict of interest.



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