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On the ρ-Caputo Impulsive p-Laplacian Boundary Problem: An Existence Analysis

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Abstract

Due to the importance of some physical systems, in this paper, we aim to investigate a generalized impulsive ρ-Caputo differential equation equipped with a p-Laplacian operator. In fact, our problem is a generalization of fractional differential equations equipped with the integral boundary conditions, impulsive forms and p-Laplacian operators under the Nemytskii operators. In this direction, we prove some theorems on the existence property along with the uniqueness of solutions under the Nemytskii operator. More precisely, we use the Schauder's and Schaefer's fixed point theorems, along with the Banach contraction principle. In the sequel, two examples are provided to show the validity of the obtained results in practical.

Keywords Nemytskii operator · ρ-Caputo derivative · Impulsive differential equation · Boundary value problem · Banach contraction principle · p-Laplacian

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1 Introduction

In this paper, our focus is on addressing the concept of existence and uniqueness of solutions for a boundary value problem involving the following nonlinear non-

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symmetric ρ-Caputo fractional p-Laplacian impulsive differential equation

$$\begin{cases} {}^{\rho_2;C}_{\ \ \xi}\mathcal{D}^{\beta}_{K^-}\left(\varkappa(\xi)\psi_p\left({}^{\rho_1;C}_{\ \ \xi}\mathcal{D}^{\nu}_{a^+}\omega\right)\right)(\xi)+t(\xi)\psi_p\left(\omega(\xi)\right)=f\left(\xi,\omega(\xi)\right) \quad a<\xi<K, \\ \Delta\left(\omega\left(\xi_q\right)\right)=I^1_q\left(\omega\left(\xi_q\right)\right),\ \Delta\psi_p\left({}^{\rho_1;C}_{\ \ \xi}\mathcal{D}^{\nu}_{a^+}\omega\right)\left(\xi_q\right)=I^2_q\left(\omega\left(\xi_q\right)\right),\ q=1,2,\ldots,k, \\ \omega(a)=u_0+\lambda\,{}^{\rho_3}{}_{\xi}\mathcal{I}^{\gamma}_{a^+}\left(\eta(\xi)|\,\omega(\xi)|^{p-1}\right)\Big|^{p^\star-1}_{\xi=K},\ \ {}^{\rho_1;C}\mathcal{D}^{\nu}_{a^+}\omega(K)=u_1, \end{cases} \tag{1.1}$$

where $p, p^\star > 1$, $0 < \nu, \beta, \gamma \leq 1$, ψ_p is an operator of p-Laplacian type, i.e., $\psi_p(\xi) = |\,\xi\,|^{p-2}\,\xi$, $t, \varkappa, \eta \in C([a, K], \mathbb{R}^\star_+)$, $f \in C([a, K] \times \mathbb{R}, \mathbb{R})$, $\omega_0, \omega_1, \lambda \in \mathbb{R}$, and for $q = 1, 2, \ldots, k$, $i = 1, 2$ we take $I^i_q \in C(\mathbb{R}, \mathbb{R})$, $a = \xi_0 < \xi_1 < \cdots < \xi_q < \cdots < \xi_k < \xi_{k+1} = K$. Moreover, $\Delta\omega\left(\xi_q\right) = \omega\left(\xi^+_q\right) - \omega\left(\xi^-_q\right)$ such that $\omega\left(\xi^+_q\right)$ and $\omega\left(\xi^-_q\right)$ denote the right and left limits of $\omega(\xi)$ at $\xi = \xi_q$ $(q = 1, 2, \ldots, k)$, respectively and $\Delta\psi_p\left({}^{\rho_1;C}_{\ \ \xi}\mathcal{D}^{\nu}_{a^+}\omega\right)\left(\xi_q\right)$ has the same meaning for $\psi_p\left({}^{\rho_1;C}_{\ \ \xi}\mathcal{D}^{\nu}_{a^+}\omega\right)\left(\xi_q\right)$.

During the past decade, theory of differential equations involving fractional derivatives of non-integer order has undergone significant development and found numerous applications in many areas such as biology, physics, rheology, mechanics, electricity, signal and image processing, control theory, aerodynamics. This field has attracted the attention of many researchers constantly to study fractional differential equations. Moreover, the fractional-order models are considered as powerful mathematical structures than the classical-order models. For more details and applications about fractional calculus, see [1–3] and references therein.

Very recently, Almeida, in [4], gave a generalized version of the Caputo fractional derivative with some interesting properties. For a special case of the increasing function ρ, one can realize that the ρ–Caputo fractional derivative can be reduced to some well-known classical kinds of the Caputo fractional operator [5–7]. Those models that employ generalized fractional derivatives could potentially give the greater accuracy than models that rely on classical derivatives.

As the operators of p-Laplacian type have natural applications in many areas of science, and are commonly used in mathematical modeling of physical and natural phenomena, such as turbulent filtration in porous media, blood flow problems, rheology, and viscoelasticity, it is important to study the fractional p-Laplacian differential equations. As it is well-known, the formulation of an ordinary operator of p-Laplacian type was done by Leibenson in 1983 [8]. In the context of the p-Laplacian differential equations, the p-Laplacian operator ψ_p is often used to model the nonlinearity in a differential equation, and the behavior of solutions of the given differential equation is influenced by the specific value of the parameter p. More precisely, these operators exhibit different behaviors depending on the value of the parameter p. Here are some characteristics based on the range of p:

1. For $p > 2$:

   - Superlinear Growth: As p increases, the function exhibits superlinear growth. This means that the function grows faster than a linear function.

- Dominance of $|\ \xi\ |^p$: The term $|\ \xi\ |^p$ dominates the behavior contributing to the superlinear growth.

2. For $1 < p < 2$:

   - Sublinear Growth: As p decreases but stays above 1, the operator exhibits sublinear growth. This means that the opertor grows more slowly than a linear function.
   - Dominance of $|\ \xi\ |$: In this range, the term $|\ \xi\ |$ dominates the behavior contributing to the sublinear growth.

3. For $p = 2$: Linear Growth: When p is exactly 2, the operator simplifies to $\psi_p(\xi) = \xi$, which shows a linear growth.

It's important to note that these observations are based on the form of the operator ψ_p itself. There are various researches of fractional boundary value problems with p-Laplacian operators published recently such as [9–19].

Numerous processes and phenomena in the real world undergo brief external influences during their evolution. However, these brief durations are insignificant when we compare to the total duration of the processes and phenomena under study. As a result, these external influences are often considered "instantaneous" in nature, manifesting in the form of impulses. Moreover, differential equations that incorporate impulsive effects are prevalent in various real-world phenomena and are utilized to model processes that involve sudden and discontinuous jumps. Many readers interested in gaining a comprehensive understanding of the fundamental theory and practical applications of impulsive differential equations are encouraged to refer to several references in the literature such as [20–22]. In particular, the existence of solutions to impulsive fractional differential equations and other types of fractional differential equations has been examined using different tools and approaches, including topological degree theory, fixed point theory, upper and lower solution methods, and monotone iterative techniques. See, for example, [23–28] and the references therein.

In 2020, Linda et al. [29] studied the existence of weak solutions for the following form of a p-Laplacian impulsive differential equation equipped with boundary conditions, employing the variational technique and theory of critical point, given by

$$\begin{cases} -\left(\varkappa(\xi)\psi_p\left(\omega'\right)\right)'(\xi) + t(\xi)\psi_p\left(\omega(\xi)\right) = f(\xi, \omega(\xi)) & 0 < \xi < K, \\ \Delta\psi_p\left(\omega'\right)\left(\xi_q\right) = I_q\left(\omega\left(\xi_q\right)\right), & q = 1, 2, \ldots, k, \\ \omega(0) = \omega(K) = 0, \end{cases}$$

where $p > 1$, ψ_p is an operator of p-Laplacian type, $t, \varkappa \in L^\infty([0, K])$, $f \in C([a, K] \times \mathbb{R}, \mathbb{R})$, for $q = 1, 2, \ldots, k$, $I_q \in C(\mathbb{R}, \mathbb{R})$, $0 = \xi_0 < \xi_1 < \cdots < \xi_q < \cdots < \xi_k < \xi_{k+1} = K$. Also, $\Delta\left(\psi_p\left(\omega'(\xi_q)\right)\right) = \psi_p\left(\omega'\left(\xi_q^+\right)\right) - \psi_p\left(\omega'\left(\xi_q^-\right)\right)$ so that $\omega\left(\xi_q^+\right)$ and $\omega\left(\xi_q^-\right)$ denote the right and left limits of $\omega(\xi)$ at $\xi = \xi_q$ ($q = 1, 2, \ldots, k$), respectively.

In [30], Liu, Lu and Szántó considered the solvability of a new form of impulsive fractional differential equation given by

$$\begin{cases} {}^{C}_{\xi}\mathcal{D}^{\beta}_{0^+}\left(\psi_p\left({}^{C}_{\xi}\mathcal{D}^{\nu}_{0^+}\omega\right)\right)(\xi) = f(\xi,\omega(\xi)) & 0<\xi<1, \\ \Delta\left(\omega\left(\xi_q\right)\right) = I_q\left(\omega\left(\xi_q\right)\right), \Delta\psi_p\left({}^{C}_{\xi}\mathcal{D}^{\nu}_{+}\omega\right)\left(\xi_q\right) = b_q, & q=1,2,\ldots,k, \\ \omega(0)=u_0, {}^{C}\mathcal{D}^{\nu}_{0^+}\omega(0)=u_1, \end{cases}$$

where $p>1$, $0<\nu,\beta,\gamma\le 1$, $1<\nu+\beta,\gamma\le 2$, ψ_p is an operator of p-Laplacian type, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$, $\omega_0,\omega_1,\lambda\in\mathbb{R}$, for $q=1,2,\ldots,k$, $i=1,2$, $I_q\in C(\mathbb{R},\mathbb{R})$, $0=\xi_0<\xi_1<\cdots<\xi_q<\cdots<\xi_k<\xi_{k+1}=1$. Also, $\Delta\omega\left(\xi_q\right)=\omega\left(\xi_q^+\right)-\omega\left(\xi_q^-\right)$ such that $\omega\left(\xi_q^+\right)$ and $\omega\left(\xi_q^-\right)$ denote the right and left limits of $\omega(\xi)$ at $\xi=\xi_q$ $(q=1,2,\ldots,k)$, respectively and $\Delta\psi_p\left({}^{C}_{\xi}\mathcal{D}^{\nu}_{a^+}\omega\right)\left(\xi_q\right)$ has a similar meaning for $\psi_p\left({}^{C}_{\xi}\mathcal{D}^{\nu}_{a^+}\omega\right)\left(\xi_q\right)$. By using the Banach contraction principle, they obtained some results on the existence and uniqueness of solutions for the given model.

The main contribution and novelty of this paper is that we try to investigate the existence results for a generalized fractional differential equation equipped with the integral boundary conditions, impulsive forms and p-Laplacian operators under the Nemytskii operators simultaneously. In this direction, we get help from the well-known fixed point theorems.

The structure of this paper is as follows. Section 2 offers a comprehensive background on fractional calculus and fixed point theory. In Section 3, we establish some theorems on the existence and uniqueness of solutions for the given p-Laplacian impulsive fractional boundary value problem (1.1) by using the Schauder's and Schaefer's fixed point theorems, as well as the principle of the Banach contraction mapping. Lastly, Section 4 presents two illustrative examples that highlight the validity and significance of our main results.

2 Preliminaries

This section of the paper deals with some preliminaries on fractional calculus and some definitions that are essential for the proofs presented later.

Throughout this paper, let $Q=[a,K]\subset(0,\infty)$ and $Q_0=[a,\xi_1]$, $Q_1=(\xi_1,\xi_2]$, $\ldots$, $Q_{k-1}=\left(\xi_{k-1},\xi_k\right]$, $Q_k=(\xi_k,K]$, $n=[\nu]+1$, where $[\nu]$ is the largest integer less than or equal to ν.

1. The Banach space of continuous functions h on Q is denoted by $C(Q)$ and is equipped with the norm

$$\|h\|_C=\max_{\upsilon\in Q}|h(\upsilon)|.$$

2. $AC(Q)$ is the space of absolutely continuous functions on Q, and $C^n(Q)$ is the set of all functions on Q with n continuous derivatives.

3. The set of all Lebesgue integrable functions on (a, K) is denoted by $L^p(a, K)$.

2.1 Fractional Calculus that Involves a Function ρ

Here, we recall the fundamental definitions and lemmas from fractional calculus theory, which involve an increasing function ρ. It is straightforward to observe that by selecting appropriate functions ρ, we can obtain several commonly used fractional operators, including the Riemann-Liouville (RL), Caputo, Hadamard, Katugampola, and Erdélyi-Kober fractional derivatives. Several works in the literature, such as [31–33], have studied and used these fractional operators.

Definition 2.1 (*PC-Function space*) Let $PC(Q, \mathbb{R}) = \{\omega : Q \to \mathbb{R} : \omega \in C\left(Q_q, \mathbb{R}\right)$ for $q = 1, 2, \ldots, k$ and there exist $\omega\left(\xi_q^+\right)$ and $\omega\left(\xi_q^-\right)$ at $\xi = \xi_q$ with $\omega\left(\xi_q^-\right) = \omega\left(\xi_q\right)\}$. Then $PC(Q, \mathbb{R})$ is a Banach space equipped with the norm $\|\omega\| = \sup_{\xi\in Q} |\omega(\xi)|$.

Definition 2.2 Let ρ be a strictly increasing and n-times differentiable function on Q. Then, $AC_\rho^n(Q) = \left\{\omega : Q \to \mathbb{R} \text{ and } \delta_\rho^{[n-1]}\omega \in AC(Q),\ \delta_\rho^{[n-1]}\omega = \left(\frac{1}{\rho'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\omega\right\}$ denotes a Banach space of n-times absolutely continuous functions with respect to the strictly increasing differentiable function ρ.

Definition 2.3 (Heaviside function H) The Heaviside function is given by

$$H(\xi) = \begin{cases} 1 & \text{if } \xi \geq 0 \\ 0 & \text{o.w.} \end{cases} \tag{2.1}$$

Proposition 2.4 *Let ω be a piecewise continuous function ($\omega \in PC(Q, \mathbb{R})$), and let $t_1, t_2, \ldots, t_k$ be the fixed moments of impulsive effect for $q = 1, 2, \ldots, k$. Denote $\varrho^q = \omega(t_k^+) - \omega(t_k^-)$ as the magnitude and direction of the impulsive effect at t_k. Then ω can be formulated as the sum of a continuous function g and a series of Heaviside functions, i.e.,*

$$\omega(\xi) = g(\xi) + \sum_{j=0}^{q} \varrho^j H(\xi - t_j), \tag{2.2}$$

such that $\varrho^0 = 0$.

Now, we recall the definitions of the ρ-Riemann-Liouville and ρ-Caputo fractional integrals and derivatives [4, 34].

Definition 2.5 [4, 34] Let $\nu > 0$ and $\rho : Q \longrightarrow \mathbb{R}$ be a strictly increasing differentiable function with $\rho'(\xi) \neq 0$ for all $\xi \in$ Q. The left and right $\rho-$Riemann-Liouville (ρ-RL) fractional integrals of order ν for an integrable function $\omega : Q \longrightarrow \mathbb{R}$ with respect to another function ρ are defined by

$$^{\rho}_{\xi}\mathcal{I}^{\nu}_{a^+}\omega(\xi) = \frac{1}{\Gamma(\nu)} \int_a^\xi \rho'(t)(\rho(\xi) - \rho(t))^{\nu-1}\omega(t)\mathrm{d}t,$$

$$ {}^{\rho}_{\xi}\mathcal{I}^{\nu}_{K-}\omega(\xi) = \frac{1}{\Gamma(\nu)}\int_{\xi}^{K} \rho'(t)(\rho(t)-\rho(\xi))^{\nu-1}\omega(t)\mathrm{d}t, $$

where Γ is the Gamma function.

Definition 2.6 [4, 34] Let $\rho \in C^{n}(\mathrm{Q},\mathbb{R})$ be a function where ρ is strictly increasing and $\rho'(\xi) \neq 0$ for all $\xi \in \mathrm{Q}$. The left and right $\rho-$Riemann-Liouville (ρ-RL) fractional derivatives of order $\nu \in (n-1,n)$ for a function $\omega : Q \to \mathbb{R}$ with respect to another function ρ are defined by

$$ {}^{\rho}_{\xi}\mathcal{D}^{\nu}_{a+}\omega(\xi) = \left(\frac{1}{\rho'(\xi)}\frac{d}{d\xi}\right)^{n} {}^{\rho}_{\xi}\mathcal{I}^{n-\nu}_{a+}\omega(\xi), \xi > a, $$
$$ {}^{\rho}_{\xi}\mathcal{D}^{\nu}_{K-}\omega(\xi) = \left(\frac{-1}{\rho'(\xi)}\frac{d}{d\xi}\right)^{n} {}^{\rho}_{\xi}\mathcal{I}^{n-\nu}_{K-}\omega(\xi), \xi < K, $$

provided that the right-hand side integrals are defined on Q.

Definition 2.7 [4, 34] For each $n \in \mathbb{N}$, let ρ and ω be two functions in $C^{n}(Q,\mathbb{R})$, where ρ is an increasing function with $\rho'(\xi) \neq 0$ for all $\xi \in Q$. The left and right ρ-Caputo (ρ-C) fractional derivatives of order ν for the function $\omega : Q \to \mathbb{R}$ with respect to another function ρ are defined by

$$ {}^{\rho;C}_{\xi}\mathcal{D}^{\nu}_{a+}\omega(\xi) = {}^{\rho}_{\xi}\mathcal{I}^{n-\nu}_{a+}\delta^{[n]}_{\rho}\omega(\xi), $$
$$ {}^{\rho;C}_{\xi}\mathcal{D}^{\nu}_{K-}\omega(\xi) = {}^{\rho}_{\xi}\mathcal{I}^{n-\nu}_{K-}(-1)^{n}\delta^{[n]}_{\rho}\omega(\xi), $$

where $n = [\nu]+1$ for $\nu \notin \mathbb{N}$, $n = \nu$ for $\nu \in \mathbb{N}$, and $\delta^{[n]}_{\rho}\omega(\xi) = \left(\frac{1}{\rho'(\xi)}\frac{d}{d\xi}\right)^{n}\omega(\xi)$.

Lemma 2.8 *The left and right ρ-Caputo fractional derivatives for the function $\omega \in C^{n}(Q)$ of order ν with respect to ρ can also be formulated as*

$$ {}^{\rho;C}_{\xi}\mathcal{D}^{\nu}_{a+}\omega(\xi) = {}^{\rho}_{\xi}\mathcal{D}^{\nu}_{a+}\left[\omega(\xi) - \sum_{q=0}^{n-1}\frac{\delta^{[q]}_{\rho}\omega(\xi)}{q!}|_{a}\,(\rho(\xi)-\rho(a))^{q}\right], \tag{2.3} $$

$$ {}^{\rho;C}_{\xi}\mathcal{D}^{\nu}_{K-}\omega(\xi) = {}^{\rho}_{\xi}\mathcal{D}^{\nu}_{K-}\left[\omega(\xi) - \sum_{q=0}^{n-1}(-1)^{q}\frac{\delta^{[q]}_{\rho}\omega(\xi)}{q!}|_{a}\,(\rho(K)-\rho(\xi))^{q}\right]. \tag{2.4} $$

Lemma 2.9 [35] *Assume that $\psi_p : \mathbb{R} \to \mathbb{R}$ is a p-Laplacian operator as $\psi_p(\upsilon) = |\upsilon|^{p-2}\upsilon$ for each $\upsilon \in \mathbb{R}$. Then $\frac{d}{d\upsilon}\psi_p(\upsilon) = (p-1)|\upsilon|^{p-2}(\upsilon \neq 0$ if $1 < p < 2)$. The following items are some fundamental properties of this operator.*

1. *The p-Laplacian operator ψ_p is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and its inverse is $\psi_{p^{\star}}(\upsilon) = |\upsilon|^{p^{\star}-2}\upsilon$ with $p^{\star} = \frac{p}{p-1}$.*

2. *Let* $1 < p < 2$, $\upsilon\zeta > 0$, *and* $|\upsilon|, |\zeta| \geq k > 0$. *Then,*

$$\left|\psi_p(\upsilon) - \psi_p(\zeta)\right| \leq (p-1)k^{p-2}|\upsilon - \zeta|.$$

3. *Let* $p \geq 2$ *and* $|\upsilon|, |\zeta| \leq M$. *Then,*

$$\left|\psi_p(\upsilon) - \psi_p(\zeta)\right| \leq (p-1)M^{p-2}|\upsilon - \zeta|.$$

Lemma 2.10 ([36], The Arzelá-Ascoli theorem of PC-type) *Let* $\Theta \subset PC(Q, \mathbb{R})$. *Then*

1. Θ *is a subset of* $PC(Q, \mathbb{R})$ *that is uniformly bounded;*
2. *If* Θ *is equicontinuous in* Q_q *for all* $q = 0, 1, 2, \ldots, k$, *then* Θ *is a subset of* $PC(Q, \mathbb{R})$ *and hence is relatively compact.*

2.2 Fixed Point Theorems

Two important theorems in fixed-point theory are given below.

Definition 2.11 A completely continuous operator preserves continuity and maps bounded sets to precompact sets.

Theorem 2.12 (The Schauder's fixed point theorem) *Let* $\mathcal{X}$ *be a Banach space and* $\Theta \subset \mathcal{X}$ *be an open, bounded subset containing a point* $\theta \in \Theta$. *Let* $\mathcal{L} : \Theta \to \mathcal{X}$ *be a completely continuous operator, where* $|\mathcal{L}\omega| \leq |\omega|$ *for all* $\omega \in \partial\Theta$. *Then there exists a fixed point of* $\mathcal{L}$ *in* Θ.

Theorem 2.13 (The Schaefer's fixed point theorem) *Consider a Banach space* $\mathcal{X}$ *and a completely continuous operator* $\mathcal{L} : \mathcal{X} \to \mathcal{X}$. *If the set*

$$E(\mathcal{L}) = \{\omega \in \mathcal{X} : \omega = \sigma\mathcal{L}\omega \text{ for } \sigma \in [0, 1]\},$$

is bounded, then $\mathcal{L}$ *has at least a fixed point.*

3 Main Results

This section of the paper deals with the investigation of the existence of solution and uniqueness results for the p-Laplacian impulsive fractional boundary value problem (1.1), which has an integral representation including the inverse of a given Nemytskii operator.

3.1 Fractional Functional Differential Equations and Integral Structure

Before starting and proving the main results, we shall provide the following lemmas.

Lemma 3.1 *Let $\nu \in (0,1)$ and $w \in C(Q,\mathbb{R})$. Then, the linear initial value problem*

$$\begin{cases} {}^{\rho;C}_{\ \ \xi}\mathcal{D}^{\nu}_{a^+}\omega(\xi) = w(\xi) & a < \xi < K, \\ \omega(\hat{a}) = \omega_a, & \hat{a} > a, \end{cases} \tag{3.1}$$

has a solution $\omega \in C(Q,\mathbb{R})$ given by the following integral equation

$$\omega(\xi) = \omega_a - \frac{1}{\Gamma(\nu)}\int_a^{\hat{a}} \rho'(t)(\rho(\hat{a}) - \rho_1(t))^{\nu-1}h(t)dt + \frac{1}{\Gamma(\nu)}\int_a^{\xi} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt. \tag{3.2}$$

Lemma 3.2 *Let $\beta \in (0,1)$ and $\varpi \in C(Q,R)$. Then, the linear initial value problem*

$$\begin{cases} {}^{\rho;C}_{\ \ \xi}\mathcal{D}^{\beta}_{K^-}\zeta(\xi) = \varpi(\xi) & a < \xi < K, \\ \zeta(\hat{K}) = y_K, & \hat{K} < K, \end{cases} \tag{3.3}$$

has a solution $\zeta \in C(Q,R)$ given by the following integral equation

$$\zeta(\xi) = \omega_K - \frac{1}{\Gamma(\beta)}\int_{\hat{K}}^{K} \rho'(t)(\rho_1(t) - \rho(\hat{K}))^{\beta-1}\varpi(t)dt + \frac{1}{\Gamma(\beta)}\int_{\xi}^{K} \rho_1'(t)(\rho_1(t) - \rho_1(\xi))^{\beta-1}\varpi(t)dt. \tag{3.4}$$

Lemma 3.3 *Suppose that $\nu \in (0,1]$ and $h : Q \to \mathbb{R}$ is continuous. Then, $\omega \in PC(Q,\mathbb{R})$ is a solution of the fractional integral equation*

$$\omega(\xi) = \begin{cases} \omega_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1}\eta(t)\,|\omega(t)|^{p-1}\,\mathrm{d}t \right|^{p^\star-1} \\ \quad + \frac{1}{\Gamma(\nu)}\int_a^{\xi} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)\mathrm{ds} \\ \quad + \sum_{j=0}^{q} I_j^1\left(\omega\left(\xi_j\right)\right) H(\xi - t_j), \quad \text{for } \forall \xi \in Q, q = 0,1,2,\ldots,k, \end{cases} \tag{3.5}$$

if and only if ω is a solution of the linear impulsive problem

$$\begin{cases} {}^{\rho_1;C}_{\quad\xi}\mathcal{D}^{\nu}_{a^+}\omega(\xi) = h(t) & a < \xi < K, \\ \Delta\left(\omega\left(\xi_q\right)\right) = I^1_q\left(\omega\left(\xi_q\right)\right), & q = 1, 2, \ldots, k \\ \omega(a) = \omega_0 + \lambda \left|{}^{\rho_3}{}_{\xi}\mathcal{I}^{\gamma}_{a^+}\eta(\xi) \mid \omega(\xi)|^{p-1}\right|^{p^\star-1}_{\xi=K}, \end{cases} \tag{3.6}$$

where H is the Heavside function and $I^1_0 = 0$.

Proof Let $h \in C(Q, \mathbb{R})$. Assume that $\omega(\xi)$ is a solution of the linear impulsive problem (3.6). If $\xi \in [a, \xi_1]$, then

$${}^{\rho_1;C}_{\quad\xi}\mathcal{D}^{\nu}_{a^+}\omega(\xi) = h(\xi). \tag{3.7}$$

By applying Lemma 3.1, we get

$$\omega(\xi) = \omega(a) + \frac{1}{\Gamma(\nu)}\int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt.$$

If $\xi \in (t_1, t_2]$, then

$${}^{\rho_1;C}_{\quad\xi}\mathcal{D}^{\nu}_{a^+}\omega(\xi) = h(\xi) \text{ with } \Delta\omega\left(\xi_1\right) = I^1_1\left(\omega\left(\xi_1\right)\right).$$

In this case, Lemma 3.1 implies that

$$\begin{aligned} \omega(\xi) &= \omega(\xi_1^+) - \frac{1}{\Gamma(\nu)}\int_a^{t_1} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt \\ &\quad + \frac{1}{\Gamma(\nu)}\int_a^{\xi} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt \\ &= \omega\left(\xi_1^-\right) + I^1_1\left(\omega\left(\xi_1\right)\right) - \frac{1}{\Gamma(\nu)}\int_a^{t_1} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt \\ &\quad + \frac{1}{\Gamma(\nu)}\int_a^{\xi} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt \\ &= \omega(a) + I^1_1\left(\omega\left(\xi_1\right)\right) + \frac{1}{\Gamma(\nu)}\int_a^{\xi} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt. \end{aligned}$$

If $\xi \in (\xi_2, \xi_3]$, then by using Lemma 3.1 one more step, we get

$$\begin{aligned} \omega(\xi) &= \omega(\xi_2^+) - \frac{1}{\Gamma(\nu)}\int_a^{t_2} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt \\ &\quad + \frac{1}{\Gamma(\nu)}\int_a^{\xi} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt \\ &= \omega\left(\xi_2^-\right) + I^1_2\left(\omega\left(\xi_1\right)\right) - \frac{1}{\Gamma(\nu)}\int_a^{t_2} \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}h(t)dt \end{aligned}$$

$$+\frac{1}{\Gamma(\nu)}\int_a^{\xi}\rho_1'(t)(\rho_1(\xi)-\rho_1(t))^{\nu-1}h(t)dt$$
$$=\omega(a)+I_1^1\left(\omega\left(\xi_1\right)\right)+I_2^1\left(\omega\left(\xi_2\right)\right)+\frac{1}{\Gamma(\nu)}\int_a^{\xi}\rho_1'(t)(\rho_1(\xi)-\rho_1(t))^{\nu-1}h(t)\mathrm{d}t.$$

If $\xi\in J_k$ for $q=1,2,\ldots,k$, then from Lemma 3.1 again and noting the boundary condition

$$\omega(a)=\omega_0+\lambda\left|{}^{\rho_3}{}_{\xi}\mathcal{I}_{a^+}^{\gamma}\eta(\xi)\mid\omega(\xi)|^{p-1}\right|_{\xi=K}^{p^\star-1},$$

one can obtain

$$\omega(\xi)=\omega(a)+\sum_{j=1}^{q}I_j^1\left(\omega\left(\xi_j\right)\right)+\int_a^{\xi}\rho_1'(t)(\rho_1(\xi)-\rho_1(t))^{\nu-1}h(t)\mathrm{d}t.$$

By a simple calculation, we get

$$\omega(a)=\omega_0+\lambda\left|\frac{1}{\Gamma(\gamma)}\int_a^K\rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\,|\omega(t)|^{p-1}\,\mathrm{d}t\right|^{p^\star-1}.$$

hence from Proposition 2.1, we get (3.5).

Now, suppose that (3.5) holds for ω. Then we have the following assumption: If $\xi\in[ca,t_1]$, then $\omega(a)=\omega_0+\lambda\left|{}^{\rho_3}{}_{\xi}\mathcal{I}_{a^+}^{\gamma}\eta(\xi)\,|\omega(\xi)|^{p-1}\right|_{\xi=K}^{p^\star-1}$ and using the fact that ${}^{\rho_1;C}{}_{\xi}\mathcal{D}_{a^+}^{\nu}$ is the left inverse of ${}^{\rho_1}{}_{\xi}\mathcal{I}_{a^+}^{\nu}$, we get (3.7). If $\xi\in J_k, q=1,2,\ldots,k$ and using the fact that the (ρ-Caputo fractional derivative of a constant equals to zero, we obtain ${}^{\rho_1;C}{}_{\xi}\mathcal{D}_{a^+}^{\nu}\omega(\xi)=h(\xi),\xi\in(t_k,\xi_{q+1}]$ and $\Delta(\omega(t_k))=I_q^1(\omega(t_k))$. So, the proof is completed. □

Lemma 3.4 *Assume that $\varphi(\xi)\in C(Q,\mathbb{R}),\nu,\beta\in(0,1]$. Then the solution of the fractional impulsive boundary value problem*

$$\begin{cases}{}^{\rho_2;C}{}_{\xi}\mathcal{D}_{K^-}^{\beta}\left(\varkappa(\xi)\psi_p\left({}^{\rho_1;C}{}_{\xi}\mathcal{D}_{a^+}^{\nu}\omega\right)\right)(\xi)=\varphi(\xi), & \xi\in Q,\xi\neq t_k\\ \Delta\left(\omega\left(\xi_q\right)\right)=I_q^1\left(\omega\left(\xi_q\right)\right), & \\ \quad\Delta\psi_p\left({}^{\rho_1;C}{}_{\xi}\mathcal{D}_{a^+}^{\nu}\omega\left(\xi_q\right)\right)=I_q^2(\omega(t_k)), & q=1,2,\ldots,k\\ \omega(a)=\omega_0+\lambda\left|{}^{\rho_3}{}_{\xi}\mathcal{I}_{a^+}^{\gamma}\eta(\xi)\mid\omega(\xi)|^{p-1}\right|_{\xi=K}^{p^\star-1}, & {}^{\rho_1;C}{}_{\xi}\mathcal{D}_{a^+}^{\nu}\omega(K)=\omega_1,\end{cases}\tag{3.8}$$

is equivalent to the following integral equation

$$\omega(\xi)=\begin{cases}\omega_0+\lambda\left|\frac{1}{\Gamma(\gamma)}\int_a^K\rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\,|\omega(t)|^{p-1}\,\mathrm{d}t\right|^{p^\star-1}\\ \quad+\frac{1}{\Gamma(\nu)}\int_a^{\xi}\rho_1'(t)(\rho_1(\xi)-\rho_1(t))^{\nu-1}\psi_{p^\star}(\mathfrak{F}\varphi(t))\mathrm{d}t\\ \quad+\sum_{j=0}^{q}I_j^1\left(\omega\left(\xi_j\right)\right)H(\xi-t_j)\quad\xi\in Q,q=0,1,\ldots,k,\end{cases}\tag{3.9}$$

where

$$\mathfrak{F}\varphi(\xi)=\frac{1}{\varkappa(\xi)}\begin{cases}\varkappa(K)\psi_p\left(u_1\right)+\frac{1}{\Gamma(\beta)}\int_{\xi}^{K}\rho_2'(u)(\rho_2(u)-\rho_2(\xi))^{\beta-1}\varphi(u)\mathrm{d}u\\ \quad+\sum_{j=0}^{q}I_j^2(\omega(t_j)H(\xi-t_j),\ \xi\in Q, q=0,1,\ldots,k,\end{cases}\tag{3.10}$$

and H is the Heavside function and $I_0^i=0$ for $i=1,2$.

Proof Let $\nu\in(0,1]$ and $\varphi\in C(Q,\mathbb{R})$. Let $v(\xi)=\psi_p\left({}^{\rho_1;C}_{\xi}\mathcal{D}^{\nu}_{a^+}\omega(\xi)\right)$, then from (3.8), we get the following linear impulsive problem

$${}^{\rho_2;C}_{\xi}\mathcal{D}^{\beta}_{K^-}\varkappa(\xi)v(\xi)=\varphi(\xi)\quad a<\xi<K,\xi\neq t_k \tag{3.11}$$

$$\Delta\left(v\left(\xi_q\right)\right)=b_q, q=1,2,\ldots,k \tag{3.12}$$

$$v(K)=\psi_p\left(u_1\right), \tag{3.13}$$

with $b_q=I_q^2(\omega(t_k), q=1,2,\ldots,k, b_0=0$.
Then by applying Lemma 3.2, (3.11)–(3.13) are equivalent to

$$\varkappa(\xi)v(\xi)=\begin{cases}\varkappa(K)\psi_p\left(u_1\right)+\frac{1}{\Gamma(\beta)}\int_{\xi}^{K}\rho_2'(u)(\rho_2(u)-\rho_2(\xi))^{\beta-1}\varphi(u)\mathrm{d}u\\ \quad+\sum_{j=0}^{q}b_j(\omega(t_j)H(\xi-t_j),\ \xi\in Q, q=1,\ldots,k.\end{cases}\tag{3.14}$$

hence by $v(\xi)=\psi_p\left({}^{\rho_1;C}_{\xi}\mathcal{D}^{\nu}_{a^+}\omega(\xi)\right)=\mathfrak{F}(\varphi(\xi)), \xi\in Q, q=0,1,2,\ldots,k$, one can find that the problem (3.8) has the equivalent form

$$\begin{cases}{}^{\rho_1;C}_{\xi}\mathcal{D}^{\nu}_{a^+}\omega(\xi)=\psi_{p^\star}\left(\mathfrak{F}(\varphi(\xi))\right), & a<\xi<K,\xi\neq t_k,\\ \Delta\left(\omega\left(\xi_j\right)\right)=I_j^1\left(\omega\left(\xi_j\right)\right), & q=1,2,\ldots,k\\ \omega(a)=\omega_0+\lambda\left|{}^{\rho_3}_{\xi}\mathcal{I}^{\gamma}_{a^+}\eta(\xi)\mid\omega(\xi)|^{p-1}\right|^{p^\star-1}_{\xi=K}. & \end{cases}\tag{3.15}$$

Then, by applying Lemma 3.1 again, we get the desired result. □

3.2 Existence and Uniqueness Results

Here, two main properties including the existence and uniqueness of solutions will be investigated.

Lemma 3.5 *If $f\in C(Q\times\mathbb{R},\mathbb{R})$, then the function $\omega(\xi)\in PC(Q,\mathbb{R})$ is a solution of the the p-Laplacian impulsive fractional boundary value problem* (1.1) *if and only* *if it satisfies the integral equation*

$$\omega(\xi) = \begin{cases} \omega_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1}\eta(t) \, |\omega(t)|^{p-1} \, dt \right|^{p^\star-1} \\ +\frac{1}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}\psi_{p^\star}(\mathfrak{F}\mathcal{N}\omega(t))dt \\ + \sum_{j=0}^q I_j^1 \left(\omega\left(\xi_j\right)\right) H(\xi - t_j) \quad \xi \in Q, q = 0, 1, \ldots, k, \end{cases} \tag{3.16}$$

where

$$\mathfrak{F}\mathcal{N}\omega(\xi) = \frac{1}{\varkappa(\xi)} \begin{cases} \varkappa(K)\psi_p(u_1) + \frac{1}{\Gamma(\beta)} \int_\xi^K \rho_2'(u)(\rho_2(u) - \rho_2(\xi))^{\beta-1}\mathcal{N}\omega(u)du \\ + \sum_{j=0}^q I_j^2(\omega(t_j)H(\xi - t_j), \ \xi \in Q, q = 1, \ldots, k, \end{cases} \tag{3.17}$$

and $\mathcal{N}$ is the Nemytskii operator associated to the p-Laplacian impulsive fractional boundary value problem (1.1) definded by

$$\mathcal{N}(\omega(\xi)) = f(\xi, \omega(\xi)) - t(\xi)\psi_p(\omega(\xi)), \quad \xi \in Q, \xi \neq t_k, \\ q = 1, 2, \ldots, k, \omega \in PC(Q, \mathbb{R}). \tag{3.18}$$

Now, we consider the integral operator $\mathcal{L} : PC(Q, \mathbb{R}) \to PC(Q, \mathbb{R})$ defined as

$$\mathcal{L}\omega(\xi) = \begin{cases} \omega_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1}\eta(t) \, |\omega(t)|^{p-1} \, dt \right|^{p^\star-1} \\ +\frac{1}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}\psi_{p^\star}(\mathfrak{F}\mathcal{N}\omega(t))dt \\ + \sum_{j=0}^q I_j^1 \left(\omega\left(\xi_j\right)\right) H(\xi - t_j) \quad \xi \in Q, q = 0, 1, \ldots, k. \end{cases} \tag{3.19}$$

It is obvious that a solution of the p-Laplacian impulsive fractional boundary value problem (1.1) will be the fixed point of the operator $\mathcal{L}$.

Lemma 3.6 *The operator $\mathcal{L} : PC(Q, \mathbb{R}) \to PC(Q, \mathbb{R})$ is completely continuous.*

Proof Firstly, let us prove the continuity of the operator $\mathcal{L} : PC(Q, \mathbb{R}) \to PC(Q, \mathbb{R})$. Assume that $\{u_n\} \subseteq PC(Q, \mathbb{R})$ is a sequence so that $u_n \to \omega$ in $PC(Q, \mathbb{R})$. In view of the continuity of $f(\xi, \omega)$, I_q^i for $i = 1, 2$ and the first item of Lemma (2.9), one can write

$$\lim_{n\to\infty} \mathcal{L}u_n(\xi) = \lim_{n\to\infty} \begin{pmatrix} \omega_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1}\eta(t) \, |u_n(t)|^{p-1} \, dt \right|^{p^\star-1} \\ +\frac{1}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1}\psi_{p^\star}(\mathfrak{F}\mathcal{N}u_n(t))dt \\ + \sum_{j=0}^q I_j^1 \left(u_n\left(\xi_j\right)\right) H(\xi - t_j), \end{pmatrix}$$

$$= \omega_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1}\eta(t) \, |u_n(t)|^{p-1} \, dt \right|^{p^\star-1}$$

$$+\frac{1}{\Gamma(\nu)}\int_a^{\xi}\rho_1'(t)(\rho_1(\xi)-\rho_1(t))^{\nu-1}\psi_{p^\star}(\mathfrak{F}\mathcal{N}u_n(t))\mathrm{d}t$$

$$+\sum_{j=0}^{q}I_j^1\left(u_n\left(\xi_j\right)\right)H(\xi-t_j)=\mathcal{L}\omega(\xi)$$

uniformly for $\xi\in\ Q$, $q=0,1,\ldots,k$. This shows the continuity of the operator $\mathcal{L}:PC(Q,\mathbb{R})\to PC(Q,\mathbb{R})$.

Next, we show that $\mathcal{L}$ is compact. Let $\Theta=\{\omega\in PC(Q,\mathbb{R}),\ \|\ \omega\ \|<\mathcal{R}\}$. Then from the continuity of f and I_q^i, there exist the constants $M_0, M_1, M_2>0$ such that $\mid f(\xi,\omega(\xi)\ \mid\leq M_0$ and $\mid I_q^i(\omega(t_k)\ \mid\leq M_i$ $(q=1,2,\ldots,k,\ i=1,2)$ for all $\xi\in\ Q$ and each $\omega\in\Theta$. We have

$$\begin{aligned}\mid\mathfrak{F}\mathcal{N}\omega(\xi)\mid=\frac{1}{|\varkappa(\xi)|}\Bigg|&\varkappa(K)\psi_p\left(u_1\right)\\&+\frac{1}{\Gamma(\beta)}\int_{\xi}^{K}\rho_2'(u)(\rho_2(u)-\rho_2(\xi))^{\beta-1}f(u,\omega(u)-t(u)\psi_p\left(\omega(u)\right)\mathrm{d}u\\&+\sum_{j=0}^{q}I_j^2(\omega(t_j)H(\xi-t_j),\ \xi\in\ Q,q=1,\ldots,k\Bigg|\end{aligned}\tag{3.20}$$

$$\begin{aligned}&\leq M_3\Bigg(\varkappa(K)\psi_p\left(\mid u_1\mid\right)+\frac{1}{\Gamma(\beta)}\int_{\xi}^{K}\rho_2'(u)(\rho_2(t)-\rho_2(\xi))^{\beta-1}\left(\mid f(u,\omega(u)\mid\right.\\&\quad\left.+t(u)\psi_p\left(\mid\omega(u)\mid\right)\right)\mathrm{d}u+\sum_{j=1}^{k}\mid I_j^2(\omega(t_j)\mid\Bigg)\\&\leq M_3\left(\varkappa(K)\mid u_1\mid^{p-1}+\frac{M_0+M_5\mathcal{R}^{p-1}}{\Gamma(\beta+1)}(\rho_2(K)-\rho_2(a))^{\beta}+mM_2\right):=L,\end{aligned}$$

where $M_3=\frac{1}{\min_{\xi\in Q}(\varkappa(\xi))}$, $M_5=\max_{\xi\in Q}\left(t(\xi)\right)$, M_0 and M_2 are given above. It implies that

$$\begin{aligned}\mid\mathcal{L}\omega(\xi)\mid\leq\mid\omega_0\mid+\mid\lambda\mid&\left|\frac{1}{\Gamma(\gamma)}\int_a^{K}\rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\left|\omega(t)\right|^{p-1}\mathrm{d}t\right|^{p^\star-1}\\&+\sum_{j=1}^{k}\mid I_j^1(\omega(t_j)\mid+\frac{1}{\Gamma(\nu)}\int_a^{\xi}\rho_1'(t)(\rho_1(\xi)-\rho_1(t))^{\nu-1}\psi_{p^\star}\left(|\mathfrak{F}\mathcal{N}\omega(t)|\right)\mathrm{d}t\\&\leq\mid\omega_0\mid+\mid\lambda\mid\frac{\mathcal{R}M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{\gamma}+kM_1\end{aligned}$$

$$+\frac{1}{\Gamma(\nu+1)}(\rho_1(K)-\rho_1(a))^{\nu}L^{p^\star-1}:=L^\star.$$

here $M_4=\max_{\xi\in Q}(\eta(\xi))$.

Therefore, we find that $|\ \mathcal{L}\omega\ |\leq L^\star$ for each $\omega\in\Theta$. Consequently, $\mathcal{L}(\Theta)$ is uniformly bounded in $PC(Q,\mathbb{R})$.

Now, we need to prove that $\mathcal{L}(\Theta)\subset PC(Q,\mathbb{R})$ is equicontinuous in J_k by the Arzelá-Ascoli theorem of PC-typ. For this, let $\omega\in\Theta$ and $\tau_1,\tau_2\in[a,t_1]$ such that $a\leq\tau_1<\tau_2\leq t_1$. We have

$$\begin{aligned}|\mathcal{L}\omega(\tau_2)-\mathcal{L}\omega(\tau_1)|=&\Bigg|\frac{1}{\Gamma(\nu)}\int_a^{\tau_1}\rho_1'(t)\left((\rho_1(\tau_2)-\rho_1(t))^{\nu-1}-(\rho_1(\tau_1)-\rho_1(t))^{\nu-1}\right)\\&\times\psi_{p^\star}(\mathfrak{F}\mathcal{N}\omega(t))\,\mathrm{d}t\\&+\frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\rho_1'(t)(\rho_1(\tau_2)-\rho_1(t))^{\nu-1}\psi_{p^\star}(\mathfrak{F}\mathcal{N}\omega(t))\,\mathrm{d}t\Bigg|\\&\leq\frac{L^{p^\star-1}}{\Gamma(\nu)}\Bigg(\int_a^{\tau_1}\left|\rho_1'(t)\left((\rho_1(\tau_2)-\rho_1(t))^{\nu-1}-(\rho_1(\tau_1)-\rho_1(t))^{\nu-1}\right)\right|\mathrm{d}t\\&+\int_{\tau_1}^{\tau_2}\rho_1'(t)(\rho_1(\tau_2)-\rho_1(t))^{\nu-1}\mathrm{d}t\Bigg)\\&=\frac{L^{p^\star-1}}{\Gamma(\nu)}\left(2(\rho_1(\tau_2)-\rho_1(\tau_1))^{\nu}-(\rho_1(\tau_2)-\rho_1(a))^{\nu}+(\rho_1(\tau_1)-\rho_1(a))^{\nu}\right).\end{aligned}$$

In other words, we have

$$|\ \mathcal{L}\omega(\tau_2)-\mathcal{L}\omega(\tau_1)\ |\leq\frac{L^{p^\star-1}}{\Gamma(\nu)}\Bigg(2(\rho_1(\tau_2)-\rho_1(\tau_1))^{\nu}-(\rho_1(\tau_2)-\rho_1(a))^{\nu}+(\rho_1(\tau_1)-\rho_1(a))^{\nu}\Bigg),$$

for Q_k, where $t_k\leq\tau_1<\tau_2\leq\xi_{q+1}$, $q=1,2,\ldots,k$.

As $\rho_1(\xi)^{\nu}$ is uniformly continuous on $\mathbb{Q}_k$ and $\tau_2\to\tau_1$, the right-hand side of the above inequality tends to zero. Therefore, $K(\Theta)$ is equicontinuous. Then, the Arzelá-Ascoli theorem of PC-typ implies that $\mathcal{L}(\Theta)$ is relatively compact in $PC(Q,\mathbb{R})$. ◻

To investigate the existence and uniqueness of solutions to the p-Laplacian impulsive fractional boundary value problem (1.1), we consider some assumptions below.

(**H**$_1$) Let $f:Q\times\mathbb{R}\to\mathbb{R}$ be continuous with the following conditions: there exist non-negative constants $r,e\in\mathbb{R}$ and $0\leq\ell<p-1$ such that $|f(\xi,\omega)|\leq r+e|\omega|^{\ell}$ for all $\xi\in Q$ and $\omega\in\mathbb{R}$.

(**H**$_2$) For $q=1,2,\ldots,k$, $i=1,2$, $I_q^i\in C(\mathbb{R},\mathbb{R})$, and there exist constants $r^i,e^i\geq0$, $0\leq\ell^1<1$ and $0\leq\ell_k^2<p-1$ such that $|I_k^i(\omega)|\leq r_q^i+e_q^i|\omega|^{\ell_k^i}$, $\omega\in\mathbb{R}$.

(**H**$_3$) Let $f:Q\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Also, there exist non-negative constants $r,e,\mathcal{R}_1\in\mathbb{R}$ and $0\leq\ell<p-1$ such that $|f(\xi,\omega)|\leq r+e|\omega|^{\ell}$, $\xi\in Q,\omega\in[0,\mathcal{R}_1]$;

(**H**$_4$) For $q=1,2,\ldots,k$, $i=1,2$, $I_q^i\in C(\mathbb{R},\mathbb{R})$, and there exist constants $r_k^i,e_k^i,\mathcal{R}_1\geq0$, $0\leq\ell_k^1<1$ and $0\leq\ell_k^2<p-1$ such that $|I_k^i(\omega)|\leq r_q^i+e_q^i|\omega|^{\ell_k^i}$, $\omega\in[0,\mathcal{R}_1]$;

($\mathbf{H}_3^\star$) $f : Q \times \mathbb{R} \to \mathbb{R}$ is continuous. Also, there exist non-negative constants $r, e, \mathcal{R}_1 \in \mathbb{R}$ such that $|f(\xi, \omega)| \le r + e|\omega|^{p-1}$, $\xi \in Q, \omega \in [0, \mathcal{R}_1]$;

($\mathbf{H}_4^\star$) For $q = 1, 2, \ldots, k$, $i = 1, 2$, $I_q^i \in C(\mathbb{R}, \mathbb{R})$, and also, there exist constants $r_k^i, e_k^i, \mathcal{R}_1 \ge 0$ and $0 \le \ell_k^1 < 1$ such that $|I_k^i(\omega)| \le r_q^i + e_q^i|\omega|^{p-1}$, $\omega \in [0, \mathcal{R}_1]$;

($\mathbf{H}_5$) $f : Q \times \mathbb{R} \to \mathbb{R}$ is continuous. Also, there exist non-negative constant $L \in \mathbb{R}$ such that

$$|f(\xi, \omega) - f(\xi, v)| \le L|\omega - v|, \qquad \xi \in Q, \omega, v \in \mathbb{R};$$

($\mathbf{H}_6$) For $q = 1, 2, \ldots, k$, $i = 1, 2$, $I_q^i \in C(\mathbb{R}, \mathbb{R})$, and also, there exist constants $L_k^i > 0$ such that

$$|I_k^i(\omega) - I_k^i(\omega)| \le L_q^i|\omega - v|, \qquad \omega, v \in \mathbb{R}$$

($\mathbf{H}_5^\star$) $f : Q \times \mathbb{R} \to \mathbb{R}$ is continuous. Also, there exist non-negative function $\Phi(\xi) \in C(Q)$ and the constant $L \in \mathbb{R}$ such that

$$\begin{array}{ll} 0 < f(\xi, \omega) - t(\xi)\psi_p(\omega(\xi)) \le \Phi(\xi), & \xi \in Q, \omega \in \mathbb{R}, \\ |f(\xi, \omega) - f(\xi, v)| \le L|\omega - v|, & \xi \in Q, \omega, v \in \mathbb{R} \\ u_0, u_1 > 0, & \lambda \ge 0; \end{array}$$

($\mathbf{H}_6^\star$) For $q = 1, 2, \ldots, k$, $i = 1, 2$, $I_q^i \in C(\mathbb{R}, \mathbb{R})$. Also, there exist the positive functions $\Psi_k^i \in C(Q, \mathbb{R})$ and constants $L_k^i, e_k^1 > 0$ such that

$$\begin{array}{ll} 0 \le I_k^1(\omega) \le \Psi_k^1(\xi) + e_k^1 \parallel \omega \parallel, & (\xi, \omega) \in \mathbb{R} \times \mathbb{R}, \\ 0 \le I_k^2(\omega) \le \Psi_k^2(\xi), & (\xi, \omega) \in Q \times \mathbb{R}, \\ |I_k^i(\omega) - I_k^i(\omega)| \le L_q^i|\omega - v|, & \omega, v \in \mathbb{R}; \end{array}$$

($\mathbf{H}_5^{\star\star}$) $f : Q \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Also, there exist non-negative constants $L \in \mathbb{R}$ and e such that

$$\begin{array}{ll} 0 < f(\xi, \omega) - t(\xi)\psi_p(\omega(\xi)), & \xi \in Q, \omega \in \mathbb{R}, \\ 0 < |f(\xi, \omega)| \le e|\omega|^{p-1}, & \xi \in Q, \omega \in \mathbb{R}, \\ |f(\xi, \omega) - f(\xi, v)| \le L|\omega - v|, & \xi \in Q, \omega, v \in \mathbb{R} \\ u_0 > u_1 > 0, & \lambda \ge 0; \end{array}$$

($\mathbf{H}_6^{\star\star}$) For $q = 1, 2, \ldots, k$, $i = 1, 2$, $I_q^i \in C(\mathbb{R}, \mathbb{R})$, and also, there exist the positive functions $\Psi_k^1 \in C(Q, \mathbb{R})$ and constants $L_k^i, e_k^i > 0$ such that

$$\begin{array}{ll} 0 \le I_k^1(\omega) \le \Psi_k^1(\xi) + e_k^1 \parallel \omega \parallel, & (\xi, \omega) \in \mathbb{R} \times \mathbb{R}, \\ 0 \le I_k^2(\omega) \le e_k^2|\omega|^{p-1}, & (\xi, \omega) \in Q \times \mathbb{R}, \\ |I_k^i(\omega) - I_k^i(\omega)| \le L_q^i|\omega - v|, & \omega, v \in \mathbb{R}; \end{array}$$

($\mathbf{H}_5^{\star\star\star}$) $f : Q \times \mathbb{R} \to \mathbb{R}$ is continuous. Also, there exist the non-negative function $\Pi(\xi) \in C(Q)$ and constant $L \in \mathbb{R}$ such that

$$\begin{aligned} &-\Pi(\xi) \leq f(\xi,\omega) - t(\xi)\psi_p(\omega(\xi)) < 0, && \xi \in Q, \omega \in \mathbb{R},\\ &|f(\xi,\omega) - f(\xi,v)| \leq L|\omega - v|, \quad \xi \in Q, \omega, v \in \mathbb{R} && \\ &u_0, u_1 < 0, && \lambda \leq 0; \end{aligned}$$

($\mathbf{H}_6^{\star\star\star}$) For $q = 1, 2, \ldots, k$, $i = 1, 2$, $I_q^i \in C(\mathbb{R}, \mathbb{R})$, and also, there exist the positive functions $\chi_k^i \in C(\mathbb{R}, \mathbf{R})$ and constants $L_k^i, e_k^1 > 0$ such that

$$\begin{aligned} &-\chi_k^1(\xi) - e_k^1 \parallel \omega \parallel \leq I_k^1(\omega) \leq 0, && (\xi,\omega) \in Q \times \mathbb{R},\\ &-\chi_k^2(\xi) \leq I_k^2(\omega) \leq 0, && (\xi,\omega) \in Q \times \mathbb{R},\\ &|I_k^i(\omega) - I_k^i(\omega)| \leq L_q^i|\omega - v|, && \omega, v \in \mathbb{R}. \end{aligned}$$

Now, we prove the first existence theorem.

Theorem 3.7 *Suppose that the assumptions* ($\boldsymbol{H}_1$) *and* ($\boldsymbol{H}_2$) *are satisfied. If*

$$\frac{|\lambda| M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\rho_3(K) - \rho_3(a))^\gamma < 1, \tag{3.21}$$

then the p-Laplacian impulsive fractional boundary value problem (1.1) *has at least one solution.*

Proof Firstly, Lemma 3.6 implies that the integral operator $\mathcal{L}(Q, \mathbb{R}) \to PC(Q, \mathbb{R})$ is completely continuous. Next, suppose that ($\mathbf{H}_1$) and ($\mathbf{H}_2$) hold. We show that the set $E(\mathcal{L}) = \{\omega \in PC(Q, \mathbb{R}) : \omega = \sigma\mathcal{L}\omega \text{ for some } \sigma \in [0, 1]\}$ is bounded.

Let $\omega \in E(\mathcal{L})$. Then we have $\omega = \sigma\mathcal{L}\omega$ for each $\xi \in Q$, $q = 0, 1, 2, \ldots, k$ and

$$\begin{aligned} |\mathfrak{F}\mathcal{N}\omega(\xi)| &\leq M_3\bigg(\varkappa(K)\psi_p(|u_1|) + \frac{1}{\Gamma(\beta)}\int_\xi^K \rho_2'(t)(\rho_2(t) - \rho_2(\xi))^{\beta-1}\\ &\quad\times\Big[|f(u,\omega(u)| + t(u)\psi_p(|\omega(u)|)\Big]du + \sum_{j=1}^k |I_j^2(\omega(t_j)|\bigg)\\ &\leq M_3\bigg(\varkappa(K)|u_1|^{p-1} + \frac{1}{\Gamma(\beta)}\int_\xi^K \rho_2'(t)(\rho_2(t) - \rho_2(\xi))^{\beta-1}\\ &\quad\times\Big[(r + e|\omega(u)|^\ell + t(u)\psi_p(|\omega(u)|)\Big]du + \sum_{j=1}^k r_q^2 + e_q^2|\omega(t_k)|^{\ell_k^2}\bigg)\\ &\leq M_3\bigg(\varkappa(K)|u_1|^{p-1} + \frac{r + e\parallel\omega\parallel^\ell + M_5\parallel\omega\parallel^{p-1}}{\Gamma(\beta+1)}(\rho_2(K) - \rho_2(a))^\beta\\ &\quad+\sum_{j=1}^k r_j^2 + e_j^2\parallel\omega\parallel^{\ell_j^2}\bigg). \end{aligned}$$

So, one can find a positive constant ϖ such that $\| \mathfrak{F}\mathcal{N}\omega \| < \varpi$. Now, we have

$$\begin{aligned}
| \omega(\xi) | &= \sigma \mid \mathcal{L}\omega(\xi) \mid \\
&= \sigma \left| \omega_0 + \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1} \eta(t) \, |\omega(t)|^{p-1} \, \mathrm{d}t \right|^{p^\star-1} \right. \\
&\quad \left. + \frac{1}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1} \psi_{p^\star}(\mathfrak{F}\mathcal{N}\omega(t)) \, \mathrm{d}t \right| \\
&\leq \sigma \mid \omega_0 \mid + \sigma \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1} \eta(t) \, |\omega(t)|^{p-1} \, \mathrm{d}t \right|^{p^\star-1} \\
&\quad + \frac{\sigma}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1} \psi_{p^\star}(|\mathfrak{F}\mathcal{N}\omega(t)|) \, \mathrm{d}t + \sigma \sum_{j=1}^{k} \mid I_j^1(\omega(t_j) \mid \\
&\leq \mid \omega_0 \mid + \mid \lambda \mid \frac{\| \omega \| \, M_4^{p^\star-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma \\
&\quad + \frac{1}{\Gamma(\nu+1)} (\rho_1(K) - \rho_1(a))^\nu \varpi^{p^\star-1} + \sum_{j=1}^{k} r_j^1 + e_j^1 \| \omega \|^{\ell_j^1} .
\end{aligned}$$

Consequently,

$$\begin{aligned}
\| \omega(\xi) \| \leq \mid \omega_0 \mid + \mid \lambda \mid \frac{\| \omega \| \, M_4^{p^\star-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma \\
+ \frac{1}{\Gamma(\nu+1)} (\rho_1(K) - \rho_1(a))^\nu \varpi^{p^\star-1} + \sum_{j=1}^{k} r_j^1 + e_j^1 \| \omega \|^{\ell_j^1}, \\
\text{for } \xi \in Q, \ q = 1, 2, \ \ldots, \ k.
\end{aligned}$$

By taking into account that $0 \leq \ell_j^1 < 1$ and $\frac{|\lambda| M_4^{p^\star-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma < 1$, we can deduce that there exists a positive constant $\varpi^\star$ such that $\| \omega \| \leq \varpi^\star$ for any solution of the functional equation $\omega = \sigma \mathcal{L}\omega$, $0 < \sigma < 1$. Therefore, by using Theorem 2.13, we obtain the existence of a fixed point for $\mathcal{L}$ implying the existence of at least one solution for the p-Laplacian impulsive fractional boundary value problem (1.1). □

Before starting the second existence theorem, we consider the following notations for the sake of convenience:

$$\begin{aligned}
C^1 &= M_3 \left[\varkappa(K) \mid u_1 \mid^{p-1} + \sum_{j=1}^{k} r_j^2 + \frac{r}{\Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^\beta \right], \\
C^2 &= \frac{4 e M_3}{\Lambda^{p-1} \Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^\beta, \ C^3 = \frac{M_5 M_3}{\Lambda^{p-1} \Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^\beta,
\end{aligned}$$

$$C_j^4 = \frac{4M_3 e_j^2 k}{\Lambda^{p-1}},\ C^5 =| \omega_0 | + \sum_{j=1}^{k} r_j^1,\ C^6 = 4 \mid \lambda \mid \frac{M_4^{p^\star -1}}{\Gamma(\gamma+1)},$$

$$C_j^7 = 4m \sum_{j=1}^{k} e_j^1,\ \Lambda = 4\frac{\Gamma(\nu+1)}{(\rho_1(K)-\rho_1(a))^\nu},\ \text{for } j = 1,\ 2,\ \ldots,\ k.$$

Now, we are ready to prove the second existence theorem.

Theorem 3.8 *Assume that the assumptions* ($\boldsymbol{H}_3$) *and* ($\boldsymbol{H}_4$) *are satisfied. If*

$$C^2, C^6 \leq 1, \tag{3.22}$$

then the p*-Laplacian impulsive fractional boundary value problem* (1.1) *has at least one solution.*

Proof We shall prove that the p-Laplacian impulsive fractional boundary value problem (1.1) has at least one solution. Suppose that ($\mathbf{H}_3$) and ($\mathbf{H}_4$) hold and C^2, C^6 satisfy (3.21), and let $\Omega_1 = \{\omega \in PC(Q,\mathbb{R}), \| \omega \| < \mathcal{R}_1\}$, where

$$\mathcal{R}_1 \geq \max\left\{ \frac{4\left(C^1\right)^{p^\star}}{\Lambda},\ \left(C^3\right)^{\frac{1}{p-1-\ell}},\ \left(C_j^4\right)^{\frac{1}{p-1-\ell_j^2}},\ 4C^5,\ \left(C_j^7\right)^{\frac{1}{1-\ell_j^1}} \right\},$$
$$\text{for } j = 1,\ 2,\ \ldots,\ k,$$

for $\forall \omega \in \Theta$, $\xi \in Q$. We have

$$\begin{aligned}
| \mathfrak{F}\mathcal{N}\omega(\xi) | &\leq M_3\Bigg(\varkappa(K)\psi_p(| u_1 |) + \frac{1}{\Gamma(\beta)}\int_\xi^K \rho_2'(t)(\rho_2(t)-\rho_2(\xi))^{\beta-1} \\
&\quad \times \Big[\ | f(u,\omega(u) | + t(u)\psi_p(| \omega(u) |)\Big]\mathrm{d}u + \sum_{j=1}^{k} | I_j^2(\omega(t_j) | \Bigg) \\
&\leq M_3\Bigg(\varkappa(K) | u_1 |^{p-1} + \frac{1}{\Gamma(\beta)}\int_\xi^K \rho_2'(t)(\rho_2(t)-\rho_2(\xi))^{\beta-1} \\
&\quad \times \Big[r + e|\omega(u)|^\ell + t(u)\psi_p(| \omega(u) |)\Big]\mathrm{d}u + \sum_{j=1}^{k} r_q^2 + e_q^2|\omega(t_k)|^{\ell_k^2}\Bigg) \\
&\leq M_3\Bigg(\varkappa(K) | u_1 |^{p-1} + \sum_{j=1}^{k} r_j^2 + \frac{r + e \| \omega \|^\ell + M_5 \| \omega \|^{p-1}}{\Gamma(\beta+1)} \\
&\quad \times (\rho_2(K)-\rho_2(a))^\beta + \sum_{j=1}^{k} e_j^2 \| \omega \|^{\ell_j^2} \Bigg) \\
&\leq C^1 + \frac{\Lambda^{p-1}}{4}C^2\mathcal{R}_1^\ell + C^3\frac{\Lambda^{p-1}}{4}\mathcal{R}_1^{p-1} + \sum_{j=1}^{k}\frac{\Lambda^{p-1}}{4m}C_j^4\Lambda^{p-1}\mathcal{R}_1^{\ell_j^2}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{(\Lambda\mathcal{R}_1)^{p-1}}{4} + \frac{\Lambda^{p-1}}{4}\mathcal{R}_1^{p-1-\ell}\mathcal{R}_1^{\ell} + C^3(\Lambda\mathcal{R}_1)^{p-1} \\
&\quad + \sum_{j=1}^{k}\frac{\Lambda^{p-1}}{4}k\mathcal{R}_1^{p-1-\ell_j^2}\mathcal{R}_1^{\ell_j^2} \\
&\le (\Lambda 4\mathcal{R}_1)^{p-1}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|\ \mathcal{L}\omega(\xi)\ | \le |\ \omega_0\ | &+ \left|\frac{\lambda}{\Gamma(\gamma)}\int_a^K \rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\,|\omega(t)|^{p-1}\,\mathrm{d}t\right|^{p^\star-1} \\
&+\sum_{j=1}^{k}|\ I_j^1\omega(t_j)\ | + \frac{1}{\Gamma(\nu)}\int_a^{\xi}\rho_1'(t)(\rho_1(\xi)-\rho_1(t))^{\nu-1}\psi_{p^\star}(|\mathfrak{F}\mathcal{N}\omega(t)|)\,\mathrm{d}t \\
&\le |\ \omega_0\ | + |\ \lambda\ |\ \frac{\|\ \omega\ \|\ M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{\gamma} + \sum_{j=1}^{k} r_j^1 + e_j^1\ \|\ \omega\ \|^{\ell_j^1} \\
&+ \frac{1}{\Gamma(\nu+1)}(\rho_1(K)-\rho_1(a))^{\nu}\Lambda\mathcal{R}_1 \\
&\le |\ \omega_0\ | + \sum_{j=1}^{k} r_j^1 + |\ \lambda\ |\ \frac{\mathcal{R}_1 M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{\gamma} + \sum_{j=1}^{k} e_j^1\mathcal{R}_1^{\ell_j^1} \\
&+ \frac{1}{\Gamma(\nu+1)}(\rho_1(K)-\rho_1(a))^{\nu}\Lambda\mathcal{R}_1 = C^5 + \frac{C^6}{4}\mathcal{R}_1 + \sum_{j=1}^{k}\frac{C_j^7}{4m}\mathcal{R}_1^{\ell_j^1} + \frac{1}{4}\mathcal{R}_1 \\
&\le \frac{\mathcal{R}_1}{4} + \frac{\mathcal{R}_1}{4} + \sum_{j=1}^{k}\frac{1}{4m}\mathcal{R}_1^{1-\ell_j^1}\mathcal{R}_1^{\ell_j^1} + \frac{1}{4}\mathcal{R}_1 \le \mathcal{R}_1.
\end{aligned}
$$

Thus,

$$
\|\ \mathcal{L}\omega(\xi)\ \| \le \mathcal{R}_1,
$$

which implies that $\mathcal{L}(\Theta) \subseteq \Theta$ for every $\omega \in \Omega_1$. Hence, from Lemma 3.6, the integral operator $\mathcal{L} : \Theta \to \Theta$ is completely continuous. By applying the Schauder's fixed point theorem, we can say that the operator $\mathcal{L}$ has a fixed point, which is also a solution to the p-Laplacian impulsive fractional boundary value problem (1.1). □

Remark 3.9 Let the assumptions ($\mathbf{H}_3^\star$) and ($\mathbf{H}_4^\star$) be satisfied. If

$$
C^2,\ \ C^3,\ \ \sum_{j=1}^{k} C_j^4,\ C^6,\ \ \sum_{j=1}^{k} C_j^7 \le 1, \tag{3.23}
$$

then, by using a similar method given in the proof of Theorem 3.8, we can follow that the p-Laplacian impulsive fractional boundary value problem (1.1) also has at least one solution.

Theorem 3.10 *Let $f(\xi,\omega)$ be continuous on $Q \times \mathbb{R}$ and $I_k^i(\omega)$ be continuous on $\mathbb{R}$. Assume that $\lim_{\omega\to 0} \frac{f(\xi,\omega)}{r+e|\omega|^{\ell}} = 0$ and $\lim_{\omega\to 0} \frac{I_k^i(\omega)}{r_k^i+e_k^i|\omega|^{\ell_k^i}} = 0$ for $i = 1, 2$, $q = 1, 2, \ldots, k$, where r, e, r_k^i, e_k^i are nonegative constants and $0 \leq \ell, \ell_k^2 < p-1$, $0 \leq \ell_k^1 < 1$. Then the p-Laplacian impulsive fractional boundary value problem (1.1) has at least one solution.*

Proof In view of $\lim_{\omega\to 0} \frac{f(\xi,\omega)}{r+e|\omega|^{\ell}} = 0$ and $\lim_{\omega\to 0} \frac{I_k^i(\omega)}{r_k^i+e_k^i|\omega|^{\ell_k^i}} = 0$, for $i = 1, 2$, $q = 1, 2, \ldots, k$, there exists a constant $\mathcal{R}_1 > 0$ such that $| f(\xi,\omega) | \leq \epsilon \left(r + e \mid \omega \mid^{\ell}\right)$ and $| I_k^i(\omega) | \leq \epsilon_k^i \left(r_k^i + e_k^i \mid \omega \mid^{\ell_k^i}\right)$ for $0 <| \omega |< \mathcal{R}_1$, where $\epsilon, \epsilon_k^i > 0$.

As $f(\xi,\omega)$ is continuous on $Q \times \mathbb{R}$ and $I_k^i(\omega)$'s are continuous on $\mathbb{R}$, we find that the conditions ($\mathbf{H}_3$) and ($\mathbf{H}_4$) hold. The proof follows a similar process as in Theorem 3.8. □

Remark 3.11 Consider the continuous functions $f(\xi,\omega)$ on $Q \times \mathbb{R}$ and $I_k^i(\omega)$ on $\mathbb{R}$ for $i = 1, 2$, $q = 1, 2, \ldots, k$. Let $\lim_{\omega\to 0} \frac{f(\xi,\omega)}{r+e|\omega|^{p-1}} = 0$, $\lim_{\omega\to 0} \frac{I_k^1(\omega)}{r_k^1+e_k^1|\omega|} = 0$ and $\lim_{\omega\to 0} \frac{I_k^2(\omega)}{r_k^2+e_k^1|\omega|^{p-1}} = 0$, where r, e, r_k^i, e_k^i are non-negative constants. The same technique used in the proof of Theorem 3.10 can be applied to show that the p-Laplacian impulsive fractional boundary value problem (1.1) has at least one solution.

In the rest of study, we will give the uniqueness results to the p-Laplacian impulsive fractional boundary value problem (1.1). For this, let us use the principle of the Banach contraction mapping. For ease of understanding, define

$$Fu =^{\rho_3} {}_{\xi}\mathcal{I}_{a^+}^{\gamma} \left(\eta(\xi)\, |\omega(\xi)|_{\xi=K}^{p-1}\right).$$

Theorem 3.12 *Suppose that there exist the constants $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6 > 0$ such that*

$$\Theta_1 \leq | \mathfrak{F}\mathcal{N}\omega(\xi) | \leq \Theta_2, \tag{3.24}$$
$$\Theta_3 \leq \| \omega \| \leq \Theta_4, \tag{3.25}$$
$$\Theta_5 \leq | Fu | \leq \Theta_6, \tag{3.26}$$

for each $\xi \in Q, \omega \in PC(Q,\mathbb{R})$. If the assumptions ($\boldsymbol{H}_5$) and ($\boldsymbol{H}_6$) hold, then the p-Laplacian impulsive fractional boundary value problem (1.1) has a unique solution.

Proof Let the assumptions ($\mathbf{H}_5$) and ($\mathbf{H}_6$) be satisfied. We only consider the case $1 < p < 2$; as the other case $p \geq 2$ is straigtforward. If $1 < p \leq 2$, we have $p^\star \geq 2$

by $\frac{1}{p}+\frac{1}{p^\star}=1$, and by applying (3.24)–(3.26) and Lemma 2.9 for every $\xi\in Q$, $\omega\in PC(Q,\mathbb{R})$, we obtain

$$
\begin{aligned}
&|\psi_{p^\star}(\mathfrak{F}\mathcal{N}\omega(\xi))-\psi_{p^\star}(\mathfrak{F}\mathcal{N}v(\xi))|\le (p^\star-1)\Theta_2^{p^\star-2}|\mathfrak{F}\mathcal{N}\omega(\xi)-\mathfrak{F}\mathcal{N}v(\xi)|\\
&\quad=(p^\star-1)\Theta_2^{p^\star-2}\frac{1}{|\varkappa(\xi)|}\left|\frac{1}{\Gamma(\beta)}\int_\xi^K\rho_2'(u)(\rho_2(u)-\rho_2(\xi))^{\beta-1}\,(f(u,\omega(u)-f(u,v(u))\,\mathrm{d}u\right.\\
&\qquad\left.-\frac{1}{\Gamma(\beta)}\int_\xi^K\rho_2'(u)(\rho_2(u)-\rho_2(\xi))^{\beta-1}t(u)(\psi_p\,(\omega(u))-\psi_p\,(\omega(u)))\mathrm{d}u\right|\\
&\quad\le(p^\star-1)\Theta_2^{p^\star-2}M_3\Bigg(\frac{1}{\Gamma(\beta)}\int_\xi^K\rho_2'(u)(\rho_2(u)-\rho_2(\xi))^{\beta-1}\mid f(u,\omega(u)-f(u,v(u)\mid\mathrm{d}u\\
&\qquad+\frac{1}{\Gamma(\beta)}\int_\xi^K\rho_2'(u)(\rho_2(u)-\rho_2(\xi))^{\beta-1}t(u)\mid\psi_p\,(\omega(u))-\psi_p\,(\omega(u))\mid\mathrm{d}u\\
&\qquad+\sum_{j=1}^{q}\mid I_j^2(\omega(t_j)-I_j^2(\omega(t_j)\mid\Bigg)\\
&\le(p^\star-1)\Theta_2^{p^\star-2}M_3\left(\frac{L+(p-1)\Theta_1^{p-2}}{\Gamma(\beta+1)}(\rho_2(K)-\rho_2(a))^\beta+\sum_{j=1}^{k}L_j^2\right)\parallel\omega-v\parallel,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\mid Fu\mid^{p^\star-1}-\mid Fv\mid^{p^\star-1}\Big|\\
&\quad=\left|\left|\frac{1}{\Gamma(\gamma)}\int_a^K(\rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\mid\omega\mid^{p-1}\mathrm{d}t\right|^{p^\star-1}\right.\\
&\qquad\left.-\left|\frac{1}{\Gamma(\gamma)}\int_a^K(\rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\mid\omega\mid^{p-1}\mathrm{d}t\right|^{p^\star-1}\right|\\
&\quad\le(p^\star-1)\Theta_6^{p^\star-2}\left|\frac{1}{\Gamma(\gamma)}\int_a^K(\rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\big[\psi_p\,(\omega(t))-\psi_p\,(v(t))\big]\mathrm{d}t\right|\\
&\quad\le(p^\star-1)\Theta_6^{p^\star-2}\left(\frac{M_4(p-1)\Theta_3^{p-2}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^\gamma\right)\parallel\omega-v\parallel.
\end{aligned}
$$

So, for every $\xi\in Q$, $\omega,v\in PC(Q,\mathbb{R})$, we obtain

$$
\begin{aligned}
&\mid\mathcal{L}\omega(\xi)-\mathcal{L}v(\xi)\mid\\
&\quad\le\mid\lambda\mid(p^\star-1)\Theta_6^{p^\star-2}\left(\frac{M_4(p-1)\Theta_3^{p-2}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^\gamma\right)\parallel\omega-v\parallel\\
&\qquad+(p^\star-1)\Theta_2^{p^\star-2}M_3\left(\frac{L+(p-1)\Theta_3^{p-2}}{\Gamma(\beta+1)}(\rho_2(K)-\rho_2(a))^\beta+\sum_{j=1}^{k}L_j^2\right)\parallel\omega-v\parallel
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{k} L_j^1 \parallel \omega - v \parallel \\
&= \Bigg[\mid \lambda \mid (p^\star - 1)\Theta_6^{p^\star-2} \left(\frac{M_4(p-1)\Theta_3^{p-2}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma \right) + \sum_{j=1}^{k} L_j^1 \\
&+(p^\star - 1)\Theta_2^{p^\star-2} M_3 \left(\frac{L+(p-1)\Theta_3^{p-2}}{\Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^\beta + \sum_{j=1}^{k} L_j^2 \right) \Bigg] \parallel \omega - v \parallel \\
&=: \Lambda^\star \parallel \omega - v \parallel .
\end{aligned}
$$

Hence, for each $\omega, v \in PC(Q, \mathbb{R})$, we get

$$\|\mathcal{L}\omega - \mathcal{L}v\| \le \Lambda^\star \parallel \omega - v \parallel;$$

otherwise,

$$\|\mathcal{L}\omega - \mathcal{L}v\| \le \Lambda^{\star\star} \parallel \omega - v \parallel,$$

for each $\omega, v \in PC(Q, \mathbb{R})$, where

$$
\begin{aligned}
\Lambda^{\star\star} = \Bigg[&\mid \lambda \mid (p^\star - 1)\Theta_5^{p^\star-2} \left(\frac{M_4(p-1)\Theta_4^{p-2}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma \right) + \sum_{j=1}^{k} L_j^1 \\
&+ (p^\star - 1)\Theta_1^{p^\star-2} M_3 \left(\frac{L+(p-1)\Theta_4^{p-2}}{\Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^\beta + \sum_{j=1}^{k} L_j^2 \right) \Bigg].
\end{aligned}
$$

Since $0 < \Lambda^\star, \Lambda^{\star\star} < 1$, we can follow that $\mathcal{L} : PC(Q, \mathbb{R}) \to PC(Q, \mathbb{R})$ is a contraction mapping. By applying the Banach contraction principle, we can say that $\mathcal{L}$ has a unique fixed point in $PC(Q, \mathbb{R})$, which is a solution of the p-Laplacian impulsive fractional boundary value problem (1.1). □

Theorem 3.13 *Assume that the assumptions* ($\boldsymbol{H}_5^\star$) *and* ($\boldsymbol{H}_6^\star$) *are satisfied. If*

$$\Lambda^\star, C^8, \Lambda^{\star\star} < 1, \tag{3.27}$$

where

$$
\begin{aligned}
&\Theta_1 =: M_3 \varkappa(K) u_1^{p-1}, \\
&\Theta_2 := M_3 \left(\varkappa(K) u_1^{p-1} + \frac{\max_{\xi \in Q}(\Phi((\xi))}{\Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^\beta + \sum_{j=1}^{k} \Psi_j^2(t_j) \right), \\
&\Theta_3 =: u_0, \quad \Theta_4 =: \frac{\omega_0 + \sum_{j=1}^{k} \Psi_j^1(t_j) + \frac{1}{\Gamma(\nu+1)} (\rho_1(K) - \rho_1(a))^\nu \Theta_2^{p^\star-1}}{1 - C^8},
\end{aligned}
$$

$$\Theta_5 =: \frac{M_6 \Theta_3^{p-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma, \ \Theta_6 =: \frac{M_5 \Theta_4^{p-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma,$$

$$C^8 =: \sum_{j=1}^{k} e_j^1 + \lambda \frac{M_4^{p^\star-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^{(p^\star-1)\gamma},$$

then there exists a unique solution for the p-Laplacian impulsive fractional boundary value problem (1.1).

Proof Assume that the assumptions ($\mathbf{H}_5^\star$) and ($\mathbf{H}_6^\star$) are satisfied and let $\omega \in PC(Q, \mathbb{R})$. Then for every $\xi \in [a, t_1]$, we obtain

$$\begin{aligned}
0 < \mathfrak{F}\mathcal{N}\omega(\xi) &\leq M_3 \Bigg(\varkappa(K)\psi_p(u_1) + \frac{1}{\Gamma(\beta)} \int_\xi^K \rho_2'(t)(\rho_2(t) - \rho_2(\xi))^{\beta-1} \\
&\quad \times \Big[f(u, \omega(u) - t(u)\psi_p(\omega(u)) \Big] \mathrm{d}u + \sum_{j=1}^{k} I_q^2 \omega(t_k) \Bigg) \\
&\leq M_3 \left(\varkappa(K) u_1^{p-1} + \frac{1}{\Gamma(\beta)} \int_\xi^K \rho_2'(t)(\rho_2(t) - \rho_2(\xi))^{\beta-1} \Phi(u) \mathrm{d}u + \sum_{j=1}^{k} \Psi_j^2(t_j) \right) \\
&\leq M_3 \left(\varkappa(K) u_1^{p-1} + \frac{\max_{\xi \in Q} \{\Phi(\xi)\}}{\Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^\beta + \sum_{j=1}^{k} \Psi_j^2(t_j) \right) =: \Theta_2,
\end{aligned}$$

and so,

$$\begin{aligned}
0 <, omega(\xi) &\leq \omega_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1} \eta(t) \left| \omega(t) \right|^{p-1} \mathrm{d}t \right|^{p^\star-1} \\
&\quad + \sum_{j=1}^{k} I_j^1 \omega(t_j) + \frac{1}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1} \psi_{p^\star}(\mathfrak{F}\mathcal{N}\omega(t)) \,\mathrm{d}t \\
&\leq \omega_0 + \lambda \frac{\| \omega \| \, M_4^{p^\star-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^\gamma + \sum_{j=1}^{k} \Psi_j^1(t_j) + e_j^1 \, \| \omega \| \\
&\quad + \frac{1}{\Gamma(\nu+1)} (\rho_1(K) - \rho_1(a))^\nu \Theta_2^{p^\star-1} \\
&= u_0 + C^8 \| \omega \| + \sum_{j=1}^{k} \Psi_j^1(t_j) + \frac{1}{\Gamma(\nu+1)} (\rho_1(K) - \rho_1(a))^\nu \Theta_2^{p^\star-1}.
\end{aligned}$$

As $C^8 < 1$, then

$$\| \omega \| \leq \frac{\omega_0 + \sum_{j=1}^{k} \Psi_j^1(t_j) + \frac{1}{\Gamma(\nu+1)} (\rho_1(K) - \rho_1(a))^\nu \Theta_2^{p^\star-1}}{1 - C^8} =: \Theta_4,$$

and

$$\begin{aligned} Fu &= \left| \frac{1}{\Gamma(\gamma)} \int_a^K (\rho_3'(t)(\rho_3(K)-\rho_3(t))^{\gamma-1}\eta(t)\psi_p(\omega(t))\,\mathrm{d}t \right| \\ &\leq \frac{M_5\Theta_4^{p-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{\gamma} =: \Theta_6. \end{aligned}$$

On the other hand, from the positivity of $\mathfrak{F}\mathcal{N}\omega(\xi),\ \lambda,\ u_0$ and I_k^i for $i=1,2,q=1,2,\ldots,k$, for each $\xi \in [a,t_1]$, $\omega \in PC(Q,\mathbb{R})$, we have

$$\begin{aligned} \omega(\xi) &\geq u_0 =: \Theta_3, \\ \mathfrak{F}\mathcal{N}\omega(\xi) &\geq M_3\varkappa(K)\psi_p(u_1) =: \Theta_1, \\ Fu &\geq \frac{M_6\Theta_4^{p-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{\gamma} =: \Theta_6. \end{aligned}$$

By using the same process above, we get

$$\begin{aligned} &\Theta_1 \leq \mid \mathfrak{F}\mathcal{N}\omega(\xi) \mid \leq \Theta_2, \\ &\Theta_3 \leq \parallel \omega \parallel \leq \Theta_4, \\ &\Theta_5 \leq \mid Fu \mid \leq \Theta_6, \end{aligned}$$

for any $\xi \in J_k, \omega \in PC(Q,\mathbb{R}),\ q=1,\ 2,\ \ldots,\ k$. Hence, by applying Theorem 3.12, one can deduce that the p-Laplacian impulsive fractional boundary value problem (1.1) has a unique solution. □

Theorem 3.14 *Suppose that the assumptions* ($\boldsymbol{H}_5^{\star\star}$) *and* ($\boldsymbol{H}_6^{\star\star}$) *are satisfied. If*

$$\Lambda^{\star},\ C^9,\ \Lambda^{\star\star} < 1, \tag{3.28}$$

where

$$\begin{aligned} &M_7 =: M_3\left(\varkappa(K) + \frac{e+M_5}{\Gamma(\beta+1)}(\rho_2(K)-\rho_2(a))^{\beta} + \sum_{j=1}^{k} e_j^2(t_j)\right), \\ &\Theta_1 =: M_3\varkappa(K)u_1^{p-1},\ \Theta_2 := M_7\Theta_4^{p-1}, \\ &\Theta_3 =: u_0,\ \ \Theta_4 =: \frac{\omega_0 + \sum_{j=1}^{k}\Psi_j^1(t_j)}{1-C^9}, \\ &\Theta_5 =: \frac{M_6\Theta_3^{p-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{\gamma},\ \Theta_6 =: \frac{M_5\Theta_4^{p-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{\gamma}, \\ &C^9 =: \sum_{j=1}^{k} e_j^1 + \lambda\frac{M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\rho_3(K)-\rho_3(a))^{(p^\star-1)\gamma}\frac{M_7^{p^\star-1}}{\Gamma(\nu+1)}(\rho_1(K)-\rho_1(a))^{\nu}, \end{aligned}$$

then a unique solution exists for the p-Laplacian impulsive fractional boundary value problem (1.1).

Proof Assume that the assumptions $(\mathbf{H}_5^{\star\star})$ and $(\mathbf{H}_6^{\star\star})$ are satisfied and let $\omega \in PC(Q, \mathbb{R})$. Then for every $\xi \in [a, t_1]$, we obtain

$$0 < \omega_0 < \omega(\xi) \leq \omega_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^K \rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1} \eta(t) \, |\omega(t)|^{p-1} \, \mathrm{d}t \right|^{p^\star - 1}$$

$$+ \sum_{j=1}^{k} I_j^1 \omega(t_j) + \frac{1}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1} \psi_{p^\star} (\mathfrak{F}\mathcal{N}\omega(t)) \, \mathrm{d}t$$

$$\leq \omega_0 + \lambda \frac{\| \omega \| \, M_4^{p^\star - 1}}{\Gamma(\gamma + 1)} (\rho_3(K) - \rho_3(a))^\gamma + \sum_{j=1}^{k} \Psi_j^1(t_j) + e_j^1 \, \| \omega \|$$

$$+ \frac{1}{\Gamma(\nu)} \int_a^\xi \rho_1'(t)(\rho_1(\xi) - \rho_1(t))^{\nu-1} \psi_{p^\star} (\mathfrak{F}\mathcal{N}\omega(t)) \, \mathrm{d}t \tag{3.29}$$

$$0 < \mathfrak{F}\mathcal{N}\omega(\xi) \leq M_3 \Bigg(\varkappa(K) \psi_p (u_1) + \frac{1}{\Gamma(\beta)} \int_\xi^K \rho_2'(t)(\rho_2(t) - \rho_2(\xi))^{\beta-1}$$

$$\times \Big[|f(u, \omega(u)| + t(u) \psi_p (|\omega(u)|) \Big] \mathrm{d}u + \sum_{j=1}^{k} I_q^2 \omega(t_k) \Bigg)$$

$$\leq M_3 \Bigg(\varkappa(K) u_0^{p-1} + \frac{\| \omega \|^{p-1}}{\Gamma(\beta)} \int_\xi^K \rho_2'(t)(\rho_2(t) - \rho_2(\xi))^{\beta-1}$$

$$\times (1 + t(u)) \, \mathrm{d}u + \| \omega \|^{p-1} \sum_{j=1}^{k} e_j^2(t_j) \Bigg)$$

$$\leq \| \omega \|^{p-1} M_3 \left(\varkappa(K) + \frac{1 + M_5}{\Gamma(\beta + 1)} (\rho_2(K) - \rho_2(a))^\beta + \sum_{j=1}^{k} e_j^2(t_j) \right)$$

$$:= M_7 \, \| \omega \|^{p-1} . \tag{3.30}$$

From(3.29) and (3.30), we have

$$u_0 \leq \omega(\xi) \leq \omega_0 + \lambda \frac{\| \omega \| \, M_4^{p^\star - 1}}{\Gamma(\gamma + 1)} (\rho_3(K) - \rho_3(a))^\gamma + \sum_{j=1}^{k} \Psi_j^1(t_j) + e_j^1 \, \| \omega \|$$

$$+ \frac{\| \omega \| \, M_7^{p^\star - 1}}{\Gamma(\nu + 1)} (\rho_1(K) - \rho_1(a))^\nu$$

$$= u_0 + C^9 \parallel \omega \parallel + \sum_{j=1}^{k} \Psi_j^1(t_j),$$

as $C^9 < 1$, then

$$\parallel \omega \parallel \leq \frac{\omega_0 + \sum_{j=1}^{k} \Psi_j^1(t_j)}{1 - C^9} =: \Theta_4,$$

and

$$\begin{aligned} Fu &= \left| \frac{1}{\Gamma(\gamma)} \int_a^K (\rho_3'(t)(\rho_3(K) - \rho_3(t))^{\gamma-1} \eta(t) \psi_p(\omega(t))\, \mathrm{d}t \right| \\ &\leq \frac{M_5 \Theta_4^{p-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^{\gamma} =: \Theta_6. \end{aligned}$$

On the other hand, from the positivity of $\mathfrak{F}\mathcal{N}\omega(\xi)$, λ, u_0 and I_k^i for $i = 1, 2$, $q = 1, 2, \ldots, k$, for each $\xi \in [a, t_1]$, $\omega \in PC(Q, \mathbb{R})$, we have

$$\begin{aligned} \omega(\xi) &\geq u_0 =: \Theta_3, \\ \mathfrak{F}\mathcal{N}\omega(\xi) &\geq M_3 \varkappa(K) \psi_p(u_1) =: \Theta_1, \\ Fu &\geq \frac{M_6 \Theta_3^{p-1}}{\Gamma(\gamma+1)} (\rho_3(K) - \rho_3(a))^{\gamma} =: \Theta_5. \end{aligned}$$

By using the same process above, we get

$$\begin{aligned} \Theta_1 &\leq \mid \mathfrak{F}\mathcal{N}\omega(\xi) \mid \leq \Theta_2, \\ \Theta_3 &\leq \parallel \omega \parallel \leq \Theta_4, \\ \Theta_5 &\leq \mid Fu \mid \leq \Theta_6, \end{aligned}$$

for each $\xi \in J_k$, $\omega \in PC(Q, \mathbb{R})$, $q = 1, 2, \ldots, k$. Hence, by applying Theorem 3.12, one can deduce that the p-Laplacian impulsive fractional boundary value problem (1.1) has a unique solution. □

Theorem 3.15 *Suppose that the assumptions* ($\boldsymbol{H}_5^{\star\star\star}$) *and* ($\boldsymbol{H}_6^{\star\star\star}$) *are satisfied. If*

$$\Lambda^{\star}, C^{10}, \Lambda^{\star\star} < 1, \tag{3.31}$$

where

$$\Theta_1 =: -M_3 \varkappa(K) \psi_p(u_1),$$

$$\Theta_2 := M_3 \left(-\varkappa(K) \psi_p(u_1) + \frac{\max_{\xi \in Q}(\Pi((\xi))}{\Gamma(\beta+1)} (\rho_2(K) - \rho_2(a))^{\beta} + \sum_{j=1}^{k} \chi_j^2(t_j) \right),$$

$$\Theta_3 =: -u_0, \quad \Theta_4 =: \frac{-\omega_0 + \sum_{j=1}^{k} \chi_j^1(t_j) + \frac{1}{\Gamma(\nu+1)}(\rho_1(K) - \rho_1(a))^{\nu} \Theta_2^{p^\star - 1}}{1 - C^{10}},$$

$$\Theta_5 =: \frac{M_5 \Theta_4^{p-1}}{\Gamma(\gamma+1)}(\rho_3(K) - \rho_3(a))^{\gamma}, \quad \Theta_6 =: \frac{M_6 \Theta_3^{p-1}}{\Gamma(\gamma+1)}(\rho_3(K) - \rho_3(a))^{\gamma},$$

$$C^{10} =: \sum_{j=1}^{k} e_j^1 - \lambda \frac{M_4^{p^\star - 1}}{\Gamma(\gamma+1)}(\rho_3(K) - \rho_3(a))^{(p^\star - 1))\gamma},$$

then the p-Laplacian impulsive fractional boundary value problem (1.1) *has a unique solution.*

Proof By following the similar steps as in the proof of Theorem 3.12, we can complete the proof of this theorem. □

4 Applications

In this section, we provide two examples to validate the applicability of the theorems proved in the previous section.

Example 4.1 Consider the following boundary value problem of impulsive differential equation

$$\begin{cases} {}^{C-H}\mathcal{D}^{\frac{3}{4}}_{e^{2-}} \left(\sqrt{\xi} \psi_p \left({}^{C-H}\mathcal{D}^{\frac{3}{4}}_{e^{1+}} \omega \right) \right) (\xi) + \ln(\xi) \psi_p (\omega(\xi)) \\ \quad = \dfrac{\sin(\xi) + \omega(\xi)}{10 \left(e^{|\omega(\xi)|} + |\, \omega(\xi) \,| \right)} \quad e^1 < \xi < e^2, \\ \Delta (\omega(u)) = |\, \omega(u) \,|^{1/2} \sin(\omega(u)), \\ \Delta \psi_p \left({}^{C-H}\mathcal{D}^{\frac{3}{4}}_{e^{1+}} \omega \right)(u) = |\, \omega(u) \,|^{1/2} \cos(\omega(u)), \\ \omega(e^1) = 1, {}^{C-H}\mathcal{D}^{\frac{3}{4}}_{e^{1+}} \omega(e^2) = 0. \end{cases} \tag{4.1}$$

here $u \in (e^1, e^2)$ and

$$\begin{array}{llll} \rho_1(\xi) = \rho_2(\xi) = t(\xi) = \ln(\xi), & \varkappa(\xi) = \sqrt{\xi}, & \nu = \beta = \frac{3}{4}, & \lambda = 0, \\ u \in (e^1, e^2), & p = 3, & p^\star = \frac{3}{2}, & u_1 = 0, \\ u_0 = 1, & t_1 = u, & k = 1. & \end{array}$$

Also, ${}^{C-H}\mathcal{D}^{\frac{3}{4}}_{e^{1+}}$ and ${}^{C-H}\mathcal{D}^{\frac{3}{4}}_{e^{2-}}$ are the left and right Caputo-Hadamard fractional derivatives.

It is easy to show that (4.1) is a special form of the p-Laplacian impulsive fractional boundary value problem (1.1).

Set

$$f(\xi,\omega(\xi))=\frac{\sin(\xi)+\omega(\xi)}{10\left(e^{|\omega(\xi)|}+\mid\omega(\xi)\mid\right)},\quad (\xi,\omega)\in[e^1,e^2]\times\mathbb{R},$$
$$\mid f(\xi,\omega(\xi))\mid\leq\frac{1}{10}+\frac{1}{10}\mid\omega(\xi)\mid,\quad (\xi,\omega)\in[e^1,e^2]\times\mathbb{R}.$$

Set

$$I^1(\omega(\xi)=\mid\omega(\xi)\mid^{1/2}\sin(\omega(\xi)),\; I^2(\omega(\xi)=\mid\omega(\xi)\mid^{1/2}\cos(\omega(\xi)).$$

Also,

$$\mid I^1(\omega(\xi)\mid\leq\mid\omega(\xi)\mid^{1/2},\quad \mid I^2(\omega(\xi)\mid\leq\mid\omega(\xi)\mid^{1/2}\quad (\xi,\omega)\in[e^1,e^2]\times\mathbb{R}.$$

- It is straightforward to show that the assumptions in Theorem 3.7 hold. Therefore, it follows that the p-Laplacian impulsive fractional boundary value problem (4.1) has a solution in $PC([e^1,e^2],\mathbb{R})$.
- It is straightforward to show that all the conditions of Theorem 3.8 satisfy the p-Laplacian impulsive fractional boundary value problem (4.1). Therefore, we can conclude that there is one solution for the p-Laplacian impulsive fractional boundary value problem (4.1) in the space of piecewise continuous functions on the interval $[e^1,e^2]$ with values in the set of real numbers.

Example 4.2 Consider the following boundary value problem of impulsive differential equation

$$\begin{cases} {}^{\sin(\upsilon/2);C}_{\xi}\mathcal{D}^{\frac{4}{5}}_{1^-}\left((18+2e^{\xi})\psi_{\frac{3}{2}}\left({}^{\ln(\upsilon+1);C}_{\xi}\mathcal{D}^{\frac{4}{5}}_{0^+}\omega\right)\right)(\xi) \\ \quad =f(\xi,\omega(\xi))-\frac{\xi^2\sin(\xi+1)}{10}\psi_{\frac{3}{2}}(\omega(\xi))\quad 0<\xi<1, \\ \Delta\left(\omega\left(\frac{1}{2}\right)\right)=\frac{1}{20}\left(\mid\sin(\omega(u))\mid+\exp\left(-\frac{1}{2}\left|\omega\left(\frac{1}{2}\right)\right|\right)\right), \\ \Delta\psi_{\frac{3}{2}}\left({}^{\ln(\upsilon+1);C}_{\xi}\mathcal{D}^{\frac{4}{5}}_{0^+}\omega\right)\left(\frac{1}{2}\right)=\frac{1}{10}\exp\left(-\frac{1}{2}\left|\omega\left(\frac{1}{2}\right)\right|\right), \\ \omega(0)=\frac{1}{2}+\lambda\left|{}^{(\upsilon+1)^2}_{\xi}\mathcal{I}^{\gamma}_{a^+}\frac{\xi^2+1}{10}\sqrt{\mid\omega(\xi)\mid}\right|^2_{\xi=1},\quad {}^{\ln(\upsilon+1);C}\mathcal{D}^{\frac{4}{5}}_{0^+}\omega(1)=\frac{1}{5}, \end{cases} \tag{4.2}$$

where

$$f(\xi,\omega)=\frac{\omega^2}{(19+e^{\xi})(1+\omega^2)}+\frac{\mid\sin(\omega)\mid\xi^2}{10}$$
$$+\frac{e^{-\xi}\mid\omega\mid}{(18+2e^{-\xi})(\omega^2+1)}+\frac{\xi^2\sin(\xi+1)}{10}\psi_{\frac{3}{2}}(\omega),$$
$$I^1(\omega)=\frac{1}{20}\left(\mid\sin(\omega(u))\mid+\exp\left(-\frac{1}{2}\left|\omega\left(\frac{1}{2}\right)\right|\right)\right),$$

$$I^2(\omega) = \frac{1}{10}\exp\left(-\frac{1}{2}\left|\omega\left(\frac{1}{2}\right)\right|\right).$$

here

$$\begin{array}{lll} \rho_1 = \ln(\upsilon+1), & \rho_2 = \sin(\frac{\upsilon}{2}), & \rho_3 = (\upsilon+1)^2, \\ \varkappa(\xi) = 18 + 2e^{\xi}, & t(\xi) = \frac{\xi^2 \sin(\xi+1)}{10}, & \eta(\xi) = \frac{\xi^2+1}{10}, \\ \nu = \beta = \frac{4}{5}, & p = \frac{3}{2}, p^{\star} = 3, & t_1 = \frac{1}{2},\ k = 1, \\ u_0 = \frac{1}{4}, & u_1 = \frac{1}{100}, & \lambda = \sqrt{\frac{\pi}{2^6}}. \end{array}$$

It is easy to show that (4.2) is a special form of the p-Laplacian impulsive fractional boundary value problem (1.1). Moreover, there exists a function $\Phi(\xi)(\xi) = \frac{1}{19+e^{\xi}} + \frac{\xi^2}{5} + \frac{1}{20}$ such that $| f(\xi, \omega(\xi)) - t(\xi)\psi_{\frac{3}{2}}(\omega(\xi)) | \leq \Phi(\xi)$.

One can see that the solution $\omega(\xi)$ of the p-Laplacian impulsive fractional boundary value problem (4.2), which is given by the integral equation (3.16), is well-defined and satisfies

$$\begin{aligned} &\Theta_1 \leq | \mathfrak{F}\mathcal{N}\omega(\xi) | \leq \Theta_2, \\ &\Theta_3 \leq \| \omega \| \leq \Theta_4, \\ &\Theta_5 \leq | Fu | \leq \Theta_6, \end{aligned}$$

where

$$\begin{aligned} &\Theta_1 = \psi_{\frac{3}{2}}\left(\frac{1}{100}\right),\ \Theta_2 = \frac{(19+2e)\sqrt{\pi} + 8\sin(\frac{1}{2})^{\frac{1}{2}}}{200\sqrt{\pi}}, \\ &\Theta_3 = \frac{1}{4},\ \Theta_4 = \frac{7\sqrt{\pi} + 40\Theta_2^2(\ln(2))^{\frac{1}{2}}}{20\sqrt{\pi}(1-C^8)},\ C^8 = \frac{6\lambda}{25\pi}, \\ &\Theta_5 = \frac{1}{5}\sqrt{\frac{3\Theta_3}{\pi}},\ \Theta_6 = \frac{2}{5}\sqrt{\frac{3\Theta_4}{\pi}}. \end{aligned}$$

It is straightforward to show that $f(\xi, \omega)$, $I_1(\omega)$ and $I_2(\omega)$ satisfy

$$\begin{aligned} &|f(\xi,\omega) - f(\xi,v)| \\ &= \left| \frac{1}{19+e^{\xi}}\left(\frac{\omega^2}{1+\omega^2} - \frac{v^2}{1+v^2}\right) + \frac{\xi^2}{10}(|\sin(\omega)| - |\sin(v)|) \right. \\ &\quad \left. + \frac{e^{-\xi}}{18+2e^{-\xi}}\left(\frac{|\omega|}{v^2+1} - \frac{|\omega|}{v^2+1}\right) + \frac{\xi^2\sin(\xi+1)}{10}\left(\psi_{\frac{3}{2}}(\omega) - \psi_{\frac{3}{2}}(v)\right)\right| \\ &\leq \frac{1}{20}\left|\frac{\omega^2}{1+\omega^2} - \frac{v^2}{1+v^2}\right| + \frac{1}{10}|\sin(\omega) - \sin(v)| \\ &\quad + \frac{1}{20}\left|\frac{\omega}{v^2+1} - \frac{\omega}{v^2+1}\right| + \frac{\sin(2)}{10}\left|\psi_{\frac{3}{2}}(\omega) - \psi_{\frac{3}{2}}(v)\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{20}\left|\frac{\omega^2}{1+\omega^2}-\frac{v^2}{1+v^2}\right|+\frac{1}{10}\left|\sin(\omega)-\sin(v)\right| \\
&\quad+\frac{1}{20}\left|\frac{\omega}{v^2+1}-\frac{\omega}{v^2+1}\right|+\frac{\sin(2)}{10}\left|\psi_{\frac{3}{2}}(\omega)-\psi_{\frac{3}{2}}(v)\right| \\
&\leq \frac{1}{10}|\omega-v|+\frac{1}{10}|\omega-v|+\frac{1}{10}|\omega-v|+\frac{\sqrt{2}\sin(2)}{20}|\omega-v| \\
&=\frac{6+\sqrt{2}\sin(2)}{20}|\omega-v|, \quad \xi\in[0,1],\ \omega,v\in PC([0,1],\mathbb{R}),
\end{aligned}
$$

$$
\begin{aligned}
&\mid I^1(\omega)-I^1(v)\mid \\
&=\frac{1}{20}\left|\left(\mid\sin(\omega)\mid-\mid\sin(\omega)\right|+\exp\left(-\frac{1}{20}\mid\omega\left(\frac{1}{2}\right)\right|\right)-\exp\left(-\frac{1}{2}\left|v\left(\frac{1}{2}\right)\right|\right)\Bigg)\Bigg| \\
&\leq \frac{1}{10}\mid\omega-v\mid, \quad (\xi,\omega)\in[0,1]\times\mathbb{R},
\end{aligned}
$$

and

$$
\begin{aligned}
\mid I^2(\omega)-I^2(v)\mid &= \frac{1}{10}\left|\exp\left(-\frac{1}{2}\left|\omega\left(\frac{1}{2}\right)\right|\right)-\exp\left(-\frac{1}{2}\left|v\left(\frac{1}{2}\right)\right|\right)\right| \\
&\leq \frac{1}{10}\mid\omega-v\mid, \quad (\xi,\omega)\in[0,1]\times\mathbb{R}.
\end{aligned}
$$

Also, we have

$$
\mid I^1(\omega(\xi)\mid\leq\frac{1}{10}, \quad \mid I^2(\omega(\xi)\mid\leq\frac{1}{10} \quad (\xi,\omega)\in[0,1]\times\mathbb{R}.
$$

Consequently, the conditions ($\mathbf{H}_5^\star$) and ($\mathbf{H}_6^\star$) hold, where $L^1 = L^2 = \frac{1}{10}$, $L = \frac{6+\sqrt{2}\sin(2)}{20}$. By some calculations, we get $\Lambda^\star \approx 0.1134991 < 1$.

Obviously, the p-Laplacian impulsive fractional boundary value problem (4.2) satisfies all the conditions of Theorem 3.13. Hence, the p-Laplacian impulsive fractional boundary value problem (4.2) has a unique solution.

5 Conclusion

Throughout this study, we explored the boundary value problem of impulsive differential equations with a nonlinear non-symmetric $\rho-$Caputo fractional derivative and an operator of p-Laplacian type. By utilizing the Schauder's and Schaefer's fixed point theorems, together with the Banach contraction principle, we established the existence and uniqueness of solutions for the impulsive p-Laplacian boundary value problem given by (1.1). Moreover, we provided two examples to show the applicability and significance of our main results. Furthermore, we derived a new representation formula for the integral solution of the p-Laplacian impulsive fractional boundary value problem (1.1) using the Heaviside function. Additionally, we established the existence and uniqueness of solutions under different conditions. These results shed light on the importance and relevance of the study of impulsive differential equations with nonlinear generalized fractional and p-Laplacian operators. Future work can be extended and delved deeper into the underlying mechanisms, potentially employing specific methodologies or techniques of functional analysis for generic p-Laplacian boundary value problems with (ψ, k)-Hilfer fractional derivatives to gain a more nuanced understanding. Additionally, investigating the long-term effects of the specific conditions on related outcomes can provide valuable insights into the persistence of the observed patterns during the time.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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