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# Asymptotic Behavior of Solutions to Difference Equations of Neutral Type

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**Abstract**
We present sufficient conditions for the existence of a solution $x$ to an equation

$$\Delta^m(x_n - u_n x_{n-k}) = a_n f(x_{n-\tau}) + b_n,$$

which is "close" to a given solution $y$ to the linear homogeneous equation of neutral type $\Delta^m(y_n - \lambda y_{n-k}) = 0$, where $\lambda$ is the limit of the sequence $u$. Closeness of solutions to above equations is understood as $x_n - y_n = \mathrm{o}(\omega_n)$, where $\omega$ is a given nonincreasing sequence with positive values. Moreover, we establish under which conditions for a given solution $x$ to $\Delta^m(x_n - u_n x_{n-k}) = a_n f(x_{n-\tau}) + b_n$ and a given nonincreasing sequence with positive values $\omega$ there exists a polynomial sequence $\varphi$ of degree less than $m$ such that $x_n = \varphi(n) + \mathrm{o}(\omega_n)$. Presented conditions strongly depend on $\lambda$.

**Keywords** Difference equation · Neutral type equation · Prescribed asymptotic behavior · Approximative solution

**Mathematics Subject Classification** 39A05 · 39A22

## 1 Introduction

Let $\mathbb{N}_0, \mathbb{N}, \mathbb{Z}, \mathbb{R}$ denote the set of nonnegative integers, the set of positive integers, the set of all integers and the set of all real numbers, respectively. In this paper we consider the difference equation of neutral type of the form

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$$\Delta^m(x_n - u_n x_{n-k}) = a_n f(x_{n-\tau}) + b_n \tag{E}$$

where

$$m \in \mathbb{N}, \quad a_n, b_n, u_n \in \mathbb{R}, \quad k \in \mathbb{N}, \quad \tau \in \mathbb{Z}, \quad f : \mathbb{R} \to \mathbb{R},$$
$$\Delta x_n := x_{n+1} - x_n, \quad \Delta^j x_n := \Delta\left(\Delta^{j-1} x_n\right), \; j = 2, 3, \ldots, m.$$

By a solution of (E) we mean a sequence $x : \mathbb{N}_0 \to \mathbb{R}$ satisfying (E) for all large $n$.

Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a solution to $\Delta^m(y_n - \lambda y_{n-k}) = 0$, with $\lim_{n\to\infty} u_n = \lambda$ and $\omega : \mathbb{N}_0 \to (0, \infty)$ be a nonincreasing, which we understand as "measure of approximation" of solutions to (E) and $\Delta^m(y_n - \lambda y_{n-k}) = 0$. In this paper we want to answer two questions. Firstly, for given $y$ and $\omega$ we construct sufficient conditions which guarantee the existence of a solution $x$ to (E) such that $x_n = y_n + \mathrm{o}(\omega_n)$. Then $y$ is called *an approximative solution* to (E) and $x$ is called *a solution with prescribed behavior*. Secondly, for a given solution $x$ to (E) and "measure of approximation" $\omega$ we show sufficient conditions which imply that there exists a polynomial sequence $\varphi$ such that $\deg \varphi < m$ and $x_n = \varphi(n) + \mathrm{o}(\omega_n)$. Note that $\varphi$ is a solution to $\Delta^m(y_n - \lambda y_{n-k}) = 0$.

Results in this paper are the continuation of studies in [15–19, 21, 22] and generalize these studies in two directions. Firstly, we consider a general class of "measures of approximation" which is defined by a nonincreasing sequence $\omega$ with positive values instead of $\mathrm{o}(n^s)$ with $s \le 0$. Let us recall that any solution $y$ to $\Delta^m(y_n - \lambda y_{n-k}) = 0$ is of the form $y_n = \varphi(n) + \mathrm{O}(\rho^n)$, where $\varphi$ is a polynomial sequence with $\deg \varphi < m$ and $\rho = \sqrt[k]{|\lambda|}$. If $|\lambda| < 1$, then the polynomial part $\varphi$ of $y$ is dominating. If $|\lambda| > 1$, then the geometric part of $y$ is dominating. Our second goal is to get results not only in the case if the polynomial part is dominating, but also in the case if the geometric part of solutions to $\Delta^m(y_n - \lambda y_{n-k}) = 0$ is dominating with nonconstant sequence $u$ in (E). It is worth noting that theorems for $|\lambda| < 1$ and $|\lambda| > 1$ differ only by one assumption on $u$. The assumption on the sequence $u$ for $|\lambda| > 1$ is stronger, then for $|\lambda| < 1$. The fixed point approach was applied to get our main results. More precisely we use the generalization of the Krasnoselskii fixed point theorem which was proved in [15]. Note the usage of Krasnoselskii's fixed point theorem excludes the case $|\lambda| \neq 1$. Moreover, we use properties of the remainder operator which were widely used in [13–15].

Asymptotic behavior of differential or difference equation of neutral type were considered in many papers. This topic can be explored in several directions. Solutions with prescribed behavior were investigated for example in [1–3, 6, 9, 10, 12, 14, 15, 17, 19, 26–29, 33]. Oscillatory solutions were studied among others in [4, 5, 8, 20, 21, 31–34]. Asymptotically polynomial solutions were considered, for example in [7, 11, 16].

The texture of this paper is as follows: after introducing our notation, Sect. 2 provides necessary information about auxiliary tools: the remainder operator and general solutions to $\Delta^m(y_n - \lambda y_{n-k}) = 0$. Section 3 is devoted to the presentation of sufficient conditions for the existence of a solution with prescribed behavior to (E). In Sect. 4, for a given solution $x$ to (E) we find conditions under which, there exists a polynomial sequence $\varphi$ which is close to $x$ according given "measure of approximation" $\omega$.

## 2 Preliminaries

Let us start with some basic definitions and notations for our paper. The space of all sequences $x : \mathbb{N}_0 \to \mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}_0}$, $|x|$ denotes the sequence defined by $|x|(n) = |x_n|$, for $x \in \mathbb{R}^{\mathbb{N}_0}$. Moreover,

$$\|x\| = \sup_{n\in\mathbb{N}_0} |x_n|,$$

and

$$c_0 = \{x \in \mathbb{R}^{\mathbb{N}_0} : \lim_{n\to\infty} x_n = 0\}, \quad \mathbb{R}^* = \mathbb{R} \setminus \{0\}, \quad \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}.$$

We say that a subset $X$ of $\mathbb{R}^{\mathbb{N}_0}$ is *ordinary* if $\|x - y\| < \infty$ for any $x, y \in X$. We regard any ordinary subset $X$ of $\mathbb{R}^{\mathbb{N}_0}$ as a metric space with metric defined by

$$d(x, y) = \|x - y\|. \tag{1}$$

Let $f : \mathbb{R} \to \mathbb{R}$. We say that a sequence $y \in \mathbb{R}^{\mathbb{N}_0}$ is *uniformly $f$-bounded* if there exists a positive number $\delta$ such that $f$ is bounded on the set

$$\bigcup_{n=0}^{\infty}[y_n - \delta, y_n + \delta].$$

For $m \in \mathbb{N}_0$ we define

$$\mathrm{Pol}(m-1) = \mathrm{Ker}\, \Delta^m = \{x \in \mathbb{R}^{\mathbb{N}_0} : \Delta^m x = 0\}.$$

Then $\mathrm{Pol}(m-1)$ is the space of all polynomial sequences of degree less than $m$. Note that $\mathrm{Pol}(-1) = \mathrm{Ker}\, \Delta^0 = 0$ is the zero space.

### 2.1 Remainder Operator

Properties of remainder operators were considered, for example in [13]. Let us recall some of them, which are crucial in our considerations. Let $m \in \mathbb{N}$,

$$\mathrm{A}(m) = \left\{ a \in \mathbb{R}^{\mathbb{N}_0} : \sum_{i_1=0}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}| < \infty \right\}.$$

For $a \in \mathrm{A}(m)$, $r^m(a)$ denotes the sequence defined by

$$r^m(a)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} a_{i_m}, \quad n \in \mathbb{N}_0. \tag{2}$$

Then

$$r^m(a)(n) = \mathrm{o}(1). \tag{3}$$

Moreover, it is known (see for example Lemma 3.1, [13]) that

$$\mathrm{A}(m) = \{a \in \mathbb{R}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} n^{m-1}|a_n| < \infty\}$$

and

$$r^m(a)(n) = \sum_{i=n}^{\infty} \binom{m-1+i-n}{m-1} a_i, \;\; \Delta^m(r^m(a))(n) = (-1)^m a_n,$$

$$|r^m(a)(n)| \leq r^m(|a|)(n), \quad r^m(|a|)(n) \leq \sum_{i=n}^{\infty} i^{m-1}|a_i| \tag{4}$$

for any $a \in \mathrm{A}(m)$ and $n \in \mathbb{N}_0$. Moreover, we recall the following general result

**Lemma 2.1** [24, Lemma 1] *Assume* $m \in \mathbb{N}$, $a \in \mathbb{R}^{\mathbb{N}_0}$, $\omega : \mathbb{N}_0 \to (0, \infty)$, $\Delta\omega \leq 0$, *and*

$$\sum_{n=0}^{\infty} \frac{n^{m-1}|a_n|}{\omega_n} < \infty.$$

*Then* $a \in \mathrm{A}(m)$ *and* $r^m(a)(n) = \mathrm{o}(\omega_n)$.

### 2.2 Fundamental Equation of Neutral Type

Let us remind some basic information about a general solution to a linear homogeneous difference equations of neutral type of the order $m$, which were considered [15]. Let $m \in \mathbb{N}$, $k \in \mathbb{Z}^*$, $\lambda \in \mathbb{R}^*$. We consider equations

$$\Delta^m(x_n - \lambda x_{n-k}) = 0 \tag{F}$$

$$x_n - \lambda x_{n-k} = 0 \tag{G}$$

which we call a *fundamental equation of neutral type* and a *geometric equation*, respectively. By a solution of (F) we mean a real sequence $x$ such that (F) is satisfied for all $n \geq \max(0, k)$. Analogously, we define solutions of (G). We denote by

$$\mathrm{PG}(m, \lambda, k), \qquad \mathrm{Geo}(\lambda, k)$$

the set of all solutions of (F) and (G), respectively. Let $x, y \in \mathbb{R}^{\mathbb{N}_0}$. If

$$x_{n+|k|} = x_n, \qquad y_{n+|k|} = -y_n$$

for any $n \in \mathbb{N}_0$, then we say that $x$ is *$k$-periodic* and $y$ is *$k$-alternating*. We denote by

$$\mathrm{Per}(k), \qquad \mathrm{Alt}(k)$$

the set of all $k$-periodic sequences and the set of all $k$-alternating sequences, respectively. Note that $\mathrm{Per}(k)$, and $\mathrm{Alt}(k)$ are linear subspaces of $\mathbb{R}^{\mathbb{N}_0}$ and

$$\dim \mathrm{Per}(k) = |k| = \dim \mathrm{Alt}(k), \qquad \mathrm{Alt}(k) \subset \mathrm{Per}(2k).$$

We define

$$n \operatorname{div} k := (\operatorname{sgn} k) \max\{j \in \mathbb{Z} : \ j|k| \le n\}, \quad n \bmod k := n - |k| \, (n \operatorname{div} |k|)\,,$$
$$\mathrm{geo}(\lambda, k), \mathrm{alt}(k) : \mathbb{N}_0 \to \mathbb{R}, \quad \mathrm{geo}(\lambda, k)(n) = \lambda^{n \operatorname{div} k}, \quad \mathrm{alt}(k) = \mathrm{geo}(-1, k).$$

Note that

$$n \operatorname{div}(-k) = -(n \operatorname{div} k), \quad n \bmod (-k) = n \bmod k, \quad \mathrm{geo}(\lambda, -k) = \mathrm{geo}(\lambda^{-1}, k).$$

Moreover, $\mathrm{geo}(\lambda, k)$ is an "expanded" geometric sequence. Note also, that for a fixed $k$, the sequence $(n \bmod k)$ is $k$-periodic.

**Lemma 2.2** (Solutions of geometric equation)[15, Theorem 3.1] *If $k \in \mathbb{Z}^*$ and $\lambda \in \mathbb{R}^*$, then a sequence $x : \mathbb{N}_0 \to \mathbb{R}$ is a solution of the geometric equation* (G) *if and only if for any $n \in \mathbb{N}_0$ we have*

$$x_n = \lambda^{n \operatorname{div} k} x_{n \bmod k}.$$

**Lemma 2.3** (Solutions of fundamental equation)[15, Theorem 3.1] *If $k \in \mathbb{Z}^*$, $\lambda \in \mathbb{R}^*$, then*

$$\mathrm{PG}(m, \lambda, k) = \mathrm{Pol}(m-1) \oplus \mathrm{Geo}(\lambda, k),$$

*Moreover, if $\rho = \sqrt[k]{|\lambda|}$, then $k(|\lambda| - 1)(\rho - 1) \ge 0$ and*

$$\mathrm{Geo}(\lambda, k) = \mathrm{geo}(\lambda, k)\,\mathrm{Per}(k) = \begin{cases} (\rho^n)\,\mathrm{Per}(k) & \text{if } \lambda > 0 \\ (\rho^n)\,\mathrm{Alt}(k) & \text{if } \lambda < 0 \end{cases},$$
$$\mathrm{Alt}(k) = \mathrm{alt}(k)\,\mathrm{Per}(k).$$

Hence any solution $y \in \mathrm{PG}(m, \lambda, k)$ of the fundamental equation

$$\Delta^m (y_n - \lambda y_{n-k}) = 0$$

is of the form

$$y_n = \varphi(n) + \omega_n \rho^n = \varphi(n) + \mathrm{O}(\rho^n), \tag{5}$$

where $\varphi \in \mathrm{Pol}(m-1)$, $\rho = \sqrt[k]{|\lambda|}$, and $\omega$ is $2k$-periodic.

Moreover, if $k(|\lambda| - 1) < 0$, then $\rho < 1$ and the polynomial part $\varphi$ of $y$ is dominating. On the other hand, if $k(|\lambda| - 1) > 0$, then $\rho > 1$ and the geometric part $\omega_n \rho^n$ is dominating (see Remark 3.3, [15]).

## 3 Approximative Solutions

Before we prove our main results we present the following lemma. For some related results see [25] and [30].

**Lemma 3.1** *Assume $k \in \mathbb{N}$, $x, z, u \in \mathbb{R}^{\mathbb{N}_0}$, $\omega : \mathbb{N}_0 \to (0, \infty)$,*

$$\limsup_{n\to\infty} \frac{|u_n|\omega_{n-k}}{\omega_n} < \alpha < 1,$$

*$z_n = x_n - u_n x_{n-k}$ for large $n$, and $z_n = \mathrm{o}(\omega_n)$. Then $x_n = \mathrm{o}(\omega_n)$.*

***Proof*** Choose $n_0 \geq k$ such that $z_n = x_n - u_n x_{n-k}$ and $|u_n|(\omega_{n-k}/\omega_n) \leq \alpha$ for $n \geq n_0$. Let

$$w_n = \frac{z_n}{\omega_n}, \text{ for } n \geq n_0; \qquad y_n = \frac{x_n}{\omega_n}, \text{ for } n \in \mathbb{N}_0; \qquad t_n = u_n \frac{\omega_{n-k}}{\omega_n}, \text{ for } n \geq k. \tag{6}$$

Then for $n \geq n_0$ we have $|t_n| \leq \alpha$ and $w_n = y_n - t_n y_{n-k}$. Hence

$$|y_{n+k}| \leq |w_{n+k}| + \alpha|y_n| \tag{7}$$

for $n \geq n_0$. Since $z_n = \mathrm{o}(\omega_n)$, then $w_n = \mathrm{o}(1)$ and there exists a positive constant $K$ such that $|w_n| \leq K$ for any $n \geq n_0$. Let

$$M = \max(y_{n_0}, y_{n_0+1}, \ldots, y_{n_0+k-1}).$$

Assume $n \geq n_0$. There exist $i \in \{n_0, \ldots, n_0 + k - 1\}$ and $l \in \mathbb{N}$ such that $n = i + lk$. Using (7) we have

$$|y_{i+2k}| \leq |w_{i+2k}| + \alpha|y_{i+k}| \leq K + \alpha(K + \alpha|y_i|) = K + \alpha K + \alpha^2 M.$$

Analogously

$$|y_{i+3k}| \leq K(1 + \alpha + \alpha^2) + \alpha^3 M.$$

After $l$ steps we obtain

$$|y_n| = |y_{i+lk}| \le K(1+\alpha+\cdots+\alpha^{l-1}) + \alpha^l M = K\frac{1-\alpha^l}{1-\alpha} + \alpha^l M.$$

Hence

$$|y_n| < \frac{K}{1-\alpha} + \alpha^l M < \frac{K}{1-\alpha} + M$$

for any $n \ge n_0$. Therefore the sequence $y$ is bounded. Choose a constant $P$ such that $|y_n| \le P$ for any $n$. Let $\varepsilon > 0$. There exist $q \in \mathbb{N}$, and $n_1 \ge n_0$ such that $\alpha^q < \varepsilon$ and $|w_n| < \varepsilon$ for $n \ge n_1$. Let $n \ge n_1 + qk$. Then there exist $i \in \{n_1, \dots, n_1+k-1\}$ and $l \in \mathbb{N}$, $l \ge q$ such that $n = i + lk$. Similarly as above one can show that

$$|y_n| \le \frac{\varepsilon}{1-\alpha} + \alpha^l P \le \left(\frac{1}{1-\alpha} + P\right)\varepsilon.$$

Hence $y_n = \mathrm{o}(1)$ and $x_n = \omega_n y_n = \omega_n \mathrm{o}(1) = \mathrm{o}(\omega_n)$. The proof is complete. □

To achieve our main results we use the following version of Krasnoselskii's fixed point theorem. For $x, y, \rho \in \mathbb{R}^{\mathbb{N}_0}$ $|x-y| \le |\rho|$ means $|x-y|(n) \le |\rho(n)|$, for $n \in \mathbb{N}_0$.

**Lemma 3.2** *Assume $y \in \mathbb{R}^{\mathbb{N}_0}$, $\rho \in c_0$, $X = \{x \in \mathbb{R}^{\mathbb{N}_0} : |x-y| \le |\rho|\}$, $A, B : X \to \mathbb{R}^{\mathbb{N}_0}$, $AX + BX \subset X$, $\alpha \in (0,1)$, $A$ is continuous and $B$ is an $\alpha$-contraction. Then there exists a point $x \in X$ such that $Ax + Bx = x$.*

***Proof*** The assertion is a consequence of [15, Lemma 2.2 and Theorem 2.2]. □

Now we are in a position to formulate and prove the first of our main theorems. We recall that asymptotic behavior of solutions to $\Delta^m(y_n - \lambda y_{n-k}) = 0$ strongly depends on $\lambda$, which means that for $|\lambda| < 1$ the polynomial part of $y$ is dominating and for $|\lambda| > 1$ the geometric part of $y$ is dominating. It is worth noting that an assumption on $\lambda$ is included in the condition

$$\limsup_{n\to\infty} \frac{|\lambda|\omega_{n-k}}{\omega_n} < 1,$$

because $\omega$ is nonincreasing with positive values.

**Theorem 3.1** *Assume $\lambda \in \mathbb{R}^*$, $k \in \mathbb{N}$, $\tau \in \mathbb{Z}$, $\omega : \mathbb{N}_0 \to (0,\infty)$ is nonincreasing,*

$$\limsup_{n\to\infty} \frac{|\lambda|\omega_{n-k}}{\omega_n} < 1, \qquad \sum_{n=0}^{\infty} \frac{n^{m-1}(|a_n| + |b_n|)}{\omega_n} < \infty, \tag{8}$$

*$f$ is continuous, and $u_n = \lambda + \mathrm{o}(n^{1-m}\omega_n)$. Then for any uniformly $f$-bounded sequence $y \in \mathrm{PG}(m, \lambda, k)$ there exists a solution $x$ of (E) such that*

$$x_n = y_n + \mathrm{o}(\omega_n).$$

***Proof*** The proof of the theorem is a nontrivial modification of the proofs of some theorems and lemmas in [16] and theorem 1 in [23]. For $x \in \mathbb{R}^{\mathbb{N}_0}$ let

$$x'_n = \begin{cases} 0 & \text{for } n < \max(0, \tau) \\ a_n f(x_{n-\tau}) + b_n & \text{for } n \geq \max(0, \tau) \end{cases}. \tag{9}$$

Let $y \in \mathrm{PG}(m, \lambda, k)$ be $f$-uniformly bounded. Choose $\delta, L > 0$ such that

$$|f(t)| \leq L \quad \text{for any} \quad t \in \bigcup_{n=0}^{\infty} [y_n - \delta, y_n + \delta]. \tag{10}$$

Since the sequence $\omega$ is nonincreasing, $\omega_{n-k}/\omega_n \geq 1$ for any $n \geq k$, and using (8) we get $|\lambda| < 1$. There exists $\xi > 1$ such that

$$\limsup_{n\to\infty} \frac{\xi|\lambda|\omega_{n-k}}{\omega_n} < 1.$$

Put $\alpha := \xi|\lambda|$. We have $|\lambda| < \alpha < 1$. Let $z', \rho', \gamma' \in \mathbb{R}^{\mathbb{N}_0}$, $\gamma'_n > 0$ for $n = 0, \dots, k-1$ and

$$z'_n = (\lambda - u_n) y_{n-k}, \qquad \rho'_n = r^m(L|a| + |b|)(n) + |z'_n|, \tag{11}$$

$$\gamma'_n = \rho'_n + \alpha\gamma'_{n-k} \tag{12}$$

for $n \geq k$. By $|\lambda| < 1$ and (5), $y_n = \mathrm{O}(n^{m-1})$ and so $y_{n-k} = \mathrm{O}(n^{m-1})$. Hence

$$z'_n = \mathrm{o}(n^{1-m}\omega_n)\mathrm{O}(n^{m-1}) = \mathrm{o}(\omega_n). \tag{13}$$

Using (8), (11), (13), and Lemma 2.1 we get $\rho'_n = \mathrm{o}(\omega_n)$. For $n \geq k$ we have

$$\rho'_n = \gamma'_n - \alpha\gamma'_{n-k}.$$

By definition and Lemma 3.1,

$$\gamma'_n > 0 \text{ for } n \in \mathbb{N}_0, \quad \gamma'_n = \mathrm{o}(\omega_n). \tag{14}$$

Choose an index $p \geq k$ such that

$$0 < \gamma'_n < \delta, \qquad |u_n| < \alpha \tag{15}$$

for $n \geq p$. Now, let $z, \rho, \gamma \in \mathbb{R}^{\mathbb{N}_0}$ and $A, B, R : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$ are defined as follows: for $n < p$ and $x \in \mathbb{R}^{\mathbb{N}_0}$

$$z_n = \rho_n = \gamma_n = B(x)(n) = R(x)(n) = 0, \tag{16}$$

for $n \geq p$ and $x \in \mathbb{R}^{\mathbb{N}_0}$

$$z_n = z'_n, \quad \rho_n = \rho'_n, \quad \gamma_n = \gamma'_n, \tag{17}$$
$$R(x)(n) = (-1)^m r^m(x')(n) \quad \text{and} \quad B(x)(n) = u_n(x_{n-k} - y_{n-k}). \tag{18}$$

Moreover, let

$$Ax = y - z + Rx \qquad \text{and} \qquad X = \{x \in \mathbb{R}^{\mathbb{N}_0} : |x - y| \leq \gamma\}. \tag{19}$$

By (14), (17) and assumptions on $\omega$, we have

$$\gamma_n = \mathrm{o}(\omega_n) \subset \mathrm{o}(1). \tag{20}$$

By (15), (16), and (17) we have $0 \leq \gamma_n < \delta$ for any $n$. Therefore, if $x \in X$, then, by (10), $|f(x_n)| \leq L$ for any $n$. Thus, using (4) and (9), we have

$$|R(x)(n)| = |r^m(x')(n)| \leq r^m(|x'|)(n) \leq r^m(L|a| + |b|)(n)$$

for $n \geq p$. Hence, using (11) and (17), we get

$$|R(x)(n)| \leq \rho'_n - |z'_n| = \rho_n - |z_n| \tag{21}$$

for $n \geq p$. If $t \in X$, then $|t - y| \leq \gamma$ and, by (15), we have

$$|B(t)(n)| = |u_n||t_{n-k} - y_{n-k}| \leq \alpha\gamma_{n-k}$$

for $n \geq p$. Hence, using (12), (17), (19), and (21), we obtain

$$|Ax + Bt - y|(n) \leq |R(x)(n)| + |z_n| + |B(t)(n)| \leq \rho_n + \alpha\gamma_{n-k} = \gamma_n$$

for $n \geq p$. Therefore, by (19), we get $Ax + Bt \in X$. Thus

$$AX + BX \subset X.$$

We will show that $A$ is continuous on $X$. Let $\varepsilon > 0$. There exist an index $q$ and $\beta > 0$ such that

$$2L \sum_{n=q}^{\infty} n^{m-1}|a_n| < \varepsilon \qquad \text{and} \qquad \beta \sum_{n=0}^{q} n^{m-1}|a_n| < \varepsilon.$$

Let $x \in X$, and

$$W = [x_0 - 1, x_0 + 1] \cup [x_1 - 1, x_1 + 1] \cup \cdots \cup [x_q - 1, x_q + 1].$$

Then $W$ is compact and $f$ is uniformly continuous on $W$. Choose $\mu_\beta \in (0, 1)$ such that for $s, t \in W$ the condition $|s - t| < \mu_\beta$ implies $|f(s) - f(t)| < \beta$. Assume $v \in X$, $\|x - v\| < \mu_\beta$. Then,

$$\|Ax - Av\| = \|Rx - Rv\| = \sup_{n\geq 0} |r^m(x' - v')(n)| \leq \sup_{n\geq 0} r^m(|x' - v'|)(n)$$

$$= r^m(|x' - v'|)(0) \leq \sum_{n=0}^{\infty} n^{m-1}|(x' - v')(n)| \leq \sum_{n=0}^{q} n^{m-1}|a_n(f(x_n) - f(v_n))|$$

$$+ \sum_{n=q}^{\infty} n^{m-1}|a_n(f(x_n) - f(v_n))| \leq \beta \sum_{n=0}^{q} n^{m-1}|a_n| + 2L \sum_{n=q}^{\infty} n^{m-1}|a_n| < \varepsilon + \varepsilon.$$

Hence $A$ is continuous on $X$. Let $x, v \in \mathbb{R}^{\mathbb{N}_0}$. By (16) and (18), we have

$$\|Bx - Bv\| = \sup_{n\geq p} |B(x)(n) - B(v)(n)| = \sup_{n\geq p} |u_n(x_{n-k} - v_{n-k})|.$$

Hence, by (15), we obtain

$$\|Bx - Bv\| \leq \alpha \sup_{n\geq p} |x_{n-k} - v_{n-k}| \leq \alpha\|x - v\|.$$

Therefore $B$ is an $\alpha$-contraction. By Lemma 3.2, there exists a point $x \in X$ such that $x = Ax + Bx$. Then, by (18) and (19) we have

$$x_n = R(x)(n) + y_n - z_n + u_n(x_{n-k} - y_{n-k}).$$

for $n \geq p$. Using (11), (17) and (18), we get

$$\begin{aligned} x_n - u_n x_{n-k} &= y_n - z_n - u_n y_{n-k} + (-1)^m r^m(x')(n) \\ &= y_n - \lambda y_{n-k} + (-1)^m r^m(x')(n) \end{aligned}$$

for $n \geq p$. Since $y \in \mathrm{PG}(m, \lambda, k)$, we have

$$0 = \Delta^m(y_n - \lambda y_{n-k})$$

Hence, by (9), we obtain

$$\begin{aligned} &\Delta^m(x_n - u_n x_{n-k}) = \Delta^m(y_n - \lambda y_{n-k}) \\ &+\Delta^m(-1)^m r^m(x')(n) = 0 + x'_n = a_n f(x_n) + b_n \end{aligned}$$

for $n \geq p$. Therefore $x$ is a solution of (E). Since $x \in X$, by (19), $|x - y| < \gamma$. Moreover, by (20), $\gamma_n = \mathrm{o}(\omega_n)$. Hence $x_n - y_n = \mathrm{o}(\omega_n)$ and we obtain

$$x_n = y_n + (x - y)_n = y_n + \mathrm{o}(\omega_n).$$

□

Before we show a theorem for $|\lambda| > 1$ we need another auxiliary lemma.

**Lemma 3.3** *If $k \in \mathbb{N}$, $\alpha \in (0, 1)$, $\rho$ is a nonincreasing sequence with positive values and*

$$\gamma_{n+k} = \rho_{n+k} + \alpha\gamma_n$$

*with $\gamma_i$, $i = 0, \ldots, k-1$ satisfying*

$$\gamma_i > (1-\alpha)^{-1}\rho_{i+k}, \quad i = 0, \ldots, k-1,$$

*then*

$$\gamma_{n+k} < \gamma_n,$$

*for $n \geq k$ and*

$$\gamma_n > (1-\alpha)^{-1}\rho_{n+k},$$

*for $n \in \mathbb{N}_0$.*

***Proof*** Let $i \in \{0, \ldots, k-1\}$. We have

$$\gamma_{k+i} = \rho_{k+i} + \alpha\gamma_i < (1-\alpha)\gamma_i + \alpha\gamma_i = \gamma_i$$

and, by monotonicity and positivity of $\rho$,

$$\gamma_{k+i} = \rho_{k+i} + \alpha\gamma_i > \rho_{k+i} + \alpha(1-\alpha)^{-1}\rho_{k+i} = \rho_{k+i}(1-\alpha)^{-1} \geq \rho_{2k+i}(1-\alpha)^{-1}.$$

Moreover,

$$\gamma_{2k+i} = \rho_{2k+i} + \alpha\gamma_{k+i} < (1-\alpha)\gamma_{k+i} + \alpha\gamma_{k+i} = \gamma_{k+i}$$

and, by monotonicity and positivity of $\rho$,

$$\begin{aligned}\gamma_{2k+i} &= \rho_{2k+i} + \alpha\gamma_{k+i} > \rho_{2k+i} + \alpha(1-\alpha)^{-1}\rho_{2k+i} \\ &= \rho_{2k+i}(1-\alpha)^{-1} \geq \rho_{3k+i}(1-\alpha)^{-1}.\end{aligned}$$

Analogously, by induction,

$$\gamma_{lk+i} < \gamma_{(l-1)k+i}, \quad \gamma_{lk+i} > (1-\alpha)^{-1}\rho_{(l+1)k+i}$$

for any $l \in \mathbb{N}$. $\square$

In the case $|\lambda| > 1$ we need a stronger assumption on $u$, then in the case $|\lambda| < 1$. The rest of the assumptions of the theorem are the same. As previously the assumption on $\lambda$ is included in the condition

$$\omega \text{ is nonincreasing and } \quad \limsup_{n\to\infty} \frac{\omega_{n-k}}{|\lambda|\omega_n} < 1.$$

**Theorem 3.2** *Assume* $\lambda \in \mathbb{R}^*$, $k \in \mathbb{N}$, $\tau \in \mathbb{Z}$, $\omega : \mathbb{N}_0 \to (0, \infty)$ *is nonincreasing,*

$$\limsup_{n\to\infty} \frac{\omega_{n-k}}{|\lambda|\omega_n} < 1, \qquad \sum_{n=0}^{\infty} \frac{n^{m-1}(|a_n| + |b_n|)}{\omega_n} < \infty, \tag{22}$$

$f$ *is continuous, and* $u_n = \lambda + \mathrm{o}\left(\left(\sqrt[k]{|\lambda|}\right)^{-n} \omega_n\right)$. *Then for any uniformly* $f$*-bounded sequence* $y \in \mathrm{PG}(m, \lambda, k)$ *there exists a solution* $x$ *of* (E) *such that*

$$x_n = y_n + \mathrm{o}(\omega_n).$$

***Proof*** For $x \in \mathbb{R}^{\mathbb{N}_0}$ let

$$x'_n = \begin{cases} 0 & \text{for } n < \max(0, \tau) \\ a_n f(x_{n-\tau}) + b_n & \text{for } n \geq \max(0, \tau) \end{cases}. \tag{23}$$

Let $y \in \mathrm{PG}(m, \lambda, k)$ be $f$-uniformly bounded. Choose $\delta, L > 0$ such that

$$|f(t)| \leq L \quad \text{for any} \quad t \in \bigcup_{n=0}^{\infty} [y_n - \delta, y_n + \delta]. \tag{24}$$

Since the sequence $\omega$ is nonincreasing, $\omega_{n-k}/\omega_n \geq 1$ for any $n \geq k$, and using (22) we get $|\lambda| > 1$. There exists $\xi > 1$ such that

$$\limsup_{n\to\infty} \frac{\xi\omega_{n-k}}{|\lambda|\omega_n} < 1.$$

Taking into account that $u_n \to \lambda$ and $|\lambda| > 1$ we assume, without loss of generality, that $\inf_{n\geq k} |u_n| > 0$. Put $\alpha := \frac{\xi}{|\lambda|}$. We have $\frac{1}{|\lambda|} < \alpha < 1$. Let $z', \tilde{z}', \rho', \gamma' \in \mathbb{R}^{\mathbb{N}_0}$, and

$$z'_n = \tfrac{\lambda - u_{n+k}}{u_{n+k}} y_n, \qquad \tilde{z}'_n = \sup_{l\geq n} \left\{|z'_l|\right\}, \quad \text{for } n \in \mathbb{N}_0; \tag{25}$$

$\rho'_i > 0$, for $i = 0, \ldots, k-1$ and

$$\rho'_n = \begin{cases} r^m(L|a| + |b|)(n-k) & \text{for } n \in \{k, \ldots, 2k-1\}; \\ r^m(L|a| + |b|)(n-k) + \tilde{z}'_{n-2k} & \text{for } n \geq 2k, \end{cases} \tag{26}$$

$\gamma_i' > (1-\alpha)^{-1}\rho_i'$ for $i = 0, \ldots, k-1$ and

$$\gamma_n' = \rho_n' + \alpha\gamma_{n-k}' \tag{27}$$

for $n \geq k$. By (5), $y_n = \mathrm{O}\left(\left(\sqrt[k]{|\lambda|}\right)^n\right)$ and so $y_{n-k} = \mathrm{O}\left(\left(\sqrt[k]{|\lambda|}\right)^n\right)$. Moreover, from the fact that $\inf\limits_{n\geq k} |u_n| > 0$ we get that

$$z_n' = \mathrm{o}\left(\left(\sqrt[k]{|\lambda|}\right)^{-n}\omega_n\right)\mathrm{O}\left(\left(\sqrt[k]{|\lambda|}\right)^n\right) = \mathrm{o}(\omega_n) \subset \mathrm{o}(1) \tag{28}$$

and $\tilde{z}'$ is well defined. Let $\varepsilon > 0$. There exists $n_\varepsilon \in \mathbb{N}$ such that

$$|z_n'| \leq \varepsilon\omega_n,$$

for any $n \geq n_\varepsilon$. By monotonicity of $\omega$ we get

$$\tilde{z}_n' = \sup_{l\geq n}\{|z_l'|\} \leq \varepsilon \sup_{l\geq n}\{\omega_l\} = \varepsilon\omega_n$$

for any $n \geq n_\varepsilon$. Hence

$$\tilde{z}_n' = \mathrm{o}(\omega_n). \tag{29}$$

Using (22), (26), (29), and Lemma 2.1 we get $\rho_n' = \mathrm{o}(\omega_n)$. For $n \geq k$ we have

$$\rho_n' = \gamma_n' - \alpha\gamma_{n-k}'.$$

By definition and Lemma 3.1,

$$\gamma_n' > 0 \text{ for } n \in \mathbb{N}, \quad \gamma_n' = \mathrm{o}(\omega_n) \subset \mathrm{o}(1). \tag{30}$$

Choose an index $p \geq k$ such that

$$0 < \gamma_n' < \delta, \qquad \left|\tfrac{1}{u_n}\right| < \alpha \tag{31}$$

for $n \geq p$. Now, let $z, \rho, \gamma \in \mathbb{R}^{\mathbb{N}_0}$ and $A, B, R : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$ are defined as follows: for $n < p$ and $x \in \mathbb{R}^{\mathbb{N}_0}$

$$z_n = \rho_n = \gamma_n = B(x)(n) = R(x)(n) = 0, \tag{32}$$

for $n \geq p$ and $x \in \mathbb{R}^{\mathbb{N}_0}$

$$z_n = z_n', \quad \rho_n = \rho_n', \quad \gamma_n = \gamma_n', \tag{33}$$

$$R(x)(n) = -\tfrac{1}{u_{n+k}}(-1)^m r^m(x')(n+k) \quad \text{and} \quad B(x)(n) = \tfrac{1}{u_{n+k}}(x_{n+k} - y_{n+k}). \tag{34}$$

Moreover, let

$$Ax = y - z + Rx \qquad \text{and} \qquad X = \{x \in \mathbb{R}^{\mathbb{N}_0} : |x - y| \le \gamma\}. \tag{35}$$

By (30) and (33) we have

$$\gamma_n = \mathrm{o}(\omega_n) \subset \mathrm{o}(1). \tag{36}$$

By (31), (32), and (33) we have $0 \le \gamma_n < \delta$ for any $n$. Therefore, if $x \in X$, then, by (24), $|f(x_n)| \le L$ for any $n$. Thus, using (4) and (23), we have

$$|R(x)(n)| = \tfrac{1}{|u_{n+k}|}|r^m(x')(n+k)| \le \alpha r^m(|x'|)(n+k) \le r^m(L|a| + |b|)(n+k)$$

for $n \ge p$. Hence, using (25) and (33), we get

$$|R(x)(n)| \le \rho'_{n+2k} - \tilde{z}'_n \tag{37}$$

for $n \ge p$. If $t \in X$, then $|t - y| \le \gamma$ and, by (31), we have

$$|B(t)(n)| = \tfrac{1}{|u_{n+k}|}|t_{n+k} - y_{n+k}| \le \alpha\gamma_{n+k}$$

for $n \ge p$. Hence by the fact $\rho'$ is the nonincreasing sequence, Lemma 3.3, (29), (33), and (37) we obtain

$$|Ax + Bt - y|(n) \le |R(x)(n)| + |z_n| + |B(t)(n)| \le \rho_{n+2k} - \tilde{z}'_n + |z_n| + \alpha\gamma_{n+k} \le$$
$$\rho_{n+2k} + \alpha\gamma_{n+k} = \gamma_{n+2k} < \gamma_{n+k} < \gamma_n$$

for $n \ge p$. Therefore, by (35), we get $Ax + Bt \in X$. Thus

$$AX + BX \subset X.$$

In analogous way to the proof of Theorem 3.1 we prove that $A$ is continuous and $B$ is $\alpha$-contraction. By Lemma 3.2, there exists a point $x \in X$ such that $x = Ax + Bx$. Then, by (34) and (35) we have

$$x_n = R(x)(n) + y_n - z_n + \tfrac{1}{u_{n+k}}(x_{n+k} - y_{n+k}).$$

for $n \ge p$. Using (25), (33) and (34), we get

$$u_{n+k}x_n - x_{n+k} = u_{n+k}y_n - y_{n+k} + (\lambda - u_{n+k})y_n - (-1)^m r^m(x')(n+k)$$
$$x_{n+k} - u_{n+k}x_n = y_{n+k} - \lambda y_n + (-1)^m r^m(x')(n+k)$$

for $n \ge p$. Hence

$$x_n - u_n x_{n-k} = y_n - \lambda y_{n-k} + (-1)^m r^m(x')(n)$$

for $n \geq p+k$. Since $y \in \mathrm{PG}(m, \lambda, k)$, we have

$$0 = \Delta^m (y_n - \lambda y_{n-k}).$$

Hence, by (23), we obtain

$$\begin{aligned}\Delta^m (x_n - u_n x_{n-k}) &= \Delta^m (y_n - \lambda y_{n-k}) + \Delta^m (-1)^m r^m (x')(n)\\ &= 0 + x'_n = a_n f(x_n) + b_n\end{aligned}$$

for $n \geq p+k$. Therefore $x$ is a solution of (E). Since $x \in X$, by (35), $|x-y| \leq \gamma$. Moreover, by (36), $\gamma_n = \mathrm{o}(\omega_n)$. Hence $x_n - y_n = \mathrm{o}(\omega_n)$ and we obtain

$$x_n = y_n + (x-y)_n = y_n + \mathrm{o}(\omega_n).$$

$\square$

In the case of $\lambda > 1$ the assumption $u_n = \lambda + \mathrm{o}\left(\left(\sqrt[k]{|\lambda|}\right)^{-n} \omega_n\right)$ can not be weakened even to $u_n = \lambda + \mathrm{O}\left(\left(\sqrt[k]{|\lambda|}\right)^{-n} \omega_n\right)$. It is worth noting that for $|\lambda| < 1$, a similar technique can not be applied.

***Example 3.1*** Let $m=1$, $k=1$, $\omega_n = 1$, $\lambda > 1$, $\tau \in \mathbb{Z}$. Let us consider

$$\Delta(y_n - \lambda y_{n-1}) = 0 \tag{38}$$

with a general solution to the form

$$y_n = c_1 + c_2 \lambda^n, \quad c_1, c_2 \in \mathbb{R}.$$

Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, bounded function and $a$, $b$ be sequences such that $\sum_{n=0}^{\infty} (|a_n| + |b_n|) < \infty$. Let us consider $u_n = \lambda + (-\frac{1}{\lambda})^n$. Notice that $u_n = \lambda + \mathrm{O}(\lambda^{-n})$ and $|u_n - \lambda| \notin \mathrm{o}(\lambda^{-n})$ and the rest of the assumptions of Theorem 3.2 are satisfied for $y_n = \lambda^n$, $n \in \mathbb{N}_0$ which is a $f$-bounded solution to (38). We prove that for $y_n = \lambda^n$, there does not exist a solution $x$ to

$$\Delta(x_n - (\lambda + (-\tfrac{1}{\lambda})^n) x_{n-1}) = a_n f(x_{n-\tau}) + b_n \tag{39}$$

with $x_n = y_n + \mathrm{o}(1)$. On the contrary, we assume that there exists a solution to (39) such that $x_n = \lambda^n + d_n$ where $d_n = \mathrm{o}(1)$. Note that the left side of (39) is equal to

$$\begin{aligned}&\Delta(x_n - (\lambda + (-\tfrac{1}{\lambda})^n) x_{n-1}) = \Delta(\lambda^n + d_n - (\lambda + (-\tfrac{1}{\lambda})^n)(\lambda^{n-1} + d_{n-1})) =\\ &\Delta(d_n - \lambda d_{n-1} + (-\tfrac{1}{\lambda})^n d_{n-1}) + \tfrac{2}{\lambda}(-1)^n.\end{aligned}$$

Under assumptions on sequences $a$, $b$ and the function $f$, the right side of (39) tends to 0 as $n \to \infty$. Taking account that $\lim\limits_{n\to\infty} d_n = 0$ and $\lambda > 1$ we have

$$\lim_{n\to\infty} \left(d_n - \lambda d_{n-1} + (-\tfrac{1}{\lambda})^n) d_{n-1}\right) = 0,$$

and

$$\lim_{n\to\infty} \Delta\left(d_n - \lambda d_{n-1} + (-\tfrac{1}{\lambda})^n d_{n-1}\right) = 0.$$

Hence the left side of (39) is divergent as $n \to \infty$, because of part $(-1)^n$, which gives a contradiction.

## 4 Approximations of Solutions

In this section we show results in which for a given solution $x$ to nonlinear equation (E) and a given measure of approximation $\omega$ we can find $\varphi \in \mathrm{Pol}(m-1)$ such that $x_n = \varphi(n) + \mathrm{o}(\omega_n)$. In this section we consider only case $|\lambda| < 1$. Before we present the main result of this section we need two auxiliary lemmas.

**Lemma 4.1** [16, Lemma 3.4] *Assume $x, z, u \in \mathbb{R}^{\mathbb{N}_0}$, $z_n = x_n - u_n x_{n-k}$ for large $n$, $\eta \in \mathbb{R}$, $\lambda \in (-1, 1)$, $\lim_{n\to\infty} u_n = \lambda$, and $z_n = \mathrm{O}(n^\eta)$. Then $x_n = \mathrm{O}(n^\eta)$.*

**Lemma 4.2** *Assume $k \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\lambda \in \mathbb{R} \setminus \{1\}$, $x, z \in \mathbb{R}^{\mathbb{N}_0}$, $\omega : \mathbb{N}_0 \to (0, \infty)$, $\omega_n = \mathrm{O}(1)$,*

$$\limsup_{n\to\infty} \frac{|\lambda|\omega_{n-k}}{\omega_n} < 1,$$

*$z_n = x_n - \lambda x_{n-k}$ for large $n$, and $z \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n)$. Then $x \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n)$.*

***Proof*** Induction on $m$. For $m = 0$ the assertion follows from Lemma 3.1. Assume it is true for certain $m \geq 0$ and let

$$z \in \mathrm{Pol}(m) + \mathrm{o}(\omega_n).$$

There exist a real number $c$ and a sequence $w \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n)$ such that

$$z_n = cn^m + w_n.$$

Since

$$1 = \frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}.$$

we have

$$\begin{aligned} w_n = z_n - cn^m &= x_n - \lambda x_{n-k} - \frac{c}{1-\lambda} n^m + \frac{\lambda c}{1-\lambda} n^m \\ &= \left(x_n - \frac{c}{1-\lambda} n^m\right) - \lambda\left(x_{n-k} - \frac{c}{1-\lambda} n^m\right) \end{aligned}$$

There exists a sequence $r \in \mathrm{Pol}(m-1)$ such that $n^m = (n-k)^m + r_n$. Hence there exists a sequence $R \in \mathrm{Pol}(m-1)$ such that

$$w_n = \left(x_n - \frac{c}{1-\lambda}n^m\right) - \lambda\left(x_{n-k} - \frac{c}{1-\lambda}(n-k)^m\right) + R_n = v_n - \lambda v_{n-k} + R_n$$

where $v$ is a sequence defined by

$$v_n = x_n - \frac{c}{1-\lambda}n^m.$$

We have

$$v_n - \lambda v_{n-k} = w_n - R_n \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n).$$

By inductive hypothesis $v \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n)$. Hence

$$x_n = v_n + \frac{c}{1-\lambda}n^m \in \mathrm{Pol}(m) + \mathrm{o}(\omega_n).$$

□

**Lemma 4.3** *Assume* $k \in \mathbb{N}$, $x, z, u \in \mathbb{R}^{\mathbb{N}_0}$, $\omega : \mathbb{N}_0 \to (0, \infty)$, $\omega_n = \mathrm{O}(1)$,

$$\lambda \in (-1, 1), \quad u_n = \lambda + \mathrm{o}(n^{1-m}\omega_n), \quad \limsup_{n\to\infty} \frac{|\lambda|\omega_{n-k}}{\omega_n} < 1,$$

$z_n = x_n - u_n x_{n-k}$ *for large* $n$, *and* $z \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n)$. *Then* $x \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n)$.

***Proof*** Let $\gamma$ be a sequence definded by $\gamma_n = \lambda - u_n$. Then

$$z_n = x_n - u_n x_{n-k} = x_n - \lambda x_{n-k} + \gamma_n x_{n-k}.$$

By Lemma 4.1, $x_n = \mathrm{O}(n^{m-1})$. Hence $x_{n-k} = \mathrm{O}(n^{m-1})$ and we get

$$\gamma_n x_{n-k} = \mathrm{o}(n^{1-m}\omega_n)\mathrm{O}(n^{m-1}) = \mathrm{o}(\omega_n).$$

There exist a polynomial sequence $\beta$ such that $\deg \beta < m$ and $z_n = \beta(n) + \mathrm{o}(\omega_n)$. Hence

$$x_n - \lambda x_{n-k} = z_n - \gamma_n x_{n-k} = \beta(n) + \mathrm{o}(\omega_n) + \mathrm{o}(\omega_n) = \beta(n) + \mathrm{o}(\omega_n).$$

Using Lemma 4.2 we obtain the result. □

**Theorem 4.1** *Assume $m, k \in \mathbb{N}$, $\omega : \mathbb{N}_0 \to (0, \infty)$ is nonincreasing,*

$$\limsup_{n\to\infty} \frac{|\lambda|\omega_{n-k}}{\omega_n} < 1, \quad \sum_{n=0}^{\infty} \frac{n^{m-1}(|a_n| + |b_n|)}{\omega_n} < \infty,$$

*$u_n = \lambda + \mathrm{o}(n^{1-m}\omega_n)$, $F : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$, and $x$ is a solution of the equation*

$$\Delta^m(x_n - u_n x_{n-k}) = a_n F(x)(n) + b_n$$

*such that the sequence $F(x)$ is bounded. Then there exists a polynomial sequence $\varphi$ such that $\deg \varphi < m$, and $x_n = \varphi(n) + \mathrm{o}(\omega_n)$.*

***Proof*** By assumption $\limsup_{n\to\infty} \frac{|\lambda|\omega_{n-k}}{\omega_n} < 1$ and monotonicity of $\omega$, we get that $|\lambda| < 1$. For $n \in \mathbb{N}_0$ let

$$g_n = a_n F(x)(n) + b_n.$$

By assumption we have

$$\sum_{n=0}^{\infty} \frac{n^{m-1}|g_n|}{\omega_n} < \infty.$$

By Lemma 2.1, $g \in \mathrm{A}(m)$ and $r^m(g)(n) = \mathrm{o}(\omega_n)$. Let $h$ be a sequence defined by $h_n = (-1)^m r^m(g)(n)$. Then $h_n = \mathrm{o}(\omega_n)$ and,

$$\Delta^m h_n = g_n = \Delta^m(x_n - u_n x_{n-k})$$

for large $n$. Hence by linearity of the operator $\Delta^m$, there exists a polynomial sequence $\beta$ such that $\deg \beta < m$ and

$$x_n - u_n x_{n-k} = \beta(n) + h_n.$$

By Lemma 4.3 there exists a polynomial sequence $\varphi$ such that $\deg \varphi < m$ and

$$x_n = \varphi(n) + \mathrm{o}(\omega_n).$$

□

Before we prove the last corollary we recall that for $x, u \in \mathbb{R}^{\mathbb{N}_0}$ and $k \in \mathbb{N}$ $x$ is said to be $(u, k)$-*nonoscillatory* if $u_n x_n x_{n-k} \geq 0$ for large $n$.

**Corollary 4.1** *Assume $m, k \in \mathbb{N}$, $\omega : \mathbb{N}_0 \to (0, \infty)$ is nonincreasing,*

$$\limsup_{n\to\infty} \frac{|\lambda|\omega_{n-k}}{\omega_n} < 1, \quad \sum_{n=0}^{\infty} \frac{n^{m-1}(|a_n| + |b_n|)}{\omega_n} < \infty,$$

$$f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}, \quad g : [0, \infty) \to [0, \infty),$$
$$g \textit{ is nondecreasing}, \ g(t_0) > 0 \textit{ for some } t_0 > 0,$$
$$|f(n,t)| \le g(n^{1-m}|t|), \ \textit{for all} t \ge 0 \textit{and for large} n, \quad \int_{t_0}^{\infty} \frac{dt}{g(t)} = \infty,$$

*$\sigma : \mathbb{N}_0 \to \mathbb{N}_0$, $\lim_{n\to\infty} \sigma(n) = \infty$, $\sigma(n) \le n$ for large $n$, $u_n = \lambda + \mathrm{o}(n^{1-m}\omega_n)$, and $x$ is a nonoscillatory solution of the equation*

$$\Delta^m(x_n - u_n x_{n-k}) = a_n f(n, x_{\sigma(n)}) + b_n.$$

*Then* $x \in \mathrm{Pol}(m-1) + \mathrm{o}(\omega_n)$.

***Proof*** Define an operator $F : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$ by $F(y)(n) = f(n, y_{\sigma(n)})$. As in the proof of [16, Theorem 1], it can be shown that the sequence $F(x)$ is bounded. Hence the assertion is a consequence of Theorem 4.1. □

**Availability of data and materials:** Not applicable.

## Declarations

**Conflicts of interest** The authors declare that they have no competing interests.

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