Qualitative Theory of Dynamical Systems (2024) 23:118
https://doi.org/10.1007/s12346-023-00907-8

A Test for Global Attractivity of Linear Dynamic Equations with Delay

Nour H. M. Alsharif[1] **· Basak Karpuz**[2]

Received: 22 August 2023 / Accepted: 4 November 2023 / Published online: 1 March 2024
© The Author(s) 2024

Abstract

In this paper, we consider the delay dynamic equation

$$x^{\Delta}(t) + p(t)x\big(\tau(t)\big) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{$*$}$$

Based on Lyapunov's method, we study global attractivity of the trivial solution of ($*$). Our new result generalizes some well-known results in the theory of difference and differential equations to dynamic equations of the form ($*$). We also present some examples on nonstandard time scales to illustrate the importance of the new result.

Keywords Delay dynamic equations · Global attractivity · Time scale

1 Introduction

The stability theory has proven to be a versatile and effective tool in comprehending the behaviour of solutions in differential equations in recent years. Over the last few decades, the theory of dynamic equations on time scales has played a pivotal role in bridging the gap between difference and differential equations [2]. Notably, significant results on uniform global asymptotic stability in differential equations [22] and difference equations [6] were extended to the theory of dynamic equations by Wu and Zou in their work [20] (also discussed in [10]). In this direction, Braverman and

Nour H. M. Alsharif and Basak Karpuz contributed equally to this work.

✉ Basak Karpuz
bkarpuz@gmail.com

Nour H. M. Alsharif
nour2791994@gmail.com

1 Department of Mathematics, Graduate School of Natural and Applied Sciences, Dokuz Eylül University, Tınaztepe Campus, Buca, 35160 Izmir, Turkey

2 Department of Mathematics, Faculty of Sciences, Dokuz Eylül University, Tınaztepe Campus, Buca, 35160 Izmir, Turkey

Karpuz [4] unified the results for differential equations [15] and for difference equations [7]. While some studies in the time scale literature have generalized classical forms like difference equations, q-difference equations, and differential equations, it is essential to note that extending results from these fields to dynamic equations is not always straightforward. In some cases, this process necessitates the development of new tools within the framework of time scale calculus.

Let us now start introducing the objective of this paper. In 1994, Yu and Cheng [21] presented a general form of the following stability test for neutral difference equations with delay.

Theorem A (Cf. [21, Theorem 2]) *Suppose that* $\{p_n\}_{n=0}^{\infty}$ *is a nonnegative sequence and* τ *is a nonnegative integer such that*

$$\sum_{j=0}^{\infty} p_j = \infty \quad \textit{and} \quad \limsup_{n\to\infty} \sum_{j=n}^{n+2\tau} p_j < 2.$$

Then, the trivial solution of

$$x(n+1) - x(n) + p(n)x(n-\tau) = 0 \quad \textit{for } n \in \mathbb{N}_0$$

is globally attracting.

A continuous counterpart of Theorem A for neutral differential equations with positive and negative coefficients is considered by Shen and Yu. They have obtained the following result as corollaries of their more general main result.

Theorem B [19, Corollary 1 and Corollary 2] *Suppose that* $p \in \mathrm{C}([0,\infty),[0,\infty))$ *and* $\tau \geq 0$ *such that*

$$\int_0^{\infty} p(\xi)\mathrm{d}\xi = \infty \quad \textit{and} \quad \limsup_{n\to\infty} \int_t^{t+2\tau} p(\xi)\mathrm{d}\xi < 2.$$

Then, the trivial solution of

$$x'(t) + p(t)x(t-\tau) = 0 \quad \textit{for } t \in [0,\infty)$$

is globally attracting.

It is a natural question to ask if Theorems A and B can be generalized to time scales for linear delay dynamic equations. In this work, we aim to give a positive answer to this question. However, the proofs on time scales require very technical computations, which may lead to some expansions in the field of dynamic equations.

In this paper, we consider the linear delay dynamic equation

$$x^{\Delta}(t) + p(t)x\big(\tau(t)\big) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{1}$$

where $\mathbb{T}$ is a time scale unbounded above, $p \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,\infty)_{\mathbb{T}})$ and τ satisfies the following two conditions.

(C1) $\tau \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ satisfies $\tau(t) \le t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ and $\lim_{t\to\infty}\tau(t) = \infty$.

(C2) $\tau \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ satisfies $\tau^{\Delta}(t) > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ and $\tau(\mathbb{T}) = \mathbb{T}$.

Due to (C2), $\tau^{-1} : \mathbb{T} \to \mathbb{T}$ exists. Let us define $t_{-1} := \inf\{\tau(t),\ t \in [t_0,\infty)_{\mathbb{T}}\}$, which is a finite value by (C1).

Next, we have two definitions for the completeness of the structure of the paper.

Definition 1 (*Solution*) A function $x : [t_{-1},\infty)_{\mathbb{T}} \to \mathbb{R}$, which is rd-continuous on $[t_{-1},t_0]_{\mathbb{T}}$ and Δ-differentiable on $[t_0,\infty)_{\mathbb{T}}$ with an rd-continuous derivative, is called a **solution** of (1) provided that it satisfies the equality (1) identically on $[t_0,\infty)_{\mathbb{T}}$.

Definition 2 (*Globally Attractivity*) The trivial solution of (1) is said to be **globally attracting** if any solution x of (1) satisfies $\lim_{t\to\infty} x(t) = 0$.

We mention here the papers [6–8, 14] and [12, 13, 15, 16, 19, 22–24], which deal with asymptotic behaviour of difference equations and differential equations, respectively. We also refer the readers to the papers [1, 3, 5, 9, 10, 17, 18] for a discussion on asymptotic behaviour of solutions of dynamic equations on time scales.

The paper is organized as follows. In Sect. 2, we state our main theorem, which generalizes Theorems A and B to time scales. Section 3 includes auxiliary lemmas required in the proof of the main theorem. The proof of the main theorem is given in Sect. 4. In Sect. 5, there are two numerical examples illustrating applicability and importance of the main result. Section 6 is dedicated to conclusions. Finally, in Appendices A and B, we give some basic results on time scales required in the proofs of our lemmas.

2 Main Result

The main result of the paper is presented below.

Theorem 1 *Assume (C1), (C2),*

$$\int_{t_0}^{\infty} p(\xi)\Delta\xi = \infty, \tag{2}$$

and

$$\limsup_{t\to\infty} \int_{t}^{\tau^{-2}(\sigma(t))} p(\xi)\Delta\xi < 2. \tag{3}$$

Then, the trivial solution of (1) *is globally attracting.*

3 Preparatory Lemmas

The main result is motivated by the following important observations.

Lemma 1 *If (C1) and* (2) *hold, then every convergent solution of* (1) *tends to zero.*

Proof Let x be a solution of (1), which tends to some constant $\ell \in \mathbb{R}$. If $\ell \neq 0$, we may assume without loss of generality that $\ell > 0$. Then, we can find $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t), x(\tau(t)) > \frac{\ell}{2} > 0$ for all $t \geq t_1$. In view of (1), we see that

$$x^{\Delta}(t) = -p(t)x(\tau(t)) \leq -\frac{\ell}{2}p(t) \tag{4}$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. Due to (2), there exists a sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$\int_{t_1}^{t_2} p(\xi)\Delta\xi > \frac{2}{\ell}x(t_1). \tag{5}$$

By integrating (4), and then using (5), we see that

$$x(t_2) \leq x(t_1) - \frac{\ell}{2}\int_{t_1}^{t_2} p(\xi)\Delta\xi < 0.$$

This contradiction completes the proof. □

In the next result, we will define a companion function to get some information on the solutions of (1).

Lemma 2 *Suppose that (C1) and (C2) hold. Let x be a solution of* (1)*, and define the companion function*

$$y(t) := x(t) - \int_{t}^{\tau^{-1}(t)} p(\xi)x\big(\tau(\xi)\big)\Delta\xi \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{6}$$

If y converges to some value, then x also converges to the same value.

Proof Suppose x is a solution of (1) and y defined by (6) converges to ℓ. We will show that x converges to ℓ. It follows that

$$\begin{aligned} y^{\Delta}(t) &= x^{\Delta}(t) - \Big[\big(\tau^{-1}\big)^{\Delta}(t)p\big(\tau^{-1}(t)\big)x(t) - p(t)x(\tau(t))\Big] \\ &= x^{\Delta}(t) + p(t)x(\tau(t)) - \frac{p\big(\tau^{-1}(t)\big)x(t)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)} \\ &= -\frac{p\big(\tau^{-1}(t)\big)x(t)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)} \end{aligned} \tag{7}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$ (see Theorem 2 and Theorem 3). Integrating (7) from t_1 to ∞, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ is sufficiently large, we have

$$\int_{t_1}^{\infty} \frac{p\big(\tau^{-1}(\eta)\big)x(\eta)}{\tau^{\Delta}\big(\tau^{-1}(\eta)\big)}\Delta\eta = -\int_{t_1}^{\infty} y^{\Delta}(\eta)\Delta\eta = y(t_1) - \lim_{t\to\infty} y(t) = y(t_1) - \ell.$$

Note that

$$\int_{t_2}^{\infty} p(\xi)x\big(\tau(\xi)\big)\Delta\xi = \int_{t_1}^{\infty} \frac{p\big(\tau^{-1}(\eta)\big)x(\eta)}{\tau^{\Delta}\big(\tau^{-1}(\eta)\big)}\Delta\eta =: L < \infty, \quad \text{where } t_2 := \tau^{-1}(t_1) \tag{8}$$

(see Theorem 4). We claim that

$$\lim_{t\to\infty} \int_{t}^{\tau^{-1}(t)} p(\xi)x\big(\tau(\xi)\big)\Delta\xi = 0. \tag{9}$$

Indeed, letting

$$z(t) := \int_{t_2}^{t} p(\xi)x\big(\tau(\xi)\big)\Delta\xi \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}},$$

we see from (8) that

$$\lim_{t\to\infty} z(t) = L \quad \text{and} \quad \lim_{t\to\infty} z\big(\tau^{-1}(t)\big) = L$$

since $\lim_{t\to\infty} \tau^{-1}(t) = \infty$ because of $\lim_{t\to\infty} \tau(t) = \infty$. Therefore, we estimate

$$\lim_{t\to\infty} \int_{t}^{\tau^{-1}(t)} p(\xi)x\big(\tau(\xi)\big)\Delta\xi = \lim_{t\to\infty}\Big[z\big(\tau^{-1}(t)\big) - z(t)\Big] = L - L = 0,$$

which proves (9). Rewriting (6) in the form

$$x(t) = y(t) + \int_{t}^{\tau^{-1}(t)} p(\xi)x\big(\tau(\xi)\big)\Delta\xi \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

and considering (9) yields that $x(t) \to \ell$ as $t \to \infty$. This completes the proof. □

4 Proof of the Main Result

In view of Lemmas 1 and 2, in order to show that the trivial solution of (1) is globally attracting, that is, every solution x tends to zero, we shall need a sufficient condition such that for every solution x of (1), the function y defined by (6) will tend to a constant.

Before proceeding to the next result, we compute

$$\begin{aligned}(y^2)^{\Delta}(t) &= y^{\Delta}(t)\Big(2y(t) + \mu(t)y^{\Delta}(t)\Big)\\ &= -\frac{p\big(\tau^{-1}(t)\big)x(t)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\left[2x(t) - 2\int_{t}^{\tau^{-1}(t)} p(\xi)x\big(\tau(\xi)\big)\Delta\xi - \mu(t)\frac{p\big(\tau^{-1}(t)\big)x(t)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\right]\\ &= \frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\left[\mu(t)\frac{p\big(\tau^{-1}(t)\big)\big(x(t)\big)^2}{\tau^{\Delta}\big(\tau^{-1}(t)\big)} - 2\big(x(t)\big)^2 + 2\int_{t}^{\tau^{-1}(t)} p(\xi)x\big(\tau(\xi)\big)x(t)\Delta\xi\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\bigg[\mu(t)\frac{p\big(\tau^{-1}(t)\big)\big(x(t)\big)^2}{\tau^{\Delta}\big(\tau^{-1}(t)\big)} - 2\big(x(t)\big)^2 \\
&\quad + \int_t^{\tau^{-1}(t)} p(\xi)\Big(\big[x\big(\tau(\xi)\big)\big]^2 + \big(x(t)\big)^2\Big)\Delta\xi\bigg] \\
&\leq \frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\bigg[\int_t^{\sigma(t)} \frac{p\big(\tau^{-1}(\xi)\big)\big(x(\xi)\big)^2}{\tau^{\Delta}\big(\tau^{-1}(\xi)\big)}\Delta\xi - 2\big(x(t)\big)^2 \\
&\quad + \int_t^{\tau^{-1}(t)} p(\xi)\big[x\big(\tau(\xi)\big)\big]^2\Delta\xi + \int_t^{\tau^{-1}(t)} p(\xi)\big(x(t)\big)^2\Delta\xi\bigg] \\
&= \frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\bigg[\int_{\tau^{-1}(t)}^{\tau^{-1}(\sigma(t))} p(\xi)\big(x(\tau(\xi))\big)^2\Delta\xi - 2\big(x(t)\big)^2 \\
&\quad + \int_t^{\tau^{-1}(t)} p(\xi)\big[x\big(\tau(\xi)\big)\big]^2\Delta\xi + \int_t^{\tau^{-1}(t)} p(\xi)\big(x(t)\big)^2\Delta\xi\bigg] \\
&= \frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_t^{\tau^{-1}(\sigma(t))} p(\xi)\big(x(\tau(\xi))\big)^2\Delta\xi - 2\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\big(x(t)\big)^2 \\
&\quad + \frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_t^{\tau^{-1}(t)} p(\xi)\big(x(t)\big)^2\Delta\xi
\end{aligned} \tag{10}
$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. Note that we have applied the inequality $2\alpha\beta \leq \alpha^2 + \beta^2$ in the fourth step above.

Lemma 3 *Assume (C1), (C2) and*

$$
\limsup_{t\to\infty} \int_t^{\tau^{-2}(\sigma(t))} p(\xi)\Delta\xi < 2. \tag{11}
$$

Then, every solution of (1) *tends to a constant.*

Proof We shall construct a nonnegative Lyapunov functional V, which is eventually nonincreasing along solutions of (1). Let x be a solution of (1), and let y be defined by (6). Define Lyapunov functional V by

$$
V\big(x(t)\big) = \big(y(t)\big)^2 + w(t) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},
$$

where

$$
w(t) := \int_t^{\tau^{-1}(t)} \int_\eta^{\tau^{-1}(t)} \frac{p\big(\tau^{-1}(\eta)\big)}{\tau^{\Delta}\big(\tau^{-1}(\eta)\big)} p(\xi)\big[x\big(\tau(\xi)\big)\big]^2\Delta\xi\,\Delta\eta \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.
$$

In view of (10), we estimate that

$$
\begin{aligned}
(V\circ x)^{\Delta}(t) &= \big(y^2\big)^{\Delta}(t) + w^{\Delta}(t) \\
&\leq \bigg[\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_t^{\tau^{-1}(\sigma(t))} p(\xi)\big(x(\tau(\xi))\big)^2\Delta\xi - 2\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\big(x(t)\big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_{t}^{\tau^{-1}(t)}p(\xi)\big(x(t)\big)^{2}\Delta\xi\Bigg]\\
&-\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_{t}^{\tau^{-1}(\sigma(t))}p(\xi)\big(x(\tau(\xi))\big)^{2}\Delta\xi\\
&+\frac{p(\tau^{-1}(t))\big(x(t)\big)^{2}}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_{\tau^{-2}(t)}^{\tau^{-2}(\sigma(t))}p(\xi)\Delta\xi\\
&+\frac{p\big(\tau^{-1}(t)\big)\big(x(t)\big)^{2}}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_{\tau^{-1}(t)}^{\tau^{-2}(t)}p\big(\xi\big)\Delta\xi\\
=&-2\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\big(x(t)\big)^{2}+\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_{t}^{\tau^{-1}(t)}p(\xi)\big(x(t)\big)^{2}\Delta\xi\\
&+\frac{p(\tau^{-1}(t))\big(x(t)\big)^{2}}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_{\tau^{-2}(t)}^{\tau^{-2}(\sigma(t))}p(\xi)\Delta\xi\\
&+\frac{p\big(\tau^{-1}(t)\big)\big(x(t)\big)^{2}}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\int_{\tau^{-1}(t)}^{\tau^{-2}(t)}p(\xi)\Delta\xi\\
=&\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\big(x(t)\big)^{2}\Bigg[-2+\int_{t}^{\tau^{-2}(\sigma(t))}p(\xi)\Delta\xi\Bigg]
\end{aligned}
$$

for $t\in[t_0,\infty)_{\mathbb{T}}$. In view of (11), there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ and $\varepsilon\in\mathbb{R}^{+}$ such that

$$2-\int_{t}^{\tau^{-2}(\sigma(t))}p(\xi)\Delta\xi\le\varepsilon\quad\text{for all } t\in[t_1,\infty)_{\mathbb{T}}.$$

Thus, we get

$$(V\circ x)^{\Delta}(t)\le-\varepsilon\frac{p\big(\tau^{-1}(t)\big)}{\tau^{\Delta}\big(\tau^{-1}(t)\big)}\big(x(t)\big)^{2}\le 0\quad\text{for all } t\in[t_1,\infty)_{\mathbb{T}}.\tag{12}$$

Therefore, $(V\circ x)$ is nonnegative and nonincreasing on $[t_1,\infty)_{\mathbb{T}}$. This shows that

$$\lim_{t\to\infty}V\big(x(t)\big)=\lim_{t\to\infty}\Big(\big(y(t)\big)^{2}+w(t)\Big)=\ell\ge 0.$$

By integrating the inequality (12) from t_1 to ∞, we have

$$\int_{t_1}^{\infty}\frac{p\big(\tau^{-1}(\xi)\big)\big(x(\xi)\big)^{2}}{\tau^{\Delta}\big(\tau^{-1}(\xi)\big)}\Delta\xi\le\frac{V\big(x(t_1)\big)-\ell}{\varepsilon}.\tag{13}$$

By using (11), we obtain

$$
\begin{aligned}
w(t) &= \int_t^{\tau^{-1}(t)} \int_t^{\sigma(\xi)} \frac{p\big(\tau^{-1}(\eta)\big)}{\tau^{\Delta}\big(\tau^{-1}(\eta)\big)} p(\xi)\big[x\big(\tau(\xi)\big)\big]^2 \Delta\eta\Delta\xi \\
&= \int_t^{\tau^{-1}(t)} \int_{\tau^{-1}(t)}^{\tau^{-1}(\sigma(\xi))} p(\eta)p(\xi)\big[x\big(\tau(\xi)\big)\big]^2 \Delta\eta\Delta\xi \\
&< \int_t^{\tau^{-1}(t)} \int_t^{\tau^{-2}(t)} p(\eta)p(\xi)\big[x\big(\tau(\xi)\big)\big]^2 \Delta\eta\Delta\xi \\
&< \int_t^{\tau^{-1}(t)} \int_t^{\tau^{-2}(\sigma(t))} p(\eta)p(\xi)\big[x\big(\tau(\xi)\big)\big]^2 \Delta\eta\Delta\xi \\
&< 2\int_t^{\tau^{-1}(t)} p(\xi)\big[x\big(\tau(\xi)\big)\big]^2 \Delta\xi,
\end{aligned}
$$

which together with (13) implies

$$
\lim_{t\to\infty} w(t) = 0. \tag{14}
$$

Combining (13) and (14), we find

$$
\lim_{t\to\infty} \mid y(t) \mid = \lim_{t\to\infty} \sqrt{V\big(x(t)\big) - w(t)} = \sqrt{\ell}.
$$

We assert further that y converges. The proof is clear if $\ell = 0$. Let $\ell > 0$. We claim that y cannot be oscillatory. Assume the contrary that y is oscillatory. Then, there exist $\delta \in (0, \sqrt{\ell})_{\mathbb{R}}$ and $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$
0 < \sqrt{\ell} - \delta <\mid y(t) \mid< \sqrt{\ell} + \delta \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}} \tag{15}
$$

and let

$$
J_+ = \{t \in [t_2, \infty)_{\mathbb{T}} : \ y(t) > 0\} \quad \text{and} \quad J_- = \{t \in [t_2, \infty)_{\mathbb{T}} : \ y(t) < 0\}. \tag{16}
$$

Since y is oscillatory, I and J are unbounded and satisfy $J_- \cup J_+ = [t_2, \infty)_{\mathbb{T}}$. Hence, we may pick an increasing divergent sequence $\{s_k\}_{k\in\mathbb{N}} \subset [t_2, \infty)_{\mathbb{T}}$ (i.e., $s_{k+1} > s_k$ for $k \in \mathbb{N}$ and $s_k \to \infty$ as $k \to \infty$) such that $\{s_k\} \subset J_+$ and $\{\sigma(s_k)\} \subset J_-$. Then, $y(s_k) > 0$ and $y^{\sigma}(s_k) < 0$ for all $k \in \mathbb{N}$. Furthermore, by (15), we easily see that

$$
\begin{aligned}
0 < \sqrt{\ell} - \delta <\mid y^{\sigma}(s_k) \mid< \sqrt{\ell} + \delta& \\
\implies \sqrt{\ell} - \delta < -y^{\sigma}(s_k) < \sqrt{\ell} + \delta& \\
\implies -\big(\sqrt{\ell} + \delta\big) < y^{\sigma}(s_k) < -\sqrt{\ell} + \delta&
\end{aligned}
$$

and

$$\begin{aligned} 0 < \sqrt{\ell} - \delta < \mid y(s_k) \mid < \sqrt{\ell} + \delta \\ \Longrightarrow \sqrt{\ell} - \delta < y(s_k) < \sqrt{\ell} + \delta \\ \Longrightarrow -(\sqrt{\ell} + \delta) < -y(s_k) < -\sqrt{\ell} + \delta, \end{aligned}$$

then

$$-2(\sqrt{\ell} + \delta) < y^{\sigma}(s_k) - y(s_k) < 2(-\sqrt{\ell} + \delta) < 0, \quad k \in \mathbb{N}.$$

It follows from (7) that

$$\begin{aligned} & -2(\sqrt{\ell} + \delta) < \int_{s_k}^{\sigma(s_k)} y^{\Delta}(\xi)\Delta\xi < 2(-\sqrt{\ell} + \delta) \\ & \Longrightarrow -2(\sqrt{\ell} + \delta) < -\int_{s_k}^{\sigma(s_k)} \frac{p(\tau^{-1}(\xi))x(\xi)}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\xi < 2(-\sqrt{\ell} + \delta) \\ & \Longrightarrow 2(\sqrt{\ell} - \delta) < \int_{s_k}^{\sigma(s_k)} \frac{p(\tau^{-1}(\xi))x(\xi)}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\xi < 2(\sqrt{\ell} + \delta) \\ & \Longrightarrow 4(\sqrt{\ell} - \delta)^2 < \left(\int_{s_k}^{\sigma(s_k)} \frac{p(\tau^{-1}(\xi))x(\xi)}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\xi\right)^2 < 4(\sqrt{\ell} + \delta)^2 \end{aligned} \tag{17}$$

for all $k \in \mathbb{N}$. By using (11), we obtain the following inequality

$$\begin{aligned} \left(\int_{s_k}^{\sigma(s_k)} \frac{p(\tau^{-1}(\xi))x(\xi)}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\xi\right)^2 &= \left(\mu(s_k)\frac{p(\tau^{-1}(s_k))x(s_k)}{\tau^{\Delta}(\tau^{-1}(s_k))}\right)^2 \\ &= \int_{s_k}^{\sigma(s_k)} \int_{\xi}^{\sigma(\xi)} \frac{p(\tau^{-1}(\eta))}{\tau^{\Delta}(\tau^{-1}(\eta))} \frac{p(\tau^{-1}(\xi))(x(\xi))^2}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\eta\Delta\xi \\ &\leq \int_{s_k}^{\sigma(s_k)} \int_{\tau^{-1}(\xi)}^{\tau^{-1}(\sigma(\xi))} p(\eta)\frac{p(\tau^{-1}(\xi))(x(\xi))^2}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\eta\Delta\xi \\ &\leq \int_{s_k}^{\sigma(s_k)} \int_{\xi}^{\tau^{-2}(\sigma(\xi))} p(\eta)\frac{p(\tau^{-1}(\xi))(x(\xi))^2}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\eta\Delta\xi \\ &\leq 2\int_{s_k}^{\sigma(s_k)} \frac{p(\tau^{-1}(\xi))(x(\xi))^2}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\xi \end{aligned} \tag{18}$$

for all $k \in \mathbb{N}$. From (17) and (18), we have

$$\int_{s_k}^{\sigma(s_k)} \frac{p(\tau^{-1}(\xi))(x(\xi))^2}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\xi > 2(\sqrt{\ell} - \delta)^2 \quad \text{for all } k \in \mathbb{N}. \tag{19}$$

From (13) and (19), we get

$$\sum_{k=1}^{\infty} 2(\sqrt{\ell} - \delta)^2 < \sum_{k=1}^{\infty} \int_{s_k}^{\sigma(s_k)} \frac{p(\tau^{-1}(\xi))(x(\xi))^2}{\tau^{\Delta}(\tau^{-1}(\xi))}\Delta\xi$$

$$\leq \int_{s_1}^{\infty} \frac{p\big(\tau^{-1}(\xi)\big)\big(x(\xi)\big)^2}{\tau^{\Delta}\big(\tau^{-1}(\xi)\big)} \Delta\xi < \infty$$

but $\sum_{k=1}^{\infty} 2\big(\sqrt{\ell} - \delta\big)^2 = \infty$, which is a contradiction. Therefore, y cannot be oscillatory. If y is either eventually positive or eventually negative, then $\lim_{t\to\infty} y(t) = \sqrt{\ell}$ or $\lim_{t\to\infty} y(t) = -\sqrt{\ell}$, respectively. By Lemma 2, every solution x of (1) converges to some value, and this completes the proof. □

Proof of Theorem 1 The proof is an immediate consequence of Lemmas 1 and 3. □

5 Some Examples

We start this section with a numerical example on a particular isolated time scale.

Example 1 Let $\mathbb{T} = \{\ldots, 0, 1, 3, 6, 7, 9, 12, \ldots\} = \cup_{k\in\mathbb{Z}}\{6k, 6k+1, 6k+3\}$, and consider the equation

$$x^{\Delta}(t) + p(t)x(\rho(t)) = 0 \quad \text{for } t \in [0, \infty)_{\mathbb{T}}, \tag{20}$$

where

$$p(t) := \begin{cases} \frac{3}{4}, & t = 6k \\ \frac{1}{2}, & t = 6k+1 \\ \frac{1}{36}, & t = 6k+3 \end{cases} \quad \text{for } t \in [0, \infty)_{\mathbb{T}}.$$

Comparing (20) with (1), we see that $\tau(t) = \rho(t) < t$ for all $t \in \mathbb{T}$, and $\tau^{-1}(t) = \sigma(t)$ for $t \in \mathbb{T}$. Further, $\tau^{\Delta}(t) = \frac{t-\rho(t)}{\mu(t)} > 0$ for all $t \in \mathbb{T}$, i.e., τ is differentiable, increasing and that $\tau(\mathbb{T}) = \mathbb{T}$. Next, we compute that

$$\begin{aligned} \limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(\xi)\Delta\xi &= \limsup_{t\to\infty} \int_{\rho(t)}^{\sigma(t)} p(\xi)\Delta\xi \\ &= \limsup_{t\to\infty} \left[\int_{\rho(t)}^{t} p(\xi)\Delta\xi + \int_{t}^{\sigma(t)} p(\xi)\Delta\xi\right] \\ &= \limsup_{t\to\infty} [\mu(\rho(t))p(\rho(t)) + \mu(t)p(t)] \\ &= \limsup_{t\to\infty} \begin{cases} 3\cdot\frac{1}{36} + 1\cdot\frac{3}{4}, & t = 6k \\ 1\cdot\frac{3}{4} + 2\cdot\frac{1}{2}, & t = 6k+1 \\ 2\cdot\frac{1}{2} + 3\cdot\frac{1}{36}, & t = 6k+3 \end{cases} \\ &= \limsup_{t\to\infty} \left\{\begin{matrix} \frac{5}{6}, & t = 6k \\ \frac{7}{4}, & t = 6k+1 \\ \frac{13}{12}, & t = 6k+3 \end{matrix}\right\} = \frac{7}{4} \not< 1 \end{aligned}$$

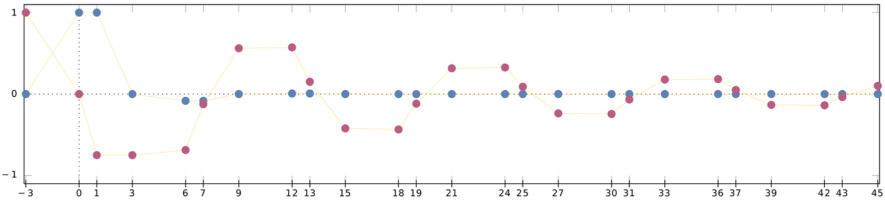

Fig. 1 Graphic of the solutions x_1 (blue) and x_2 (red) on $[-3, 45]_{\mathbb{T}}$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(\xi)\Delta\xi = \frac{7}{4} \not< \frac{3}{2} + \frac{1}{3+3} = \frac{5}{3}.$$

Hence, conditions of [4, §6] and [20, Theorem 1.2] (see also [10]) fail to hold. Fortunately,

$$\begin{aligned}\limsup_{t\to\infty} \int_{t}^{\tau^{-2}(\sigma(t))} p(\xi)\Delta\xi &= \limsup_{t\to\infty} \int_{t}^{\sigma^3(t)} p(\xi)\Delta\xi \\ &= \limsup_{t\to\infty} \left[\int_{t}^{\sigma(t)} p(\xi)\Delta\xi + \int_{\sigma(t)}^{\sigma^2(t)} p(\xi)\Delta\xi + \int_{\sigma^2(t)}^{\sigma^3(t)} p(\xi)\Delta\xi\right] \\ &= \limsup_{t\to\infty} [\mu(t)p(t) + \mu(\sigma(t))p(\sigma(t)) + \mu(\sigma^2(t))p(\sigma^2(t))] \\ &= \limsup_{t\to\infty} \begin{Bmatrix} 1\cdot\frac{3}{4} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{36}, & t = 6k \\ 2\cdot\frac{1}{2} + 3\cdot\frac{1}{36} + 1\cdot\frac{3}{4}, & t = 6k+1 \\ 3\cdot\frac{1}{36} + 1\cdot\frac{3}{4} + 2\cdot\frac{1}{2}, & t = 6k+3 \end{Bmatrix} = \frac{11}{6} < 2,\end{aligned}$$

which shows by Theorem 1 that the trivial solution of (20) is globally attracting.

In Fig. 1, we have the plots of the solutions x_1 and x_2, which have the initial conditions $x(-3) = 0$, $x(0) = 1$, and $x(-3) = 1$, $x(0) = 0$, respectively.

We can simply compute x_1 as

$$x_1(t) = \begin{cases} \left(-\frac{1}{12}\right)^k, & t = 6k \text{ or } t = 6k+1 \\ 0, & t = 6k+3, \end{cases}$$

which satisfies $\lim_{t\to\infty} x(t) = 0$. But, it is not very easy to compute the solution x_2 explicitly. Due to the superposition principle, we can say that any solution x of (20) is of the form $x(t) = x(-3)x_1(t) + x(0)x_2(t)$ for $t \in [-3, \infty)_{\mathbb{T}}$. It can be also seen from Fig. 1 that $\lim_{t\to\infty} x(t) = 0$.

Next, we give an example on a time scale, which is a union of closed intervals of length 1.

Example 2 Let $\mathbb{T} = \mathbb{P}_{1,1} = \cup_{k\in\mathbb{Z}}[2k, 2k+1]_{\mathbb{R}}$, and consider the equation

$$x^{\Delta}(t) + p(t)x(t-2) = 0 \quad \text{for } t \in [0, \infty)_{\mathbb{T}}, \tag{21}$$

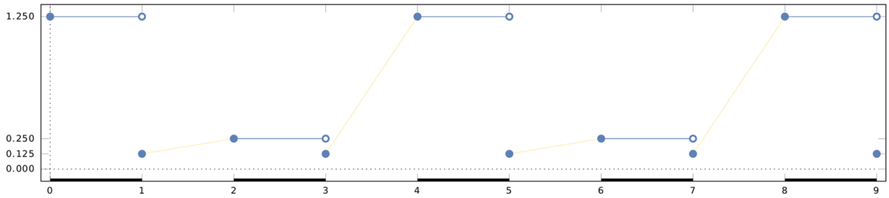

Fig. 2 Graphic of the coefficient p on $[0, 9]_{\mathbb{T}}$

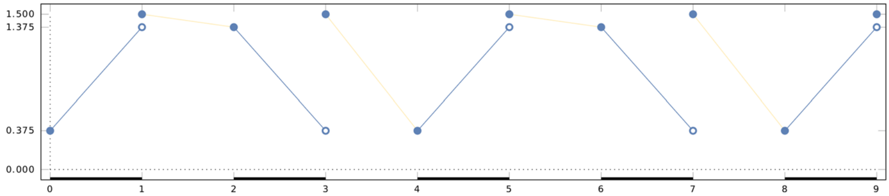

Fig. 3 Graphic of the integral of the coefficient p from $\tau(t)$ to $\sigma(t)$ for $t \in [0, 9]_{\mathbb{T}}$

where

$$p(t) := \begin{cases} \frac{5}{4}, & t \in [4k, 4k+1)_{\mathbb{R}} \\ \frac{1}{8}, & t = 2k+1 \\ \frac{1}{4}, & t \in [4k+2, 4k+3)_{\mathbb{R}} \end{cases} \qquad \text{for } t \in [0, \infty)_{\mathbb{T}}.$$

Comparing (21) with (1), we see that $\tau(t) = t-2 < t$ for all $t \in \mathbb{T}$, and $\tau^{-1}(t) = t+2$ for $t \in \mathbb{T}$. Further, $\tau^{\Delta}(t) \equiv 1 > 0$ for all $t \in \mathbb{T}$, i.e., τ is differentiable, increasing and that $\tau(\mathbb{T}) = \mathbb{T}$.

It should be mentioned here that $\min_{t\in\mathbb{T}}\{\mu(t)\} = 0$, $\max_{t\in\mathbb{T}}\{\mu(t)\} = 1$ and $\max_{t\in\mathbb{T}}\{t - \tau(t)\} = 2$.

Now, we compute

$$\begin{aligned} \limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(\xi)\Delta\xi &= \limsup_{t\to\infty} \left[\int_{\tau(t)}^{t} p(\xi)\Delta\xi + \int_{t}^{\sigma(t)} p(\xi)\Delta\xi \right] \\ &= \limsup_{t\to\infty} \left[\int_{t-2}^{t} p(\xi)\Delta\xi + \mu(t)p(t) \right] \\ &= \limsup_{t\to\infty} \left\{ \begin{array}{ll} t - 4k + \frac{3}{8}, & 4k \le t < 4k+1 \\ \frac{3}{2}, & t = 4k+1 \\ -t + 4k + \frac{27}{8}, & 4k+2 \le t < 4k+3 \\ \frac{1}{2}, & t = 4k+3 \end{array} \right\} = \frac{3}{2} \end{aligned}$$

showing that conditions of [4, §6] and [20, Theorem 1.2] (see also [10]) fail to hold.

On the other hand, we compute that

$$\limsup_{t\to\infty} \int_{t}^{\tau^{-2}(\sigma(t))} p(\xi)\Delta\xi = \limsup_{t\to\infty} \int_{t}^{\sigma(t)+4} p(\xi)\Delta\xi$$

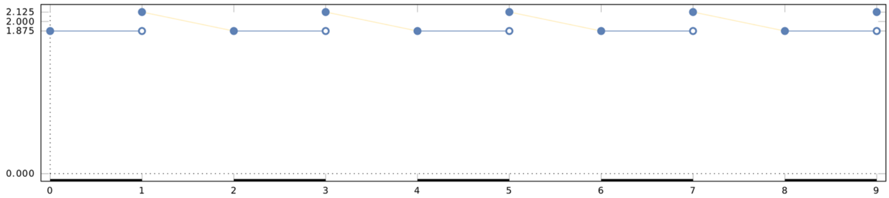

Fig. 4 Graphic of the integral of the coefficient p from t to $\tau^{-2}(\sigma(t))$ for $t \in [0, 9]_{\mathbb{T}}$

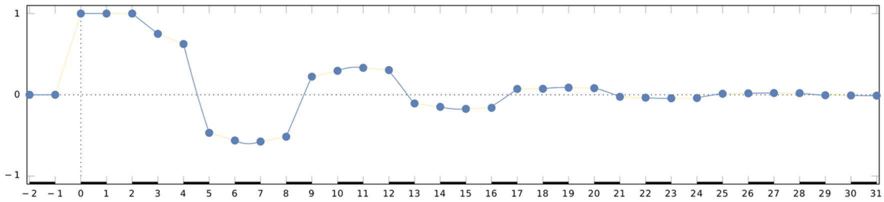

Fig. 5 Graphic of a solution x on $[-2, 31]_{\mathbb{T}}$ under the initial condition $x(t) = 0$ for $t \in [-2, -1]_{\mathbb{T}}$ and $x(0) = 1$

$$= \limsup_{t\to\infty} \left\{ \begin{matrix} \frac{7}{4}, & 4k \leq t < 4k+1 \\ \frac{15}{8}, & t = 4k+1 \\ \frac{7}{4}, & 4k+2 \leq t < 4k+3 \\ \frac{15}{8}, & t = 4k+3 \end{matrix} \right\} = \frac{15}{8} < 2.$$

This shows that the conditions of Theorem 1 hold, i.e., the trivial solution of (21) is globally attracting.

Due to Theorem 1, the trivial solution of (21) is globally attracting. A solution of (21) is plotted in Fig. 2 showing that $\lim_{t\to\infty} x(t) = 0$ (Figs. 3, 4, 5).

6 Final Comments

In this section, we make our final comments to conclude the paper. We will start with a remark that holds trivially for the well-known time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ with a delay in the linear form.

Remark 1 Suppose that $\tau \circ \sigma = \sigma \circ \tau$, then

$$\begin{aligned} &\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta < 1 \\ &\implies \limsup_{t\to\infty} \int_{t}^{\sigma(\tau^{-1}(t))} p(\eta)\Delta\eta < 1 \text{ and } \limsup_{t\to\infty} \int_{\tau^{-1}(t)}^{\sigma(\tau^{-2}(t))} p(\eta)\Delta\eta < 1 \\ &\implies \limsup_{t\to\infty} \int_{t}^{\tau^{-1}(\sigma(t))} p(\eta)\Delta\eta < 1 \text{ and } \limsup_{t\to\infty} \int_{\tau^{-1}(t)}^{\tau^{-2}(\sigma(t))} p(\eta)\Delta\eta < 1 \end{aligned}$$

$$\Longrightarrow \limsup_{t\to\infty}\int_{t}^{\tau^{-1}(t)} p(\eta)\Delta\eta < 1 \text{ and } \limsup_{t\to\infty}\int_{\tau^{-1}(t)}^{\tau^{-2}(\sigma(t))} p(\eta)\Delta\eta < 1$$
$$\Longrightarrow \limsup_{t\to\infty}\int_{t}^{\tau^{-2}(\sigma(t))} p(\eta)\Delta\eta < 2.$$

Hence, the condition in [4, §6 (i)] implies Theorem 1 under the condition $\tau\circ\sigma=\sigma\circ\tau$.

The key point in this paper is to assume conditions such that $\tau(\mathbb{T})=\mathbb{T}$ holds. This enables us to use the substitution formula in integrals on time scales (see Theorem 4). Note that for the classical types of equations

$$x(n+1)-x(n)+p(n)x(n-\tau)=0 \quad \text{for } n\in\mathbb{N}_0,\ \mathbb{T}=\mathbb{Z}$$

and

$$x'(t)+p(t)x(t-\tau)=0 \quad \text{for } t\in[0,\infty),\ \mathbb{T}=\mathbb{R},$$

the condition $\tau(\mathbb{T})=\mathbb{T}$ holds trivially. For more general special types of dynamic equations

$$\Delta_h x(t)+p(t)x(t-kh)=0 \quad \text{for } t\in h\mathbb{Z},\ k\in\mathbb{N}_0,\ \Delta_h x(t):=\frac{x(t+h)-x(t)}{h},$$
$$x'(t)+p(t)x(\alpha t)=0 \quad \text{for } t\in\mathbb{R}_0^+,\ \alpha\in(0,1),$$

and

$$\mathrm{D}_q x(t)+p(t)x(t/q^k)=0 \quad \text{for } t\in q^{\mathbb{Z}}\cup\{0\},\ k\in\mathbb{N}_0,\ \mathrm{D}_q x(t):=\frac{x(qt)-x(t)}{(q-1)t},$$

one can easily see that (C2) holds too. h-difference equations and q-difference equations can be considered under a more general class, that is, dynamic equations on isolated time scales. In this case, with a delay term of the form $\tau(t)=\rho^k(t):=(\rho\circ\rho^{k-1})(t)$ and $\rho^0(t):=t$ for $t\in\mathbb{T}$ and $k\in\mathbb{N}_0$, (C2) also holds.

We would like to emphasize that Theorem 1 generalizes Theorems A and B to arbitrary time scales. It is also important to mention that Theorem 1 cannot be compared with the results in [4] and [20]. Hence, it is new in the stability theory of dynamic equations.

Finally, we would like to mention that the main result of this paper can be extended to dynamic equations of the neutral form

$$\big[x(t)-r(t)x\big(\gamma(t)\big)\big]^{\Delta}+p(t)x\big(\alpha(t)\big)-q(t)x\big(\beta(t)\big)=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{22}$$

where the coefficients p,q,r and the delays α,β,γ satisfy suitable conditions (cf. [11]). However, we prefer demonstrating our results on the simple equation (1) because the key point behind the idea can be presented without getting lost in computational details.

Author Contributions N.A. and B.K. wrote the main manuscript text and prepared all of the graphics together. All authors also reviewed the manuscript.

Funding Open access funding provided by the Scientific and Technological Research Council of Türkiye (TÜBİTAK).

Declarations

Conflict of interest The authors declare no competing interests.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Appendix A Basics of Time Scales

A time scale which is represented by the symbol $\mathbb{T}$, is a nonempty closed subset of reals $\mathbb{R}$. The intervals with a subscript $\mathbb{T}$ are used to indicate the ordinary interval intersects with $\mathbb{T}$, i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. The forward and backward jump operators, $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$, are respectively defined by $\sigma(t) = \inf(t, \infty)_{\mathbb{T}}$ and $\rho(t) = \sup(t, \infty)_{\mathbb{T}}$. A point t is called right or left-dense, if $\sigma(t) = t$ or $\rho(t) = t$, respectively, meanwhile, it is called right or left-scattered, if $\sigma(t) > t$ or $\rho(t) < t$, respectively. The graininess or forward step size map $\mu : \mathbb{T} \to [0, \infty)_{\mathbb{R}}$ is given by $\mu(t) = \sigma(t) - t$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, the Δ-derivative $f^{\Delta}(t)$ of f at the point t is defined to be the number, provided it exists with the property that, for given $\varepsilon \in \mathbb{R}^+$, there is a neighborhood U_t of t such that

$$\mid (f^{\sigma}(t) - f(s)) - f^{\Delta}(t)(\sigma(t) - s) \mid \ \leq \varepsilon \mid \sigma(t) - s \mid \quad \text{for all } t \in U_t,$$

where $f^{\sigma} = f \circ \sigma$ on $\mathbb{T}$. A function f is said to be an *rd-continuous* function if it is continuous at right-dense points in $t \in \mathbb{T}$ and has a finite limit at left-dense points. The set of *rd-continuous* and *differentiable rd-continuous* functions are respectively denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. For a function f, the so-called *simple useful formula* holds

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t) \quad \text{for all } t \in \mathbb{T}^{\kappa},$$

where

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

For $s, t \in \mathbb{T}$ and a function $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\xi)\Delta\xi = F(t) - F(s) \quad \text{for } s, t \in \mathbb{T},$$

Table 1 Forward jump, Δ-derivative, Δ-integral on some particular time scales

$\mathbb{T}$	$\mathbb{R}$	$h\mathbb{Z}$ $(h > 0)$	$q^{\mathbb{Z}}$ $(q > 1)$
$\sigma(t)$	t	$t + h$	qt
$\mu(t)$	0	h	$(q-1)t$
$f^{\Delta}(t)$	$f'(t)$	$\frac{f(t+h)-f(t)}{h}$	$\frac{f(qt)-f(t)}{(q-1)t}$
$\int_s^t f(\xi)\Delta\xi$	$\int_s^t f(\xi)\mathrm{d}\xi$	$\sum_{\xi=\frac{s}{h}}^{\frac{t}{h}-1} f(h\xi)h$	$\sum_{\xi=\log_q(s)}^{\log_q(t)-1} f(q^{\xi})q^{\xi}$

where $F \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ is an antiderivative of f, i.e., $F^{\Delta} = f$ on $\mathbb{T}^{\kappa}$. Assume that $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and $t \in \mathbb{T}^{\kappa}$, then

$$\int_t^{\sigma(t)} f(\xi)\Delta\xi = \mu(t)f(t).$$

Table 1 gives the explicit forms for the forward jump, Δ-derivative, and Δ-integral on the well-known time scales of reals, integers, and the quantum set, respectively.

Appendix B Advanced Results on Time Scales

Some advanced results on time scales such as chain rule, derivative of inverse, substitution, Leibniz rule, and change of integration order are given in this section.

Theorem 2 ([2, Theorem 1.93] Chain Rule) *Assume that $f \in \mathrm{C}^1(\mathbb{T}, \mathbb{R})$ is strictly increasing and $f(\mathbb{T}) =: \widetilde{\mathbb{T}}$ is a time scale. Further assume that $g \in \mathrm{C}^1(\widetilde{\mathbb{T}}, \mathbb{R})$. Then*

$$(g \circ f)^{\Delta}(t) = g^{\widetilde{\Delta}}\big(f(t)\big)f^{\Delta}(t) \quad \textit{for } t \in \mathbb{T}^{\kappa}.$$

Remark 2 If $\widetilde{\mathbb{T}} = \mathbb{T}$ holds in Theorem 2, then we get the familiar form $(g \circ f)^{\Delta}(t) = g^{\Delta}(f(t))f^{\Delta}(t)$ for $t \in \mathbb{T}$.

Theorem 3 ([2, Theorem 1.97] Derivative of Inverse) *Assume that $f \in \mathrm{C}^1(\mathbb{T}, \mathbb{R})$ is strictly increasing and $f(\mathbb{T}) =: \widetilde{\mathbb{T}}$ is a time scale. Then*

$$\big(f^{-1}\big)^{\widetilde{\Delta}}(t) = \frac{1}{f^{\Delta}\big(f^{-1}(t)\big)} \quad \textit{provided that } f^{\Delta}\big(f^{-1}(t)\big) \neq 0.$$

Remark 3 If $\widetilde{\mathbb{T}} = \mathbb{T}$ holds in Theorem 3, then we get the familiar form $(f^{-1})^{\Delta}(t) = \frac{1}{f^{\Delta}(f^{-1}(t))}$ provided that $f^{\Delta}(f^{-1}(t)) \neq 0$.

Theorem 4 (Cf. [2, Theorem 1.98] Substitution) *Assume that $a, b \in \mathbb{T}$ with $b > a$ and $\nu \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ is strictly increasing and $\nu(\mathbb{T}) =: \widetilde{\mathbb{T}}$ is a time scale. Furthermore,* *assume that g is a function such that $g \circ \nu \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. Then*

$$\int_a^b g\big(\nu(\eta)\big)\nu^{\Delta}(\eta)\Delta\eta = \int_{\nu(a)}^{\nu(b)} g(\xi)\widetilde{\Delta}\xi.$$

Theorem 5 ([2, Theorem 1.117] Leibniz Rule) *Let $a \in \mathbb{T}^{\kappa}$ and $f : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$. Assume that*

(i) f is continuous at (t, t) where $t \in \mathbb{T}^{\kappa}$ with $t > a$.
(ii) $f^{\Delta_1}(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]_{\mathbb{T}}$, where $\cdot^{\Delta_1}$ denotes the derivative with respect to the first component.
(iii) for every $\varepsilon \in \mathbb{R}^{+}$, there exists a neighborhood U_t of t, independent of $r \in [a, \sigma(t)]_{\mathbb{T}}$, such that

$$\mid f(\sigma(t), r) - f(s, r) - f^{\Delta_1}(t, r)(\sigma(t) - s) \mid \leq \varepsilon \mid \sigma(t) - s \mid \; quad for\ all\ s \in U_t.$$

Then

$$g^{\Delta}(t) = \int_a^t f^{\Delta_1}(t, \xi)\Delta\xi + f(\sigma(t), t), \quad where \quad g(t) = \int_a^t f(t, \xi)\Delta\xi.$$

Theorem 6 ([8, Theorem 3.2] Change of Integration Order) *Let $a \in \mathbb{T}^{\kappa}$, $b \in \mathbb{T}$ and $f : \{(t, s) \in \mathbb{T} \times \mathbb{T} : \ a \leq s \leq t \leq b\} \to \mathbb{R}$ be an rd-continuous function. Let $a \in \mathbb{T}^{\kappa}$, $b \in \mathbb{T}$ and $f : \{(t, s) \in \mathbb{T} \times \mathbb{T}^{\kappa} : \ a \leq s \leq t \leq b\} \to \mathbb{R}$. Assume that*

(i) f is bounded function,
(ii) $f(\cdot, \cdot)$, $\int_{\cdot}^{b} f(\xi, \cdot)\Delta\xi$ and $\int_a^{\cdot} f(\cdot, \eta)\Delta\eta$ are rd-continuous on $[a, b]_{\mathbb{T}^{\kappa}}$,
(iii) for every $\varepsilon \in \mathbb{R}^{+}$, there exists a neighborhood U_t of t, independent of $r \in [a, b]_{\mathbb{T}}$ with $r \leq s \leq t$, such that

$$\mid f(t, r) - f(s, r) \mid \; < \varepsilon \;\; for\ all\ s \in U_t.$$

Then,

$$\int_a^b \int_{\eta}^b f(\xi, \eta)\Delta\xi\Delta\eta = \int_a^b \int_b^{\sigma(\xi)} f(\xi, \eta)\Delta\eta\Delta\xi.$$

References

1. Akin-Bohner, E., Raffoul, Y.N., Tisdell, C.: Exponential stability in functional dynamic equations on time scales. Commun. Math. Anal. **9**(1), 93–108 (2010)
2. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser Inc, Boston, MA (2001)
3. Braverman, E., Karpuz, B.: Uniform exponential stability of first-order dynamic equations with several delays. Appl. Math. Comput. **218**(21), 10468–10485 (2012)
4. Braverman, E., Karpuz, B.: On different types of stability for linear delay dynamic equations. Z. Anal. Anwend. **36**(3), 343–375 (2017)
5. Du, N.H., Tien, L.H.: On the exponential stability of dynamic equations on time scales. J. Math. Anal. Appl. **331**(2), 1159–1174 (2007)

6. Erbe, L.H., Xia, H., Yu, J.S.: Global stability of a linear nonautonomous delay difference equation. J. Differ. Equ. Appl. **1**(2), 151–161 (1995)
7. Győri, I., Pituk, M.: Asymptotic stability in a linear delay difference equation. In: Proceedings of SICDEA, Veszprém, Hungary, 1995, Adv. Differ. Equ. Gordon and Breach, Amsterdam, 295–299 (1997)
8. Karpuz, B.: Volterra theory on time scales. Res. Math. **65**(3–4), 263–292 (2014)
9. Karpuz, B.: Asymptotic stability of the complex dynamic equation $u^{\Delta} - z \cdot u + w \cdot u^{\sigma} = 0$. J. Math. Anal. Appl. **421**(1), 925–937 (2015)
10. Karpuz, B., Öcalan, Ö.: Erratum to: "Stability for first order delay dynamic equations on time scales" [Comput. Math. Appl. 53, 1820–1831(2007)] by H. Wu and Z. Zhou. Comput. Math. Appl. **56**(4), 1157–1158 (2008)
11. Karpuz, B., Öcalan, Ö.: Oscillation and nonoscillation of first-order dynamic equations with positive and negative coefficients. Dyn. Syst. Appl. **18**(3–4), 363–374 (2009)
12. Krisztin, T.: On stability properties for one-dimensional functional–differential equations. Funkcial. Ekvac. **34**(2), 241–256 (1991)
13. Kulenović, M.R.S., Ladas, G., Meimaridou, A.: Stability of solutions of linear delay differential equations. Proc. Am. Math. Soc. **100**(3), 433–441 (1987)
14. Ladas, G., Qian, C., Vlahos, P.N., Yan, J.: Stability of solutions of linear nonautonomous difference equations. Appl. Anal. **41**(1–4), 183–191 (1991)
15. Ladas, G., Sficas, Y.G., Stavroulakis, I.P.: Asymptotic behavior of solutions of retarded differential equations. Proc. Am. Math. Soc. **88**(2), 247–253 (1983)
16. Muroya, Y.: On Yoneyama's $3/2$ stability theorems for one-dimensional delay differential equations. J. Math. Anal. Appl. **247**(1), 314–322 (2000)
17. Peterson, A.C., Raffoul, Y.N.: Exponential stability of dynamic equations on time scales. Adv. Differ. Equ. **2**, 133–144 (2005)
18. Rashford, M., Siloti, J., Wrolstad, J.: Exponential stability of dynamic equations on time scales. PanAm. Math. J. **17**, 39–51 (2007)
19. Shen, J.H., Yu, J.S.: Asymptotic behavior of solutions of neutral differential equations with positive and negative coefficients. J. Math. Anal. Appl. **195**(2), 517–526 (1995)
20. Wu, H., Zhou, Z.: Stability for first order delay dynamic equations on time scales. Comput. Math. Appl. **53**(12), 1820–1831 (2007)
21. Yu, J.S., Cheng, S.S.: A stability criterion for a neutral difference equation with delay. Appl. Math. Lett. **7**(6), 75–80 (1994)
22. Yoneyama, T.: On the $\frac{3}{2}$ stability theorem for one-dimensional delay-differential equations. J. Math. Anal. Appl. **125**(1), 161–173 (1987)
23. Yoneyama, T.: The $3/2$ stability theorem for one-dimensional delay-differential equations with unbounded delay. J. Math. Anal. Appl. **165**(1), 133–143 (1992)
24. Yorke, J.A.: Asymptotic stability for one dimensional differential-delay equations. J. Differ. Equ. **7**, 189–202 (1970)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.