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Solution Sets for Young Differential Inclusions

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Abstract

The paper deals with some properties of solutions of differential inclusions driven by set-valued integrals of a Young type. The existence of solutions, boundedness, closedness of the set of solutions and continuous dependence type results are considered. These inclusions contain as a particular case set-valued stochastic inclusions with respect to a fractional Brownian motion (fBm), and therefore, their properties are crucial for investigation the properties of solutions of fBm stochastic differential inclusions.

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1 Introduction

Control and optimal control problems inspired on intensive expansion of the differential and stochastic differential inclusions theory [1, 15, 16] and [28]. Set-valued Aumann or Itô type integrals are necessary tools in the investigation of properties of solutions of differential or stochastic differential inclusions [13, 17]. Controlled differential equations driven by Young integrals were initiated by T. Lions in [21]. A more advanced approach to controlled differential equations was developed in [7, 10, 11] and [20]. Thus it seems reasonable to investigate differential inclusions driven by set-valued Young type integrals. To the best of our knowledge, such integrals were introduced and investigated in [8, 22] and [23] only. Moreover, in [4] and [8] the authors considered Young type differential inclusions with solutions understood as Young integrals of appropriately regular selections of set-valued right-hand side. Motivated by this, the aim of this work is to investigate the properties of set-valued

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Young integrals and study Young differential inclusions driven by such Young type integrals.

Set-valued Young integrals considered in the paper concern the class of set-valued functions having Hukuhara derivatives. We investigate differential inclusions driven by such set-valued integrals. We prove that the set of solutions of such an inclusion is nonempty, closed and bounded in the space of α-Hölder continuous functions. We also consider the problem of continuous dependence of solutions from the right-hand side of the inclusion.

Such an inclusion contains as a particular case a stochastic differential inclusion driven by a fractional Brownian motion. Therefore, in our opinion, the results obtained herein are useful for investigating the existence of solutions of stochastic inclusions driven by fBm and their properties.

The paper is organized as follows. In Sect. 2, we define a space of β-Hölder set-valued functions. Section 3 deals with properties of sets of appropriately regular selections of set-valued functions together with definitions of Aumann and a set-valued Young integrals. In Sect. 4 we investigate properties of set-valued Young integral and Young differential inclusions driven by set-valued integrals of an Aumann and a Young type. We prove the existence of solutions to such a differential inclusion and some properties of the set of its solutions.

2 Hölder Continuous Set-Valued Functions

Let $(R^n, |\cdot|)$ be an Euclidean space. The absolute value norm in R^1 is denoted by $|\cdot|$ instead of $|\cdot|$. Denote by $Comp(R^n)$ and $Conv(R^n)$ the families of all nonempty and compact, and nonempty compact and convex subsets of R^n, respectively. The Hausdorff metric H in $Comp(R^n)$ is defined by

$$H(B, C) = \max\{\overline{H}(B, C), \overline{H}(C, B)\},$$

where $\overline{H}(B, C) = \sup_{b\in B} \mathrm{dist}(b, C) = \sup_{b\in B} \inf_{c\in C} \mid c - b \mid$. The space $(Comp(R^n), H)$ is a Polish metric space and $(Conv(R^n), H)$ is its closed subspace. For $B, C, D, E \in Comp(R^n)$ we have,

$$H(B + C, D + E) \leq H(B, D) + H(C, E) \tag{1}$$

where $B + C := \{b + c : b \in B, c \in C\}$ denotes the Minkowski sum of B and C. Moreover, for $B, C, D \in Conv(R^n)$ the equality

$$H(B + D, C + D) = H(B, C), \tag{2}$$

holds, see e.g., [19] for details.

We use the notation

$$\|A\| := H(A, \{0\}) = \sup_{a\in A} \mid a \mid \text{ for } A \in Conv(R^n).$$

Let $\beta \in (0, 1]$. For every function $f : R^n \supset [a, b]^n \to R^n$ we define

$$\|f\|_\infty = \sup_{x\in[a,b]^n} |\, f(x)\,| \text{ and } M_\beta(f) = \sup_{x\neq y\in[a,b]^n} \frac{|\, f(x) - f(y)\,|}{|\, x - y\,|^\beta}.$$

By $\mathcal{C}^\beta([a, b]^n, R^n)$ we denote the space of β-Hölder-continuous functions with a finite norm

$$\|f\|_\beta := \|f\|_\infty + M_\beta(f).$$

It can be shown that $\mathcal{C}^\beta([a, b]^n, R^n)$ is a Banach space.

Similarly, for a set-valued function $F : [a, b]^n \to Conv(R^n)$ let

$$\|F\|_\beta := \|F\|_\infty + M_\beta(F)$$

where

$$\|F\|_\infty = \sup_{x\in[a,b]^n} \|F(x)\| \text{ and } M_\beta(F) = \sup_{x\neq y\in[a,b]^n} \frac{H(F(x), F(y))}{|\, x - y\,|^\beta}.$$

A set-valued function F is said to be β-Hölder if $\|F\|_\beta < \infty$. The space of β-Hölder set-valued functions having compact and convex values will be denoted by $\mathcal{C}^\beta\big([a, b]^n, Conv(R^n)\big)$. We say that $F : R^n \to Conv(R^n)$ is locally β-Hölder if for every $[a, b]^n \subset R^n$ a set-valued function $F_{|[a,b]^n}$ with its domain restricted to a cube $[a, b]^n$ belongs to $\mathcal{C}^\beta([a, b]^n, Conv(R^n))$. If $F : [0, T] \times R^n \to Conv(R^n)$ then we say that $F(t, x) \in \mathcal{C}^{\beta\times\gamma}([0, T] \times R^n, Conv(R^n))$ if F is β-Hölder in t and γ-Hölder in x.

Let $A, B \in \mathrm{Conv}(R^n)$. The set $C \in \mathrm{Conv}(R^n)$ is said to be *the Hukuhara difference* $A \div B$ if $A = B + C$.

Definition 1 Consider a set-valued mapping $G : R^n \to \mathrm{Conv}(R^n)$. For $k = 1, \ldots n$, let $e_k = (e_k^1, \ldots e_k^n)$ be the vector such that $e_k^j = 0$ for $k \neq j$ and $e_k^k = 1$.

We say that G admits *a Hukuhara derivative* at $x_0 \in R^n$, if there exists a set $DH^k(G)(x_0)$ such that the limits

$$\lim_{h\to 0+} \frac{G(x_0 + he_k) \div G(x_0)}{h}$$

and

$$\lim_{h\to 0+} \frac{G(x_0) \div G(x_0 - he_k)}{h}$$

exist with respect to a Hausdorff metric in $Conv(R^n)$ and are equal to the set $DH^k(G)(x_0)$, $k = 1, \ldots, n$, (see e.g., [6]).

We define a set $DH(G)(x_0)$ by the formula

$y \in DH(G)(x_0)$ if and only if $y = (y_1, y_2, \dots, y_n)$, where $y_k \in DH^k(G)(x_0)$, $k = 1, \dots, n$.

If $G = \{g\}$ is a single-valued function then

$$DH(G)(x_0) = g'(x_0) = (g_1'(x_0), g_2'(x_0), \dots, g_d'(x_0)) = \left(\frac{\partial g(x_0)}{\partial x_0^1}, \dots, \frac{\partial g(x_0)}{\partial x_0^n}\right)$$

is a gradient of a function g at point x_0. A set-valued function G has a locally bounded Hukuhara derivative if for every $[a, b]^n \subset R^n$ there exists $N_{a,b} > 0$ such that $\sup_{x\in[a,b]^n} \sup_{k=1,\dots,n} \|DH^k(G)(x)\| < N_{a,b}$.

Proposition 1 *Let $G : R^n \to Conv(R^d)$ be a set-valued function. If G has a locally bounded Hukuhara derivative then G is locally β-Hölder (i.e., for every $[a, b]^n \subset R^n$, $G_{|[a,b]^n}$ belongs to $\mathcal{C}^\beta([a, b]^n, Conv(R^d))$ for every $\beta \in (0, 1]$. Moreover,*

$$\|G_{|[a,b]^n}\|_\beta = \|G_{|[a,b]^n}\|_\infty + M_\beta(G_{|[a,b]^n}) \le \|G(\mathbf{a})\| + nN_{a,b}\big((b-a) + (b-a)^{1-\beta}\big),$$

where $\mathbf{a} = (a, \dots, a) \in R^n$.

Proof Let $[a, b]^n \subset R^n$ be fixed and such that there exists $N_{a,b} > 0$ satisfying $\sup_{x\in[a,b]^n} \sup_{i=1,\dots,n} \|DH^i(G)(x)\| < N_{a,b}$. For every $i = 1, \dots, n$ and $a \le t_0 \le t \le b$ we have

$$\begin{aligned} &G(x_1, \dots x_{i-1}, t, x_{i+1}, \dots x_n) \\ &\quad = G(x_1, \dots x_{i-1}, t_0, x_{i+1}, \dots x_n) + \int_{t_0}^t DH^i(G)(x_1, \dots x_{i-1}, \tau, x_{i+1}, \dots x_n)d\tau \end{aligned}$$

by [19].

Therefore, using formula (2), we obtain for every $a \le t_0 \le s \le t \le b$

$$\begin{aligned} &H\left(G(x_1, \dots x_{i-1}, t, \dots, x_n), G(x_1, \dots x_{i-1}, s, \dots, x_n)\right) \\ &\quad = H\left(\int_{t_0}^t DH^i(G)(x_1, \dots x_{i-1}, \tau, \dots, x_n)d\tau, \int_{t_0}^s DH^i(G)(x_1, \dots x_{i-1}, \tau, \dots, x_n)d\tau\right) \\ &\quad = H\left(\int_s^t DH^i(G)(x_1, \dots x_{i-1}, \tau, \dots, x_n)d\tau, \{0\}\right) \le N_{a,b}(t-s). \end{aligned}$$

Dividing both sides of the above inequality by $(t-s)^\beta$ we obtain

$$M_\beta^i(G_{|[a,b]^n}) \le N_{a,b}(b-a)^{1-\beta}. \tag{3}$$

Moreover, for every $x, y \in [a, b]^n$ we get

$$\begin{aligned} H\big(G(x), G(y)\big) &= H\big(G(x_1, \dots, x_n), G(y_1, \dots, y_n)\big) \\ &\le H\big(G(x_1, \dots, x_n), G(y_1, x_2, \dots, x_n)\big) \\ &\quad + H\big(G(y_1, x_2, \dots, x_n), G(y_1, y_2, x_3, \dots, x_n)\big) \\ &\quad + \dots + H\big(G(y_1, y_2, \dots y_{n-1}, x_n), G(y_1, y_2, \dots, y_{n-1}, y_n)\big). \end{aligned}$$

Since $| x - y |^{\beta} \geq | x_i - y_i |^{\beta}$ then we have

$$M_\beta(G_{|[a,b]^n}) = \sup_{x \neq y \in [a,b]^n} \frac{H(G(x), G(y))}{| x - y |^\beta}$$
$$\leq \sum_{i=1}^{n} M_\beta^i(G_{|[a,b]^n}) \leq nN_{a,b}(b-a)^{1-\beta} < \infty \tag{4}$$

by formula (3).

Now we calculate the upper bound for $\|G_{|[a,b]^n}\|_\infty = \sup_{x \in [a,b]^n} \|G(x)\|$.

Let us note that for every $x \in [a,b]^n$ we have

$$G(x) = G(x_1, \ldots, x_n) = G(a, x_2, \ldots, x_n) + \int_a^{x_1} DH^1(G)(\tau, x_2, \ldots, x_n)d\tau$$
$$= G(a, a, x_3, \ldots, x_n) + \int_a^{x_2} DH^2(G)(a, \tau, x_3, \ldots, x_n)d\tau$$
$$+ \int_a^{x_1} DH^1(G)(\tau, x_2, \ldots, x_n)d\tau$$
$$= \ldots = G(a, a, \ldots, a) + \sum_{i=1}^{n} \int_a^{x_i} DH^i(G)(a, a, \ldots, \tau, x_{i+1}, \ldots, x_n)d\tau.$$

Therefore,

$$\sup_{x \in [a,b]^n} \|G(x)\| \leq \|G(\mathbf{a})\| + \sum_{i=1}^{n} N_{a,b}(x_i - a) \leq \|G(\mathbf{a})\| + nN_{a,b}(b-a) < \infty, \tag{5}$$

what together with (4) proves the desired inequality. □

Remark 1 In the proof of formula (5) one can take any point $\mathbf{x_0} \in [a,b]^n$ instead of a point $\mathbf{a}$ to obtain the inequality

$$\sup_{x \in [a,b]^n} \|G_{|[a,b]^n}\|_\beta \leq \|G(\mathbf{x_0})\| + nN_{a,b}\Big((b-a) + (b-a)^{1-\beta}\Big).$$

For a detailed discussion of the properties and applications of Hukuhara differentiable set-valued functions we refer the reader to [19].

3 Hölder Set-Valued Functions and Set-Valued Young Integrals

We recall the notion of a Young integral in a single valued case introduced by L.S. Young in [29]. For details see also [10]. Let $f : R^1 \to R^d$ and $w : R^1 \to R^1$ be given functions. For the partition $\Pi_m : a = t_0 < t_1 < \ldots < t_m = b$ of the interval $[a, b]$ we consider the Riemann sum of f with respect to w

$$S(f, w, \Pi_m) := \sum_{i=1}^{m} f(t_{i-1})(w(t_i) - w(t_{i-1})).$$

Let $|\ \Pi_m\ | := \max\{t_i - t_{i-1} : 1 \leq i \leq m-1\}$. Then the following result holds (see e.g., [11] and [26]).

Proposition 2 *Let $f_{|[a,b]} \in \mathcal{C}^\beta([a, b], R^d)$ and $w_{|[a,b]} \in \mathcal{C}^\alpha([a, b], R^1)$ where $\alpha, \beta \in (0, 1]$, $\beta + \alpha > 1$. Then the limit*

$$\lim_{|\Pi_m| \to 0} S(f_{|[a,b]}, w, \Pi_m) =: \int_a^b f(\tau) dw_\tau$$

exists and the inequality

$$\left| \int_s^t f(\tau) dw_\tau - f(s)(w(t) - w(s)) \right| \leq C(\alpha, \beta) M_\beta(f) M_\alpha(w)(t-s)^{\alpha+\beta} \tag{6}$$

holds for every $a \leq s < t \leq b$, where the constant $C(\alpha, \beta)$ depends only on β and α.

In the case $f_{|[a,b]} \in \mathcal{C}^\beta([a, b], R^d)$, $w_{|[a,b]} \in \mathcal{C}^\alpha([a, b], R^1)$ and $\alpha, \beta \in (0, 1]$ with $\alpha + \beta > 1$, one can express the Young integral by fractional derivatives. Namely, let

$$f_{a+}(t) = (f(t) - f(a+)) I_{(a,b)}(t) \text{ and } f_{b-}(t) = (f(t) - f(b-)) I_{(a,b)}(t),$$

where $I_{(a,b)}(\cdot)$ denotes the characteristic function of the interval (a, b). The right-sided and left-sided fractional derivatives of order $0 < \rho < 1$ for the function $f_{|[a,b]}$ are defined by

$$D_{a+}^{\rho} f(t) = \frac{1}{\Gamma(1-\rho)} \left(\frac{f(t)}{(t-a)^\rho} + \rho \int_a^t \frac{f(t) - f(s)}{(t-s)^{\rho+1}} ds \right)$$

and

$$D_{b-}^{\rho} f(t) = \frac{(-1)^\rho}{\Gamma(1-\rho)} \left(\frac{f(t)}{(b-t)^\rho} + \rho \int_t^b \frac{f(t) - f(s)}{(s-t)^{\rho+1}} ds \right).$$

Then we get by [26]

$$\int_a^b f_\tau dw_\tau = (-1)^\rho \int_a^b D_{a+}^{\rho} f_{a+}(t) D_{b-}^{1-\rho} w_{b-}(t) dt + f(a)(w(b) - w(a))$$

for every $\rho \in (1 - \alpha, \beta)$.

Let $F : R^n \to Conv(R^d)$ be a measurable set-valued function. For $1 \leq p < \infty$, define the set

$$S_{L^p}(F) = \{f \in L^p : \ f(x) \in F(x) \text{ for a.e. } x \in R^n\},$$

where $L^p = L^p(R^n, R^d)$. Elements of the set $S_{L^p}(F)$ are called *integrable selections of* F. Since values of F are closed in R^d then $S_{L^p}(F)$ is a closed subset of L^p (see e.g., [14], formula (1.1)). It is nonempty if F is p-integrably bounded i.e., if there exists $h \in L^p$ such that $\|F(x)\| \leq h(x)$ for a.e. $x \in R^n$ ([14], Theorem 3.2).

A set-valued Aumann integral of F over the measurable set $A \subset R^n$ was defined in [3] by the formula

$$\int_A F(x)\, d\mu = \left\{ \int_A f(x)\, d\mu : \ f \in S_{L^1}(F) \right\}.$$

For properties of measurable set-valued functions and their measurable selections see e.g., [2].

Let $G : R^1 \to Conv(R^d)$ be a Hukuhara differentiable set-valued function with p-integrably bounded Hukuhara derivative $DH(G)$. Assume that G belongs to $\mathcal{C}^\beta(R^1, Conv(R^d))$ locally. By $S_\beta(G)$ we denote the set of all locally β-Hölder selections of G, i.e.,

$$S_\beta(G) = \{g : g_{|[a,b]} \in \mathcal{C}^\beta([a,b], R^d) \text{ for every } a < b, \ g(t) \in G(t) \text{ for } t \in R^1\}$$

Definition 2 We define the set

$$\mathcal{IS}(G) = \{g : g \in S_\beta(G) \text{ and } \ g' \in S_{L^p}(DH(G))\} \tag{7}$$

and *a set-valued Young integral* of a locally β-Hölder and Hukuhara differentiable set-valued function G with respect to a function $w \in \mathcal{C}^\alpha(R^1, R^1)$, $\beta + \alpha > 1$, by the formula

$$(\mathcal{IS}) \int_a^b G(\tau) dw_\tau := \left\{ \int_a^b g(\tau) dw_\tau : \ g \in \mathcal{IS}(G) \right\}. \tag{8}$$

The above definition of set-valued integral is proper if the set $\mathcal{IS}(G)$ is nonempty. Conditions assuring the nonemptiness of this set can be found in [23].

By Proposition 2 we obtain

$$\|(\mathcal{IS}) \int_s^t G(\tau) dw_\tau\| \leq M_\alpha(w) \|G\|_\beta (1 + C(\alpha, \beta)(b-a)^\beta)(t-s)^\alpha$$

for $a \leq s \leq t \leq b$. Since G and $D_H(G)$ take on convex values then the sets $S_\beta(G)$ and $S_{L^p}(D_H(G))$ are convex and therefore, $\mathcal{IS}(G)$ and $(\mathcal{IS}) \int_s^t G(\tau) dw_\tau$ for every

$s, t \in [a, b]$, $s < t$, are convex subsets of $\mathcal{C}^{\beta}([a, b], R^d)$ and R^d, respectively. It was proved in [23] Theorem 7 that the set $(\mathcal{IS}) \int_a^b G(\tau)dw_\tau$ is bounded in R^d and $(\mathcal{IS}) \int_a^{\cdot} G(\tau)dw_\tau$ is a bounded set in $\mathcal{C}^{\alpha}(R^1, Conv R^d)$.

The following result is a variant of Lemma 1 from [22].

Lemma 3 *Let $w \in \mathcal{C}^{\alpha}(R^1, R^1)$ locally. Then, for every $\rho \in (1-\alpha, \beta)$, $a, b \in R^1$, $a < b$, there exists a positive constant $C(\rho)$ such that for every $g^1, g^2 \in \mathcal{C}^{\beta}(R^1, R^d)$ locally and $\theta \in (0, 1]$ the inequality*

$$\begin{aligned} &\left| \int_a^b g^1(\tau)dw_\tau - \int_a^b g^2(\tau)dw_\tau \right| \\ &\leq C(\rho) \left[\|g^1 - g^2\|_\infty + (M_\beta(g^1_{|[a,b]}) + M_\beta(g^2_{|[a,b]}))\theta^\beta \right] \theta^{-\rho} \\ &\quad + |\ g^1(a) - g^2(a)\ | \cdot |\ w(b) - w(a)\ | \end{aligned}$$

holds.

4 Young Differential Inclusion, Existence and Properties of Solutions

Let $\alpha \in (0, 1]$, $\beta > 1-\alpha$ and $\gamma > \frac{1-\alpha}{\alpha}$ be given. Let $w : [0, T] \to R^m$ be an α-Hölder function while $F : [0, T] \times R^d \to Conv(R^d)$ and $G : [0, T] \times R^d \to Conv(R^{d\times m})$ be set-valued functions. Assume the following hypotheses on set-valued functions F and G:

$(H1)$ F is a Carathéodory set-valued function, i.e., product measurable in (t, x) and lower semicontinuous in x.

$(H2)$ There exists a function $b \in L^{\frac{1}{1-\alpha}}([0, T])$ and a constant $L > 0$ such that for every $(t, x) \in [0, T] \times R^d$

$$\|\ F(t, x)\ \| \leq L \mid\ x\ \mid + b(t).$$

$(H3)$ G is measurable in (t, u) and G has locally bounded Hukuhara derivative $DH(G)$, i.e., if for every $[[0, T] \times [a, b]^n \subset R^{n+1}$ there exists $N_{a,b} > 0$ such that $\sup_{(t,u)\in[0,T]\times[a,b]^n} \|DH(G)(t, u)\| < N_{a,b}$.

We define the set

$$\mathcal{IS}(G) = \{g \in \mathcal{C}^{\beta\times\gamma}([0, T] \times R^d, R^{d\times m}) : g(t, x) \in G(t, x),\ g' \in S_{L^p}(DH(G))\}$$

Definition 3 ([23]) For every $x \in \mathcal{C}^{\alpha}([0, T], R^d)$ and a set-valued function G from $\mathcal{C}^{\beta\times\gamma}([0, T] \times R^d, Conv(R^{d\times m}))$ and being Hukuhara differentiable we define *a set-valued Young integral* of $G \circ x$ with respect to a function $w \in \mathcal{C}^{\alpha}([0, T], R^m)$ by the formula

$$(\mathcal{IS}) \int_s^t G(\tau, x(\tau))dw_\tau := \left\{ \int_s^t g(\tau, x(\tau))dw_\tau :\ g \in \mathcal{IS}(G) \right\} \tag{9}$$

for every $0 \leq s < t \leq T$.

This integral is nonempty under condition $(H3)$ above.

Proposition 4 *Let $G_1, G_2 : [0,T] \times R^d \to Conv(R^{d\times m})$ be set-valued functions with locally bounded Hukuhara derivatives. Let $w \in \mathcal{C}^\alpha([0,T], R^m)$ while $x \in \mathcal{C}^\alpha([0,T], R^d)$, $x(0) = x_0$. Then, for every $\delta \in (0,1)$, $\rho \in (1-\alpha, \delta)$, there exist positive constants $C(\rho)$, $P_{T,x,\epsilon(x)}$ such that for every $\theta \in (0,1]$, $0 \leq s < t \leq T$ the inequality*

$$
\begin{aligned}
&H\Big((\mathcal{IS})\int_s^t G_1(\tau, x(\tau))dw(\tau), (\mathcal{IS})\int_s^t G_2(\tau, x(\tau))dw(\tau/)\Big)\\
&\leq C(\rho)\Bigg\{2(d+1)(\epsilon(x)\vee T)\sup_{(\tau,v)\in A(x)}\bar{H}\Big(DH(G_1)(\tau,v), DH(G_2)(\tau,v)\Big)\\
&\quad +(T+T^{1-\delta})P_{T,x,\epsilon(x)}\Bigg(\sup_{(t,u)\in A(x)}\|DH(G_1)(t,u)\| + \sup_{(t,u)\in A(x)}\|DH(G_2)(t,u)\|\Bigg)\theta^\delta\\
&\quad +H(G_1(0,x_0), G_2(0,x_0))\Bigg\}\theta^{-\rho}\\
&\quad +M_\alpha(w)T^\alpha(2d\epsilon(x)+T)\sup_{(\tau,v)\in A(x)}\bar{H}\Big(DH(G_1)(\tau,v), DH(G_2)(\tau,v)\Big)\\
&\quad +H(G_1(0,x_0), G_2(0,x_0)) \qquad (10)
\end{aligned}
$$

holds, where $A(x) = [0,T] \times [-\epsilon(x), \epsilon(x)]^d$ is such that $x(t) \in [-\epsilon(x), \epsilon(x)]^d$ for every $t \in [0,T]$.

Proof Since $x \in \mathcal{C}^\alpha([0,T], R^d)$ then there exists $\epsilon(x) > 0$ such that $\mid x(t) \mid < \epsilon(x)$ for $t \in [0,T]$ and therefore, the set $A(x)$ exists. Moreover, there exists $N_{\epsilon(x)} > 0$ such that $\|DH(G)(t,u)\| \leq N_{\epsilon(x)}$ for every $(t,u) \in A(x)$ by $(H3)$. Then, for every $\delta \in (0,1)$, $G_{|A(x)}$ is δ-Hölder with a Hölder constant dependent only on $\epsilon(x)$ and

$$
\|G_{|A(x)}\|_\delta \leq \|G(0,x_0)\| + N_{\epsilon(x)}(T+T^{1-\delta}) + dN_{\epsilon(x)}\big(2\epsilon(x) + (2\epsilon(x))^{1-\delta}\big) < \infty
$$

by Proposition 1. Let $(\mathcal{IS})(G_i)\circ x = \{g\circ x : g \in (\mathcal{IS})(G_i)\}$ for $i = 1, 2$.

Using Lemma 3 we get

$$
\begin{aligned}
&\bar{H}\Big((\mathcal{IS})\int_s^t G_1(\tau, x(\tau))dw(\tau), (\mathcal{IS})\int_s^t G_2(\tau, x(\tau))dw(\tau)\Big)\\
&\leq C(\rho)[\bar{H}_\infty\Big((\mathcal{IS})(G_1)\circ x, (\mathcal{IS})(G_2)\circ x\Big)\theta^{-\rho}\\
&\quad +\Big(\sup_{g_1\in\mathcal{IS}(G_1)} M_\sigma(g_1\circ x) + \sup_{g_2\in\mathcal{IS}(G_2)} M_\sigma(g_2\circ x)\Big)\theta^\sigma\theta^{-\rho}]\\
&\quad +M_\alpha(w)(t-s)^\alpha\bar{H}_\infty\Big((\mathcal{IS})(G_1)\circ x, (\mathcal{IS})(G_2)\circ x\Big) \qquad (11)
\end{aligned}
$$

for every $0 \le s < t \le T$. First we will establish the estimation of $\bar{H}_\infty\big((\mathcal{IS})(G_1) \circ x, (\mathcal{IS})(G_2) \circ x\big)$. For simplicity of notation let us denote the variable t as a "zero" variable i.e., let $g(t, x(t)) = g(x_0(t), x_1(t), \dots, x_d(t))$, where $x_0(t) = t,\ x(t) = (x_1(t), \dots, x_d(t)),\ x_0(0) = 0 \in R^1,\ x(0) = x_0 = (x_1(0), \dots, x_d(0)) \in R^d$.

Let us note that for every differentiable function $f : R^{d+1} \to R^d$ with $f' \in L^p$, the following formula

$$f(z_0, z_1, \dots, z_d) = f(y_0, y_1, \dots, y_d) + \int_{y_0}^{z_0} f_0'(s_0, z_1, \dots, z_d)ds_0 + \sum_{i=1}^{d} \int_{y_i}^{z_i} f_i'(y_0, \dots, y_{i-1}, s_i, z_{i+1}, \dots, z_d)ds_i \qquad (12)$$

holds.

Let us take an arbitrary $g_1 \in (\mathcal{IS})(G_1)$ and $g_2 \in (\mathcal{IS})(G_2)$. Then $(g_1)_i' \in S_{L^p}(DH^i(G_1))$ while $(g_2)_i' \in S_{L^p}(DH^i(G_2))$ for $i = 0, 1, \dots, d$. Since $(t, x(t)) \in A(x)$ then taking $u = (u_1, \dots u_d)$ we get by formula (12)

$$\begin{aligned} &\inf_{g_2 \in \mathcal{IS}(G_2)} \sup_{t \in [0,T]} \mid g_1(t, x(t)) - g_2(t, x(t)) \mid \\ &\le \inf_{g_2 \in \mathcal{IS}(G_2)} \sup_{(t,u) \in A(x)} \mid g_1(t, u) - g_2(t, u) \mid \\ &= \inf_{g_2 \in \mathcal{IS}(G_2)} \sup_{(t,u) \in A(x)} \mid (g_1 - g_2)(0, x(0)) + \int_{x_0(0)}^{t} (g_1 - g_2)_0'(s_0, u_1, \dots, u_d)ds_0 \\ &\quad + \sum_{i=1}^{d} \int_{x_i(0)}^{u_i} (g_1 - g_2)_i'(x_0(0), \dots, x_{i-1}(0), s_i, u_{i+1}, \dots, u_d)ds_i \mid . \end{aligned}$$

Since the formula $y(u) = y(u_0) + \int_{u_0}^{u} y'(s)ds$ uniquely determines the function y then $\inf_{g_2 \in \mathcal{IS}(G_2)}$ is the same as $\inf_{g_2' \in S_{L^p}(DH(G_2))} \inf_{\bar{y}_0 \in G_2(0, x(0))}$, where $\bar{y}_0$ means $g_2(0, x(0))$. Therefore,

$$\begin{aligned} &\inf_{g_2 \in \mathcal{IS}(G_2)} \sup_{(t,u) \in A(x)} \mid g_1(t, u) - g_2(t, u) \mid \\ &\le \inf_{g_2' \in S_{L^p}(DH(G_2))} \inf_{\bar{y}_0 \in G_2(0, x(0))} \sup_{(t,u) \in A(x)} \Bigg| g_1(0, x(0)) - \bar{y}_0 \\ &\quad + \int_{x_0(0)}^{t} (g_1 - g_2)_0'(s_0, u_1, \dots, u_d)ds_0 \\ &\quad + \sum_{i=1}^{d} \int_{x_i(0)}^{u_i} (g_1 - g_2)_i'(x_0(0), \dots, x_{i-1}(0), s_i, u_{i+1}, \dots, u_d)ds_i \Bigg| \\ &\le \inf_{g_2' \in S_{L^p}(DH(G_2))} \inf_{\bar{y}_0 \in G_2(0, x(0))} \sup_{(t,u) \in A(x)} \Bigg\{ \mid g_1(0, x(0)) - \bar{y}_0 \mid \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \mid (g_1-g_2)_0'(s_0,u_1,\dots,u_d)\mid ds_0 \\
&+\sum_{i=1}^d \int_{-\epsilon(x)}^{\epsilon(x)} \mid (g_1-g_2)_i'(x_0(0),\dots,x_{i-1}(0),s_i,u_{i+1},\dots,u_d)\mid ds_i \Bigg\} \\
&\le \inf_{\bar{y}_0\in G_2(0,x(0))} \mid g_1(0,x(0))-\bar{y}_0\mid \\
&+\inf_{(g_2)_0'\in S_{L^p}(DH^0(G_2))}\sup_{(t,u)\in A(x)}\sup_{s_0\in[0,T]} \mid (g_1-g_2)_0'(s_0,u_1,\dots,u_d)\mid \cdot T \\
&+\sum_{i=1}^d \inf_{(g_2)_i'\in S_{L^p}(DH^i(G_2))}\sup_{(t,u)\in A(x)}\sup_{s_i\in[-\epsilon(x),\epsilon(x)]} \\
&\mid (g_1-g_2)_i'(x_0(0),\dots,s_i,u_{i+1},\dots,u_d)\mid \cdot 2\epsilon(x) \\
&\le dist_{R^d}\big(g_1(0,x(0)),G_2(0,x(0))\big) \\
&+T\inf_{(g_2)_0'\in S_{L^p}(DH^0(G_2))}\sup_{(t,u)\in A(x)} \mid (g_1)_0'(t,u)-(g_2)_0'(t,u)\mid \\
&+2\epsilon(x)\sum_{i=1}^d \inf_{(g_2)_i'\in S_{L^p}(DH^i(G_2))}\sup_{(t,u)\in A(x)} \mid (g_1)_i'(t,u)-(g_2)_i'(t,u)\mid \\
&\le dist_{R^d}\big(g_1(0,x(0)),G_2(0,x(0))\big)+T dist_\infty\big((g_1)_0',DH^0(G_2)\big) \\
&+2\epsilon(x)\sum_{i=1}^d dist_\infty\big((g_1)_i',DH^i(G_2)\big) \\
&\le \bar{H}\big(G_1(0,x(0)),G_2(0,x(0))\big)+T\bar{H}_\infty\big(DH^0(G_1),DH^0(G_2)\big) \\
&+2\epsilon(x)\sum_{i=1}^d \bar{H}_\infty\big(DH^i(G_1),DH^i(G_2)\big) \\
&\le \bar{H}\big(G_1(0,x(0)),G_2(0,x(0))\big)+(2d\epsilon(x)+T)\bar{H}_\infty\big(DH(G_1),DH(G_2)\big)
\end{aligned}
$$

because (g_1) and $(g_1)_i'$ are arbitrary elements from G_1 and $DH^i(G_1)$, respectively. Hence

$$
\begin{aligned}
\bar{H}_\infty\big((\mathcal{IS})(G_1)\circ x,(\mathcal{IS})(G_2)\circ x\big) &\le \bar{H}\big(G_1(0,x(0)),G_2(0,x(0))\big) \\
&\quad+(2d\epsilon(x)+T)\bar{H}_\infty\big(DH(G_1),DH(G_2)\big).
\end{aligned} \tag{13}
$$

By formula (19) we have

$$
\sup_{g_i\in\mathcal{IS}(G_i)} M_\sigma(g_i\circ x)\le P_{T,x,\epsilon(x)} \text{ for } i=1,2,
$$

where $P_{T,x,\epsilon(x)}=N_{T,\epsilon(x)}\big(T^{1-\delta}+d(2\epsilon(x))^{1-\delta}\big)\big(T^{1-\alpha}+M_\alpha(x)^\delta\big)$. Then by (13) and (11) we obtain desired inequality (10). □

Corollary 5 *Let $G : [0,T] \times R^d \to Conv(R^{d\times m})$ be a set-valued function with locally bounded Hukuhara derivative. Let $w, x \in \mathcal{C}^\alpha([0,T], R^d)$, $x(0) = x_0$. Then, $(\mathcal{IS})\int_s^t G(\tau, x(\tau))dw(\tau)$ is a bounded set in R^d for every $0 \le s < t \le T$.*

Proof Let $G_1 := G$, $G_2 := \{0\} \subset R^{d\times m}$ and $0 \le s \le t \le T$. Then by Proposition 4 we get

$$\left\|(\mathcal{IS})\int_s^t G(\tau, x(\tau))dw(\tau)\right\|$$
$$\le C(\rho)\left\{2(d+1)(\epsilon(x)\vee T)\sup_{(\tau,v)\in A(x)}\|DH(G)(\tau,v)\| + \left(T+T^{1-\delta}\right)P_{T,x,\epsilon(x)}\right.$$
$$\left.\cdot \sup_{(\tau,v)\in A(x)}\|DH(G)(\tau,v)\|\,\theta^\delta + \|G(0,x_0)\|\right\}\theta^{-\rho}$$
$$+M_\alpha(w)T^\alpha\left(2d\epsilon(x)+T\right)\sup_{(\tau,v)\in A(x)}\|DH(G)(\tau,v)\| + \|G(0,x_0)\|,$$

where $\epsilon(x)$, $A(x) = [0,T]\times[-\epsilon(x),\epsilon(x)]^d$, $\delta\in(0,1)$, $\alpha+\delta>1$, $\rho\in(1-\alpha,\delta)$ and $\theta\in(0,1]$ are the same as in Proposition 4. Since G has compact values in $R^{d\times m}$, then $\|G(0,x_0)\| < \infty$. Moreover, since it has also a locally bounded Hukuhara derivative, then $\sup_{(\tau,v)\in A(x)}\|DH(G)(\tau,v)\| < \infty$. Hence it follows that $\left\|(\mathcal{IS})\int_s^t G(\tau,x(\tau))dw(\tau)\right\| < \infty$ what proves boundedness of Young integral $(\mathcal{IS})\int_s^t G(\tau,x(\tau))dw(\tau)$ in R^d. □

Corollary 6 *Let $w, x \in \mathcal{C}^\alpha([0,T], R^d)$, $x(0) = x_0$ and $A(x)$ be the same as in Proposition 4. Let $G_n, G : [0,T]\times R^d \to Conv(R^{d\times m})$ be set-valued functions with locally bounded Hukuhara derivatives satisfying*

$$H(G_n(0,x_0), G(0,x_0)) \to 0 \text{ as } n\to\infty$$

and

$$\sup_{(t,v)\in[0,T]\times A(x)} H(DH(G_n)(t,v), DH(G)(t,v)) \to 0 \text{ as } n\to\infty.$$

Then,

$$\sup_{0\le s<t\le T} H\left((\mathcal{IS})\int_s^t G_n(s,x(s))dw(s), (\mathcal{IS})\int_s^t G(s,x(s))dw(s)\right) \to 0 \quad (14)$$

as $n\to\infty$.

Proof Let us note that for every $n\ge 1$ we have

$$\left|\sup_{(t,v)\in[0,T]\times A(x)}\|DH(G_n)(t,v)\| - \sup_{(t,v)\in[0,T]\times A(x)}\|DH(G)(t,v)\|\right|$$

$$\leq \sup_{(t,v)\in[0,T]\times A(x)} H(DH(G_n)(t,v), DH(G)(t,v)).$$

Thus,

$$\sup_{(t,v)\in[0,T]\times A(x)} \|DH(G_n)(t,v)\| \to \sup_{(t,v)\in[0,T]\times A(x)} \|DH(G)(t,v)\| \text{ as } n \to \infty.$$

Therefore, the sequence $(\sup_{(t,v)\in[0,T]\times A(x)} \|DH(G_n)(t,v)\|)_{n\geq 1}$ is bounded. Hence we get by Proposition 4

$$\limsup_n \sup_{0\leq s<t\leq T} H\left((\mathcal{IS})\int_s^t G_n(s,x(s))dw(s), (\mathcal{IS})\int_s^t G(s,x(s))dw(s)\right)$$
$$\leq C(\rho)(T+T^{1-\delta})P_{T,x,\epsilon(x)}\left(\sup_n \sup_{(t,v)\in[0,T]\times A(x)} \|DH(G_n)(t,v)\|\right.$$
$$\left. + \sup_{(t,v)\in[0,T]\times A(x)} \|DH(G)(t,v)\|\right)\theta^{\delta-\rho}.$$

Since $\delta > \rho$ and $\theta \in (0,1]$ is arbitrarily taken, we obtain formula (14). □

Consider the Young differential inclusion:

$$(YDI)\ x(t)-x(s) \in \int_s^t F(\tau,x(\tau))d\tau + (\mathcal{IS})\int_s^t G(\tau,x(\tau))dw_\tau, \quad x(0)=x_0$$

for every $0 \leq s \leq t < T$.

An α-Hölder function $x : [0,T] \to R^d$ is called a solution to inclusion (YDI) if $x(0) = x_0$ and

$$x(t)-x(s) \in \int_s^t F(\tau,x(\tau))d\tau + (\mathcal{IS})\int_s^t G(\tau,x(\tau))dw_\tau$$

for every $0 \leq s \leq t < T$.

Theorem 7 *Under conditions $(H1)-(H3)$ the Young differential inclusion (YDI) admits solutions.*

Proof Since F satisfies $(H1)$ then there exist selections $f : [0,T]\times R^d \to R^d$ of F which are of a Carathéodory type (i.e., measurable in (t,x) and continuous in x). Moreover, for every $(t,x) \in [0,T]\times R^d$ $F(t,x) = cl_{R^d}\{f(t,x) : f$ are Carathéodory selections of $F\}$ by [12]. For every continuous function $x : [0,T] \to R^d$ we have

$$| f(t,x(t)) | \leq \|F(t,x(t))\| \leq L \mid x(t) \mid + b(t) \leq L\|x\|_\infty + b(t).$$

It means that every $f(\cdot, x(\cdot))$ is $1/(1-\alpha)$-integrably bounded and therefore, $f \circ x \in S_{L^1}(F \circ x)$. From this we deduce that the Aumann integral $\int_s^t F(u, x(u))du$ is a nonempty set for every $0 \leq s < t \leq T$.

Let $C \in Conv(R^{d\times m})$ and let $\sigma_C(p) := \sigma(C, p) := \sup_{y\in C} < p, y > \in R^1 \cup \{+\infty\}$. The function $\sigma_C : R^{d\times m} \to R^1 \cup \{+\infty\}$ is a support function of C. Let Σ denote the unit sphere in $R^{d\times m}$ and let V denote a Lebesgue measure of a closed unit ball $B(0, 1)$ in $R^{d\times m}$, i.e., $V = \pi^{dm/2}/\Gamma(1 + dm/2)$ with Γ being the Euler function. Let p_V be a normalized Lebesgue measure on $B(0, 1)$, i.e., $dp_V = dp/V$. Let

$$\mathcal{M} = \Big\{\mu : \mu \text{ is a probability measure on } B(0,1) \text{ having the } C^1-\text{density } d\mu/dp_V \text{ with respect to measure } p_V\Big\}.$$

Let $\xi_\mu := d\mu/dp_V$ and let $\nabla\xi_\mu$ denote the gradient of ξ_μ. By ω we denote a Lebesgue measure on Σ. The function $St_\mu : Conv(R^{d\times m}) \to R^{d\times m}$ called a generalized Steiner center, and given by the formula

$$St_\mu(C) = V^{-1}\left(\int_\Sigma p\sigma(p, C)\xi_\mu(p)d\omega(p) - \int_{B(0,1)} \sigma(p, C) \bigtriangledown \xi_\mu(p)dp\right) \tag{15}$$

for every $\mu \in \mathcal{M}$, has the following properties.

For $A, B, C \in Conv(R^{d\times m})$ and $a, b \in R^1$ the following properties hold

$$\begin{aligned} &St_\mu(C) \in C, \\ &St_\mu(aA + bB) = aSt_\mu(A) + bSt_\mu(B), \\ &\mid St_\mu(A) - St_\mu(B) \mid \leq L_\mu \cdot H(A, B), \end{aligned} \tag{16}$$

where $L_\mu = dm \max_{p\in\Sigma} \xi_\mu(p) + \max_{p\in B(0,1)} \mid \nabla\xi_\mu(p) \mid$ (see, [5], [9]).

Moreover, it was proved in [9] that every set $C \in Conv(R^{d\times m})$ has a representation

$$C = cl_{R^{d\times m}}\{St_\mu(C)\}_{\mu\in\mathcal{M}}. \tag{17}$$

Since $G : [0, T] \times R^d \to Conv(R^{d\times m})$ then we define the generalized Steiner selections $St_\mu(G) : [0, T] \times R^d \to R^{d\times m}$ by formula

$$(t, x) \to St_\mu(G(t, x)).$$

It follows by (17) that for every $(t, x) \in [0, T] \times R^d$ the set $\{St_\mu(G(t, x))\}_{\mu\in\mathcal{M}}$ is dense in $G(t, x)$. Using $(H3)$ together with inequality (16) we get

$$\begin{aligned} \mid St_\mu(G(t, x)) - St_\mu(G(s, y)) \mid &\leq L_\mu \cdot H\big(G(t, x), G(s, y)\big) \\ &\leq L_\mu M(\mid t - s \mid^\beta + \mid x - y \mid^\gamma). \end{aligned}$$

It means that every generalized Steiner selection $St_\mu(G)$ is β-Hölder in t and γ-Hölder in x i.e.,

$$St_\mu(G) \in \mathcal{C}^{\beta\times\gamma}([0,T]\times R^d, R^{d\times m}). \tag{18}$$

Let $\Delta_h^0 G(t,x) = G(t+h,x) \div G(t,x)$, $\Delta_h^i G(t,x) = G(t,x+he_i) \div G(t,x)$ for $i = 1,\dots,d$.

Then $h^{-1}\Delta_h^0 G(t,x) \to DH^0(G)(t,x)$, $h^{-1}\Delta_h^i G(t,x) \to DH^i(G)(t,x)$ for $i = 1,\dots,d$ and $h^{-1}St_\mu(\Delta_h^j G(t,x)) \to St_\mu(DH^j(G)(t,x))$ for $j = 0,1,\dots,d$, as $h \to 0^+$, by continuity of St_μ. Since $A \div B = C \iff A = B + C$ then

$St_\mu(G(t+h,x)) = St_\mu(\Delta_h^0 G(t,x)) + St_\mu(G(t,x))$ and

$St_\mu(G(t,x+he_i)) = St_\mu(\Delta_h^i G(t,x)) + St_\mu(G(t,x))$ for $i = 1,\dots,d$.

Therefore,

$h^{-1}\big(St_\mu(G(t+h,x)) - St_\mu(G(t,x))\big) = St_\mu(h^{-1}\Delta_h^i G(t,x))$ and

$h^{-1}\big(St_\mu(G(t,x+he_i)) - St_\mu(G(t,x))\big) = St_\mu(h^{-1}\Delta_h^i G(t,x))$ for $i = 1,\dots,d$ because of linearity of St_μ. Taking $h \to 0$ we obtain

$\frac{\partial St_\mu(G)}{\partial t}(t,x) = St_\mu(DH^0(G)(t,x))$ and $\frac{\partial St_\mu(G)}{\partial x_i}(t,x) = St_\mu(DH^i(G)(t,x))$, for $i = 1,\dots,d$.

Therefore, $St_\mu(G)'(t,x) \in DH(G)(t,x)$ and taking in mind formula (7) we deduce that every $St_\mu(G) \in \mathcal{IS}(G)$.

Let $\tilde f$ be an arbitrary Carathéodory selection of F and $St_\mu(G)$ be any generalized Steiner selection of G. It was proved above that $St_\mu(G) \in \mathcal{IS}(G)$.

Consider the Young differential equation

$$x(t) = x_0 + \int_0^t \tilde f(u,x(u))du + \int_0^t St_\mu(G)(u,x(u))dw(u).$$

Since $\tilde f$ and $St_\mu(G)$ satisfy all assumptions of Theorem 4.1 of [24] then this equation admits a solution $\tilde x \in \mathcal{C}^\alpha([0,T],R^d)$. But $\tilde f$ and $St_\mu(G)$ are appropriate selections of F and G, respectively. Therefore, $\int_s^t \tilde f(\tau,x(\tau))d\tau \in \int_s^t F(\tau,x(\tau))d\tau$ and

$\int_s^t St_\mu(G)(\tau,x(\tau))dw_\tau \in (\mathcal{IS})\int_s^t G(\tau,x(\tau))dw_\tau$.

From this we get

$$\tilde x(t) - \tilde x(s) \in \int_s^t F(\tau,\tilde x(\tau))d\tau + (\mathcal{IS})\int_s^t G(\tau,\tilde x(\tau))dw_\tau,\ \tilde x(0) = x_0.$$

It means that $\tilde x$ is a solution of Young differential inclusion (YDI). □

Let $Sol(x_0,F,G,w)$ denote the set of all solutions of the inclusion (YDI). First we will present conditions imposed on $F,G : [0,T]\times R^d \to Conv(R^d)(resp., Conv(R^{d\times m})$ assuring the closedness of the set $Sol(x_0,F,G,w)$ in $\mathcal{C}^\alpha([0,T],R^d)$.

Assume the following condition

$(H2')$ F is locally δ-Hölder in x for some $\delta \in (0,1)$.

Theorem 8 *Under conditions* $(H1)-(H3)$ *and* $(H2')$*, the set* $Sol(x_0, F, G, w)$ *of all solutions of the Young differential inclusion (YDI) is closed in* $\mathcal{C}^{\alpha}([0,T], R^d)$.

Proof Suppose that $(x_k)_{k=1}^{\infty} \subset Sol(x_0, F, G, w)$ is a sequence convergent to some limit x in $\mathcal{C}^{\alpha}([0,T], R^d)$. We have to prove that $x \in Sol(x_0, F, G, w)$. Since x_k are solutions of (YDI) then $x_k(0) = x_0$ for every k. But $\|x_k - x\|_{\infty} \to 0$ and from this we deduce $x(0) = x_0$. Moreover, $\mid x(s) - x(0) \mid \le M_{\alpha}(x) \mid s - 0 \mid^{\alpha}$. Therefore, $\mid x(s) \mid \le \mid x_0 \mid + M_{\alpha}(x)T^{\alpha} := \epsilon_1(x)$. Since x_k tends to x in $\mathcal{C}^{\alpha}([0,T], R^d)$ then for every $\epsilon_2(x) > 0$ there exists N_0 such that for every $k > N_0$ and every $s \in [0,T]$ $\mid x_k(s) - x(s) \mid < \epsilon_2(x)$. Therefore, $\mid x_k(s) \mid \le \mid x(s) \mid + \epsilon_2(x) \le \epsilon_2(x) + \epsilon_1(x) := \epsilon(x)$. From this we deduce that there exists $\epsilon(x)$ depending only on $M_{\alpha}(x), \alpha, T, x_0$ and such that $x(s), x_k(s) \in [-\epsilon(x), \epsilon(x)]^d$ for every $s \in [0,T]$ and $k > N_0$. Let us take a set $A(x) := [0,T] \times [-\epsilon(x), \epsilon(x)]^d$ and consider set-valued functions $F_{|A(x)}$ and $G_{|A(x)}$ with their domains restricted to the set $A(x)$. Then $F_{|A(x)}(t, \cdot)$ is δ-Hölder with a Hölder constant $L_{\epsilon(x)}$ by $(H2')$. From the other side there exists $N_{\epsilon(x)} > 0$ such that $\|DH^i(G)(t,u)\| \le N_{\epsilon(x)}$ for every $(t,u) \in A(x)$ and each $i = 0, 1, \ldots, d$ by $(H3)$. Then, for every $\delta \in (0,1)$, $G_{|A(x)}$ is δ-Hölder with a Hölder constant dependent only on $\epsilon(x)$ and

$$\|G_{|A(x)}\|_{\delta} \le \|G(0,x_0)\| + N_{\epsilon(x)}(T + T^{1-\delta}) + dN_{\epsilon(x)}\big(2\epsilon(x) + (2\epsilon(x))^{1-\delta}\big) < \infty$$

by Proposition 1.

From the convergence of x_k to x in $\mathcal{C}^{\alpha}([0,T], R^d)$ it follows $M_{\alpha}(x_k - x) \to 0$. From this $\mid M_{\alpha}(x_k) - M_{\alpha}(x) \mid \to 0$ and therefore, $M_{\alpha}(x_k) \to M_{\alpha}(x)$. Then there exists a constant $C > 0$ such that $\sup_k M_{\alpha}(x_k) < C + M_{\alpha}(x) < \infty$.

If $g \in \mathcal{IS}(G)$ then

$$sup_{(t,u)\in A(x)} \mid g'(t,u) \mid \le sup_{(t,u)\in A(x)} \|DH(G)(t,u)\| \le N_{\epsilon(x)}.$$

Similarly as in the proof of Proposition 1 we deduce that $g_{|A(x)}$ is δ-Hölder with a Hölder constant $M_{\delta}(g_{|A(x)}) \le N_{\epsilon(x)}\big(T^{1-\delta} + d(2\epsilon(x))^{1-\delta}\big)$. Moreover,

$$\begin{aligned} \mid g(t,x(t)) - g(s,x(s)) \mid &\le M_{\delta}(g_{|A(x)})\big(\mid t-s \mid^{\delta} + \mid x(t) - x(s) \mid^{\delta}\big) \\ &\le N_{\epsilon(x)}\big(T^{1-\delta} + d(2\epsilon(x))^{1-\delta}\big)\big(\mid t-s \mid^{\delta} + M_{\alpha}(x)^{\delta} \mid t-s \mid^{\alpha\delta} \big). \end{aligned}$$

Taking $\sigma = \alpha\delta$ and dividing both sides of the above inequality by $\mid t-s \mid^{\sigma}$ we get

$$\begin{aligned} M_{\sigma}(g \circ x) &\le M_{\delta}(g_{|A(x)})\big(T^{1-\alpha} + M_{\alpha}(x)^{\delta}\big) \\ &\le N_{\epsilon(x)}\big(T^{1-\delta} + d(2\epsilon(x))^{1-\delta}\big)\big(T^{1-\alpha} + M_{\alpha}(x)^{\delta}\big) < \infty. \end{aligned} \tag{19}$$

Therefore, $g \circ x$ is σ-Hölder. Similarly, $g \circ x_k$ are σ-Hölder and

$$\begin{aligned} \sup_k M_{\sigma}(g \circ x_k) &\le M_{\delta}(g_{|A(x)})\big(T^{1-\alpha} + \sup_k M_{\alpha}(x_k)^{\delta}\big) \\ &\le M_{\delta}(g_{|A(x)})\big(T^{1-\alpha} + (C + M_{\alpha}(x))^{\delta}\big) \end{aligned}$$

$$\leq N_{\epsilon(x)}\big(T^{1-\delta} + d(2\epsilon(x))^{1-\delta}\big)\big(T^{1-\alpha} + (C + M_\alpha(x))^\delta\big) < \infty. \tag{20}$$

From Theorem 4.1(1) in [14] we get

$$\begin{aligned} &H\left(\int_s^t F(\tau, x_k(\tau))d\tau, \int_s^t F(\tau, x(\tau))d\tau\right) \\ &\leq \int_s^t H\big(F(\tau, x_k(\tau)), F(\tau, x(\tau))d\tau\big) \leq L_{\epsilon(x)}\|x_k - x\|_\infty^\delta T \to 0 \end{aligned}$$

for every $s, t \in [0, T]$, $s \leq t$, as $k \to \infty$. We wish to show that $x \in Sol(x_0, F, G, w)$. For this let us estimate

$$\begin{aligned} &dist_{R^d}\left(x(t) - x(s), \int_s^t F(\tau, x(\tau))d\tau + (\mathcal{IS})\int_s^t G(\tau, x(\tau))dw_\tau\right) \\ &\leq |\ x(t) - x(s) - (x_k(t) - x_k(s))\ | \\ &\quad + dist_{R^d}\left(x_k(t) - x_k(s), \int_s^t F(\tau, x_k(\tau))d\tau + (\mathcal{IS})\int_s^t G(\tau, x_k(\tau))dw_\tau\right) \\ &\quad + H\big(\int_s^t F(\tau, x_k(\tau))d\tau, \int_s^t F(\tau, x(\tau))d\tau\big) \\ &\quad + H\left((\mathcal{IS})\int_s^t G(\tau, x_k(\tau))dw_\tau, (\mathcal{IS})\int_s^t G(\tau, x(\tau))dw_\tau\right) \\ &= I + II + III + IV. \end{aligned}$$

Since $x_k \in Sol(x_0, F, G, w)$ then $II = 0$. I and III tend to 0 because of $\|x_k - x\|_\infty \to 0$. We have to prove that $IV \to 0$. For this we will estimate both semimetrics

$$\bar{H}\left((\mathcal{IS})\int_s^t G(\tau, x_k(\tau))dw_\tau, (\mathcal{IS})\int_s^t G(\tau, x(\tau))dw_\tau\right)$$

and

$$\bar{H}\left((\mathcal{IS})\int_s^t G(\tau, x(\tau))dw_\tau, (\mathcal{IS})\int_s^t G(\tau, x_k(\tau))dw_\tau\right).$$

Using Lemma 3, for every $s, t \in [0, T]$, $s \leq t$, it holds

$$\begin{aligned} &\sup_{g_1 \in \mathcal{IS}(G)} \mathrm{dist}_{R^d}\left(\int_s^t g_1(\tau, x_k(\tau))dw_\tau, (\mathcal{IS})\int_s^t G(\tau, x(\tau))dw_\tau\right) \\ &= \sup_{g_1 \in \mathcal{IS}(G)} \inf_{g_2 \in \mathcal{IS}(G)} \left|(\int_s^t g_1(\tau, x_k(\tau))dw_\tau - \int_s^t g_2(\tau, x(\tau))dw_\tau\right| \\ &\leq C(\rho) \sup_{g_1 \in \mathcal{IS}(G)} \inf_{g_2 \in \mathcal{IS}(G)} \Big\{ \|g_1 \circ x_k - g_2 \circ x\|_\infty \end{aligned}$$

$$
\begin{aligned}
&+\Big(M_\sigma(g_1\circ x_k)+M_\sigma(g_2\circ x)\Big)\theta^\sigma\Big\}\theta^{-\rho}\\
&+\sup_{g_1\in\mathcal{IS}(G)}\inf_{g_2\in\mathcal{IS}(G)}\|g_1\circ x_k-g_2\circ x\|_\infty M_\alpha(w)T^\alpha\\
&=C(\rho)\Big\{\sup_{g_1\in\mathcal{IS}(G)}\operatorname{dist}_\infty(g_1\circ x_k,\mathcal{IS}(G)\circ x)\\
&+\Big(\sup_{g_1\in\mathcal{IS}(G)}M_\sigma(g_1\circ x_k)+\sup_{g_2\in\mathcal{IS}(G)}M_\sigma(g_2\circ x)\Big)\theta^\sigma\Big\}\theta^{-\rho}\\
&+\sup_{g_1\in\mathcal{IS}(G)}\operatorname{dist}_\infty(g_1\circ x_k,\mathcal{IS}(G)\circ x)M_\alpha(w)T^\alpha,
\end{aligned}
$$

where $\mathcal{IS}(G)\circ x:=\{g\circ x : g\in\mathcal{IS}(G)\}$.

Therefore,

$$
\begin{aligned}
&\bar{H}\Big((\mathcal{IS})\int_s^t G(\tau,x_k(\tau))dw_\tau,(\mathcal{IS})\int_s^t G(\tau,x(\tau))dw_\tau\Big)\\
&\le C(\rho)\bar{H}_\infty\Big((\mathcal{IS})(G)\circ x_k,(\mathcal{IS})(G)\circ x\Big)\theta^{-\rho}\\
&+C(\rho)\Big(\sup_{g_1\in\mathcal{IS}(G)}M_\sigma(g_1\circ x_k)+\sup_{g_2\in\mathcal{IS}(G)}M_\sigma(g_2\circ x)\Big)\theta^\sigma\theta^{-\rho}\\
&+M_\alpha(w)T^\alpha\bar{H}_\infty\big((\mathcal{IS})(G)\circ x_k,(\mathcal{IS})(G)\circ x\big)\\
&=I(a)+II(a)+III(a).
\end{aligned}
$$

We will prove that $\bar{H}_\infty\big((\mathcal{IS})(G)\circ x_k,(\mathcal{IS})(G)\circ x\big)\to 0$ as $k\to\infty$.

Really, since $g_1\in\mathcal{IS}(G)$ then $g_{1|A(x)}$ is δ-Hölder with a Hölder constant $M_\delta(g_{1|A(x)})\le N_{\epsilon(x)}\big(T^{1-\delta}+d(2\epsilon(x))^{1-\delta}\big)$. Then we get

$$
\begin{aligned}
&\bar{H}_\infty\big((\mathcal{IS})(G)\circ x_k,(\mathcal{IS})(G)\circ x\big)\\
&=\sup_{g_1\in\mathcal{IS}(G)}\inf_{g_2\in\mathcal{IS}(G)}\|(g_1(\cdot,x_k(\cdot))-g_2(\cdot,x(\cdot))\|_\infty\\
&\le\sup_{g_1\in\mathcal{IS}(G)}\|(g_1(\cdot,x_k(\cdot))-g_1(\cdot,x(\cdot))\|_\infty\\
&\le N_{\epsilon(x)}\big(T^{1-\delta}+d(2\epsilon(x))^{1-\delta}\big)\|x_k(\cdot)-x(\cdot)\|_\infty^\delta\to 0\text{ as }k\to\infty.
\end{aligned}
$$

From this we deduce that $I(a)$ and $III(a)$ tend to 0 as $k\to\infty$. Since $\sigma-\rho>0$ and $\theta\in(0,1)$ can be arbitrarily taken then $II(a)$ is arbitrarily small because of formulas (19) and (20). Therefore,

$$
\bar{H}\big((\mathcal{IS})\int_s^t G(u,x_k(u))dw(u),(\mathcal{IS})\int_s^t G(u,x(u))dw(u)\big)\to 0.
$$

Similarly, we deduce that the reversed semimetric satisfies

$$\bar{H}\big((\mathcal{IS})\int_s^t G(u,x(u))dw(u),(\mathcal{IS})\int_s^t G(u,x_k(u))dw(u)\big)\to 0.$$

It means that $IV \to 0$. By this we obtain

$$dist_{R^d}\big(x(t)-x(s),\int_s^t F(u,x(u))du+(\mathcal{IS})\int_s^t G(u,x(u))dw(u)\big)=0.$$

Therefore,

$$x(t)-x(s)\in\int_s^t F(u,x(u))du+(\mathcal{IS})\int_s^t G(u,x(u))dw(u).$$

It was proved previously that $x(0)=x_0$, so x belongs to $Sol(x_0,F,G,w)$. □

Now we discuss the continuous dependence of solutions with respect to the right hand side of inclusion.

Consider Young differential inclusions:

(YDI_n) $\quad x_n(t)-x_n(s)\in(\mathcal{IS})\int_s^t G_n(\tau,x_n(\tau))dw_n(\tau),\quad x_n(0)=a_n$ for every $0\le s\le t<T$

and

(YDI) $\quad x(t)-x(s)\in(\mathcal{IS})\int_s^t G(\tau,x_n(\tau))dw(\tau),\quad x(0)=a$

for every $0\le s\le t<T$.

The following version of Proposition 4 from [20] will be used in the proof of next result.

Proposition 9 ([20]) *Assume that $f:[0,T]\times R^d\to R^{d\times m}$ is γ-Hölder, differentiable and its derivative is γ-Hölder, $w\in\mathcal{C}^\alpha([0,T],R^m)$, $\alpha>1/2$, $\gamma>1/\alpha-1$. Then the solution to*

$$x(t)=x_0+\int_0^t f(s,x(s))dw(s)$$

is unique. In addition the map $w\to x$, called the Itô map, is locally Lipschitz continuous with respect to (a,f,w). More precisely, let x (resp. $\hat{x}$) be the solution to $x(t)=a+\int_0^t f(s,x(s))dw(s)$ (resp. $\hat{x}(t)=\hat{a}+\int_0^t \hat{f}(s,\hat{x}(s))d\hat{w}(s)$). We assume that $\|x\|_\infty\le R$, $M_\alpha(x)\le R$, $|a|\le R$, $M_\gamma(\nabla f)\le R$, $\|\nabla f\|_\infty\le R$, $M_\alpha(w)\le R$ and the same holds true when (a,f,w,x) is replaced by $(\hat{a},\hat{f},\hat{w},\hat{x})$. Then there exists a constant C depending only on T and R such that

$$M_\alpha(x-\hat{x})\le C\big(M_\alpha(w-\hat{w})+\|(f-\hat{f})_{|B(0,R)}\|_\infty+\|(\nabla f-\nabla\hat{f})_{|B(0,R)}\|_\infty+|a-\hat{a}|\big).$$

Let $G_n, G : [0, T] \times R^d \to R^{d\times m}$ be such that:

$(H1'')$ G_n and G are γ-Hölder for some $\gamma > (1-\alpha)/\alpha$, $\alpha > 1/2$ and satisfy $\sup_n M_\gamma(G_n) \leq K_1$ for some $K_1 > 0$,

$(H2'')$ G_n and G have γ-Hölder and bounded Hukuhara derivatives and $\sup_n M_\gamma(DH(G_n)) \leq K_2$ for some $K_2 > 0$,

Theorem 10 *Let $(H1'')$ and $(H2'')$ hold. Moreover, assume*

(i) $M_\alpha(w_n - w) \to 0$ *and* $\mid a_n - a \mid \to 0$,
(ii) $\sup_{(t,v)\in[0,T]\times R^d} H(G_n(t, v), G(t, v)) \to 0$,
(iii) $\sup_{(t,v)\in[0,T]\times R^d} H(DH(G_n)(t, v), DH(G)(t, v)) \to 0$

as $n \to \infty$.

Then every sequence of solutions (x_n) of (YDI_n) satisfying

$$(YDE_n) \quad x_n(t) = a_n + \int_0^t St_\mu(G_n)(u, x_n(u))dw_n(u),$$

is convergent in $C^\alpha([0, T], R^d)$ norm to solution x of (YDI) satisfying

$$(YDE) \quad x(t) = a + \int_0^t St_\mu(G)(u, x(u))dw(u).$$

Proof $DH(G)$ is bounded by $(H2'')$. Then, similarly as in Corollary 6, there exist $K_3 > 0$ and $N_0 > 0$ such that

$$\sup_{(t,v)\in[0,T]\times R^d} \|DH(G_n)(t, v)\| \leq K_3$$

for $n > N_0$ by assumption (iii).

Therefore, (G_n) have uniformly bounded Hukuhara derivatives for $n > N_0$. Similarly as in formula (18) of the proof of Theorem 7 we deduce that $St_\mu(G_n)$ and $St_\mu(G)$ are γ-Hölder with Hölder constants $M_\gamma(St_\mu(G_n)) \leq L_\mu K_1$ and $St_\mu(G_n)'(t, v) \in DH(G_n)(t, v)$, $St_\mu(G)'(t, v) \in DH(G)(t, v)$ with $M_\gamma(St_\mu(G_n)') \leq L_\mu K_2$ by $(H1'')$ and $(H2'')$. Therefore, equations (YDE_n) and (YDE) admit solutions x_n and x, which are solutions to (YDI_n) and (YDI), respectively, by Theorem 7.

It was proved in Proposition 1 of [20] that there exist constants R_n depending only on $T, \alpha, \gamma, M_\alpha(w_n), M_\gamma(St_\mu(G_n)), a_n$ and such that $\|x_n\|_\infty \leq R_n$ and $M_\alpha(x_n) \leq R_n$. Since $\sup_n M_\gamma(St_\mu(G_n)) \leq L_\mu K_1$ and $\sup_n \mid a_n \mid < \infty$ then there exists one constant $P_1 = \sup_n R_n < \infty$ such that $\|x_n\|_\infty \leq P_1$ and $M_\alpha(x_n) \leq P_1$ for every n. The same holds for solution x of (YDE), i.e., there exists a constant P_2 depending only on $T, \alpha, \gamma, M_\alpha(w), M_\gamma(St_\mu(G))$, a and such that $\|x\|_\infty \leq P_2$ and $M_\alpha(x) \leq P_2$. Moreover, we have $\|St_\mu(G_n)'\|_\infty \leq K_3$ and $\sup_n M_\gamma(St_\mu(G_n)') \leq L_\mu K_2$. Taking $R = \max\{P_1, P_2, K_3, L_\mu K_2\}$ all assumptions of Proposition 9 are satisfied and we get

$$M_\alpha(x_n - x) \leq C \Bigg(M_\alpha(w_n - w) + \|(St_\mu(G_n) - St_\mu(G))_{|B(0,R)}\|_\infty$$

$$+\|(St_{\mu}(G_n)' - St_{\mu}(G)')_{|B(0,R)}\|_{\infty} + \mid a_n - a \mid$$
$$\le C\left(M_{\alpha}(w_n - w) + L_{\mu} \sup_{(t,v)\in[0,T]\times R^d} H(G_n(t,v), G(t,v))\right.$$
$$\left. + L_{\mu} \sup_{(t,v)\in[0,T]\times R^d} H(DH(G_n)(t,v), DH(G)(t,v)) + \mid a_n - a \mid\right).$$

Thus, by assumptions $(i) - (iii)$ we get $M_{\alpha}(x_n - x) \to 0$ as $n \to \infty$.

Since $\sup_{t\in[0,T]} \mid x_n(t) - x(t) \mid \le \mid a_n - a \mid + T^{\alpha} M_{\alpha}(x_n - x)$ then we obtain convergence of x_n to x in $\mathcal{C}^{\alpha}([0,T], R^d)$. □

Remark 2 All the results are easily adapted to deal with the stochastic differential inclusion

$$x(t) - x(s) \in \int_s^t F(u, x(u))du + (\mathcal{IS}) \int_s^t G(u, x(u))dB^H(u), \quad x(0) = x_0.$$

where B^H is a fractional Brownian motion. Last theorem needs additional property that Hurst index of fBm satisfies inequality $H > 1/2$.

Let us conclude with applications to stochastic inclusions. Consider a 1-dimensional fractional Brownian motion $B^H = (B^H(t))_{t\in[0,T]}$ of Hurst index $H \in (1/2, 1)$, which is a centered Gaussian process such that

$$E(B^H(s)B^H(t)) = 1/2(t^{2H} + s^{2H} - \mid t - s \mid^{2H})$$

for every $s, t \in [0, T]$. Let (Ω, Σ, P) be the associated canonical probability space. By the Garcia-Rodemich-Rumsey lemma (see Nualart [25], Lemma A.3.1), the paths of B^H are α-Hölder continuous for any $\alpha \in (0, H)$ and therefore, the integral $(\mathcal{IS}) \int_s^t G(u, x(u))dB^H(u)$ can be treated as a Young integral.

Let us consider the following medical model. Let $x : [0, T] \to R_+$ be a function for which the value $x(t)$ denotes the number of cancerous cells at time $t \in [0, T]$. The disease is diagnosed if $x(t) \ge a_0 > 0$ where a_0 is a given reference level. In practice we may assume that a_0 is a level at which some therapy starts. One can also assume that there is a maximum level of cancer cells $a_1 > a_0$ at which no further treatment (dose of drugs) can help. So in the presence of active treatment $x(t) \in [a_0, a_1]$. Let $u : [0, T] \to [0, c]$ be a function for which $u(t)$ represents the dose of the drug at time t and let $d : [0, c] \to R_+$ be a given continuous function describing destroying rate per tumor cell. In [27] the following controlled cancer growth model under the influence of drugs was considered

$$dx(t) = \left(\lambda \ln\left(\frac{\mu}{x(t)}\right) - d(u(t))\right) \cdot x(t)dt, \quad x(0) = a_0, \tag{21}$$

where λ and μ are some positive parameters. In [18] the function d was taken in particular as $d(u) = k_1 u/(k_2 + u)$ with some positive parameters $k_1,\ k_2$. Since therapy effects may be different in cases of different patients it seems reasonable to generalize model (21) adding a stochastic perturbation governed by the fractional Brownian motion B^H with Hurst parameter $H \in (0, 1)$, i.e., to consider the model

$$dx(t) = \left(\lambda \ln\left(\frac{\mu}{x(t)}\right) - d(u(t))\right) \cdot x(t)dt + v(u(t))x(t)dB_t^H, \quad x(0) = a_0,$$

or in its integral form

$$x(t) = a_0 + \int_0^t \left(\lambda \ln\left(\frac{\mu}{x(\tau)}\right) - d(u(\tau))\right) \cdot x(\tau)d\tau + \int_0^t v(u(\tau))x(\tau)dB_\tau^H \tag{22}$$

instead of model (1). The term $v(u(t))$ can be interpreted as unforeseen (random) reaction of the patient to treatment. It is assumed that v is a continuous function.

Let us define the following set-valued functions: $F(x) := f(x) + F_1(x)$ with $f(x) = \lambda x \ln(\mu/x)$, $F_1(x) = -d([0, c]) \cdot x$, and $G(x) := v([0, c]) \cdot x$ for $x \in [a_0, a_1]$. Then the controlled model (22) can be rewritten in the framework of Young integral inclusion as

$$x(t) - x(s) \in \int_s^t (f + F_1)(x(\tau)d\tau + (\mathcal{IS}) \int_s^t G(x(\tau))dB_\tau^H, \quad x(0) = a_0. \tag{23}$$

Let us note that both $f + F_1$ and G take on nonempty, compact and convex values in R^1. By standard calculations and Mean Value Theorem one can show that f is Lipschitz continuous with a Lipschitz constant

$$K = \lambda(|\ln(\mu)| + \max\{|\ln(a_0)|, \ |\ln(a_1)|\} + 1).$$

It is also easy to see that F_1 is Lipschitz continuous with a Lipschitz constant $\|d([0, c])\|$. Thus F is Lipschitz continuous too. It can be also verified that $\|F(x)\| \leq L \mid x \mid$ with $L = \lambda \max\{|\ln(\mu/a_0)|, \ |\ln(\mu/a_1)|\} + \|d([0, c])\|$. Similarly, G is Lipschitz continuous with a bounded Hukuhara derivative $DH(G)(x) = v([0, c])$. Hence F satisfies all conditions $(H1)$, $(H2)$ and $(H2')$ while G satisfies $(H3)$. Therefore, using Theorems 7 and 8 we deduce that inclusion (23) admits solutions and the set of its solutions is closed in $\mathcal{C}^\alpha([a_0, a_1], R^1)$.

It is known, that classical Steiner point can be interpreted as a center of a mass of a given set. Therefore, Theorem 10 applied to a system of equations

$$x_n(t) = a_n + \int_0^t StG_n(u(\tau)) \cdot x_n(\tau)dB_\tau^{H,n}$$

and

$$x(t) = a + \int_0^t StG(u(\tau)) \cdot x(\tau)dB_\tau^H$$

allows us to deduce that a mean unforced reaction of a patient to treatment $v(u(t))$ is stable with respect to a convergent sequence of individual mean unforced reactions $v_n(u(t))$, initial reference levels a_n and a sequence of different fractional Brownian motions $B_\tau^{H,n}$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Aubin, J.P., Cellina, A.: Differential inclusions. Springer, Berlin (1984)
2. Aubin, J.P., Frankowska, H.: Set-valued analysis. Birkhäuser, Basel (1990)
3. Aumann, R.J.: Integrals of set-valued functions. J. Math. Anal. Appl. **12**, 1–12 (1965)
4. Bailleul, I., Brault, A., Coutin, L.: Young and rough differential inclusions, Revista Mat. Iberoamer. Eur. Math. Soc. **37**(4), 1489–1512 (2021)
5. Baier, R., Farkhi, E.: Regularity and integration of set-valued maps represented by generalized Steiner points. Set-valued Anal. **15**, 185–207 (2007)
6. Bhaskar, T.G., Lakshmikantham, V., Devi, J.V.: Nonlinear variation of parameters formula for set differential equations in metric space. Nonlin. Anal. TMA **63**, 735–744 (2005)
7. Coutin, L., Qian, Z.: Stochastic analysis, rough paths analysis and fractional Brownian motions. Probab. Theory Related Fields **122**, 108–140 (2002)
8. Coutin, L., Nicolas, M., DeFitte, P.: On a set-valued Young integral with applications to differential inclusions. J. Math. Anal. Appl. **512**(1), 126104 (2022)
9. Dentcheva, D.: Differentiable selections and Castaing representations of multifunctions. J. Math. Anal. Appl. **223**, 371–396 (1998)
10. Friz, P.K., Victoir, N.B.: Multidimensional stochastic processes as rough paths: theory and applications, Cambridge Studies in Advanced Mathematics (2010)
11. Friz, P.K., Zhang, H.: Differential equations driven by rough paths with jumps. J. Differ. Equ. **264**, 6226–6301 (2018)
12. Fryszkowski, A.: Carathéodory type Selectors of Set-valued Maps of Two Variables. Bull. de L'Acad. Polonaise des Sciences ser. Math. **25**(1), 41–46 (1977)
13. Djebali, S., Górniewicz, L., Ouahab, A.: Solutions sets for differential equations and inclusions. Series in Nonlin. Anal. Appl, De Gruyter GmbH (2013)
14. Hiai, F., Umegaki, H.: Integrals, conditional expectations and martingales for multivalued functions. J. Multivar. Anal. **7**(1), 149–182 (1977)
15. Kisielewicz, M.: Differential inclusions and optimal control. Kluwer Acad. Publ, Dordrecht (1991)
16. Kisielewicz, M.: Stochastic differential inclusions and applications. Springer, Berlin (2013)
17. Kisielewicz, M.: Set-valued stochastic integrals and applications. Springer, Berlin (2020)
18. Krabs, W.: Matematische Modellierung: eine Eienführung in die Problematik. Springer, Berlin (1997)

19. Lakshmikantham, V., Gnana Bhaskar, T., Vasundhara Devi, J.: Theory of set differential equations in a metric space. Cambridge Scientific Publishers, Cambridge (2006)
20. Lejay, A.: Controlled differential equations as Young integrals: a simple approach. J. Differ. Equ. **249**, 1777–1798 (2010)
21. Lyons, T.: Differential equations driven by rough signals. Rev. Mat. Iberoam. **14**, 215–310 (1998)
22. Michta, M., Motyl, J.: Selection properties and set-valued young integrals of set-valued functions. Results Math. **75**, 164 (2020)
23. Michta, M., Motyl, J.: Set-valued functions of bounded generalized variation and set-valued young integrals. J. Theor. Probab. **35**, 528–549 (2022)
24. Naghshineh, O., Zangeneh, B.Z.: Existence and measurability of the solution of the stochastic differential equations driven by fractional Brownian motion. Bull. Iranian Math. Soc. **35**(2), 47–68 (2009)
25. Nualart, D.: The Malliavin calculus and related topics. Springer, Berlin (2006)
26. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional integrals and derivatives. Theory and applications. Gordon and Breach Science Publishers, Philadelphia (1993)
27. Swan, G.W.: Applications of optimal control in biomedicine (1984)
28. Tolstonogov, A.A.: Differential inclusions in a Banach space. Kluwer Acad. Publl, Dordrecht (2000)
29. Young, L.S.: An inequality of the Hölder type connected with Stieltjes integration. Acta Math. **67**, 251–282 (1936)