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Existence of Solutions to Nonlinear Fourth-Order Beam Equation

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Abstract
This paper studies the boundary value problem for a fourth-order difference equation with three quasidifferences. The new existence criterion of at least one solution to the issues considered is obtained using the theory of variational methods. The main result is illustrated in some examples.

Keywords Boundary value problem · Fourth-order difference equation · Existence criterion · Critical point theory

Mathematics Subject Classification 39A27 · 39A12

1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, all integers, and real numbers, respectively. We consider the existence of solutions to the following boundary value problem (BVP) for the fourth-order difference equation with quasidifferences of the form

$$L(x(t)) + r(t)g(x(t)) + f(t) = 0, \tag{1}$$

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where

$$L(x(t)) = \Delta(p_2(t-1)\Delta(p_1(t-2)\Delta(p_2(t-2)\Delta x(t-2)))) \\ -\Delta(q(t-1)\Delta x(t-1)),$$

and $t \in [1, T]_{\mathbb{Z}}$, with the following boundary value conditions

$$\Delta^k x(-1) = \Delta^k x(T-1), \quad k = 0, 1, 2, 3. \tag{2}$$

Here T is a natural number greater or equal to 2. We denote the forward difference operator defined in a standard way

$$\Delta x(t) = x(t+1) - x(t) \;\text{ and }\; \Delta^k x(t) = \Delta(\Delta^{k-1} x(t)), \;\text{ for }\; k \geq 2.$$

Moreover, we assume that functions $p_1, p_2, q, r, f : [1, T]_{\mathbb{Z}} \to \mathbb{R}$ satisfy

$$p_i(-1) = p_i(T-1), p_i(0) = p_i(T), p_i(1) = p_i(T+1), \;\text{ for }\; i = 1, 2 \tag{3}$$

and

$$q(0) = q(T).$$

Our interests are focused on the widespread problem related to fourth-order equations, namely the existence of a solution to the boundary problem for a fourth-order equation. Those kinds of problems are generally of considerable practical importance. Such equations arise in many areas of science and technology, such as boundary value problems for the deflection of a horizontal beam. The beam equation is widely used in engineering, primarily civil and mechanical engineering, to determine the deflection or strength of a bending beam. Notice that various beam theories can be used to describe the motion of the beam.

Fourth-order ordinary differential equation (ODE) is a tool to predict the deflection for a beam problem, bending moment, modeling of viscoelastic flows, and soil settlement. For this reason, vast literature deals with the existence and the multiplicity of solutions for such a problem.

Yao [1] established several local existence theorems concerned with positive solutions for a fourth-order two-point boundary value problem, where the nonlinear term may be singular. The problem describes the deflection of an elastic beam rigidly fastened on the left and supported on the right. Khanfer and Bougoffa [2] established an existence and uniqueness theorem for the nonlocal fourth-order nonlinear beam differential equations with a parameter.

Galewski et al. [3, 4] investigated a fourth-order Dirichlet problem connected with the elastic beam equation with supported ends via a direct variational approach. Bonanno and Bella [5] presented multiplicity results for a fourth-order nonlinear boundary value problem. The existence of at least one nontrivial solution to a boundary value problem for fourth-order elastic beam equations is established by Bonanno et al. [6] based on a local minimum theorem for differentiable functionals. Grossinhoa et al.

[7] considered the existence and multiplicity of nontrivial periodic solutions for a semi-linear fourth-order equation arising in studying spatial patterns for bistable systems using variational tools such as the Brezis-Nirenberg Theorem and Clark Theorem. Moghadam [8] established the existence of a nontrivial solution for a class of fourth-order elastic beam equations involving Lipschitz non-linearity with Navier boundary value condition using variational methods. The author ensured the exact collections of the parameters in which the problem possesses at least a nontrivial solution. Drábek and Holubová [9] showed that the usual limitations on the coefficient in the linear problem with Navier boundary conditions are not necessary to get the existence of positive or negative solutions whenever the coefficient is a nonconstant function.

In our considerations, we deal with the fourth-order discrete equation, which could describe the case where the beam is fixed at both ends, it means the boundary conditions for a cantilevered beam are given. However, more is needed to know about the existence of solutions to such boundary value problems. He and Yu [10] considered the existence of positive solutions of fourth-order difference equation

$$\Delta^4 u(t-2) - ra(t)f(u(t)) = 0,$$

satisfying the boundary conditions

$$u(0) = \Delta^2 u(0) = u(T+2) = \Delta^2 u(T) = 0.$$

Using a symmetric Green's function approach, Anderson and Minhós [11] presented various existence, multiplicity, and nonexistence results for nontrivial solutions to a nonlinear discrete fourth-order Lidstone boundary value problem

$$\begin{aligned} &\Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = \lambda f(y(t)), \ \ t \in \{a+1, \ldots, b-1\}, \\ &y(a) = 0 = \Delta^2 y(a-1), \ \ y(b) = 0 = \Delta^2 y(b-1), \end{aligned}$$

with dependence on two parameters.

Ma et al. [12] investigated the existence of positive solutions to the nonlinear boundary value problem of fourth-order difference equation

$$\begin{aligned} &\Delta^4 u(t-2) - ra(t)f(u(t)) = 0, \\ &u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0. \end{aligned}$$

The authors' approaches are based on the Krein-Rutman Theorem and Global Bifurcation Theorem.

In [13], Sang et al. presented sufficient conditions for the existence and uniqueness of positive solutions for a discrete fourth-order beam equation under Lidstone boundary conditions of the form

$$\begin{aligned} &\Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = h(t)[f_1(y(t)) + f_2(y(t))], \ \ t \in \{a+1, \ldots, b-1\}, \\ &y(a) = 0 = \Delta^2 y(a-1), \ \ y(b) = 0 = \Delta^2 y(b-1). \end{aligned}$$

By using the critical point theory, Huang and Zhou [14] established various sets of sufficient conditions on the nonexistence and existence of solutions for the boundary value problem

$$\begin{aligned}&\Delta^2(p(t-1)\Delta^2 u(t-2))+\Delta(q(t)\Delta u(t-1))=f(t,u(t)),\ t\in\{1,\ldots,k\},\\&u(-1)=u(0)=0=u(k+1)=u(k+2).\end{aligned}$$

In 2019, Heidarkhani et al. [15] obtained the multiplicity results for discrete fourth-order boundary value problem with four parameters

$$\begin{aligned}&\Delta^4 u(t-2)+\delta\Delta^2 u(t-1)-\xi u(t)=\lambda f(t,u(t))+\mu g(t,u(t))+h(u(t),\\&u(a)=\Delta^2 u(a-1)=0,\ u(b+2)=\Delta^2 u(b+1)=0.\end{aligned}$$

By employing two critical point theorems, one due to Averna and Bonanno, and another one due to Bonanno, the authors guarantee the existence of two and three solutions for the problem.

Ousbika and El Allali [16] proved the existence of three solutions for the discrete nonlinear fourth-order boundary value problem

$$\begin{aligned}&\Delta^4 u(t-2)+\alpha\Delta^2 u(t-1)-\beta u(t)=\lambda f(t,u(t))+\mu g(t,u(t)),\ t\in[2,T]_{\mathbb{Z}},\\&u(1)=\Delta u(0)=\Delta u(T)=\Delta^3 u(0)=\Delta^3 u(T-1)=0,\end{aligned}$$

with four parameters. The methods used in the proofs are based on the critical point theory.

In 2022, Dhar and Kong, in [17], by using variational methods and critical point theory, obtained criteria for the existence of at least three solutions for a generalized fourth-order nonlinear difference equation together with periodic boundary conditions of the form

$$\begin{aligned}&\Delta^2(r(t-2)\Delta^2 u(t-2))+\Delta(p(t-1)(\Delta u(t-1))^{\delta})-Q(t)(u(t))^{\sigma}\\&\quad=\lambda f(t,u(t))+\mu g(t,u(t)),\quad t\in[1,T]_{\mathbb{Z}},\\&\quad\Delta^i u(-1)=\Delta^i u(T-1),\quad i+0,1,2,3.\end{aligned}$$

Long and Zhang [18], by combining the method of the invariant sets of descending flow with variational technique, gave a series of criteria in terms of different values of λ to ensure that a discrete fourth-order Lidstone problem with three parameters

$$\begin{aligned}&\Delta^4 u(t-2)+\alpha\Delta^2 u(t-1)-\beta u(t)=\lambda f(t,u(t)),\ t\in[a+1,b+1]_{\mathbb{Z}},\\&u(a)=\Delta^2 u(a-1)=0,\ u(b+2)=\Delta^2 u(b+1)=0,\end{aligned}$$

possesses at least four solutions. It is further shown that these four solutions consist of one sign-changing solution, one positive solution, one negative solution and one trivial solution.

For the background of the higher-order difference equations, see [19–21]. For the theory of variational methods, see [22, 23] and the references therein.

Compared to the previous results, our results concern a more general equation than in the earlier studied boundary problems since a second sequence was introduced to the quasidifference of the fourth-order, while earlier studies dealt with boundary problems with one sequence in this quasidifference.

The remainder of the paper is organized as follows. In Sect. 2, we describe the variational structure. In Sect. 3, we state and prove the existence and uniqueness of a solution to the problem. Several examples are provided in Sect. 4 to demonstrate our main results. The last section is a conclusion.

2 Variational Structure

The variational method will be applied to show the existence of solutions to BVP (1), (2). That is why we should introduce a suitable variational structure.

Let H be a vector space defined by

$$H := \{x : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R} : \Delta^k x(-1) = \Delta^k x(T-1), k = 0, 1, 2, 3\}.$$

Rewriting the conditions given by (2) as follows

$$\begin{aligned} x(-1) &= x(T-1), \\ x(0) &= x(T), \\ x(1) &= x(T+1), \\ x(2) &= x(T+2), \end{aligned}$$

we obtain that H is isomorphic to $\mathbb{R}^T$ and can be equipped with the scalar product and norm defined in the standard way, i.e., for $x, y \in H$

$$(x, y) := \sum_{t=1}^{T} x(t)y(y)$$

and

$$\|x\| := \left(\sum_{t=1}^{T} (x(t))^2\right)^{\frac{1}{2}}.$$

Let us define the functional on H in the following way

$$F(x) := \frac{1}{2}\sum_{t=1}^{T} p_1(t)\Big(\Delta(p_2(t)\Delta x(t))\Big)^2 + \frac{1}{2}\sum_{t=1}^{T} q(t)(\Delta x(t))^2 + \sum_{t=1}^{T} r(t)\int_0^{x(t)} g(s)ds + \sum_{t=1}^{T} f(t)x(t). \tag{4}$$

If g is a locally integrable function on $\mathbb{R}$, then $F \in C^1(H, \mathbb{R})$ and it is easy to check that

$$\frac{\partial F}{\partial x(t)} = \Delta(p_2(t-1)\Delta(p_1(t-2)\Delta(p_2(t-2)\Delta x(t-2)))) \\ -\Delta(q(t-1)\Delta x(t-1)) + r(t)g(x(t)) + f(t), \quad t \in [1, T]_{\mathbb{Z}}. \qquad (5)$$

It means that $\frac{\partial F}{\partial x(t)} = 0$ if and only if the Eq. (1) holds. In other words, function $x \in H$ is a solution of the BVP given by (1), (2) if and only if x is a critical point of the functional F on H.

We can rewrite the functional F using the matrix notation. Let A and B be square matrices of the size T defined as follows. For $T = 2$, let

$$A = \begin{bmatrix} (p_1(2)+p_1(1))(p_2(2)+p_2(1))^2 & -(p_1(2)+p_1(1))(p_2(2)+p_2(1))^2 \\ -(p_1(2)+p_1(1))(p_2(2)+p_2(1))^2 & (p_1(2)+p_1(1))(p_2(2)+p_2(1))^2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} q(2)+q(1) & -q(2)-q(1) \\ -q(2)-q(1) & q(2)+q(1) \end{bmatrix}.$$

For $T = 3$, let

$$A = [a_{ij}]_{3\times 3},$$

where

$$\begin{aligned} a_{11} &= p_1(2)(p_2(3))^2 + p_1(3)(p_2(3)+p_2(1))^2 + p_1(1)(p_2(1))^2, \\ a_{22} &= p_1(2)(p_2(2))^2 + p_1(3)(p_2(1))^2 + p_1(1)(p_2(1)+p_2(2))^2, \\ a_{33} &= p_1(2)(p_2(3)+p_2(2))^2 + p_1(3)(p_2(3))^2 + p_1(1)(p_2(2))^2, \\ a_{12} = a_{21} &= p_1(2)p_2(3)p_2(2) - p_1(3)p_2(1)(p_2(3)+p_2(1)) \\ &\quad -p_1(1)p_2(1)(p_2(1)+p_2(2)), \\ a_{23} = a_{32} &= -p_1(2)p_2(2)(p_2(2)+p_2(3)) + p_1(3)p_2(3)p_2(1) \\ &\quad -p_1(1)p_2(2)(p_2(1)+p_2(2)), \\ a_{13} = a_{31} &= -p_1(2)p_2(3)(p_2(2)+p_2(3)) - p_1(3)p_2(3)(p_2(3)+p_2(1)) \\ &\quad +p_1(1)p_2(1)p_2(2). \end{aligned}$$

For $T = 4$, let

$$A = [a_{ij}]_{4\times 4},$$

where

$$\begin{aligned} a_{11} &= p_1(3)(p_2(4))^2 + p_1(4)[p_2(1)+p_2(4)]^2 + p_1(1)(p_2(1))^2, \\ a_{22} &= p_1(4)(p_2(1))^2 + p_1(1)[p_2(1)+p_2(2)]^2 + p_1(2)(p_2(2))^2 \\ a_{33} &= p_1(1)(p_2(2))^2 + p_1(2)[p_2(2)+p_2(3)]^2 + p_1(3)(p_2(3))^2 \\ a_{44} &= p_1(2)(p_2(3))^2 + p_1(3)[p_2(3)+p_2(4)]^2 + p_1(4)(p_2(4))^2 \\ a_{12} = a_{21} &= -p_1(4)p_2(1)[p_2(1)+p_2(4)] - p_1(1)p_2(1)[p_2(2)+p_2(1)] \end{aligned}$$

$$\begin{aligned}
a_{13} = a_{31} &= p_1(3)p_2(4)p_2(3) + p_1(1)p_2(2)p_2(1)\\
a_{14} = a_{41} &= -p_1(3)p_2(4)[p_2(4) + p_2(3)] - p_1(4)p_2(4)[p_2(1) + p_2(4)]\\
a_{23} = a_{32} &= -p_1(1)p_2(2)[p_2(2) + p_2(1)] - p_1(2)p_2(2)[p_2(3) + p_2(2)]\\
a_{24} = a_{42} &= p_1(4)p_2(1)p_2(4) + p_1(2)p_2(3)p_2(2)\\
a_{34} = a_{43} &= -p_1(3)p_2(3)[p_2(4) + p_2(3)] - p_1(2)p_2(3)[p_2(3) + p_2(2)]
\end{aligned}$$

For $T \geq 5$, let

$$A = \begin{bmatrix}
m(1) & n(1) & p(1) & 0 & \dots & 0 & p(T-1) & n(T)\\
n(1) & m(2) & n(2) & p(2) & \dots & 0 & 0 & p(T)\\
p(1) & n(2) & m(3) & n(3) & \dots & 0 & 0 & 0\\
0 & p(2) & n(3) & m(4) & \dots & 0 & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & 0 & \dots & n(T-3) & p(T-3) & 0\\
0 & 0 & 0 & 0 & \dots & m(T-2) & n(T-2) & p(T-2)\\
p(T-1) & 0 & 0 & 0 & \dots & n(T-2) & m(T-1) & n(T-1)\\
n(T) & p(T) & 0 & 0 & \dots & p(T-2) & n(T-1) & m(T)
\end{bmatrix},$$

where

$$\begin{aligned}
m(k) &= p_1(k-2)(p_2(k-1))^2 + p_1(k-1)[p_2(k) + p_2(k-1)]^2 + p_1(k)(p_2(k))^2,\\
n(k) &= -p_1(k-1)p_2(k)[p_2(k) + p_2(k-1)] - p_1(k)p_2(k)[p_2(k) + p_2(k+1)],\\
p(k) &= p_1(k)p_2(k)p_2(k+1),
\end{aligned}$$

$k = 1, 2, \dots, T$.

For $T \geq 3$, let

$$B = \begin{bmatrix}
q(T)+q(1) & -q(1) & 0 & \dots & -q(T)\\
-q(1) & q(1)+q(2) & -q(2) & \dots & 0\\
0 & -q(2) & q(2)+q(3) & \dots & 0\\
\dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & \dots & -q(T-1)\\
-q(T) & 0 & 0 & \dots & q(T-1)+q(T)
\end{bmatrix}.$$

Introducing $M := A + B$, we are in a position to rewrite the functional F as follows

$$F(x) = \frac{1}{2}x^* Mx + \sum_{t=1}^{T} r(t) \int_0^{x(t)} g(s)ds + \sum_{t=1}^{T} f(t)x(t), \tag{6}$$

whereby x^* we denote the transpose of a vector x. One can easily check the following property of matrices A and M.

Notice 1 For each row of the matrix A, its elements' sum equals zero.

Proof For $T = 2$, the thesis is obvious. Let us now consider the case $T \geq 5$. Since, by assumption (3), the following equalities hold

$$\begin{aligned} &m(1) = m(T+1), \quad n(1) = n(T+1), \ p(1) = p(T+1), \\ &p(-1) = p(T-1), \ p(0) = p(T), \qquad n(0) = n(T), \end{aligned}$$

then from the definition of matrix A, we obtain that

$$\sum_{j=1}^{T} a_{k+2,j} = p(k) + n(k+1) + m(k+2) + n(k+2) + p(k+2)$$

for $k = 0, 1, \ldots, T-1$. Here $\sum_{j=1}^{T} a_{T+1,j}$ denotes $\sum_{j=1}^{T} a_{1j}$. Let us rewrite the above sum as follows

$$\sum_{j=1}^{T} a_{k+2,j} = p_1(k)a(k) + p_1(k+1)a(k+1) + p_1(k+2)a(k+2),$$

where

$$\begin{aligned} a(k) &= p_2(k)p_2(k+1) - p_2(k+1)(p_2(k+1) + p_2(k)) + (p_2(k+1))^2 \\ &= 0, \\ a(k+1) &= (p_2(k+2) + p_2(k+1))^2 - p_2(k+1)(p_2(k+1) + p_2(k+2)) \\ &\quad - p_2(k+2)(p_2(k+1) + p_2(k+2)) \\ &= 0, \\ a(k+2) &= (p_2(k+2))^2 - p_2(k+2)(p_2(k+2) + p_2(k+3)) \\ &\quad + p_2(k+3)p_2(k+3) \\ &= 0, \end{aligned}$$

which ends the proof in this case.

For $T = 3, 4$, we can show the thesis by similar direct calculations. To illustrate that, let us consider the first row of the matrix A when $T = 3$. The sum of its elements can be written as follows

$$\sum_{j=1}^{3} a_{1j} = p_1(1)a(1) + p_1(2)a(2) + p_1(3)a(3),$$

where

$$\begin{aligned} a(1) &= p_2(1)p_2(2) - p_2(1)(p_2(1) + p_2(2)) + (p_2(1))^2 \\ &= 0, \\ a(2) &= (p_2(3))^2 + p_2(3)p_2(2) - p_2(3)(p_2(2) + p_2(3)) \end{aligned}$$

$$\begin{aligned} &= 0, \\ a(3) &= (p_2(3) + p_2(1))^2 - p_2(1)(p_2(3) + p_2(1)) - p_2(3)(p_2(3) + p_2(1)) \\ &= 0. \end{aligned}$$

□

Having the above fact, we immediately conclude that the same property is also valid for matrix M.

Notice 2 For each row of the matrix M, the elements' sum equals 0.

Lemma 1 *Assume that* $p_1(t) \geq 0$, $p_2(t) \geq 0$, $q(t) \geq 0$ *for any* $t \in [1, T]_{\mathbb{Z}}$. *Then* M *is positive semidefinite matrix.*

Proof Let $x \in \mathbb{R}^T$ and let x^* denote its transpose.

For $T = 2, 3, 4$, it is easy to check that A and B are positive semidefinite matrices by direct calculations.

For $T \geq 5$, we have

$$x^*Ax = \sum_{i=1}^{T} p_1(i) \left(p_2(i)x_i - (p_2(i) + p_2(i+1))\, x_{i+1} + p_2(i+1)x_{i+2}\right)^2 \geq 0,$$

and

$$x^*Bx = \sum_{i=1}^{T} q(i)(x_i - x_{i+1})^2 \geq 0.$$

The sum of any two positive semidefinite matrices of the same size is positive semidefinite. Hence, M is the positive semidefinite matrix for any T. □

In 2009, Stehlík in Lemma 1 [24] showed that matrix A is positive semidefinite, but his proof is based not on the definition but Gerschgorin Circle Theorem.

3 Main Results

As we mentioned, the critical point of the functional F is a solution of BVP given by (1), (2). Therefore, let us recall the following suitable result.

Theorem 1 *Let* H *be a Hilbert space. Let* $F : H \to \mathbb{R}$ *be a weakly sequentially lower semi-continuous and weakly coercive functional. Then* F *is bounded from below on* H *and there exists* $x_0 \in H$ *such that* $F(x_0) = \min F(x)$. *Moreover, if the Frechet derivative exists, then*

$$F'(x_0) = 0. \tag{7}$$

Theorem 2 *Assume that the following conditions hold*

(A0) $p_1(t) > 0$, $p_2(t) > 0$, $q(t) \geq 0$ *for* $[1, T]_{\mathbb{Z}}$, (A1) *r is a positive function on* $[1, T]_{\mathbb{Z}}$,

(A2) $g \in L^1_{loc}(\mathbb{R})$ *and there exists constant* $\alpha > 0$ *such that for any* $w \in \mathbb{R}$ *the function g satisfies*

$$\int_0^w g(s)ds \geq \alpha w^2.$$

Then the BVP (1), (2) has a solution.

Proof As we observe, the integrability of function g yields that the functional F is continuous. That is why it only remains to show the weak coercivity of the functional F. Since matrix M is positive semidefinite, we have that for any $x \in H$ we have $x^* Mx \geq 0$, then

$$\begin{aligned}\lim_{\|x\|\to\infty} F(x) &\geq \lim_{\|x\|\to\infty} \left(\sum_{t=1}^{T} r(t) \int_0^{x(t)} g(s)ds + \sum_{t=1}^{T} f(t)x(t) \right) \\ &\geq \lim_{\|x\|\to\infty} \left(\sum_{t=1}^{T} r_{min} \int_0^{x(t)} g(s)ds - \sum_{t=1}^{T} |f(t)x(t)| \right),\end{aligned}$$

where

$$r_{min} = \min\{r(1), r(2), \dots, r(T)\}.$$

The Hölder inequality gives us

$$\sum_{t=1}^{T} |f(t)x_n(t)| \leq \left(\sum_{t=1}^{T} f^2(t) \right)^{\frac{1}{2}} \left(\sum_{t=1}^{T} x_n^2(t) \right)^{\frac{1}{2}}.$$

Applying the above and the assumption (A2) with $\alpha > 0$, we get

$$\begin{aligned}\lim_{\|x\|\to\infty} F(x) &\geq \lim_{\|x\|\to\infty} \left(\sum_{t=1}^{T} \alpha r_{min} x^2(t) - \left(\sum_{t=1}^{T} f^2(t) \right)^{\frac{1}{2}} \|x\| \right) \\ &= \lim_{\|x\|\to\infty} \left(\alpha r_{min} \|x\|^2 - \left(\sum_{t=1}^{T} f^2(t) \right)^{\frac{1}{2}} \|x(t)\| \right).\end{aligned}$$

Finally, by assumption (A1) we know that $r_{min} > 0$ and it implies that the functional F is weakly coercive, which by Theorem 1 ends the proof. □

To obtain the uniqueness of the solution, let us recall the following theorem.

Theorem 3 *If, in addition to assumptions of Theorem 1, the functional F is continuous and strictly convex, then x_0 satisfying (7) is uniquely determined.*

The above theorem will be crucial in proving the uniqueness of the solution to the BVP.

Theorem 4 *If assumptions* (A0)-(A2) *hold and* (A3) *g is strictly increasing function, then there exists a unique solution to the BVP (1), (2).*

Proof To apply Theorem 3, we have to prove that the functional F given by (6) is strictly convex. Since M is positive semidefinite, we can conclude that x^*Mx is convex. Obviously, the sum $\sum_{t=1}^{T} f(t)x(t)$ is linear function of x, so in particular is convex. Note that the functional F is strictly convex if the functional

$$\widetilde{F}(x) = \sum_{t=1}^{T} r(t) \int_0^{x(t)} g(s)ds$$

is strictly convex. And that is what is only left to show. Now let us introduce notation $\widetilde{g}(w) = \int_0^w g(s)ds$. The assumption (A3) implies that $\widetilde{g}$ is strictly convex. Hence, for any $x, y \in H$ and $u \in (0, 1)$ we have

$$\begin{aligned}
\widetilde{F}(ux + (1-u)y) &= \sum_{t=1}^{T} r(t) \int_0^{ux(t)+(1-u)y(t)} g(s)ds \\
&= \sum_{t=1}^{T} r(t)\widetilde{g}(ux(t) + (1-u)y(t)) \\
&< \sum_{t=1}^{T} r(t)[u\widetilde{g}(x(t)) + (1-u)\widetilde{g}(y(t))] \\
&= u\sum_{t=1}^{T} r(t)\widetilde{g}(x(t)) + (1+u)\sum_{t=1}^{T} r(t)\widetilde{g}(y(t)) \\
&= u\widetilde{F}(x) + (1-u)\widetilde{F}(y).
\end{aligned}$$

□

Corollary 1 *Assume that $p_1(t) \geq 0, p_2(t) \geq 0, q(t) \geq 0$ for any $t \in [1, T]_{\mathbb{Z}}$. If for any $t \in [1, T]_{\mathbb{Z}}$ there exists $g^{-1}(f(t))$ and conditions (A1)-(A3) hold, then the thesis of Theorem 4 is satisfied.*

Remark 1 If in Theorem 4 condition (A0) is replaced by

$$p_1(t) > 0, p_2(t) \neq 0, q(t) \geq 0 \text{ for any } t \in [1, T]_{\mathbb{Z}},$$

the thesis of Theorem 4 holds.

Remark 2 If in Theorem 4 condition (A0) and (A1) are replaced by

$$p_1(t) < 0,\, p_2(t) \neq 0,\, q(t) \leq 0 \text{ for any } t \in [1, T]_{\mathbb{Z}},$$

and

$$r \text{ is a negative function on } [1, T]_{\mathbb{Z}}$$

respectively, the thesis of Theorem 4 also holds.

Notice that presented here Theorems 3 and 4 improved and generalized the results published by Liu, Zhou and Shi [25] in 2018. Taking $p_1 \equiv 1$, $p_2(t) = p(t)$ for $t \in [1, T]_{\mathbb{Z}}$, and $g(t) \equiv t$ for $t \in \mathbb{R}$ we obtain Theorem 3 from that paper. So more, condition (M) from Theorem 3 [25] is always satisfied as a particular case of our Notice 2, and condition (27) is redundant.

4 Examples

In the last section, we present six examples.

Example 1 Consider the difference equation

$$\Delta^4 x(t-2) - \Delta^2 x(t-1) + 2x(t) + \sqrt[3]{x(t)} + f(t) = 0, \tag{8}$$

where

$$f(t) = -\frac{43}{6}\left(t - 4\left\lfloor\frac{t}{4}\right\rfloor\right)^3 + \frac{15}{2}\left(t - 4\left\lfloor\frac{t}{4}\right\rfloor\right)^2 + \frac{41}{3}\left(t - 4\left\lfloor\frac{t}{4}\right\rfloor\right) - 5$$

with $T = 4$. One can check that all assumptions of the Theorem 4 hold (with $p_1 \equiv 1$, $p_2 \equiv 1$, $q \equiv 1$ and $r \equiv 1$). It is easy to verify that $x(1) = 0$, $x(2) = 0$, $x(3) = 1$, $x(4) = 0$ is a solution of equation (8). Here function $g(x) = 2x + \sqrt[3]{x}$ is a strictly increasing function, thus the given solution is the unique solution of the considered BVP.

Example 2 For $T = 5$ consider Eq. (1) with coefficients

$$\begin{aligned}
&p_1(1) = 1,\ p_1(2) = 2,\ p_1(3) = 3,\ p_1(4) = 2,\ p_1(5) = 1,\\
&p_2(1) = 4,\ p_2(2) = 3,\ p_2(3) = 4,\ p_2(4) = 6,\ p_2(5) = 5,\\
&q(1) = 5,\ q(2) = 2,\ q(3) = 3,\ q(4) = 5,\ q(5) = 4,
\end{aligned}$$

satisfying assumptions (A0), and

$$\begin{aligned}
&r(1) = 1, r(2) = 2, r(3) = 4, r(4) = 3, r(5) = 1,\\
&g(t) = 2t + \pi \sin(\pi t),
\end{aligned}$$

$$f(t) = \frac{-1903}{6}t^4 + \frac{10942}{3}t^3 - \frac{43012}{3}t^2 + \frac{133981}{6}t - 11323.$$

We have

$$A = \begin{bmatrix} 147 & -64 & 12 & 60 & -155 \\ -64 & 83 & -63 & 24 & 20 \\ 12 & -63 & 155 & -176 & 72 \\ 60 & 24 & -176 & 404 & -312 \\ -155 & 20 & 72 & -312 & 375 \end{bmatrix}, B = \begin{bmatrix} 9 & -5 & 0 & 0 & -4 \\ -5 & 7 & -2 & 0 & 0 \\ 0 & -2 & 5 & -3 & 0 \\ 0 & 0 & -3 & 8 & -5 \\ -4 & 0 & 0 & -5 & 9 \end{bmatrix},$$

and positive semidefinite matrix

$$M = \begin{bmatrix} 156 & -69 & 12 & 60 & -159 \\ -69 & 90 & -65 & 24 & 20 \\ 12 & -65 & 160 & -179 & 72 \\ 60 & 24 & -179 & 412 & -317 \\ -159 & 20 & 72 & -317 & 384 \end{bmatrix}.$$

Note that all assumptions of Theorem 2 hold. Moreover, one can check that one of the solutions of the boundary value problem (1), (2) is $x(1) = 3, x(2) = 4, x(3) = 5, x(4) = 1, x(5) = 2$.

Example 3 Let $T = 7$, and assume the following coefficients of equation (1)

$$p_1(1) = 5,\ p_1(2) = 4,\ p_1(3) = 3,\ p_1(4) = 2,\ p_1(5) = 1,\ p_1(6) = 7,\ p_1(7) = 6,$$
$$p_2(1) = 4,\ p_2(2) = 6,\ p_2(3) = 7,\ p_2(4) = 8,\ p_2(5) = 10\ p_2(6) = 2,\ p_2(7) = 3,$$
$$q(1) = 1,\ q(2) = 2,\ q(3) = 3,\ q(4) = 2,\ q(5) = 1\ q(6) = 2,\ q(7) = 2,$$

and

$$\begin{aligned} r(t) &= 1, \quad g(t) = 2^t, \\ f(t) &= \frac{18127}{180}t^6 - \frac{47593}{20}t^5 + \frac{795275}{36}t^4 - \frac{1223993}{12}t^3 \\ &\quad + \frac{21916469}{90}t^2 - \frac{4193324}{15}t + 117408. \end{aligned}$$

Then

$$A = \begin{bmatrix} 437 & -368 & 120 & 0 & 0 & 42 & -231 \\ -368 & 740 & -612 & 168 & 0 & 0 & 72 \\ 120 & -612 & 1003 & -679 & 168 & 0 & 0 \\ 0 & 168 & -679 & 999 & -648 & 160 & 0 \\ 0 & 0 & 168 & -648 & 940 & -480 & 20 \\ 42 & 0 & 0 & 160 & -480 & 372 & -94 \\ -231 & 72 & 0 & 0 & 20 & -94 & 233 \end{bmatrix},$$

$$B = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 & -2 \\ -1 & 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 5 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 5 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -2 \\ -2 & 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix},$$

and

$$M = \begin{bmatrix} 440 & -369 & 120 & 0 & 0 & 42 & -233 \\ -369 & 743 & -614 & 168 & 0 & 0 & 72 \\ 120 & -614 & 1008 & -682 & 168 & 0 & 0 \\ 0 & 168 & -682 & 1004 & -650 & 160 & 0 \\ 0 & 0 & 168 & -650 & 943 & -481 & 20 \\ 42 & 0 & 0 & 160 & -481 & 375 & -96 \\ -233 & 72 & 0 & 0 & 20 & -96 & 237 \end{bmatrix}.$$

All assumptions of Theorem 2 hold (for $\alpha = \frac{1}{2}$). The solution of considered BVP is $x(1) = 4$, $x(2) = 2$, $x(3) = 1$, $x(4) = 2$, $x(5) = 1$, $x(6) = 3$, $x(7) = 2$. Since function g is strictly increasing function, thus the solution is unique.

Example 4 Consider Eq. (1) for $T = 100$, taking $p_1(t) = 2 + \sin \frac{2\pi t}{100}$, $p_2(t) = 2 + \cos \frac{2\pi t}{100}$, $q(t) = 1 - 2t(t - 100)$, $g(t) = 4t^3 + t$, $r(t) = t^2$, $f(t) = -34t^2$. The elements of matrices A and B are given by

$$\begin{aligned} m(k) &= \left(2 + \sin \frac{2(k-2)\pi}{100}\right)\left(2 + \cos \frac{2(k-1)\pi}{100}\right)^2 \\ &\quad + \left(2 + \sin \frac{2(k-1)\pi}{100}\right)\left[\left(2 + \cos \frac{2k\pi}{100}\right) + \left(2 + \cos \frac{2(k-1)\pi}{100}\right)\right]^2 \\ &\quad + \left(2 + \sin \frac{2k\pi}{100}\right)\left(2 + \cos \frac{2k\pi}{100}\right)^2, \\ n(k) &= -\left(2 + \sin \frac{2(k-1)\pi}{100}\right)\left(2 + \cos \frac{2k\pi}{100}\right)\left[\left(2 + \cos \frac{2k\pi}{100}\right) + \left(2 + \cos \frac{2(k-1)\pi}{100}\right)\right] \\ &\quad - \left(2 + \sin \frac{2k\pi}{100}\right)\left(2 + \cos \frac{2k\pi}{100}\right)\left[\left(2 + \cos \frac{2k\pi}{100}\right) + \left(2 + \cos \frac{2(k+1)\pi}{100}\right)\right], \\ p(k) &= \left(2 + \sin \frac{2k\pi}{100}\right)\left(2 + \cos \frac{2k\pi}{100}\right)\left(2 + \cos \frac{2(k+1)\pi}{100}\right), \\ q(k) &= 1 - 2k(k - 100), \ k = 1, 2, \ldots, 100. \end{aligned}$$

It is easy to verify that the assumptions of Theorem 4 hold and $x(t) = 2$ is the unique solution to BVP.

Example 5 Consider Eq. (1) for $T = 1000$, where $p_1(t) = 5 + (-1)^t$, $p_2(t) = 2 + (-1)^t$, $q(t) \equiv 0$, $g(t) = t$, $r(t) = 320$, $f(t) = 640\cos(t\pi) - 1600$. Here

$$m(k) = \begin{cases} 136, & k \equiv 1 \pmod 2 \\ 124, & k \equiv 0 \pmod 2, \end{cases}$$

$$n(k) = \begin{cases} -40, & k \equiv 1 \pmod 2 \\ -120, & k \equiv 0 \pmod 2, \end{cases}$$

$$p(k) = \begin{cases} 12, & k \equiv 1 \pmod 2 \\ 18, & k \equiv 0 \pmod 2, \end{cases}$$

for $k = 1, 2, \ldots, 1000$. The assumptions of Theorem 4 hold and $x(t) = 5 + (-1)^{t+1}$ is the unique solution to the considered equation.

Example 6 Consider Eq. (1) where $T = 10,000$, $p_1(t) = p_2(t) = q(t) = r(t) \equiv 1$, $g(t) = t^{10001} + t$, $f(t) = 22\cos((t+1)\pi)$. Here $m(k) = 6$, $n(k) = -4$, $p(k) = 1$ for $k = 1, 2, \ldots, 10,000$. It is easy to verify that the assumptions of Theorem 4 hold and $x(t) = (-1)^t$ is the unique solution to the considered problem.

Conclusion

This paper studies the boundary value problem (1), (2). We obtain sufficient conditions for the existence and uniqueness of a solution to consider BVP using variational methods. Our results concern a more general equation than in the earlier studied boundary problems since we introduce a second sequence to the quasidifference of the fourth-order. At the same time, earlier studies dealt with boundary problems with one sequence in this quasidifference. The examples illustrate the presented results.

An interesting topic is a possibility of generalizing the presented results to equations of higher-orders with one or more quasidifferences. The issues of providing sufficient conditions for the existence of the BVP solution with the imposition of other assumptions on nonlinear functions than those assumed in this work also require further research.

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Declarations

Conflict of interest The authors declare no competing interests.

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