



# Analysis of Stochastic SIRC Model with Cross Immunity Based on Ornstein–Uhlenbeck Process

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## Abstract

In this paper, we analyze a stochastic SIRC model with Ornstein–Uhlenbeck process. Firstly, we give the existence and uniqueness of global solution of stochastic SIRC model and prove it. In addition, the existence of ergodic stationary distributions for stochastic SIRC system is proved by constructing a suitable series of Lyapunov functions. A quasi-endemic equilibrium related to endemic equilibrium of deterministic systems is defined by considering randomness. And we obtain the probability density function of the linearized system near the equilibrium point. After the proof of probability density function, the sufficient condition of disease extinction is given and proved. We prove the theoretical results in the paper by numerical simulation at the end of the paper.

**Keywords** Stochastic SIRC epidemic model · Cross immunity · Ornstein–Uhlenbeck process · Stationary distribution

## 1 Introduction

Until now, COVID-19 has been considered the most devastating infectious disease to affect humans worldwide. Infected persons transmit the disease to others mainly through person-to-person transmission [1], including airborne transmission and contact transmission. Therefore, it is essential to study the spread and control of COVID-19 and determine the conditions for the extinction of the epidemic [2].

Since 1972, Kermack and Mckendrick began to link the spread of infectious diseases to mathematics [3]. At present, the construction of mathematical models of infectious

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diseases and epidemics has become an important tool for analyzing and investigating the characteristics of diseases. Accurate and precise mathematical model plays an important part in decision making [4]. For the spread of COVID-19, we consider the most standard epidemic-susceptible-recovered(SIR) epidemic model [5].

If you're vaccinated against a particular virus and you develop immunity to another virus, we call this cross-immunity. In the spread of COVID-19, some vaccines also provide cross-immunity to the novel coronavirus. Casagnandi [6] and Rihan et al. [7] introduced the SIRC model to analyze different infectious behavior by inserting a new state C between Susceptible (S) and Recovered (R). Compartment C is used to describe COVID-19 infected persons in a state of cross-immunity. In terms of cross-immunity, Kwang et al. [8] developed a two-strain SIR model for inferring the cross-immune responses of different strains. Shrock et al. [9] suggest that perhaps previous CoV infection infers an "immune response memory". Therefore, we study a stochastic SIRC epidemiological model that includes cross-immune states. Based on the existing literature, we know that few scholars have considered the impact of cross-immune status on COVID-19 transmission [10].

The rest of this article is structured as follows. In the part 2, we show the stochastic SIRC model including the mean reversion process Ornstein–Uhlenbeck process. The existence and uniqueness of stochastic systems (4) are proved in the part 3. In the part 4, the existence of ergodic stationary distribution of stochastic system (4) is proved by constructing a suitable Lyapunov function. In the part 5, we use the corresponding Foker-Planck equation to derive the explicit expression of the probability density function for the stationary distribution  $\pi(\cdot)$  of the stochastic SIRC system (4). Sufficient conditions for the extinction of infectious diseases are given and proved in Sect. 6. Finally, we prove the theoretical results through numerical simulation and summarize the work of the whole paper at the end of the paper, and further give the research direction of related problems.

## 2 Model Formulation and Local Stability of the Equilibrium States

### 2.1 Mathematical Model

In this paper, we use the SIRC model that includes cross-immune status because it can effectively describe the transmission mechanism of COVID-19 [11]. SIRC model adopts the following form: [7, 10]

$$\begin{cases} \dot{S}(t) = \eta(1 - S(t)) - \xi S(t)I(t) + \beta C(t), \\ \dot{I}(t) = \xi S(t)I(t) + \sigma \xi C(t)I(t) - (\eta + \alpha)I(t), \\ \dot{R}(t) = (1 - \sigma)\xi C(t)I(t) + \alpha I(t) - (\eta + \gamma)R(t), \\ \dot{C}(t) = \gamma R(t) - \xi C(t)I(t) - (\eta + \beta)C(t), \end{cases} \quad (1)$$

where  $S(t)$ ,  $I(t)$ ,  $R(t)$ ,  $C(t)$  represent the density of susceptible individuals, infected individuals, recovered individuals and cross-immune individuals from the disease at time  $t$ , respectively.  $\eta$  denotes the natural mortality rate, which is equal to the rate of newborns.  $\beta$  represents the rate of cross-immune individuals  $C(t)$  becomes susceptible

individuals  $S(t)$ .  $\xi$  denotes the infection rate of susceptible individuals  $S(t)$  infected with infectious disease to become infected individuals  $I(t)$ .  $\sigma$  is the average reinfection probability of a cross-immune individuals, where we assume that  $\sigma < 1$ .  $\alpha$  denotes the recovery rate of infected individuals  $I(t)$  to recovered individuals  $R(t)$ .  $\gamma$  is the rate at which the recovered individuals  $R(t)$  becomes the cross-immune individuals  $C(t)$  and moves from total to partial immunity. All of the parameters  $\beta, \xi, \sigma, \alpha, \gamma$  are assumed to be positive.

For the system (1) mentioned above, we can get

$$d(S(t) + I(t) + R(t) + C(t)) = [\eta - \eta(S(t) + I(t) + R(t) + C(t))]dt.$$

Then we obtain

$$S(t) + I(t) + R(t) + C(t) \equiv 1 \quad \text{if} \quad S(0) + I(0) + R(0) + C(0) = 1.$$

Let  $\mathbb{R}^n$  be an n-dimensional Euclidean space and  $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_i > 0, 1 \leq i \leq n\}$ . Thus the region

$$\Gamma_0 = \{(S, I, R, C) \in \mathbb{R}_+^4 : S + I + R + C = 1\}$$

is a positively invariant set of system (1), then, model (1) is simplified to analyze a three-dimensional system as follows:

$$\begin{cases} \dot{I}(t) = \xi(1 - I(t) - R(t) - C(t))I(t) + \sigma\xi C(t)I(t) - (\eta + \alpha)I(t), \\ \dot{R}(t) = (1 - \sigma)\xi C(t)I(t) + \alpha I(t) - (\eta + \gamma)R(t), \\ \dot{C}(t) = \gamma R(t) - \xi C(t)I(t) - (\eta + \beta)C(t). \end{cases} \quad (2)$$

The above system has two equilibria, where the basic regeneration number is defined as  $R_0 = \frac{\xi}{\eta + \alpha}$ , the disease-free equilibrium is defined as  $E_0 = (I_0, R_0, C_0) = (0, 0, 0)$ . In addition, if  $R_0 > 1$ , we can easily get a unique endemic equilibrium point  $E_* = (I_*, R_*, C_*)$ ,  $I_*, R_*, C_*$  are defined as follows:

$$R_* = \frac{\alpha I_*(\xi I_* + \eta + \beta)}{[(\eta + \gamma) - (1 - \sigma)\gamma]\xi I_* + (\eta + \beta)(\eta + \gamma)},$$

$$C_* = \frac{\gamma \alpha I_*}{[(\eta + \gamma) - (1 - \sigma)\gamma]\xi I_* + (\eta + \beta)(\eta + \gamma)},$$

and the value of  $I_*$  is the root of the following equation  $h_1 I_*^2 + h_2 I_* + h_3 = 0$ , where

$$h_1 = \eta\xi(\eta + \alpha + \sigma\gamma),$$

$$h_2 = \eta\xi[\alpha(2\eta + \gamma + \beta) + (\eta + \beta)(\eta + \gamma) + (\eta + \sigma\gamma)(\eta - \xi)],$$

$$h_3 = \eta(\eta + \beta)(\eta + \gamma)(\eta + \alpha)(1 - R_0).$$

## 2.2 Stochastic Model

However, in real life, according to the existing literature, the spread of infectious diseases and the formulation of control policies are affected by random factors [12]. For example, Nadjat et al. [13] showed that population density is an important factor affecting the spread of COVID-19. In [14], Wilfried Allaerts proposed that national public media and government advisory bodies play an important role in policy formulation. Therefore, we have to consider the influence of random factors on the mathematical model. The stochastic model has gradually become the core of various scholars' research on the behavior of infectious diseases. To simulate the stochastic effects, we usually introduce stochastic models with linear perturbations [15–20].

For the linear perturbation, we assume that both  $\xi(t)$  and  $\beta(t)$  are influenced by the linear white noises: so we get

$$\xi(t) = \bar{\xi} + \frac{\sigma_1 dB_1(t)}{dt}, \quad \beta(t) = \bar{\beta} + \frac{\sigma_2 dB_2(t)}{dt}. \quad (3)$$

By calculating two equations  $\mathbb{E}[\xi(t)] = \bar{\xi}$  and  $\mathbb{E}[\beta(t)] = \bar{\beta}$ , give the average values of  $\xi(t)$  and  $\beta(t)$  respectively.  $B_i(t)$  ( $i = 1, 2$ ) are two independent standard Brownian motions, and  $\sigma_i$  ( $i = 1, 2$ ) are the strength of each Brownian motion.

However, linear perturbation and (3) are something irrational. Our biological explanation for this is as follows: for any time interval  $[0, t]$ , integrating (3) from 0 to  $t$ , then we have

$$\begin{aligned} \frac{1}{t} \int_0^t \xi(s) ds &= \bar{\xi} + \sigma_1 \frac{B_1(t)}{t} \sim \mathbb{N}\left(\bar{\xi}, \frac{\sigma_1^2}{t}\right), \\ \frac{1}{t} \int_0^t \beta(s) ds &= \bar{\beta} + \sigma_2 \frac{B_2(t)}{t} \sim \mathbb{N}\left(\bar{\beta}, \frac{\sigma_2^2}{t}\right). \end{aligned}$$

This states that as  $t \rightarrow 0$ , the variance of  $\xi(t)$  and  $\beta(t)$  will go to infinity, which produces the unreasonable result that  $\xi(t)$  and  $\beta(t)$  fluctuations will become larger in a small time interval.

In this paper, a new method for simulating random disturbances is presented. Ornstein–Uhlenbeck process [21–25] plays an important role in the exchange rate of money, sale price of commodities and interest rate in economics. Dixit and Pindyck [26] analyzed Ornstein–Uhlenbeck process (Dixit & Pindyck model) for the first time in finance, which was inspired by it. We use the Ornstein–Uhlenbeck process in this paper to simulate the interference of random factors in the mathematical model in the biological population. In [27], Zhou et al., in their paper, studied a stochastic non-autonomous population model of two mean-reverting Ornstein–Uhlenbeck processes. Song et al. [28] studied a stochastic SVEIS infectious disease model with Ornstein–Uhlenbeck processes and presented the conditions for the stationary distribution and extinction of the stochastic model, which also inspired me. Here we assume that the contact or transmission rate  $\xi$  and the rate at which the cross-immune population becomes susceptible again  $\beta$  in the system (2) satisfy Ornstein–Uhlenbeck process.

Then we have the results of  $\xi(t)$  and  $\beta(t)$  under the influence of the mean-reverting Ornstein–Uhlenbeck process.

$$d\xi(t) = \theta_1(\bar{\xi} - \xi(t))dt + \sigma_1dB_1(t), \quad d\beta(t) = \theta_2(\bar{\beta} - \beta(t))dt + \sigma_2dB_2(t).$$

According to the conclusion in reference [29], we obtain that  $\xi(t)$  and  $\beta(t)$  have unique explicit solutions of the form defined as follows

$$\begin{aligned} \xi(t) &= \bar{\xi} + (\xi(0) - \bar{\xi})e^{-\theta_1 t} + \sigma_1 \int_0^t e^{-\theta_1(t-s)}dB_1(s), \\ \beta(t) &= \bar{\beta} + (\beta(0) - \bar{\beta})e^{-\theta_2 t} + \sigma_2 \int_0^t e^{-\theta_2(t-s)}dB_2(s). \end{aligned}$$

According to the above formula and references [29], it shows that  $\xi(t)$  follows the Gaussian distribution  $\mathbb{N}(\mathbb{E}(\xi(t)), \text{Var}(\xi(t)))$  on  $[0, t]$ , where  $\mathbb{E}(\xi(t)) = \bar{\xi} + (\xi(0) - \bar{\xi})e^{-\theta_1 t}$  and  $\text{Var}(\xi(t)) = \frac{\sigma_1^2}{2\theta_1}(1 - e^{-2\theta_1 t})$ . Because of  $\lim_{t \rightarrow \infty} \mathbb{E}(\xi(t)) = \bar{\xi}$  and  $\lim_{t \rightarrow \infty} \text{Var}(\xi(t)) = \frac{\sigma_1^2}{2\theta_1}$ . Obviously,  $\xi(t)$  follows the Gaussian distribution  $\mathbb{N}(\bar{\xi}, \frac{\sigma_1^2}{2\theta_1})$  as  $t$  tends to infinity.

Similarly, the correlation analysis of  $\beta(t)$  is similar to that of  $\xi(t)$ , thus we can conclude that  $\beta(t)$  also follows the Gaussian distribution  $\mathbb{N}(\bar{\beta}, \frac{\sigma_2^2}{2\theta_2})$  as  $t$  tends to infinity.

In summary, in this article, it is reasonable to characterize random factors in infectious disease models by introducing Ornstein–Uhlenbeck processes. Therefore, define an invariant set

$$\Gamma = \left\{ (I, R, C, \xi, \beta) \in \mathbb{R}_+^3 \times \mathbb{R}^2 : I + R + C < 1 \right\},$$

then we get the following stochastic IRC system incorporating Ornstein–Uhlenbeck process:

$$\begin{cases} \dot{I}(t) = \xi^+(1 - I(t) - R(t) - C(t))I(t) + \sigma\xi^+C(t)I(t) - (\eta + \alpha)I(t), \\ \dot{R}(t) = (1 - \sigma)\xi^+C(t)I(t) + \alpha I(t) - (\eta + \gamma)R(t), \\ \dot{C}(t) = \gamma R(t) - \xi^+C(t)I(t) - (\eta + \beta^+)C(t), \\ d\xi(t) = \theta_1(\bar{\xi} - \xi(t))dt + \sigma_1dB_1(t), \\ d\beta(t) = \theta_2(\bar{\beta} - \beta(t))dt + \sigma_2dB_2(t), \end{cases} \tag{4}$$

where  $\xi^+(t) = \max\{\xi(t), 0\}$ , in the same way,  $\beta^+(t) = \max\{\beta(t), 0\}$ . In addition, we write  $a \wedge b$  as  $\min\{a, b\}$ , similarly, we write  $a \vee b$  as  $\max\{a, b\}$ .

### 3 Existence and Uniqueness of the Global Positive Solution

**Theorem 3.1** *For any initial value  $(I(0), R(0), C(0), \xi(0), \beta(0)) \in \Gamma$ , there exists a unique solution  $(I(t), R(t))$ ,*

$C(t), \xi(t), \beta(t)$  to the system (4) on  $t \geq 0$  and the solution will remain in  $\Gamma$  with probability one, namely, the solution  $(I(t), R(t), C(t), \xi(t), \beta(t)) \in \Gamma$  for all  $t \geq 0$  almost surely (a.s.).

**Proof** For any initial value  $(I(0), R(0), C(0), \xi(0), \beta(0)) \in \Gamma$ , the coefficients of system (4) are locally Lipschitz continuous [30], so there is a unique local solution  $(I(t), R(t), C(t), \xi(t), \beta(t)) \in \Gamma$  on  $t \in [0, \tau_e]$ , where  $\tau_e$  is an explosion time [31]. To testify that  $(I(t), R(t), C(t), \xi(t), \beta(t))$  is global, we need to prove that  $\tau_e = \infty$  a.s. Set  $s_0 > 0$  big enough to make that each element of  $(I(0), R(0), C(0), \xi(0), \beta(0))$  are all on the interval  $[\frac{1}{s_0}, s_0]$ . For each integer  $s \geq s_0$ , let  $Z(t) = (I(t), R(t), C(t))$ , and let's define the stopping time  $\tau_s$  as follows [32]

$$\tau_s = \inf \left\{ t \in (0, \tau_e) \mid \min\{Z(t), e^{\xi(t)}, e^{\beta(t)}\} \leq \frac{1}{s} \text{ or } \max\{Z(t), e^{\xi(t)}, e^{\beta(t)}\} \geq s \right\},$$

where throughout our paper, we set  $\inf\{\emptyset\} = \infty$ . Obviously,  $\tau_s$  is increasing as  $s \rightarrow \infty$ . Set  $\tau_\infty = \lim_{s \rightarrow \infty} \tau_s$ , then we derive that  $\tau_\infty \leq \tau_e$  a.s. If we show that  $\tau_\infty = \infty$  a.s., then  $\tau_e = \infty$  a.s., then  $(I(t), R(t), C(t), \xi(t), \beta(t)) \in \Gamma$  a.s. for all  $t \geq 0$ .

If  $\tau_\infty < \infty$ , suppose there is that a pair of constants  $\tilde{t} > 0$  and  $\varepsilon \in (0, 1)$  such that  $\mathbb{P}\{\tau_\infty \leq \tilde{t}\} > \varepsilon$ , so there is an integer  $s_1 \geq s_0$ , which makes

$$\mathbb{P}\{\tau_{s_1} \leq \tilde{t}\} \geq \varepsilon, \tag{5}$$

for all  $s \geq s_1$ . We define a non-negative  $C^2$ -function  $V: \mathbb{R}_+^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$ ,

$$\begin{aligned} V(I(t), R(t), C(t), \xi(t), \beta(t)) &= (I - 1 - \ln I) + (R - 1 - \ln R) + (C - 1 - \ln C) \\ &\quad + [(1 - I - R - C) - 1 - \ln(1 - I - R - C)] \\ &\quad + \frac{\xi^2}{2} + \frac{\beta^2}{2}. \end{aligned}$$

Using Itô's formula [30], we get

$$dV = \mathcal{L}V dt + \sigma_1 \xi dB_1(t) + \sigma_2 \beta dB_2(t).$$

In Appendix B of the article by Zhou et al. [15], the definition of differential operators  $\mathcal{L}$  is given, then

$$\begin{aligned} \mathcal{L}V &= \left(-\frac{1}{I}\right) [\xi^+(1 - I - R - C)I + \sigma \xi^+ CI - (\eta + \alpha)I] \\ &\quad + \left(-\frac{1}{R}\right) [(1 - \sigma)\xi^+ CI + \alpha I - (\eta + \gamma)R] + \left(-\frac{1}{C}\right) [\gamma R - \xi^+ CI - (\eta + \beta^+)C] \\ &\quad + \left(\frac{1}{1 - I - R - C}\right) [\xi^+(1 - I - R - C) - \eta(I + R + C) - \beta^+ C] \\ &\quad + \theta_1 \xi (\bar{\xi} - \xi) + \frac{\sigma_1^2}{2} + \theta_2 \beta (\bar{\beta} - \beta) + \frac{\sigma_2^2}{2} \end{aligned}$$

$$\begin{aligned}
 &\leq 3\eta + 2\xi^+ + \alpha + \gamma + \beta^+ + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \theta_1\xi(\bar{\xi} - \xi) + \theta_2\beta(\bar{\beta} - \beta) \\
 &\leq 3\eta + 2|\xi| + |\beta| + \alpha + \gamma + \frac{\sigma_1^2 + \sigma_2^2}{2} - \theta_1\xi^2(t) + \theta_1\bar{\xi}|\xi| - \theta_2\beta^2 + \theta_2\bar{\beta}|\beta| \\
 &\leq 3\eta + \alpha + \gamma + \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{(\theta_1\bar{\xi} + 2)^2}{4\theta_1} + \frac{(\theta_2\bar{\beta} + 1)^2}{4\theta_2} := K,
 \end{aligned}$$

where  $K$  is a constant satisfying the condition independent of  $S, I, R, C$  and  $t$ . The rest of the proof is similar to Mao et al. [33]’s article and hence is omitted. So we have completed the proof of Theorem 3.1. □

### 4 Existence of a Stationary Distribution

In this part, we are concerned about the existence of a stationary distribution in the system (4), which matters whether infectious diseases will continue to spread. To prove the existence of a stationary distribution for system (4), we first introduce Lemma 4.1 before the proof.

Let  $X(t)$  be a homogeneous Markov process in  $\Omega_l$  ( $\Omega_l$  denotes the  $l$ -dimensional Euclidean space) satisfying the following stochastic differential equation.

$$dX(t) = b(t, X(t))dt + \sum_{j=1}^k \sigma_j(t, X(t))dB_j(t). \tag{6}$$

**Lemma 4.1** (Khasminskii [34]). *Let the vectors  $b(s, x), \sigma_1(s, x), \dots, \sigma_k(s, x)$  ( $s \in [t_0, T], x \in \mathbb{R}^k$ ) be continuous functions of  $(s, x)$ , such that for some constant  $B$  the following conditions hold in the entire domain of definition:*

$$\begin{aligned}
 |b(s, x) - b(s, y)| + \sum_{j=1}^k |\sigma_j(s, x) - \sigma_j(s, y)| &\leq B|x - y|, \\
 |b(s, x)| + \sum_{j=1}^k |\sigma_j(s, x)| &\leq B(1 + |x|).
 \end{aligned}$$

Moreover, there exists a non-negative function  $V^*(x)$  such that

$$\mathcal{L}V^*(x) \leq -1, \quad \forall x \in \mathbb{R}^k \setminus \mathbb{H},$$

where  $\mathbb{H}$  is a compact subset defined on  $\mathbb{R}^k$ .

Then the Markov process (6) has at least one stationary solution  $X(t)$ , which has a stationary distribution  $\varpi(\cdot)$  on  $\mathbb{R}^k$ .

Define a critical value,

$$R_0^h = \frac{\bar{\xi}}{\eta + \alpha + \frac{\sigma_1}{\sqrt{2\theta_1}}}.$$

**Theorem 4.2** *Assume that  $R_0^h > 1$ . For any initial value  $(I(0), R(0), C(0), \xi(0), \beta(0)) \in \Gamma$ , the system (4) has a stationary distribution  $\pi$ .*

**Proof** First we give a definition of a  $C^2$ -function  $V_1$  as follows

$$V_1 = -\ln I + a_1(R + C) + a_2\left(\frac{(\xi - \bar{\xi})^2}{2}\right),$$

where  $a_1$  and  $a_2$  are positive constants, and they will be fixed later. Using Itô's formula [30], we have

$$\begin{aligned} \mathcal{L}V_1 &= -\xi^+(1 - I - R - C) - \sigma\xi^+C + (\eta + \alpha) + a_1(\alpha I - \eta R - \eta C - \sigma\xi^+CI) \\ &\quad + a_2\left(\theta_1(\xi - \bar{\xi})^2 + \frac{\sigma_1^2}{2}\right) \\ &\leq -\bar{\xi}(1 - I - R - C) + |\bar{\xi} - \xi^+|(1 - I - R - C) + \eta + \alpha \\ &\quad + a_1(\alpha I - \eta R - \eta C) + a_2\left(-\theta_1(\xi - \bar{\xi})^2 + \frac{\sigma_1^2}{2}\right), \end{aligned}$$

where  $a_1 = \frac{\bar{\xi}}{\eta}$ ,  $a_2 = \frac{1}{\sigma_1\sqrt{2\theta_1}}$ , then

$$\begin{aligned} \mathcal{L}V_1 &\leq -\bar{\xi} + \eta + \alpha + \bar{\xi}I + a_1\alpha I + |\xi - \bar{\xi}| - a_2\theta_1(\xi - \bar{\xi})^2 + \frac{a_2\sigma_1^2}{2} \\ &\leq -\bar{\xi} + \eta + \alpha + \bar{\xi}I + a_1\alpha I + \frac{1}{4a_2\theta_1} + \frac{a_2\sigma_2^2}{2} \\ &\leq -(R_0^h - 1)\left(\eta + \alpha + \frac{\sigma_1}{\sqrt{2\theta_1}}\right) + (\bar{\xi} + a_1\alpha)I. \end{aligned}$$

Denote

$$\begin{aligned} V_2 &= -\ln R, \quad V_3 = -\ln C, \quad V_4 = -\ln(1 - I - R - C), \quad V_5 = \frac{\xi^2}{2} + \frac{\beta^2}{2}. \\ \mathcal{L}V_2 &= -\frac{\alpha I}{R} + (\eta + \gamma) - \frac{(1 - \sigma)\xi^+CI}{R} \leq -\frac{\alpha I}{R} + \eta + \gamma, \\ \mathcal{L}V_3 &= -\frac{\gamma R}{C} + \xi^+I + \eta + \beta^+, \\ \mathcal{L}V_4 &= \xi^+I - \frac{\beta^+C}{1 - (I + R + C)} - \frac{\eta(I + R + C)}{1 - (I + R + C)} \leq \xi^+I - \frac{\eta I}{1 - (I + R + C)}, \end{aligned}$$



$$\mathcal{L}V_5 = \theta_1 \xi (\bar{\xi} - \xi) + \theta_2 \beta (\bar{\beta} - \beta) + \frac{\sigma_1^2 + \sigma_2^2}{2}.$$

Define a  $C^2$ -function  $W: \mathbb{R}_+^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$\begin{aligned} W(I, R, C, \xi, \beta) &= M V_1 - \ln R - \ln C - \ln(1 - I - R - C) + \frac{\xi^2}{2} + \frac{\beta^2}{2} \\ &:= M V_1 + V_2 + V_3 + V_4 + V_5, \end{aligned}$$

where  $M > 0$  is a sufficiently large number such that the inequality (7) holds.

We give a definition of a non-negative  $C^2$ -function  $\bar{W}$  as follows

$$\bar{W}(I, R, C, \xi, \beta) = W(I, R, C, \xi, \beta) - W(\bar{X}_0),$$

where  $W(X)$  is a continuous function,  $\bar{X}_0$  is a minimum point  $(\bar{I}_0, \bar{R}_0, \bar{C}_0, \bar{\xi}_0, \bar{\beta}_0)$  in the interior domain of  $\Gamma$ . Using Itô's formula, we obtain

$$\begin{aligned} \mathcal{L}\bar{W} &\leq -M(R_0^h - 1) \left( \eta + \alpha + \frac{\sigma_1}{\sqrt{2\theta_1}} \right) + M(\bar{\xi} + a_1\alpha)I + 2\xi^+I + 2\eta + \gamma \\ &\quad - \frac{\alpha I}{R} - \frac{\gamma R}{C} + \beta^+ - \frac{\eta I}{1 - (I + R + C)} - \theta_1 \xi (\xi - \bar{\xi}) - \theta_2 \beta (\beta - \bar{\beta}) \\ &\quad + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} \\ &\leq -M(R_0^h - 1) \left( \eta + \alpha + \frac{\sigma_1}{\sqrt{2\theta_1}} \right) + M(\bar{\xi} + a_1\alpha)I + 2\eta + \gamma + 2|\xi| + |\beta| \\ &\quad - \frac{\alpha I}{R} - \frac{\gamma R}{C} \\ &\quad + \beta^+ - \frac{\eta I}{1 - (I + R + C)} - \theta_1 \xi^2 + \theta_1 |\xi| \bar{\xi} - \theta_2 \beta^2 + \theta_2 |\beta| \bar{\beta} + \frac{\sigma_1^2 + \sigma_2^2}{2} \\ &\leq -M(R_0^h - 1) \left( \eta + \alpha + \frac{\sigma_1}{\sqrt{2\theta_1}} \right) + M(\bar{\xi} + a_1\alpha)I + 2\eta + \gamma - \frac{\alpha I}{R} - \frac{\gamma R}{C} \\ &\quad - \frac{\eta I}{1 - (I + R + C)} + \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{(\theta_1 \bar{\xi} + 2)^2}{2\theta_1} \\ &\quad + \frac{(\theta_2 \bar{\beta} + 1)^2}{2\theta_2} - \frac{\theta_1 \xi^2}{2} - \frac{\theta_2 \beta^2}{2}. \end{aligned}$$

We choose a positive constant  $M$  that satisfies the following conditions

$$-M\mu_1 + \mu_2 \leq -2, \tag{7}$$

where

$$\mu_1 = (R_0^h - 1) \left( \eta + \alpha + \frac{\sigma_1}{\sqrt{2\theta_1}} \right),$$

$$\mu_2 = 2\eta + \gamma + \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{(\theta_1\bar{\xi} + 2)^2}{2\theta_1} + \frac{(\theta_2\bar{\beta} + 1)^2}{2\theta_2}.$$

Define  $\mu_3 = M(\bar{\xi} + a_1\alpha)$ . Next, in order for the conditions of Lemma 4.1 to hold, we construct a compact subset  $D_\varepsilon$ . The form of bounded closed set we establish is as follows

$$D_\varepsilon = \left( I, R, C, \xi, \beta \in \Gamma \mid I \geq \varepsilon, R \geq \varepsilon^2, C \geq \varepsilon^3, I + R + C \leq 1 - \varepsilon^2, |\xi| \leq \frac{1}{\varepsilon}, |\beta| \leq \frac{1}{\varepsilon} \right),$$

where  $\varepsilon$  is a small enough constant to satisfy the following conditions

$$-2 + \mu_3\varepsilon \leq -1, \tag{8}$$

$$-2 + \mu_3 - \frac{\alpha}{\varepsilon} \leq -1, \tag{9}$$

$$-2 + \mu_3 - \frac{\gamma}{\varepsilon} \leq -1, \tag{10}$$

$$-2 + \mu_3 - \frac{\eta}{\varepsilon} \leq -1, \tag{11}$$

$$-2 + \mu_3 - \frac{\theta_1}{2\varepsilon^2} \leq -1, \tag{12}$$

$$-2 + \mu_3 - \frac{\theta_2}{2\varepsilon^2} \leq -1. \tag{13}$$

For the sake of clarity, we split  $(\Gamma) \setminus D_\varepsilon$  into the following six domains,

$$\begin{aligned} D_1^c &= \{(I, R, C, \xi, \beta) \in \Gamma \mid I < \varepsilon\}, \\ D_2^c &= \{(I, R, C, \xi, \beta) \in \Gamma \mid I \geq \varepsilon, R < \varepsilon^2\}, \\ D_3^c &= \{(I, R, C, \xi, \beta) \in \Gamma \mid R \geq \varepsilon^2, C < \varepsilon^3\}, \\ D_4^c &= \{(I, R, C, \xi, \beta) \in \Gamma \mid I \geq \varepsilon, I + R + C > 1 - \varepsilon^2\}, \\ D_5^c &= \left\{ (I, R, C, \xi, \beta) \in \Gamma \mid |\xi| > \frac{1}{\varepsilon} \right\}, \\ D_6^c &= \left\{ (I, R, C, \xi, \beta) \in \Gamma \mid |\beta| > \frac{1}{\varepsilon} \right\}. \end{aligned}$$

Obviously,  $D_\varepsilon^c = \bigcup_{i=1}^6 D_i^c$ , Case 1: if  $(I, R, C, \xi, \beta) \in D_1^c$ , according to (8), we get

$$\mathcal{L}\bar{W} \leq -2 + \mu_3\varepsilon \leq -1.$$

Case 2: if  $(I, R, C, \xi, \beta) \in D_2^c$ , according to (9), we get

$$\mathcal{L}\bar{W} \leq -2 + \mu_3 - \frac{\alpha}{\varepsilon} \leq -1.$$

Case 3: if  $(I, R, C, \xi, \beta) \in D_3^c$ , according to (10), we get

$$\mathcal{L}\bar{W} \leq -2 + \mu_3 - \frac{\gamma}{\varepsilon} \leq -1.$$

Case 4: if  $(I, R, C, \xi, \beta) \in D_4^c$ , according to (11), we get

$$\mathcal{L}\bar{W} \leq -2 + \mu_3 - \frac{\eta}{\varepsilon} \leq -1.$$

Case 5: if  $(I, R, C, \xi, \beta) \in D_5^c$ , according to (12), we get

$$\mathcal{L}\bar{W} \leq -2 + \mu_3 - \frac{\theta_1}{2\varepsilon^2} \leq -1.$$

Case 6: if  $(I, R, C, \xi, \beta) \in D_6^c$ , according to (13), we get

$$\mathcal{L}\bar{W} \leq -2 + \mu_3 - \frac{\theta_2}{2\varepsilon^2} \leq -1.$$

By the above proof, obviously, there admits a sufficiently small  $\varepsilon$ , such that

$$\mathcal{L}\bar{W}(I, R, C, \xi, \beta) \leq -1 \text{ for all } (I, R, C, \xi, \beta) \in D_\varepsilon^c.$$

So that proves the conditions in the Lemma 4.1. Thus, system (4) has a stationary distribution  $\pi(\cdot)$ . We have completed the proof of Theorem 4.2. □

### 5 Density Function of Stochastic Epidemic Model

In Sect. 4, we prove that the system (4) has at least one stationary distribution. Next, we analyze the local probability density function of the system (4).

Firstly, we give the following definition of the quasi-positive equilibrium  $E^*$  of system (4), where

$$E^* = (I^*, R^*, C^*, \bar{\xi}, \bar{\beta}) \in \Gamma,$$

the quasi-endemic equilibrium  $E^*$  is the same as the endemic equilibrium  $E_*$  in the ordinary differential equation if there is no fluctuations.

Next, we perform an equilibrium offset transformation, let  $y_1 = I - I^*$ ,  $y_2 = R - R^*$ ,  $y_3 = C - C^*$ ,  $y_4 = \xi - \bar{\xi}$ ,  $y_5 = \beta - \bar{\beta}$ , we get the linearized system (14) of system (4) [35, 36],

$$\begin{cases} dy_1 = -a_{11}y_1 - a_{12}y_2 - a_{13}y_3 + a_{14}y_4, \\ dy_2 = a_{21}y_1 - a_{22}y_2 + a_{23}y_3 + a_{24}y_4, \\ dy_3 = -a_{31}y_1 + a_{32}y_2 - a_{33}y_3 - a_{34}y_4 - a_{35}y_5, \\ dy_4 = -a_{44}y_4 + \sigma_1 dB_1(t), \\ dy_5 = -a_{55}y_5 + \sigma_2 dB_2(t), \end{cases} \tag{14}$$

where

$$\begin{aligned}
 a_{11} &= \bar{\xi}I^*, & a_{13} &= (1 - \sigma)\bar{\xi}I^*, & a_{14} &= (1 - I^* - R^* - C^*)I^* + \sigma C^*I^*, \\
 a_{21} &= \alpha + (1 - \sigma)\bar{\xi}C^*, & a_{22} &= \eta + \gamma, & a_{24} &= (1 - \sigma)I^*C^*, & a_{31} &= \bar{\xi}C^*, \\
 a_{32} &= \gamma, & a_{33} &= \bar{\xi}I^* + (\eta + \bar{\beta}), & a_{34} &= C^*I^*, & a_{35} &= C^*, & a_{44} &= \theta_1, & a_{55} &= \theta_2.
 \end{aligned}$$

According to the definition of the quasi-endemic equilibrium point  $E^*$ , we can conclude that the values of all the above parameters are positive.

**Theorem 5.1** *For any initial value  $I(0), R(0), C(0), \xi(0), \beta(0) \in \Gamma$ , when*

$$\begin{aligned}
 &(1 - \sigma) \left[ \xi^2 C^* I^* + \gamma \xi I^* + \xi C^* (\eta + \gamma) - [(1 - \sigma)\xi C^* + \alpha] \gamma \right] \\
 &< [\eta + \gamma + \alpha + (1 - \sigma)\xi C^*] (\xi I^* + \eta + \beta), \tag{15}
 \end{aligned}$$

$$\begin{aligned}
 &(1 - \sigma)\xi I^* \\
 &< \frac{(\xi I^* + \eta + \gamma) \left( (\xi I^* + \eta + \beta)(2\xi I^* + 2\eta + \gamma + \beta) + \xi I^* (\eta + \gamma + \alpha + (1 - \sigma)\xi C^*) \right)}{(\eta + \gamma)\gamma + \xi C^* (\xi I^* + \eta + \beta) + (\xi I^* + \eta + \beta)\gamma + ((1 - \sigma)\xi C^* + \alpha)\gamma}, \tag{16}
 \end{aligned}$$

$$(1 - \sigma)\xi C^* > \frac{(\eta + \gamma)(1 - \sigma)I^*C^*}{(1 - I^* - R^* - C^*)I^* + \sigma C^*I^*} - \alpha, \tag{17}$$

system (4) has a probability density  $\Psi(y_1, y_2, y_3, y_4, y_5)$  around the quasi-endemic equilibrium  $E^*$ .  $\Sigma$  is a positive definite matrix, and the special form of  $\Sigma$  is given as follows.

Case 1, if  $w_4 \neq 0$ , we get

$$\begin{aligned}
 \Sigma &= \rho_1^2 (M_1 J_4 J_3 J_2 J_1)^{-1} \Sigma_1 [(M_1 J_4 J_3 J_2 J_1)^{-1}]^T \\
 &\quad + \rho_2^2 (M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^{-1} \Sigma_2 [(M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^{-1}]^T.
 \end{aligned}$$

Case 2, if  $w_4 = 0$ , we get

$$\begin{aligned}
 \Sigma &= \rho_{1w}^2 (M_2 J_4 J_3 J_2 J_1)^{-1} \tilde{\Sigma}_1 [(M_2 J_4 J_3 J_2 J_1)^{-1}]^T \\
 &\quad + \rho_2^2 (M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^{-1} \Sigma_2 [(M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^{-1}]^T,
 \end{aligned}$$

where

$$\begin{aligned}
 w_1 &= a_{21} + \frac{a_{11}a_{24}}{a_{14}} + (-a_{22} + \frac{a_{11}a_{24}}{a_{14}}) \frac{a_{24}}{a_{14}}, & w_2 &= -a_{31} + \frac{a_{32}a_{24}}{a_{14}} - \frac{a_{34}a_{11}}{a_{24}} + \frac{a_{33}a_{34}}{a_{14}}, \\
 w_3 &= a_{21} - \frac{a_{22}a_{24}}{a_{14}}, & w_4 &= a_{32} - \frac{a_{11}a_{34}}{a_{14}} - \frac{w_2}{w_3} \left( -a_{22} + \frac{a_{11}a_{24}}{a_{14}} \right) \\
 &\quad - \frac{w_2}{w_3} \left( a_{33} + \frac{a_{13}a_{34}}{a_{14}} + \frac{w_2}{w_3} (a_{13} + \frac{a_{13}a_{24}}{a_{14}}) \right)
 \end{aligned}$$

$$\rho_1 = w_3 w_4 a_{14} \sigma_1, \quad \rho_{1w} = w_3 a_{14} \sigma_1, \quad \rho_2 = -a_{35} a_{13} g_3 \sigma_2,$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{a_{24}}{a_{14}} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{a_{34}}{a_{14}} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{w_2}{w_3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tilde{J}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{J}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} w_3 w_4 a_{14} & w_3 w_4 (f_1 + f_3 + f_5) & e_1 & e_2 & e_3 \\ 0 & w_3 w_4 & w_4 (f_3 + f_5) & w_4 f_4 + f_5^2 & -f_5 a_{35} + a_{35} a_{55} \\ 0 & 0 & w_4 & f_5 & -a_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$e_1 = w_3 w_4 f_2 + w_4 f_3 (f_3 + f_5) + w_4 (w_4 f_4 + f_5^2),$$

$$e_2 = -w_3 w_4 a_{13} + w_4 f_4 (f_3 + f_5) + f_5 (w_4 f_4 + f_5^2),$$

$$e_3 = -a_{35} (w_4 f_4 + f_5^2) + a_{35} a_{55} f_5 - a_{35} a_{55}^2,$$

$$f_1 = -a_{11} - \frac{a_{11} a_{24}}{a_{14}} + \frac{a_{13} a_{34}}{a_{14}}, \quad f_2 = -a_{11} - \frac{a_{13} w_2}{w_3},$$

$$f_3 = -a_{22} + \frac{a_{11} a_{24}}{a_{14}} + \frac{w_2}{w_3} \left( a_{13} + \frac{a_{13} a_{24}}{a_{14}} \right), \quad f_4 = a_{13} + \frac{a_{13} a_{24}}{a_{14}},$$

$$f_5 = -a_{33} - \frac{a_{13} a_{34}}{a_{14}} - \frac{w_2}{w_3} \left( a_{13} + \frac{a_{13} a_{24}}{a_{14}} \right),$$

$$M_2 = \begin{pmatrix} a_{14} w_3 & w_3 (f_1 + f_3) & w_3 f_2 + f_3^2 & -a_{13} w_3 + f_4 (f_3 + f_5) & -a_{35} f_4 \\ 0 & w_3 & f_3 & f_4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -a_{35} a_{13} g_3 & g_3 a_{13} (-a_{33} + g_2 + g_4) & h_1 & h_2 & h_3 \\ 0 & g_3 a_{13} & g_3 (g_2 + g_4) & g_3 a_{21} + g_4^2 & g_3 a_{24} + g_4 g_5 - a_{44} g_5 \\ 0 & 0 & g_3 & g_4 & g_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned}
 g_1 &= a_{32} + a_{31}, & g_2 &= -a_{22} - a_{21}, & g_3 &= -a_{22} - a_{21}, \\
 g_4 &= -a_{11} + a_{21}, & g_5 &= -a_{34} + a_{24}. \\
 h_1 &= g_1 a_{13} g_3 + g_2 g_3 (g_2 + g_4) + g_3 (g_3 a_{21} + g_4^2), \\
 h_2 &= -g_3 a_{13} a_{31} + g_3 a_{21} (g_2 + g_4) + g_4 (g_3 a_{21} + g_4^2), \\
 h_3 &= -g_3 a_{13} a_{34} + g_3 a_{24} (g_2 + g_4) + g_5 (g_3 a_{21} + g_4^2) - a_{44} (g_3 a_{24} + g_4 g_5 - g_5 a_{44}), \\
 \Sigma_1 &= \begin{pmatrix} \frac{a_1 a_4 - a_2 a_3}{\Lambda_1} & 0 & \frac{a_3}{\Lambda_1} & 0 & 0 \\ 0 & -\frac{a_3}{\Lambda_1} & 0 & \frac{a_1}{\Lambda_1} & 0 \\ \frac{a_3}{\Lambda_1} & 0 & -\frac{a_1}{\Lambda_1} & 0 & 0 \\ 0 & \frac{a_1}{\Lambda_1} & 0 & \frac{a_3 - a_1 a_2}{\Lambda_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

and

$$\begin{aligned}
 \Lambda_1 &= 2(a_4 a_1^2 - a_2 a_1 a_3 + a_3^2), \\
 \tilde{\Sigma}_1 &= \begin{pmatrix} -\frac{b_2}{2(b_3 - b_1 b_2)} & 0 & \frac{1}{2(b_3 - b_1 b_2)} & 0 & 0 \\ 0 & -\frac{1}{2(b_3 - b_1 b_2)} & 0 & 0 & 0 \\ \frac{1}{2(b_3 - b_1 b_2)} & 0 & -\frac{b_1}{2(b_3 - b_1 b_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 \Sigma_2 &= \begin{pmatrix} \frac{c_1 c_4 - c_2 c_3}{\Lambda_2} & 0 & \frac{c_3}{\Lambda_2} & 0 & 0 \\ 0 & -\frac{c_3}{\Lambda_2} & 0 & \frac{c_1}{\Lambda_2} & 0 \\ \frac{c_3}{\Lambda_2} & 0 & -\frac{c_1}{\Lambda_2} & 0 & 0 \\ 0 & \frac{c_1}{\Lambda_2} & 0 & \frac{c_3 - c_1 c_2}{c_4 \Lambda_2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

among them

$$\Lambda_2 = 2(c_4 c_1^2 - c_2 c_1 c_3 + c_3^2),$$

$a_1, a_2, a_3, a_4, c_1, c_2, c_3, c_4, b_1, b_2$  and  $b_3$  will be described below.

**Proof** For convenience, we define the matrix as follows

$$A = \begin{pmatrix} -a_{11} & -a_{11} & -a_{13} & a_{14} & 0 \\ a_{21} & -a_{22} & a_{13} & a_{24} & 0 \\ -a_{31} & a_{32} & -a_{33} & -a_{34} & -a_{35} \\ 0 & 0 & 0 & -a_{44} & 0 \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix}.$$

$G = \text{diag}(0, 0, 0, \sigma_1, \sigma_2)$ ,  $B(t) = (0, 0, 0, B_1(t), B_2(t))$ , then the linearized system (14) can be simplified to

$$dY = AYdt + GdB(t).$$

By the relevant theory of Gardiner and the continuous Markov processes theory [37], the probability density function can be given by the following Fokker–Planck equation [38],

$$\begin{aligned} & -\frac{\sigma_1^2}{2} \frac{\partial^2 \Psi}{\partial y_4^2} - \frac{\sigma_2^2}{2} \frac{\partial^2 \Psi}{\partial y_5^2} + \frac{\partial}{\partial y_1} [(-a_{11}y_1 - a_{11}y_2 - a_{13}y_3 + a_{14}y_4)\Psi] \\ & + \frac{\partial}{\partial y_2} [(a_{21}y_1 - a_{22}y_2 + a_{13}y_3 + a_{24}y_4)\Psi] \\ & + \frac{\partial}{\partial y_3} [(-a_{31}y_1 + a_{32}y_2 - a_{33}y_3 - a_{34}y_4 - a_{35}y_5)\Psi] \\ & + \frac{\partial}{\partial y_4} (-a_{44}y_4\Psi) + \frac{\partial}{\partial y_5} (-a_{55}y_5\Psi) = 0 \end{aligned}$$

and it can be approximated by a Gaussian distribution

$$\Psi(y_1, y_2, y_3, y_4, y_5) = \phi_0 e^{-\frac{1}{2}(y_1, y_2, y_3, y_4, y_5)Q(y_1, y_2, y_3, y_4, y_5)^T},$$

where  $\phi_0$  is a positive constant, which is obtained by the normalized condition  $\int_{R^5} \Psi(Y)dy_1dy_2dy_3dy_4dy_5 = 1$ , and  $Q$  is a real symmetric matrix satisfying the following equation

$$QG^2Q + A^TQ + QA = 0.$$

If  $Q$  is still an inverse matrix, let  $Q^{-1} = \Sigma$ , the above equation can be rewritten as

$$G^2 + A\Sigma + \Sigma A^T = 0.$$

According to the finite independent superposition principle in [39], the above equation is equivalent to the sum of the following two equations

$$G_i^2 + A\Sigma_i + \Sigma_i A^T = 0 \quad (i = a, b),$$

where  $G_a = (0, 0, 0, \sigma_1, 0)$ ,  $G_b = (0, 0, 0, 0, \sigma_2)$ ,  $\Sigma = \Sigma_a + \Sigma_b$  and  $G^2 = G_a^2 + G_b^2$ . Before proving the positive definiteness of  $\Sigma$ , we firstly verify that all the eigenvalues of  $A$  have negative real parts. The corresponding characteristic polynomial of  $A$  is defined by

$$\varphi_A(\lambda) = (\lambda + a_{44})(\lambda + a_{55})(\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3),$$

where  $q_1$ ,  $q_2$  and  $q_3$  has been defined as follows:

$$\begin{aligned}q_1 &= a_{11} + a_{22} + a_{33}, \\q_2 &= a_{11}a_{21} + a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}(a_{31} + a_{32}), \\q_3 &= a_{11}a_{22}a_{33} + a_{11}a_{21}a_{33} + a_{13}(a_{21}a_{32} - a_{22}a_{31} - a_{11}a_{32} - a_{11}a_{31}),\end{aligned}$$

By simple calculation, we can have  $q_1 > 0$  and  $q_2 > a_{11}a_{22} + a_{22}a_{33} > 0$ . According to (15), then we can simplify  $q_3$  to get  $q_3 > 0$ , and similarly, from (16) we can simplify to get  $q_1q_2 > q_3$ . Then we have that matrix A has all negative real-part eigenvalues.

Next, we prove the positive characterization of  $\Sigma$  in two steps. **Step1** Consider the algebraic equation

$$G_a^2 + A\Sigma_a + \Sigma_aA^T = 0. \quad (18)$$

By letting  $A_1 = J_1AJ_1^{-1}$ , we have

$$A_1 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 & 0 \\ a_{14} & -a_{11} & -a_{11} & -a_{13} & 0 \\ a_{24} & a_{21} & -a_{22} & a_{13} & 0 \\ -a_{34} & -a_{31} & a_{32} & -a_{33} & -a_{35} \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix}.$$

Next, let  $A_2 = J_4J_3J_2A_1(J_4J_3J_2)^{-1}$ , we get

$$A_2 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 & 0 \\ a_{14} & f_1 & f_2 & -a_{13} & 0 \\ 0 & w_3 & f_3 & f_4 & 0 \\ 0 & 0 & w_4 & f_5 & -a_{35} \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix},$$

where according to (17), we have  $w_3 = a_{21} - \frac{a_{22}a_{24}}{a_{14}} > 0$ .

Dependent on the value of  $w_4$ , so we're going to have two cases.

**Case(I)** If  $w_4 \neq 0$ , based on the method given in the references [36], let  $A_3 = M_1A_2M_1^{-1}$ , where the standardized transformation matrix

$$M_1 = \begin{pmatrix} w_3w_4a_{14} & w_3w_4(f_1 + f_3 + f_5) & e_1 & e_2 & e_3 \\ 0 & w_3w_4 & w_4(f_3 + f_5) & w_4f_4 + f_5^2 & -f_5a_{35} + a_{35}a_{55} \\ 0 & 0 & w_4 & f_5 & -a_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$



then we have

$$A_3 = \begin{pmatrix} -a_1 & -a_2 & -a_3 & -a_4 & -a_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix},$$

where

$$\begin{aligned} a_1 &= q_1 + a_{44} > 0, & a_2 &= q_2 + q_1 a_{44} > 0, \\ a_3 &= q_3 + q_2 a_{44} > 0, & a_4 &= q_3 a_{44} > 0, \end{aligned}$$

So with conditions (15) and (16), we verify that

$$a_1(a_2 a_3 - a_1 a_4) - a_3^2 > 0.$$

Since the value of  $a_5$  has no influence on the results of this paper, we omit the value of  $a_5$  here. Equation (18) can be equivalently transformed into

$$\begin{aligned} (M_1 J_4 J_3 J_2 J_1) G_a^2 (M_1 J_4 J_3 J_2 J_1)^T + A_3 [(M_1 J_4 J_3 J_2 J_1) \Sigma_a (M_1 J_4 J_3 J_2 J_1)^T] \\ + [(M_1 J_4 J_3 J_2 J_1) \Sigma_a (M_1 J_4 J_3 J_2 J_1)^T] A_3^T = 0. \end{aligned}$$

Let  $\Sigma_1 = \rho_1^{-2} (M_1 J_4 J_3 J_2 J_1) \Sigma_a (M_1 J_4 J_3 J_2 J_1)^T$ , where  $\rho_1 = w_3 w_4 a_{14} \sigma_1$ , it can be simplified as

$$G_0^2 + A_3 \Sigma_1 + \Sigma_1 A_3^T = 0.$$

By solving the equation above, we get

$$\Sigma_1 = \begin{pmatrix} \frac{a_1 a_4 - a_2 a_3}{\Lambda_1} & 0 & \frac{a_3}{\Lambda_1} & 0 & 0 \\ 0 & -\frac{a_3}{\Lambda_1} & 0 & \frac{a_1}{\Lambda_1} & 0 \\ \frac{a_3}{\Lambda_1} & 0 & -\frac{a_1}{\Lambda_1} & 0 & 0 \\ 0 & \frac{a_1}{\Lambda_1} & 0 & \frac{a_3 - a_1 a_2}{a_4 \Lambda_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma_x & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\Lambda_1 = 2(a_4 a_1^2 - a_2 a_1 a_3 + a_3^2).$$

From the reference [40], we can conclude that  $\Sigma_1$  is a semi-positive definite matrix. In the same way, we also get that  $\Sigma_x$  is a positive definite matrix. Therefore,

$$\Sigma_{ax} = \rho_1^2 (M_1 J_4 J_3 J_2 J_1)^{-1} \Sigma_x [(M_1 J_4 J_3 J_2 J_1)^{-1}]^T.$$

It is also a positive definite matrix. Similarly,

$$\Sigma_a = \rho_1^2 (M_1 J_4 J_3 J_2 J_1)^{-1} \Sigma_1 [(M_1 J_4 J_3 J_2 J_1)^{-1}]^T.$$

It is also a semi-positive definite matrix.

Let  $n_1$  be the minimum eigenvalue of the matrix  $\Sigma_x$ , then

$$\Sigma_{ax} \geq n_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and so,

$$\Sigma_a \geq n_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Case(II)** If  $w_4 = 0$ , let  $A_{3w} = M_2 A_2 M_2^{-1}$ , where the standardized transformation matrix is

$$M_2 = \begin{pmatrix} a_{14}w_3 & w_3(f_1 + f_3) & w_3f_2 + f_3^2 & -a_{13}w_3 + f_4(f_3 + f_5) & -a_{35}f_4 \\ 0 & w_3 & f_3 & f_4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the same method, we can get

$$A_{3w} = \begin{pmatrix} -b_1 & -b_2 & -b_3 & -b_4 & -b_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_5 & -a_{35} \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix},$$

where

$$\begin{aligned} b_1 &= q_1 + a_{44} + f_5 > 0, & b_2 &= q_2 + q_1 a_{44} + f_5 b_1 > 0, \\ b_3 &= q_3 + q_2 a_{44} + f_5 b_2 > 0. \end{aligned}$$

By direct calculation, we have

$$b_1 b_2 - b_3 > 0.$$

Since the value of  $b_4$  and  $b_5$  has no influence on the results of this paper, we omit the value of  $b_4$  and  $b_5$  here. Meanwhile, equation (18) can be equivalently transformed into

$$(M_2 J_4 J_3 J_2 J_1) G_a^2 (M_2 J_4 J_3 J_2 J_1)^T + A_{3w} [(M_2 J_4 J_3 J_2 J_1) \Sigma_a (M_2 J_4 J_3 J_2 J_1)^T] + [(M_2 J_4 J_3 J_2 J_1) \Sigma_a (M_2 J_4 J_3 J_2 J_1)^T] A_{3w}^T = 0.$$

Let  $\tilde{\Sigma}_1 = \rho_{1w}^{-2} (M_2 J_4 J_3 J_2 J_1) \Sigma_a (M_2 J_4 J_3 J_2 J_1)^T$ , where  $\rho_{1w} = w_3 a_{14} \sigma_1$ , we similarly obtain

$$G_0^2 + A_{3w} \tilde{\Sigma}_1 + \tilde{\Sigma}_1 A_{3w}^T = 0.$$

By solving the equation above, we have

$$\tilde{\Sigma}_1 = \begin{pmatrix} -\frac{b_2}{2(b_3-b_1b_2)} & 0 & \frac{1}{2(b_3-b_1b_2)} & 0 & 0 \\ 0 & -\frac{1}{2(b_3-b_1b_2)} & 0 & 0 & 0 \\ \frac{1}{2(b_3-b_1b_2)} & 0 & -\frac{b_1}{2(b_3-b_1b_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the reference [40], we can conclude that  $\tilde{\Sigma}_1$  is a semi-positive definite matrix. Hence,

$$\Sigma_a = \rho_{1w}^2 (M_2 J_4 J_3 J_2 J_1)^{-1} \tilde{\Sigma}_1 [(M_2 J_4 J_3 J_2 J_1)^{-1}]^T.$$

It is also a semi-positive definite matrix. Therefore, we get that  $\Sigma_a$  is semi-positive definite and there is a positive constant  $n_2$  such that

$$\Sigma_a \geq n_2 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Step2** For the following algebraic equation

$$G_b^2 + A \Sigma_b + \Sigma_b A^T = 0. \tag{19}$$

We first let  $B_1 = \tilde{J}_1 A \tilde{J}_1^{-1}$ , and then define  $B_2 = (\tilde{J}_3 \tilde{J}_3) B_1 (\tilde{J}_3 \tilde{J}_3)^{-1}$ , where

$$\tilde{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a_{55} & 0 & 0 & 0 & 0 \\ 0 & -a_{11} & -a_{11} & -a_{13} & a_{14} \\ 0 & a_{21} & -a_{22} & a_{13} & a_{24} \\ -a_{35} & -a_{31} & a_{32} & -a_{33} & -a_{34} \\ 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix},$$

$$\tilde{J}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{J}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -a_{55} & 0 & 0 & 0 & 0 \\ -a_{35} & -a_{33} & g_1 & -a_{31} & -a_{34} \\ 0 & a_{13} & g_2 & a_{21} & a_{24} \\ 0 & 0 & g_3 & g_4 & g_5 \\ 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix}.$$

Let  $B_3 = M_3 B_3 M_3^{-1}$ , we construct the following standardized transformation matrix

$$M_3 = \begin{pmatrix} -a_{35}a_{13}g_3 & g_3a_{13}(-a_{33} + g_2 + g_4) & h_1 & h_2 & h_3 \\ 0 & g_3a_{13} & g_3(g_2 + g_4) & g_3a_{21} + g_4^2 & g_3a_{24} + g_4g_5 - a_{44}g_5 \\ 0 & 0 & g_3 & g_4 & g_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

According to the same method above, we can get directly

$$B_3 = \begin{pmatrix} -c_1 & -c_2 & -c_3 & -c_4 & -c_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix},$$

where

$$c_1 = q_1 + a_{55} > 0, \quad c_2 = q_2 + q_1a_{55} > 0, \\ c_3 = q_3 + q_2a_{55} > 0, \quad c_4 = q_3a_{55} > 0.$$

Similarly, we have

$$c_1(c_2c_3 - c_1c_4) - c_5^2 > 0.$$

Since the value of  $c_5$  has no influence on the results of this paper, we omit the value of  $c_5$  here. Similarly to the method in **Step1**, equation (19) can be equalized to

$$(M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1) G_b^2 (M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^T + B_3 [(M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1) \Sigma_b (M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^T] \\ + [(M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1) \Sigma_b (M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^T] B_3^T = 0.$$

Let  $\Sigma_2 = \rho_2^{-2} (M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1) \Sigma_b (M_3 \tilde{J}_3 \tilde{J}_2 \tilde{J}_1)^T$ , where  $\rho_2 = -a_{35}a_{13}g_3\sigma_2$ , we similarly obtain

$$G_0^2 + B_3 \Sigma_2 + \Sigma_2 B_3^T = 0.$$

After direct calculation of the above equation, we have

$$\Sigma_2 = \begin{pmatrix} \frac{c_1 c_4 - c_2 a_3}{\Lambda_2} & 0 & \frac{c_3}{\Lambda_2} & 0 & 0 \\ 0 & -\frac{c_3}{\Lambda_2} & 0 & \frac{c_1}{\Lambda_2} & 0 \\ \frac{c_3}{\Lambda_2} & 0 & -\frac{c_1}{\Lambda_2} & 0 & 0 \\ 0 & \frac{c_1}{\Lambda_2} & 0 & \frac{c_3 - c_1 c_2}{c_4 \Lambda_2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

According to reference [40] and its proof, we get  $\Sigma_2$  is a semi-positive definite matrix. It can be concluded from the above proof,  $\Sigma_b = \rho_2^2 [(M_3 \text{ title } J_3 \tilde{J}_2 \tilde{J}_1)^{-1} \Sigma_2 (M_3 \text{ title } J_3 \tilde{J}_2 \tilde{J}_1)^{-1}]^T$  is also a semi-positive definite matrix and there is a positive constant  $n_3$  such that

$$\Sigma_b \geq n_3 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now let's discuss the above situation in two cases. **Case(I)** If  $w_4 \neq 0$ , then the covariance matrix

$$\begin{aligned} \Sigma = \Sigma_a + \Sigma_b &\geq n_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + n_3 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &\geq (n_1 \wedge n_3) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

And then we can prove that  $\Sigma$  is positive definite.

**Case(II)** If  $w_4 = 0$ , similarly, then the covariance matrix

$$\Sigma = \Sigma_a + \Sigma_b \geq n_2 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + n_3 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\geq (n_2 \wedge n_3) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we can show that  $\Sigma$  is positive definite.

In summary, we have now shown that  $\Sigma$  is a positive definite matrix, and that the form of  $\Sigma = \Sigma_a + \Sigma_b$  is determined by the steps above. By the positive definiteness of  $\Sigma$ , we can compute that  $\phi_0 = (2\pi)^{-\frac{5}{2}} |\Sigma|^{-\frac{1}{2}}$ . System (4) has a probability density  $\Psi(y_1, y_2, y_3, y_4, y_5)$  around the quasi-endemic equilibrium  $E^*$ .

We have completed the proof of Theorem 5.1. □

### 6 Extinction of the Disease

Next, we'll talk about the extinction of diseases.

**Theorem 6.1** *For any initial value  $(I(0), R(0), C(0), \xi(0), \beta(0)) \in \Gamma$ , if  $R_0^d < 1$ , then the solution  $(I(t), R(t), C(t), \xi(t), \beta(t))$  of system (4) follows*

$$\lim_{t \rightarrow \infty} \sup \frac{\ln I(t)}{t} \leq (\eta + \alpha)(R_0^d - 1) < 0, \text{ a.s.}$$

where

$$R_0^d = \frac{\bar{\xi} \Phi\left(\frac{\bar{\xi}\sqrt{2\theta_1}}{\sigma_1}\right) + \frac{\sigma_1}{2\sqrt{\theta_1}\pi} e^{-\frac{\bar{\xi}^2\theta_1}{\sigma_1^2}}}{\eta + \alpha}.$$

**Proof** Employing Itô's formula to  $\ln I(t)$ , we get

$$\begin{aligned} d \ln I(t) &= [\xi^+(1 - I - R - C) + \sigma \xi^+ C - (\eta + \alpha)]dt \\ &= [\xi^+(1 - I - R) + (\sigma - 1)\xi^+ C - (\eta + \alpha)]dt \\ &\leq [\xi^+ - (\eta + \alpha)]dt. \end{aligned} \tag{20}$$

Let's integrate both sides of this inequality (20) from 0 to t, and divide by t, we get

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &\leq \frac{1}{t} \int_0^t [\xi^+(u) - (\eta + \alpha)]du \\ &= \frac{1}{t} \int_0^t \xi^+(u)du - (\eta + \alpha). \end{aligned} \tag{21}$$

Taking the superior limit of  $t \rightarrow \infty$  on both sides of the inequality (21), because of  $\xi^+ \sim N(\bar{\xi}, \frac{\sigma_1^2}{2\theta_1})$ . By the strong law of large numbers [41], we get

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \xi^+(u) du &= \int_0^\infty x \frac{\sqrt{\theta_1}}{\sqrt{\pi} \sigma_1} e^{-\frac{\theta_1 x^2}{\sigma_1^2}} dx \\ &= \frac{\sigma_1}{2\sqrt{\theta_1 \pi}} e^{-\frac{\bar{\xi}^2 \theta_1}{\sigma_1^2}} + \bar{\xi} \left[ \Phi(\infty) - \Phi\left(-\frac{\bar{\xi} \sqrt{2\theta_1}}{\sigma_1}\right) \right] \\ &= \bar{\xi} \Phi\left(\frac{\bar{\xi} \sqrt{2\theta_1}}{\sigma_1}\right) + \frac{\sigma_1}{2\sqrt{\theta_1 \pi}} e^{-\frac{\bar{\xi}^2 \theta_1}{\sigma_1^2}}, \end{aligned}$$

where  $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$ , according to  $\lim_{t \rightarrow \infty} \frac{\ln I(t)}{t} = 0$ , then we get

$$\lim_{t \rightarrow \infty} \sup \frac{\ln I(t)}{t} \leq (\eta + \alpha)(R_0^d - 1),$$

where

$$R_0^d = \frac{\bar{\xi} \Phi\left(\frac{\bar{\xi} \sqrt{2\theta_1}}{\sigma_1}\right) + \frac{\sigma_1}{2\sqrt{\theta_1 \pi}} e^{-\frac{\bar{\xi}^2 \theta_1}{\sigma_1^2}}}{\eta + \alpha}.$$

If  $R_0^d < 1$ , then  $\lim_{t \rightarrow \infty} \sup \frac{\ln I(t)}{t} < 0$ , which means that the epidemic of system (4) will go to extinction with probability 1.

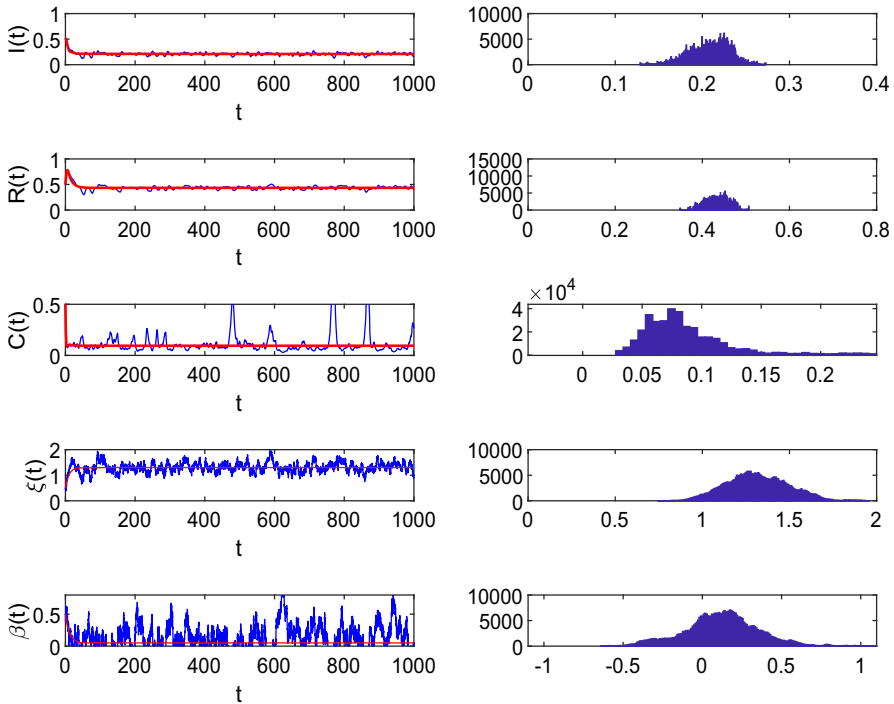
Thus the Theorem 6.1 is proved. □

### 7 Numerical Simulations

In order to verify the theoretical results in the paper, we conduct numerical simulation on it in this section. Making use of the common higher-order method developed by Milstein’s Higher Order Method mentioned in [42], we have the discretization equation of system (4),

$$\begin{cases} I^{i+1} = I^i + [\xi^i(1 - I^i - R^i - C^i)I^i + \sigma \xi^i C^i I^i - (\eta + \alpha)I^i] \Delta t, \\ R^{i+1} = R^i + [(1 - \sigma)\xi^i C^i I^i + \alpha I^i - (\eta + \gamma)R^i] \Delta t, \\ C^{i+1} = C^i + [\gamma R^i - \xi^i C^i I^i - (\eta + \beta^+)C^i] \Delta t, \\ \xi^{i+1} = \xi^i + \theta_1(\bar{\xi} - \xi^i)\Delta t + \sigma_1 \zeta_i \sqrt{\Delta t} + \frac{\sigma_1^2}{2}(\zeta_i^2 - 1)\Delta t, \\ \beta^{i+1} = \beta^i + \theta_2(\bar{\beta} - \beta^i)\Delta t + \sigma_2 \iota_i \sqrt{\Delta t} + \frac{\sigma_2^2}{2}(\iota_i^2 - 1)\Delta t, \end{cases} \tag{22}$$

where  $\Delta t > 0$  denotes the time increment,  $\zeta_i$  and  $\iota_i$  are the Gaussian random variables which follows the distribution  $N(0, 1)$ .



**Fig. 1** When  $R_0^h = 1.9299 > 1$ , simulations of the solution in the stationary distribution of stochastic system (4). On the left, the blue and red lines represent simulations of the stochastic system (4) and the deterministic system (2), respectively, where the stochastic system is perturbed by the Ornstein–Uhlenbeck process. The picture on the right means the density of the stochastic system (4) (color figure online)

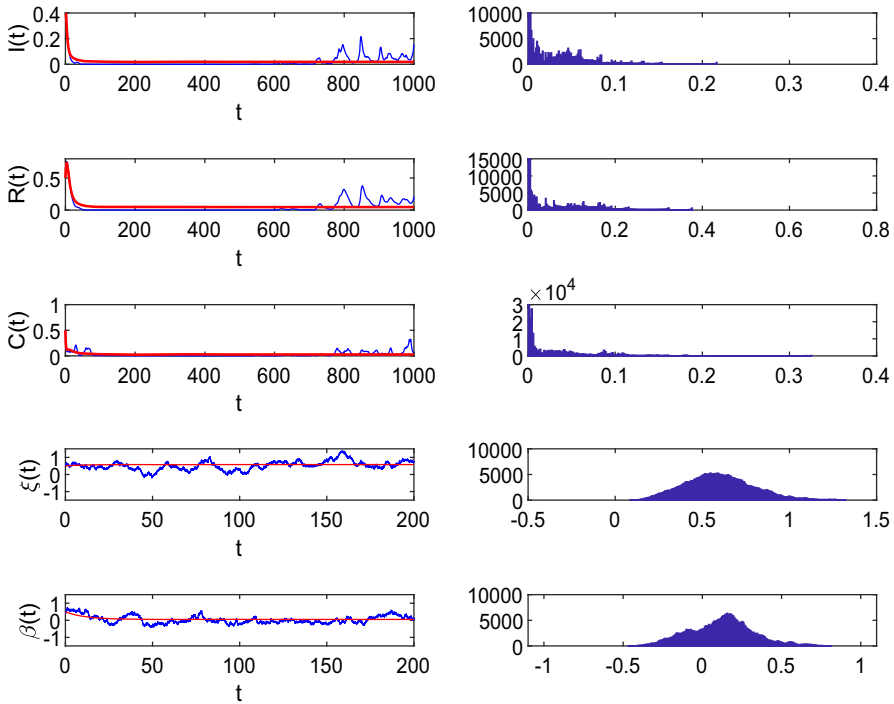
**Case(1)** Consider the parameters  $\theta_1 = 0.1, \theta_2 = 0.09, \sigma_1 = 0.1, \sigma_2 = 0.1, \eta = 0.09, \beta = 0.05, \sigma = 0.9, \gamma = 0.09, \alpha = 0.36, \xi = 1.3$ . According to the Theorem 4.2, system (4) has a stationary distribution which can be proved by numerical simulation as shown in Fig. 1.

**Case(2)** Consider the parameters  $\theta_1 = 0.1, \theta_2 = 0.09, \sigma_1 = 0.1, \sigma_2 = 0.1, \eta = 0.09, \beta = 0.05, \sigma = 0.9, \gamma = 0.09, \alpha = 0.45, \xi = 0.58$ . By the Theorem 5.1, it verifies the conclusion that the disease will be extinct.

To sum up, we verified the conclusion through numerical simulation in Figs. 1, 2 and 3. In Fig. 1, the blue and red lines represent the simulations for the stochastic system (4) and the deterministic system (2), respectively. The picture on the right shows the density of the stochastic system (4). In Fig. 2, when  $R_0^d < 1$ , system (4) will be extinct with the data of **Case(2)**. From the right of the Fig. 3, it displays the curve tend to zero point. It also means that the system (4) will be extinct.

In the process of numerical simulation, the change of the important parameters  $\bar{\xi}, \alpha, \eta$  have a great influence on the change of the curve of the system (4). We apply the method of control variables, by changing the different values of an important parameter, with the change of the curve to show the effect of simulation, examples can be found in Figs. 4 and 5.





**Fig. 2** When  $R_0^d = 0.5830 < 1$ , simulations of the system (4) will be extinct. On the left, The blue and red lines represent simulations of the stochastic system (4) and the deterministic system (2), respectively, where the stochastic system is perturbed by the Ornstein–Uhlenbeck process. The picture on the right means the density of the stochastic system (4) (color figure online)

**Case(3)** Consider the parameters  $\theta_1 = 0.1, \theta_2 = 0.09, \sigma_1 = 0.1, \sigma_2 = 0.1, \eta = 0.09, \beta = 0.05, \sigma = 0.9, \gamma = 0.09, \alpha = 0.36$ . And

$$R_0^h = 2.9691, \quad \text{when } \bar{\xi} = 2.0.$$

$$R_0^h = 2.5237, \quad \text{when } \bar{\xi} = 1.7.$$

$$R_0^h = 1.9299, \quad \text{when } \bar{\xi} = 1.3.$$

$$R_0^d = 0.7442, \quad \text{when } \bar{\xi} = 0.3.$$

Which implies that for  $\bar{\xi} = 2.0, \bar{\xi} = 1.7, \bar{\xi} = 1.3$ , the diseases will persist, and then when  $\bar{\xi} = 0.3$ , the diseases will be extinct. In Fig. 4, we can clearly observe the image changes caused by parameters change.

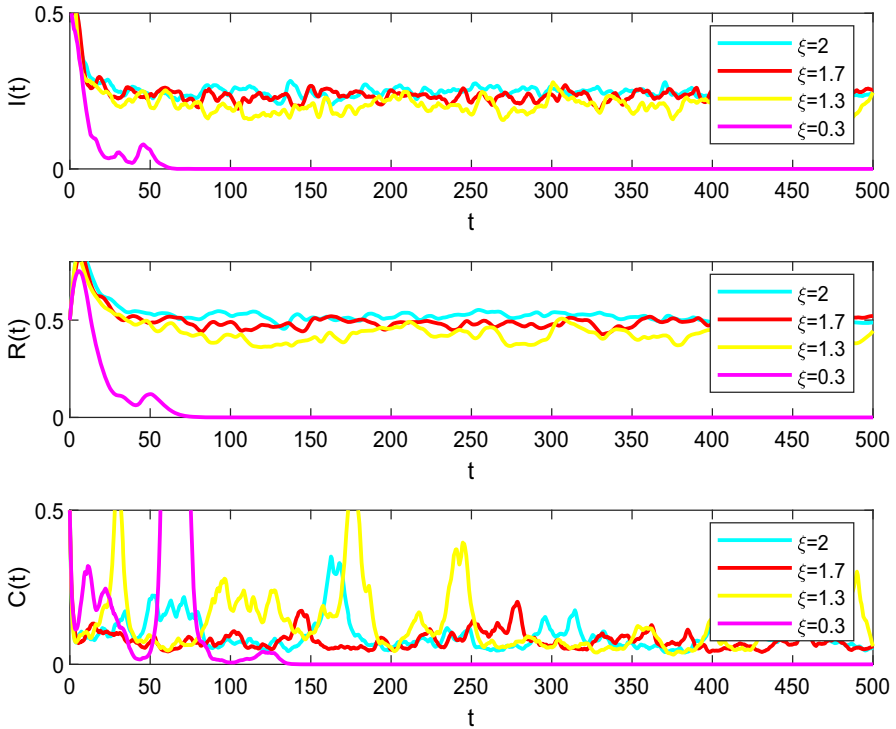
**Case(4)** Consider the parameters  $\theta_1 = 0.1, \theta_2 = 0.09, \sigma_1 = 0.1, \sigma_2 = 0.1, \eta = 0.09, \beta = 0.05, \sigma = 0.9, \gamma = 0.09, \bar{\xi} = 1.3$ . And

$$R_0^d = 0.8457, \quad \text{when } \alpha = 1.44.$$

$$R_0^h = 1.2577, \quad \text{when } \alpha = 0.72.$$

$$R_0^h = 1.9299, \quad \text{when } \alpha = 0.36.$$

$$R_0^h = 2.6337, \quad \text{when } \alpha = 0.18.$$



**Fig. 3** Simulations of the solution  $I(t)$ ,  $R(t)$ , and  $C(t)$  in system (4) under the different value of  $\bar{\xi}$

Similarly, which implies that for  $\alpha = 0.72$ ,  $\alpha = 0.36$ ,  $\alpha = 0.18$ , the diseases will persist, and then when  $\alpha = 1.44$ , the diseases will be extinct. The variation of these parameters can be clearly observed in Fig. 5.

**Case(5)** Consider the parameters  $\theta_1 = 0.1$ ,  $\theta_2 = 0.09$ ,  $\sigma_1 = 0.1$ ,  $\sigma_2 = 0.1$ ,  $\alpha = 0.36$ ,  $\beta = 0.05$ ,  $\sigma = 0.9$ ,  $\gamma = 0.09$ ,  $\bar{\xi} = 1.3$ . And

$$R_0^d = 0.9028, \quad \text{when } \eta = 1.08.$$

$$R_0^h = 1.5229, \quad \text{when } \eta = 0.27.$$

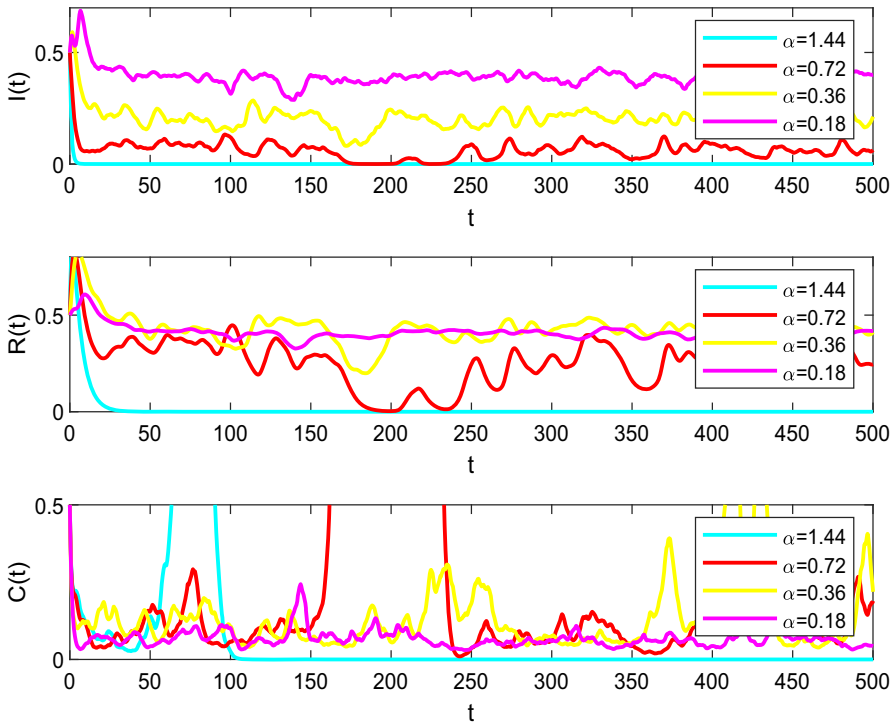
$$R_0^h = 1.9299, \quad \text{when } \eta = 0.09.$$

$$R_0^h = 2.0681, \quad \text{when } \eta = 0.045.$$

Similarly, which implies that for  $\eta = 0.27$ ,  $\eta = 0.09$ ,  $\eta = 0.045$ , the diseases will persist, and then when  $\eta = 1.08$ , the diseases will be extinct. The effect of different changes in the value of  $\eta$  is clearly reflected in Fig. 5.

## 8 Conclusions

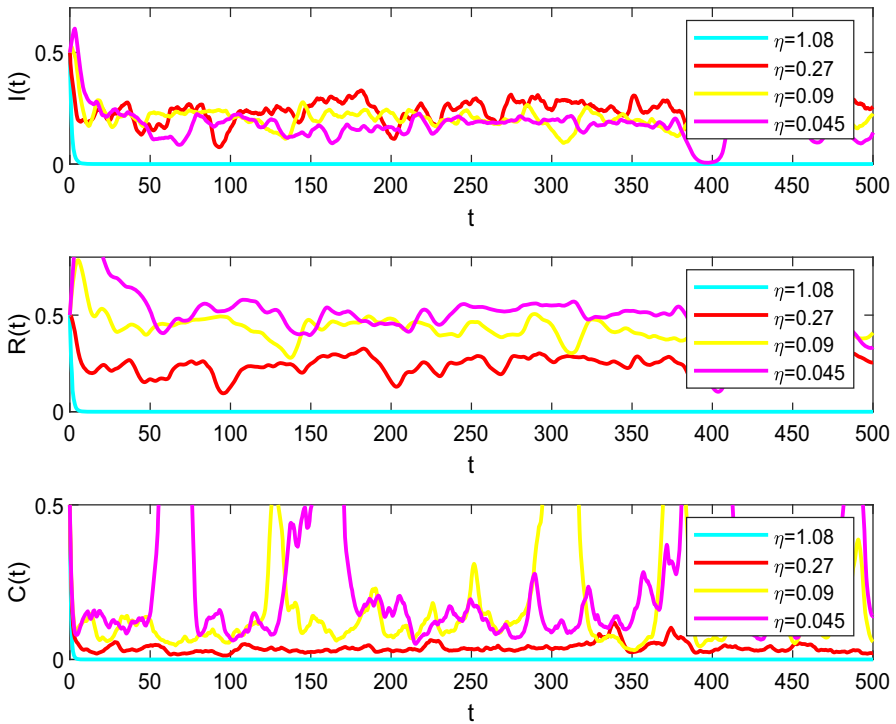
In this article, we analyze a stochastic SIRC model with Ornstein–Uhlenbeck process. We first give the theoretical result that the stochastic SIRC system (4) has a unique



**Fig. 4** Simulations of the solution  $I(t)$ ,  $R(t)$ , and  $C(t)$  in system (4) under the different value of  $\alpha$

global positive solution and prove it. Then, the theoretical results on the existence of ergodic stationary distributions for random SIRC systems (4) are given. We prove the theoretical results by constructing a series of suitable Lyapunov functions. It reveals the influence of Ornstein–Uhlenbeck process on the existence of stationary distribution. In addition, the theoretical results of probability density function are obtained and proved. In addition, we derive the sufficient conditions for the extinction of the disease, providing a theoretical basis for the prevention and control of infectious diseases. Finally, we use numerical simulation to simulate the theoretical results in order to verify the theoretical results in the paper.

However, there are still many questions about this article to be resolved. In this paper, we only get the local probability density expression of stochastic systems, and the global probability density expression is still a problem to be considered. In addition, whether other types of random disturbances are more suitable for solving the problem also needs further verification.



**Fig. 5** Simulations of the solution  $I(t)$ ,  $R(t)$ , and  $C(t)$  in system (4) under the different value of  $\eta$

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**Data availability** The data that support the findings of this study are available from the corresponding author upon reasonable request.

## Declarations

**Conflict of interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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