



Uncountable Family of 0-Rigid Continua that are Homeomorphic to Their Inverse Limits

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Abstract

It is a well-known fact that there are continua X such that the inverse limit of any inverse sequence $\{X, f_n\}$ with surjective continuous bonding functions f_n is homeomorphic to X . The pseudoarc or any Cook continuum are examples of such continua. Recently, a large family of continua X was constructed in such a way that X is $\frac{1}{m}$ -rigid and the inverse limit of any inverse sequence $\{X, f_n\}$ with surjective continuous bonding functions f_n is homeomorphic to X by Banič and Kac. In this paper, we construct an uncountable family of pairwise non-homeomorphic continua X such that X is 0-rigid and prove that for any sequence (f_n) of continuous surjections on X , the inverse limit $\varprojlim \{X, f_n\}$ is homeomorphic to X .

Keywords Continua · Cook continua · Rigid continua · Degree of rigidity · Stars of continua · Simple fan of Cook continua · Inverse limits

Mathematics Subject Classification 54H25 · 37C25 · 37B45 · 54F15 · 54F65

1 Introduction

The rigidity of topological spaces has been studied in various ways by different authors, e.g., [1–8, 10–12]. Cook continua are basic examples of non-degenerate continua that

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are rigid, and therefore, they play an important role in the study of rigid continua, also in continuum theory and dynamical systems in general. There are many rigid continua or continua that are not 0-rigid that are constructed using Cook continua, see [2, 6, 10], where more references may be found. For example, in [2], Cook continua are used to construct stars, paths, and cycles of Cook continua. They are all examples of continua that are not 0-rigid. It was shown in [2] that for each positive integer m , X such that

1. X is $\frac{1}{m}$ -rigid, and
2. for any inverse sequence $\{X, f_n\}$ with surjective continuous bonding functions f_n the inverse limit $\varprojlim\{X, f_n\}$ is homeomorphic to X .

Note that for any Cook continuum X and for any sequence (f_n) of continuous surjections on X , the inverse limit $\varprojlim\{X, f_n\}$ is homeomorphic to X (since each f_n is the identity on X), while $\text{degrig}(\varprojlim X) \neq \infty$ (since $\text{degrig}(X) = 1$).

The following problem is also presented in that paper.

Problem 1 [2, Problem 5.7] *Find all non-degenerate continua X such that*

1. $\text{degrig}(X) = \infty$,
2. $\varprojlim\{X, f_n\}$ is homeomorphic to X for any sequence (f_n) of continuous surjections $\varprojlim f_n : X \rightarrow X$.

Note that the pseudoarc is one such continuum (see [2, Sect. 5] for a detailed explanation).

The main goal of family of pairwise non-homeomorphic continua X such that

1. $\text{degrig}(X) = \infty$ (or, equivalently, X is 0-rigid), and
2. $\varprojlim\{X, f_n\}$ is homeomorphic to X for any sequence (f_n) of continuous surjections $\varprojlim f_n : X \rightarrow X$.

We organize the paper in the following way. In Sect. 2, definitions and notations that are needed in the paper are given. In Sect. 3, we construct a large family of continua, and we call each of them a simple fan of Cook continua. They are then used in Sect. 4, where an uncountable family of pairwise non-homeomorphic continua X such that

1. X is 0-rigid, and
2. $\varprojlim\{X, f_n\}$ is homeomorphic to X for any sequence (f_n) of continuous surjections $\varprojlim f_n : X \rightarrow X$

is constructed. This partially answers Problem 1.

2 Definitions and Notation

Definitions and notation mostly follow [2, 9]. Note that all functions in the paper are assumed to be continuous.

A continuum is a non-empty connected compact metric space. A continuum is degenerate if it consists of only a single point. Otherwise, it is non-degenerate. A subcontinuum is a subspace of a continuum which itself is also a continuum.

Let (X, d) be a metric space. By $B_X(x, \varepsilon)$ we denote the open ball of radius $\varepsilon > 0$ with center $x \in X$.

Definition 1 Let n be a positive integer, X a metric space, and $f : X \rightarrow X$ a function. We use f^n to denote the composition:

$$f^n = \underbrace{f \circ f \circ \dots \circ f}_n.$$

Definition 2 An inverse sequence is a double sequence $\{X_n, f_n\}_{n=1}^\infty$ of metric spaces X_n and functions $f_n : X_{n+1} \rightarrow X_n$. The spaces X_n are called the coordinate spaces and the functions f_n are called the bonding functions. The inverse limit of an inverse sequence $\{X_n, f_n\}_{n=1}^\infty$ is the subspace of the product space $\prod_{n=1}^\infty X_n$ that consists of all points $\mathbf{x} = (x_1, x_2, x_3, \dots) \in \prod_{n=1}^\infty X_n$ such that $x_n = f_n(x_{n+1})$ for each positive integer n :

$$\varprojlim \{X_n, f_n\}_{n=1}^\infty = \left\{ (x_1, x_2, x_3, \dots) \in \prod_{n=1}^\infty X_n \mid x_n = f_n(x_{n+1}) \text{ for each } n \right\}.$$

Definition 3 Let n be a positive integer and let $(x_1, x_2, \dots, x_n, \dots) \in \prod_{n=1}^\infty [0, 1]$. A function $\pi_n : \prod_{n=1}^\infty [0, 1] \rightarrow [0, 1]$ is defined as $\pi_n(x_1, x_2, \dots, x_n, \dots) = x_n$.

Definition 4 A metric d on a Hilbert cube $\prod_{n=1}^\infty [0, 1]$ is defined as

$$d(x, y) = \sum_{n=1}^\infty \frac{|x_n - y_n|}{2^n},$$

for each $x = (x_1, x_2, x_3, \dots), y = (y_1, y_2, y_3, \dots) \in \prod_{n=1}^\infty [0, 1]$.

Definition 5 Let X and Y be any metric spaces, S a subspace of X , and let $f : X \rightarrow Y$ be a function. We use $f|_S$ to denote the restriction $f|_S : S \rightarrow Y$, defined by $f|_S(x) = f(x)$ for any $x \in S$.

Definition 6 A non-degenerate continuum X is a Cook continuum if for any non-degenerate subcontinuum S of X and any non-constant continuous function $f : S \rightarrow X$, we have that

$$f(x) = x$$

for any $x \in S$.

The identity function on a metric space X will be denoted by 1_X .

Definition 7 Let n be a positive integer. A non-degenerate continuum X is $\frac{1}{n}$ -rigid, if

$$f^n = 1_X$$

for any continuous surjective function $f : X \rightarrow X$. If a continuum X is not $\frac{1}{n}$ -rigid for any positive integer n , then we say that X is 0-rigid. If a continuum X is $\frac{1}{1}$ -rigid, then we also say that X is 1-rigid.

Definition 8 Let X be a continuum. The degree of rigidity, $\text{degrig}(X)$, of the continuum X is:

$$\text{degrig}(X) = \min (\{n \in \mathbb{N} \mid X \text{ is } \frac{1}{n}\text{-rigid}\} \cup \{\infty\}) .$$

In [2], examples of continua with the degree of rigidity equal to n for each positive integer n can be found.

Definition 9 Let X be a continuum and let $x_0 \in X$. The point x_0 is called a cut-point if $X \setminus \{x_0\}$ is not connected. Otherwise, it is called a non-cut point for the continuum X .

In [2], one can find the following result.

Lemma 1 [2, Lemma 3.2] *Let K be a Cook continuum and let $x \in K$ be any point. Then x is a non-cut point for K .*

The following lemma is proved in [10].

Lemma 2 [10, Lemma 2.12]

1. *Let K be a Cook continuum and S be a non-degenerate subcontinuum of K . Then S is also a Cook continuum.*
2. *If two Cook continua K_1 and K_2 are homeomorphic, then they are uniquely homeomorphic, i.e., there exists a unique homeomorphism $\varphi : K_1 \rightarrow K_2$. Moreover, if $\psi : K_1 \rightarrow K_2$ is a continuous surjection, then $\psi = \varphi$.*

We also use the following two propositions.

Proposition 1 *Let K be any Cook continuum and let S be any non-degenerate subcontinuum of K , such that $S \neq K$. If $f : K \rightarrow S$ is a continuous function, then f is a constant function.*

Proof Let $g : S \rightarrow K$ be a continuous function such that $g(x) = x$ for each $x \in S$. Then $g \circ f : K \rightarrow K$ is a continuous function. Since K is a Cook continuum, then $g \circ f$ is either 1_K or a constant function.

Since $g(x) = x$ for each $x \in S$, then $g(f(x)) = f(x)$ for each $x \in K$. Let $y \in K \setminus S$. Then $g(f(y)) = f(y) \neq y$, therefore $g \circ f$ is a constant function. It follows that $g(f(x)) = f(x) = f(y)$ for each $x \in K$. Consequently, f is a constant function. \square

The following proposition follows directly from Definition 6.

Proposition 2 *Let K be any Cook continuum and let S be any non-degenerate subcontinuum of K , such that $S \neq K$. Then there does not exist a continuous surjective function from S to K .*

The following corollary follows from Propositions 1 and 2.

Corollary 1 *Let K be a Cook continuum and let $(K_n)_{n=1}^\infty$ be a sequence of continua satisfying $K_1 = K$ and K_{n+1} is a proper subcontinuum of K_n for each positive integer n . Then for any positive integers m and n , $m \neq n$, it holds that there does not exist a continuous surjective function $f : K_m \rightarrow K_n$.*

Proof Let m and n be positive integers.

If $n < m$, then $K_m \subseteq K_n$ and $K_n \neq K_m$, therefore, by Proposition 1, there does not exist a continuous surjective function $f : K_m \rightarrow K_n$.

If $m < n$, then $K_n \subseteq K_m$ and $K_n \neq K_m$, therefore, by Proposition 2, there does not exist a continuous surjective function $f : K_m \rightarrow K_n$. □

3 A Construction of Simple Fans of Cook Continua

In Construction 1, simple fans of Cook continua are constructed using any Cook continuum K . Each simple fan of Cook continua will be obtained from special sequence (K_n) of non-degenerate subcontinua of K . The existence of such a sequence will be guaranteed by the Boundary Bumping Theorem (its formulation and its proof may be found in [9]).

Construction 1 *Let $(A_n)_{n=1}^\infty$ be a sequence of subsets of positive integers such that*

1. $\bigcup_{n=1}^\infty A_n$ is equal to the set of all positive integers,
2. for all positive integers m and n it holds that: $m \neq n$ if and only if $A_m \cap A_n = \emptyset$, and
3. the set A_n is infinite for each positive integer n .

Moreover, each A_n we split into n sets $A_n^1, A_n^2, \dots, A_n^n$ such that

1. $\bigcup_{p=1}^n A_n^p = A_n$,
2. for all positive integers p and q it holds that: $p \neq q \Leftrightarrow A_n^p \cap A_n^q = \emptyset$, and
3. the set A_n^p is infinite for each positive integer p .

Now, for each positive integer n and each $p \in \{1, 2, \dots, n\}$, let $Q_n^p \subseteq \prod_{n=1}^\infty [0, 1]$ such that

$$\pi_k(Q_n^p) = \begin{cases} \{0\} & ; k \notin A_n^p, \\ [0, \frac{1}{n}] & ; k \in A_n^p, \end{cases}$$

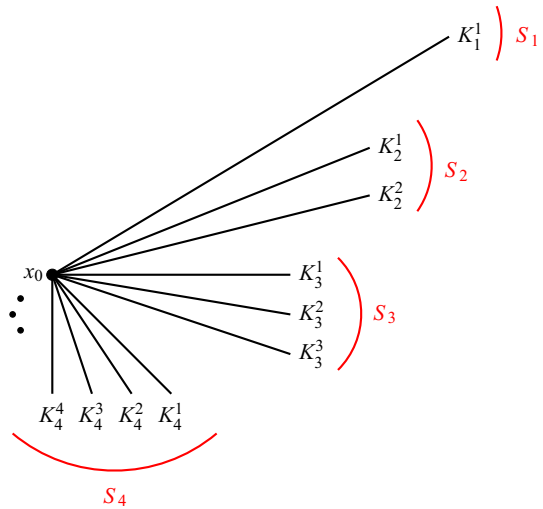
for each positive integer k .

Let K be a Cook continuum and let (K_n) be a sequence of non-degenerate continua, such that

- $K_1 = K$,
- K_{n+1} is a proper subcontinuum of K_n for each positive integer n , and
- $\lim_{n \rightarrow \infty} \text{diam}(K_n) = 0$.

By Lemma 2, K_n is a Cook continuum for each positive integer n . The existence of such a sequence of continua follows from Boundary Bumping Theorem.

Fig. 1 A simple fan of Cook continua



Now, let $\{t_0\} = \bigcap_{n=1}^{\infty} K_n$.

For each positive integer n and for each $p \in \{1, 2, \dots, n\}$, let $H_n^p : K_n \rightarrow Q_n^p$ be an embedding such that $H_n^p(t_0) = (0, 0, \dots)$. Such embeddings do exist since $\prod_{n=1}^{\infty} [0, 1]$ is a homogeneous continuum.

Let $X = \bigcup_{n=1}^{\infty} \bigcup_{p=1}^n H_n^p(K_n)$. Then X is called a simple fan of Cook continua.

To simplify the study of simple fans of Cook continua, we introduce the following notation, which we use throughout the paper.

Notation: Let X be a simple fan of Cook continua. We denote

1. $K_n^p = H_n^p(K_n)$,
2. $S_n = \bigcup_{q=1}^n K_n^q$,
3. h_n^p is the homeomorphism from K_n to K_n^p (note that there is precisely one such homeomorphism), and
4. $x_0 = (0, 0, \dots)$,

for each positive integer n and for each $p \in \{1, 2, \dots, n\}$. The continuum S_n is also called a star of continua $K_n^1, K_n^2, \dots, K_n^n$ (Fig. 1).

Lemma 3 Let X be a simple fan of Cook continua. Then

$$\lim_{n \rightarrow \infty} \text{diam}(S_n) = 0.$$

Proof Since for each positive integer n and for each $p \in \{1, 2, \dots, n\}$ it holds $S_n \subseteq \bigcup_{q=1}^n Q_n^q$, $\text{diam}(Q_n^p) \leq \frac{1}{n}$, and $x_0 \in \bigcap_{q=1}^n Q_n^q$, therefore $\lim_{n \rightarrow \infty} \text{diam}(S_n) = 0$. \square

The following observation follows directly from Construction 1 and Lemma 3.

Observation 2 Let X be a simple fan of Cook continua. Then X is a continuum.

Observation 3 *Let X be an arbitrary simple fan of Cook continua. Then x_0 is the only cut point for X .*

The following corollary follows directly from Corollary 1 and Construction 1.

Corollary 2 *Let K_m^p and K_n^r be continua from Construction 1 for some positive integers m and n such that $m \neq n$, and for some $p \in \{1, 2, 3, \dots, m\}$ and $r \in \{1, 2, 3, \dots, n\}$. Then there does not exist a continuous surjection from K_m^p to K_n^r .*

4 Properties of Simple Fan of Cook Continua

In this section, we partially answer Problem 1. We start with some results that are needed to prove the main results, Theorem 2 and Corollary 5.

Lemma 4 *Let X be any simple fan of Cook continua and let $f : X \rightarrow X$ be any continuous surjection. Then*

$$f(x_0) = x_0.$$

Proof Suppose $f(x_0) \neq x_0$. Let n be a positive integer and $p \in \{1, 2, \dots, n\}$ such that $f(x_0) \in K_n^p$. Let $\varepsilon = \frac{d(x_0, f(x_0))}{3}$. Since f is a continuous function, there exists $0 < \delta < \frac{d(x_0, f(x_0))}{3}$ such that $f(B_X(x_0, \delta)) \subseteq B_X(f(x_0), \varepsilon)$, where $B_X(x_0, \delta)$ and $B_X(f(x_0), \varepsilon)$ are open balls in X .

Let m_0 be a positive integer such that for each positive integer $m \geq m_0$, it holds that $\text{diam}(S_m) < \delta$. By Construction 1, $\text{diam}(K_m^p) < \delta$ for each $p \in \{1, 2, \dots, m\}$. Moreover, by Construction 1 and since f is continuous, the following holds:

- $n < m_0$,
- $K_m \subseteq K_n$,
- $h_n^p(K_m) \subseteq K_n^p$,
- $\text{diam}(h_n^p(K_m)) < \delta$, and
- $f(h_n^p(K_m)) \subseteq B_X(f(x_0), \varepsilon)$,

for each positive integer $m \geq m_0$.

For each positive integer t and for each $s \in \{1, 2, \dots, t\}$, we define a function $\varphi_t^s : X \rightarrow K_t^s$ by

$$\varphi_t^s(x) = \begin{cases} (h_t^s \circ (h_a^b)^{-1})(x) & ; x \in K_a^b, a \geq t, b \in \{1, 2, \dots, a\}, \\ x_0 & ; x \in K_a^b, a \leq t - 1, b \in \{1, 2, \dots, a\} \end{cases}$$

for each $x \in X$. Obviously, φ_t^s is a continuous function. Then $\varphi_t^s \circ f|_{K_t^s} : K_t^s \rightarrow K_t^s$ is a continuous function from a Cook continuum to itself. Therefore, $\varphi_t^s \circ f|_{K_t^s}$ is a constant function or an identity function on K_t^s . We consider the following three cases for t .

Case 1: $t \leq n$. Let $s \in \{1, 2, \dots, t\}$. Since $(\varphi_t^s \circ f|_{K_t^s})(x_0) = (h_t^s \circ (h_n^p)^{-1})(f(x_0)) \neq x_0$, it follows that $\varphi_t^s \circ f|_{K_t^s}$ is a constant function and

$$(\varphi_t^s \circ f|_{K_t^s})(x) = (h_t^s \circ (h_n^p)^{-1})(f(x_0))$$

for each $x \in K_t^s$. Since K_t^s is a connected space and f is a continuous function, the image $f(K_t^s)$ is also a connected space. Therefore,

$$f(K_t^s) = \{f(x_0)\}.$$

Case 2: $n < t < m_0$. Let $s \in \{1, 2, \dots, t\}$ and let $y \in K_t^s$ such that $d(x_0, y) < \delta$ and $y \neq x_0$. Then $f(y) \in B_X(f(x_0), \varepsilon) \subseteq K_n^p$. Therefore, $(\varphi_t^s \circ f|_{K_t^s})(y) = x_0$ and it follows that, $\varphi_t^s \circ f|_{K_t^s}$ is a constant function. Therefore,

$$f(K_t^s) \subseteq \bigcup_{i=1}^{t-1} S_i.$$

Case 3: $t \geq m_0$. Let $s \in \{1, 2, \dots, t\}$. Since $K_t^s \subseteq B_X(x_0, \delta)$, it holds that

$$f(K_t^s) \subseteq B_X(f(x_0), \varepsilon) \subseteq K_n^p.$$

It follows from the above cases that

$$f(X) \subseteq \bigcup_{i=1}^{m_0-1} S_i,$$

which is a contradiction, since f is a surjective function. □

Proposition 3 *Let X be any simple fan of Cook continua and let $f : X \rightarrow X$ be any continuous surjective function. Suppose S is a subcontinuum of K_n^p for some positive integer n and some $p \in \{1, 2, \dots, n\}$, such that $x_0 \notin S$. If $x_0 \notin f(S)$, then there exist exactly one positive integer m and exactly one $r \in \{1, 2, \dots, m\}$ such that $f(S) \subseteq K_m^r$.*

Proof Since S is a continuum, f is a continuous function, and x_0 is a cut-point of X , the proposition follows. □

Lemma 5 *Let X be any simple fan of Cook continua and let $f : X \rightarrow X$ be any continuous surjective function. Then for each positive integer n and for each $p \in \{1, 2, \dots, n\}$, it holds that*

$$f(K_n^p) \subseteq S_n.$$

Proof By Lemma 4, $f(x_0) = x_0$ and we use this fact until the end of this proof. We prove the lemma inductively.

First, we prove that $f(K_1^1) \subseteq S_1 = K_1^1$. Suppose, $f(K_1^1) \not\subseteq S_1$. Then there exists $y \in K_1^1 \setminus \{x_0\}$ such that $f(y) \notin S_1$. We define a function $\varphi_1^1 : X \rightarrow K_1^1$, where for each positive integer m , for each $r \in \{1, 2, \dots, m\}$, and for each $x \in K_m^r$,

$$\varphi_1^1(x) = (h_1^1 \circ (h_m^r)^{-1})(x).$$

It is easy to see that φ_1^1 is a continuous function. Therefore, $\varphi_1^1 \circ f|_{K_1^1} : K_1^1 \rightarrow K_1^1$ is also a continuous function, and since it maps from a Cook continuum to itself, it is a constant function or an identity function on K_1^1 .

Obviously, $(\varphi_1^1 \circ f|_{K_1^1})(x_0) = x_0$.

Let $s \in h_1^1(K_1 \setminus K_2) \setminus \{y\}$ be any point. We distinguish two cases for $f(s)$.

Case 1: $f(s) = s$. Note that by Lemma 1, x_0 is a non-cut point for K_1^1 , and by Observation 3, x_0 is a cut point for X . Since $K_1^1 \setminus \{x_0\}$ is connected and f continuous, $f(K_1^1 \setminus \{x_0\})$ is also connected. Consequently, there is $z \in K_1^1 \setminus \{x_0, s, y\}$ such that $f(z) = x_0$.

Then $(\varphi_1^1 \circ f|_{K_1^1})(z) = x_0 \neq z$, and therefore, $\varphi_1^1 \circ f|_{K_1^1}$ is a constant function with the image $\{x_0\}$. On the other hand, $(\varphi_1^1 \circ f|_{K_1^1})(s) = s \neq x_0$, a contradiction.

Case 2: $f(s) \neq s$. From Construction 1 and the definition of φ_1^1 , it follows that $(\varphi_1^1 \circ f|_{K_1^1})(s) \neq s$. Therefore, $\varphi_1^1 \circ f|_{K_1^1}$ is a constant function. Consequently, $(\varphi_1^1 \circ f|_{K_1^1})(K_1^1) = \{x_0\}$ and

$$f(K_1^1) = \{x_0\} \subseteq S_1.$$

Let n be a positive integer and suppose $f(K_m^r) \subseteq S_m$ for each positive integer $m \leq n$ and each $r \in \{1, 2, \dots, m\}$. Now we prove that $f(K_{n+1}^p) \subseteq S_{n+1}$ for each $p \in \{1, 2, \dots, n+1\}$.

Let $p \in \{1, 2, \dots, n+1\}$ be arbitrarily chosen. Suppose, $f(K_{n+1}^p) \not\subseteq S_{n+1}$. Then there exists $y \in K_{n+1}^p \setminus \{x_0\}$ such that $f(y) \notin S_{n+1}$. For each $q \in \{1, 2, \dots, n+1\}$, we define the function $\varphi_{n+1}^q : X \rightarrow K_{n+1}^q$ by

$$\varphi_{n+1}^q(x) = \begin{cases} (h_{n+1}^q \circ (h_{n+1}^r)^{-1})(x) & ; x \in K_{n+1}^r, r \in \{1, 2, \dots, n+1\}, \\ x_0 & ; x \in X \setminus S_{n+1}, \end{cases}$$

for each $x \in X$. Obviously, φ_{n+1}^q is a continuous function for each $q \in \{1, 2, \dots, n+1\}$. Thus, $\varphi_{n+1}^p \circ f|_{K_{n+1}^p} : K_{n+1}^p \rightarrow K_{n+1}^p$ is a continuous function and it maps from a Cook continuum to itself. Therefore it is a constant function or an identity function on K_{n+1}^p . Since $(\varphi_{n+1}^p \circ f|_{K_{n+1}^p})(y) = x_0$, it follows that $\varphi_{n+1}^p \circ f|_{K_{n+1}^p}$ is a constant function.

Let $a \in h_{n+1}^p(K_{n+1} \setminus K_{n+2}) \setminus \{y\}$. Since f is surjective, there is $b \in X$ such that $f(b) = a$. From the induction assumption, it follows that $b \notin \bigcup_{m=1}^n S_m$. We distinguish the remaining cases for b .

Case A: There is a positive integer $\ell > n+1$ and $q \in \{1, 2, \dots, \ell\}$ such that $b \in K_\ell^q$.

Let $g : K_\ell \rightarrow K_{n+1}$ be the function defined by

$$g(x) = \left((h_{n+1}^p)^{-1} \circ \varphi_{n+1}^p \circ f|_{K_\ell^q} \circ h_\ell^q \right)(x).$$

Obviously, g is a continuous function. Moreover, $g(t_0) = t_0$, $g((h_\ell^q)^{-1}(b)) = (h_{n+1}^p)^{-1}(a) \neq t_0$, and $g((h_\ell^q)^{-1}(b)) = (h_{n+1}^p)^{-1}(a) \in K_{n+1} \setminus K_\ell$, a contradiction, since $K_\ell \subseteq K_{n+1}$, K_{n+1} is a Cook continuum, and g is a continuous function.

Case B: There is a positive integer $q_1 \in \{1, 2, \dots, n + 1\}$ such that $b \in K_{n+1}^{q_1}$. Denote $b_1 = b$. Note that $\varphi_{n+1}^{q_1} \circ f|_{K_{n+1}^{q_1}} : K_{n+1}^{q_1} \rightarrow K_{n+1}^{q_1}$ is a continuous function and $(\varphi_{n+1}^{q_1} \circ f|_{K_{n+1}^{q_1}})(x_0) = x_0$. Moreover, $(\varphi_{n+1}^{q_1} \circ f|_{K_{n+1}^{q_1}})(b_1) = \varphi_{n+1}^{q_1}(a) = (h_{n+1}^{q_1} \circ (h_{n+1}^p)^{-1})(a) \neq x_0$, therefore $\varphi_{n+1}^{q_1} \circ f|_{K_{n+1}^{q_1}}$ is the identity function. Thus, $(\varphi_{n+1}^{q_1} \circ f|_{K_{n+1}^{q_1}})(b_1) = (h_{n+1}^{q_1} \circ (h_{n+1}^p)^{-1})(a) = b_1$.

Since f is surjective, there is $b_2 \in X$ such that $f(b_2) = b_1$. From the induction assumption and Case A it follows that $b_2 \in K_{n+1}^{q_2} \subseteq S_{n+1}$, for some $q_2 \in \{1, 2, \dots, n + 1\}$, and $(h_{n+1}^{q_2} \circ (h_{n+1}^{q_1})^{-1})(b_1) = b_2$.

Now we construct a finitely many elements $b_3, \dots, b_n \in S_{n+1}$ such that

1. $f(b_i) = b_{i-1}$,
2. $b_i \in K_{n+1}^{q_i}$, for some $q_i \in \{1, 2, \dots, n + 1\}$,
3. $(h_{n+1}^{q_i} \circ (h_{n+1}^{q_{i-1}})^{-1})(b_{i-1}) = b_i$,

for each $i \in \{3, \dots, n\}$.

We denote $b_0 = a$ and $q_0 = p$. First, if there is $i \in \{1, 2, \dots, n\}$ such that $b_i = a$, then $f(a) = b_{i-1}$ and consequently, $f(a) \in S_{n+1}$. Otherwise, for each $i \in \{1, 2, \dots, n\}$ it holds that $b_i \neq a$. It is easy to see that $b_i \neq b_j$ for each $i, j \in \{0, 1, \dots, n\}$ such that $i \neq j$. Since f is surjective, there is $b_{n+1} \in X$ such that $f(b_{n+1}) = b_n$. From the induction assumption and Case A it follows that $b_{n+1} \notin S_k$ for each positive integer $k \neq n + 1$. Then there exists $q_{n+1} \in \{1, 2, \dots, n + 1\}$ satisfying $b_{n+1} \in K_{n+1}^{q_{n+1}}$, and $(h_{n+1}^{q_{n+1}} \circ (h_{n+1}^{q_n})^{-1})(b_n) = b_{n+1}$. Clearly, there is $j \in \{0, 1, 2, \dots, n\}$ such that $q_{n+1} = q_j$. Since

$$\begin{aligned} f(b_{n+1}) &= b_n = \left(h_{n+1}^{q_n} \circ (h_{n+1}^{q_{n-1}})^{-1}\right)(b_{n-1}) \\ &= \left(h_{n+1}^{q_n} \circ (h_{n+1}^{q_{n-1}})^{-1} \circ h_{n+1}^{q_{n-1}} \circ (h_{n+1}^{q_{n-2}})^{-1}\right)(b_{n-2}) \\ &= \dots \\ &= \left(h_{n+1}^{q_n} \circ (h_{n+1}^{q_j})^{-1}\right)(b_j) \end{aligned}$$

and $(h_{n+1}^{q_{n+1}} \circ (h_{n+1}^{q_{n+1}})^{-1})(b_{n+1}) = b_{n+1}$, it follows that $b_j = b_{n+1}$.

If $j = 0$, then $f(a) = f(b_0) = f(b_{n+1}) = b_n \in S_{n+1}$. Otherwise, if $j \neq 0$, then $f(b_{n+1}) = b_n$ and $f(b_{n+1}) = f(b_j) = b_{j-1}$, a contradiction, since f is a function and $b_{j-1} \neq b_n$.

From the above, we conclude that there exists $j \in \{0, 1, 2, \dots, n\}$ such that $f(a) = (h_{n+1}^{q_j} \circ (h_{n+1}^p)^{-1})(a) = b_j$.

Recall that, $\varphi_{n+1}^p \circ f|_{K_{n+1}^p}$ is a constant function. On the other hand,

$$\left(\varphi_{n+1}^p \circ f|_{K_{n+1}^p}\right)(a) = \varphi_{n+1}^p(b_j) = \left(h_{n+1}^p \circ (h_{n+1}^{q_j})^{-1} \circ h_{n+1}^{q_j} \circ (h_{n+1}^p)^{-1}\right)(a) = a,$$

which is a contradiction since $a \neq x_0$.

Thus, $f(y) \in S_{n+1}$ and consequently, $f(K_{n+1}^p) \subseteq S_{n+1}$.

□

The following corollary follows directly from Lemma 5.

Corollary 3 *Let X be any simple fan of Cook continua and let $f : X \rightarrow X$ be any continuous surjective function. Then for each positive integer n ,*

$$f(S_n) \subseteq S_n.$$

Theorem 1 *Let X be any simple fan of Cook continua and let $f : X \rightarrow X$ be any continuous surjective function. Then for each positive integer n and for each $p \in \{1, 2, \dots, n\}$, there exists $r \in \{1, 2, \dots, n\}$ such that*

$$f(K_n^p) = K_n^r.$$

Moreover, $f(x) = (h_n^r \circ (h_n^p)^{-1})(x)$ for each $x \in K_n^p$.

Proof The theorem follows directly from Construction 1, Corollary 3, and [2, Theorem 3.7].

□

Corollary 4 *Let X be any simple fan of Cook continua and let $f : X \rightarrow X$ be any continuous surjective function. Then f is a homeomorphism.*

Proof By Theorem 1, it holds that for each positive integer n and for each $p \in \{1, 2, \dots, n\}$ there exists $r \in \{1, 2, \dots, n\}$ such that $f(x) = (h_n^r \circ (h_n^p)^{-1})(x)$ for each $x \in K_n^p$. Consequently, f is a homeomorphism.

□

Theorem 2 *Let X be any simple fan of Cook continua. Then $\varprojlim\{X, f_n\}$ is homeomorphic to X for any sequence $(f_n)_{n=1}^\infty$ of continuous surjective functions.*

Proof Let $(f_n)_{n=1}^\infty$ be an arbitrarily chosen sequence of continuous surjective functions. By Corollary 4, f_n is a homeomorphism for each positive integer n . Therefore, $\varprojlim\{X, f_n\}$ is homeomorphic to X .

□

Theorem 3 *Let X be any simple fan of Cook continua. Then*

$$\text{degrig}(X) = \infty.$$

Proof We prove that X is not $\frac{1}{n}$ -rigid for each positive integer n .

Let n be any positive integer. We define a function $f : X \rightarrow X$ in the following way:

$$f(x) = \begin{cases} (h_{n+1}^{r(\text{mod}(n+1))+1} \circ (h_{n+1}^r)^{-1})(x) & ; x \in K_{n+1}^r, r \in \{1, 2, 3, \dots, n+1\}, \\ x & ; x \in X \setminus S_{n+1}. \end{cases}$$

Obviously, f is a continuous surjective function. We prove that $f^n \neq 1_X$.

Let $x \in K_{n+1}^1 \setminus \{x_0\}$.

$$\begin{aligned} f^n(x) &= \left(f^{n-1} \circ h_{n+1}^2 \circ (h_{n+1}^1)^{-1} \right) (x) \\ &= \left(f^{n-2} \circ h_{n+1}^3 \circ (h_{n+1}^2)^{-1} \circ h_{n+1}^2 \circ (h_{n+1}^1)^{-1} \right) (x) \\ &= \left(f^{n-2} \circ h_{n+1}^3 \circ (h_{n+1}^1)^{-1} \right) (x) \\ &= \dots \\ &= \left(h_{n+1}^{n+1} \circ (h_{n+1}^1)^{-1} \right) (x). \end{aligned}$$

Therefore, $f^n(x) \in K_{n+1}^{n+1}$ and $f^n(x) \notin K_{n+1}^1$. Consequently, $f^n(x) \neq x$ and $f^n \neq 1_X$.

We proved that X is not $\frac{1}{n}$ -rigid for each positive integer n . Thus, $\text{degrig}(X) = \infty$. □

Corollary 5 *There exists an uncountable family \mathcal{F} of pairwise non-homeomorphic continua such that for each $X \in \mathcal{F}$ it holds*

1. $\text{degrig}(X) = \infty$,
2. $\varprojlim \{X, f_n\}$ is homeomorphic to X for any sequence (f_n) of continuous surjections $\varprojlim f_n : X \rightarrow X$.

Proof It is known that there are uncountably many pairwise non-homeomorphic Cook continua. Let K and \tilde{K} be any non-homeomorphic Cook continua. Let X be a simple fan of Cook continua obtained from K and let \tilde{X} be a simple fan of Cook continua obtained from \tilde{K} .

With \sim we denote the corresponding homeomorphisms, Cook continua, and \tilde{x}_0 for \tilde{X} . We prove that X and \tilde{X} are non-homeomorphic.

Suppose X and \tilde{X} are homeomorphic and let $F : X \rightarrow \tilde{X}$ be a homeomorphism. First we show, that $F(x_0) = \tilde{x}_0$.

If $F(x_0) \neq \tilde{x}_0$, then there exists $z_0 \in X \setminus \{x_0\}$ such that $F(z_0) = \tilde{x}_0$. Since z_0 is non-cut point in X and F is a homeomorphism, then $F(X \setminus \{z_0\})$ is connected and $F(X \setminus \{z_0\}) = \tilde{X} \setminus \{\tilde{x}_0\}$, a contradiction. Thus, $F(x_0) = \tilde{x}_0$. Moreover, from this it also follows that for each positive integer n and for each $p \in \{1, 2, \dots, n\}$, there are positive integers m and $r \in \{1, 2, \dots, m\}$ such that $F(K_n^p) \subseteq \tilde{K}_m^r$.

Suppose n and m are positive integers, $p \in \{1, 2, \dots, n\}$, and $r \in \{1, 2, \dots, m\}$ such that $F(K_n^p) \subseteq \tilde{K}_m^r$ and $F^{-1}(\tilde{K}_1^r) \subseteq K_m^r$. Now, we define $\alpha = (\tilde{h}_n^p)^{-1} \circ F \circ h_1^1 : K_1 \rightarrow \tilde{K}_1$ and $\beta = (h_m^r)^{-1} \circ F^{-1} \circ \tilde{h}_1^r : \tilde{K}_1 \rightarrow K_1$. Obviously, α and β are continuous injective functions, and therefore, $\alpha \circ \beta$ and $\beta \circ \alpha$ are also continuous injective functions. Since K_1 and \tilde{K}_1 are Cook continua, it follows that $\alpha \circ \beta = 1_{\tilde{K}_1}$ and $\beta \circ \alpha = 1_{K_1}$.

Note that if α was surjective, then α would be a homeomorphism. Since $K_1 = K$, $\tilde{K}_1 = \tilde{K}$, K and \tilde{K} are not homeomorphic, it follows that α is not surjective. Therefore, there is $y \in \tilde{K}$ such that for each $x \in K$ it holds that $\alpha(x) \neq y$. Moreover, $(\alpha \circ \beta)(y) \neq y$, a contradiction, since $\alpha \circ \beta = 1_{\tilde{K}_1}$.

By Theorems 2 and 3, each simple fan of Cook continua X satisfies

1. $\text{degrig}(X) = \infty$,
2. $\varprojlim\{X, f_n\}$ is homeomorphic to X for any sequence (f_n) of continuous surjections $f_n : X \rightarrow X$,

and the corollary is proven. \square

Declarations

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