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Continuous Dependence in a Problem of Convergence of Random Iteration

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Abstract
We show that an invariant measure of a Markov operator that is contracting in the Hutchison distance, acting on a space of Borel probability measures on a Polish space, depends continuously on the given operator. In addition, we establish an estimate for a distance between invariant measures. Some applications to the weak limit of iterates of random-valued functions (in particular, the so-called random affine maps, occuring, e.g., in perpetuities analysis) are also given.

Keywords Random iteration · Markov operator · Invariant measure · Hutchinson distance · Fortet–Mourier distance · Continuous dependence on the given function · Perpetuities

Mathematics Subject Classification Primary 37H12 · 60J05; Secondary 37A50 · 47J26

1 Introduction

Markov operators play important role for analyzing random dynamical systems (shortly RDS, see [4, Chapter 1] for precise definition) including such systems with random perturbations. Randomness of RDS allows us to consider movements of points via the evolution of probability measures describing the distribution of points on the state space X being a metric space, in general. Consequently, we can define a (linear) transformation P acting on the space $\mathcal{M}_1(X)$ of all Borel probability measures on X into itself. A probability measure μ_* such that $P\mu_* = \mu_*$ is said to be an *invariant*

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measure or *stationary* distribution with respect to P. In case of compactness of X, the space $\mathcal{M}_1(X)$ is compact with respect to the weak topology of measures (formally weak* topology, since the space of measures is identified, via Riesz representation, with the dual of some space of continuous functions), which follows from the tightness of $\mathcal{M}_1(X)$, due to the Prokhorov theorem, and we can use the Markov–Kakutani fixed point theorem or the Krylov-Bogolyubov theorem to obtain an invariant measure for any continuous operator $P : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$. The existence of an invariant measure for RDS has many important consequences, for example, in ergodic theory. There are many conditions which imply the uniqueness as well as ergodicity of invariant measures. For a fuller context we refer the reader to [24, 31] and the references given there.

Let $\mathbf{\Phi}$ be a family of Markov operators and assume that μ_P^* denotes a (unique) invariant measure for $P \in \mathbf{\Phi}$. By changing (in some sense) operators $P \in \mathbf{\Phi}$ it is of interest to know whether μ_P^* is changing continuously. Thus we shall consider an operator

$$\mathbf{\Phi} \ni P \longmapsto \mu_P^* \tag{1.1}$$

and its continuity. Moreover, the aim of the paper is to establish some estimation of a distance between stationary distributions μ_P^*, μ_Q^* of operators P and Q, respectively. Our research problem is related to the study of strongly stable Markov chains, which were first examined by Aïssani and Kartashov [1]. (The interested reader is referred to [21] for further information; cf. also [22]). However, it should be emphasized that our viewpoint sheds some new light on the problem of the stability of solutions to linear iterative functional equations. Such a problem is investigated by Baron in [7, 8]. Generalizations of his results, based on theorems of our work, will appear in a forthcoming paper written by the second author.

Recently in [11] the continuous dependence of an invariant measure of piecewise-deterministic Markov processes has been established. Other studies of this type were conducted in [6], where it was proved that the limit in law of the sequence of iterates f^n of contracting in average random-valued functions f depends continuously in the Fortet–Mourier metric on the given function. (To be more precise see Corollary 4.1 and Remark 4.2 given below.) These iterates f^n are prototype of RDS; see Sect. 2 for details. One of the aims of the work is to obtain generalizations of the main results of [6], by introducing the Hutchinson distance in the space $\{P\mu : P \in \mathbf{\Phi}, \mu$ has the first moment finite$\}$ in case when $\mathbf{\Phi}$ is a family of contractive Markov operators.

The organization of our paper goes as follows. In Sect. 2 some basic notions and facts concernig Markov operators as well as the iterates of random-valued functions are indicated. Main results in general settings are contained in Sect. 3. It will be shown that a parameterized version of the Banach fixed-point theorem allows us to get continuity of operator (1.1). However, the classical Banach fixed-point theorem will bring additional information, which is important from the applications point of view. These apllications presented in the last two sections concern random iterations and perpetuities.

2 Notions and Basic Facts

Throughout the work we assume that (X, ϱ) is a *Polish* space, i.e. a separable and complete metric space. Let $\mathcal{B}(X)$ stand for the σ-algebra of all Borel subsets of X. By $\mathcal{M}_1(X)$ we denote the space of all probability measures on $\mathcal{B}(X)$. Let $B(X)$ denote the space of all real-valued bounded Borel–measurable functions equipped with the supremum norm $||\cdot||_\infty$ and $C(X)$ be the subspace of bounded continuous functions. As abbreviation we will write $\int \varphi d\mu$ instead of $\int_X \varphi d\mu$, where $\varphi \in B(X)$ and $\mu \in \mathcal{M}_1(X)$. Recall that a sequence of measures (μ_n) *converges weakly* to μ, if $\int \varphi d\mu_n \xrightarrow[n\to\infty]{} \int \varphi d\mu$ for every $\varphi \in C(X)$. It is well known (see [13, Theorem 11.3.3]) that this convergence is metrizable by the Lévy–Prokhorov metric or by the Fortet–Mourier metric (known also as the bounded Lipschitz distance) [14]

$$d_{FM}(\mu, \nu) = \sup\left\{\left|\int \varphi d\mu - \int \varphi d\nu\right| : \varphi \in Lip_1(X), ||\varphi||_\infty \leq 1\right\},$$

where

$$Lip_1(X) = \{\varphi : X \to \mathbb{R} : |\varphi(x) - \varphi(y)| \leq \varrho(x, y), x, y \in X\}.$$

Putting $Lip_1^b(X) = Lip_1(X) \cap B(X)$ define now

$$d_H(\mu, \nu) = \sup\left\{\left|\int \varphi d\mu - \int \varphi d\nu\right| : \varphi \in Lip_1^b(X)\right\}.$$

Clearly, d_H is a distance function, called the *Hutchinson metric* [18] (also called the 1-Wasserstein distance or Kantorovich–Rubinstein distance [30]), however for some arguments it may be infinite. Moreover $d_{FM}(\mu, \nu) \leq d_H(\mu, \nu)$ and in the case when the space X is bounded we have $d_H(\mu, \nu) \leq \mathrm{diam}(X) \times d_{FM}(\mu, \nu)$ for any $\mu, \nu \in \mathcal{M}_1(X)$, i.e. metrics d_{FM}, d_H are equivalent. The classical work on metrics on measures are [9, 10, 13].

Throughout this paper we shall consider a regular *Markov operator* $P : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$, i.e. P is a linear operator (here linearity is restricted to coefficients that are nonnegative and sum up to one only) and there exists (adjoint or dual) operator $P^* : B(X) \to B(X)$ such that $\int \varphi dP\mu = \int P^*\varphi d\mu$ for any $\varphi \in B(X)$ and $\mu \in \mathcal{M}_1(X)$. Moreover, if $P^* : B(X) \to B(X)$ is a linear operator, $P^*\mathbf{1}_X = \mathbf{1}_X$, $P^* f_n \searrow 0$ for $f_n \searrow 0$, and $P^*\varphi \geq 0$ if $\varphi \geq 0$, then the operator P given by $P\mu(A) = \int P^*\mathbf{1}_A(x)\mu(dx)$, $A \in \mathcal{B}(X)$, is a Markov operator (with adjoint P^*). An operator P is called *asymptotically stable* if there exists a stationary distribution μ_* such that

$$P^n\mu \xrightarrow[n\to\infty]{} \mu_* \quad \text{weakly for every } \mu \in \mathcal{M}_1(X). \tag{2.1}$$

Clearly, the stationary distribution μ_* satisfying (2.1) is unique.

Assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space. A function $f : X \times \Omega \to X$ is said to be *random-valued function* (shortly *rv-function*) if it is measurable with respect to the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{A}$. Having an rv-function f we will examine a regular Markov operator $P : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ defined by

$$P\mu(A) = \int_X \int_\Omega \mathbf{1}_A(f(x,\omega))\mathbb{P}(d\omega)\mu(dx), \quad \mu \in \mathcal{M}_1(X), A \in \mathcal{B}(X). \tag{2.2}$$

One can show that P is a transition operator for a sequence of iterates of rv-functions in the sense of Baron and Kuczma [5]; cf. [12]. More precisely, $P\pi_n(x,\cdot) = \pi_{n+1}(x,\cdot)$, where

$$\pi_n(x, B) = \mathbb{P}^\infty(f^n(x,\cdot) \in B), \quad B \in \mathcal{B}(X),$$

denotes the distribution of the nth-iterate of f defined inductively as follows

$$f^0(x,\omega) = x, \quad f^n(x,\omega) = f(f^{n-1}(x,\omega), \omega_n), \tag{2.3}$$

for x from X and $\omega = (\omega_1, \omega_2, \dots)$ from Ω^∞ being $\Omega^{\mathbb{N}}$. Note that $f^n : X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, \mathbb{P}^\infty)$. More exactly, the n-th iterate f^n is $\mathcal{B}(X) \otimes \mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the σ-algebra of all the sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \omega_2, \dots, \omega_n) \in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$. The iterates f^n are prototype of RDS; see [4, Definition 1.1.1]. Namely, the formula $\varphi(n,\omega,x) = f^n(x,\omega)$ defines a special RDS $\varphi : \mathbb{N} \times \Omega^\infty \times X \to X$ over the flow $\theta(n) : \Omega^\infty \to \Omega^\infty$ of iterates of the shift $\theta(n)(\omega_1, \omega_2, \dots) = (\omega_{n+1}, \omega_{n+2}, \dots)$, $n \in \mathbb{N}$.

3 General Approach

As it was mentioned in Sect. 1 some kind of continuity of operator (1.1) can be obtained using a parameterized version of Banach's fixed-point theorem for contractive maps. For the convenience of the reader we quote this claim from [16, p. 18], thus making our exposition self-contained. Recall only that a family $\{H_\gamma : \gamma \in \Gamma\}$ of maps acting on a metric space $(\mathcal{X}, d)$ into itself is called λ-*contractive*, where $0 \le \lambda < 1$, provided for some $M > 0$ and some $0 < \kappa \le 1$, we have

(i) $d(H_\gamma(x), H_\gamma(y)) \le \lambda d(x,y)$ for all $\gamma \in \Gamma$ and $x, y \in \mathcal{X}$
(ii) $d(H_{\gamma_1}(x), H_{\gamma_2}(x)) \le M\big(\sigma(\gamma_1,\gamma_2)\big)^\kappa$ for all $x \in \mathcal{X}$ and $\gamma_1, \gamma_2 \in \Gamma$,

where σ stands for a metric on Γ. (Here and in the sequel all metric spaces are assumed implicitly to be non-empty.)

Theorem 3.1 *Let* $(\mathcal{X}, d)$ *be a complete metric space,* (Γ, σ) *be a metric space and let* $\{H_\gamma : \gamma \in \Gamma\}$ *be a family of contractive maps of* $\mathcal{X}$ *into itself, i.e. for any* γ *the map* H_γ *is Lipschitzian with contraction constant less than 1. Assume that for every* $x \in \mathcal{X}$

the map $\Gamma \ni \gamma \longmapsto H_\gamma(x) \in \mathcal{X}$ *is continuous and for every* $\gamma \in \Gamma$ *let* x_γ *be the unique fixed point of* H_γ. *Then the map*

$$\Gamma \ni \gamma \longmapsto x_\gamma \in \mathcal{X} \tag{3.1}$$

is continuous whenever at least one of the following two conditions holds:

(i) *the space* $\mathcal{X}$ *is locally compact,*
(ii) *the family* $\{H_\gamma : \gamma \in \Gamma\}$ *is* λ*-contractive.*

Remark 3.1 The continuity of (3.1) in case of (i) follows from [27, Theorem 2]. To see that (ii) implies the continuity of (3.1) it is enough to observe that:

$$\begin{aligned} d(x_{\gamma_1}, x_{\gamma_2}) &\le d\big(H_{\gamma_1}(x_{\gamma_1}), H_{\gamma_2}(x_{\gamma_1})\big) + d\big(H_{\gamma_2}(x_{\gamma_1}), H_{\gamma_2}(x_{\gamma_2})\big) \\ &\le M\big(\sigma(\gamma_1, \gamma_2)\big)^\kappa + \lambda d(x_{\gamma_1}, x_{\gamma_2}), \end{aligned}$$

and, in consequence,

$$d(x_{\gamma_1}, x_{\gamma_2}) \le \frac{M}{1-\lambda}\big(\sigma(\gamma_1, \gamma_2)\big)^\kappa.$$

It is well known that the space $(\mathcal{M}_1(X), d_{FM})$ is compact if and only if X is. (Note however that the local compactness X is not inherited by the space $\mathcal{M}_1(X)$ in general. To see this let us consider $X = \mathbb{R}$ for simplicity, and fix weak neighbourhood $\mathcal{O}$ of the Dirac measure δ_0. Then we can find $p, q \in (0, 1)$ such that $p+q = 1$ and $p\delta_0 + q\delta_x \in \mathcal{O}$ for every $x \in X$. Consequently, $\mathcal{O} \ni p\delta_0 + q\delta_n \xrightarrow[n\to\infty]{} p\delta_0 + q\delta_\infty \notin \mathcal{M}_1(\mathbb{R})$, i.e. the closure of $\mathcal{O}$ is not compact. Cf. [26, 28] and [3, Remark 7.1.9].) According to this fact we may use part (i) of Theorem 3.1 to get directly the following result concerning some kind of continuity of operator (1.1).

Theorem 3.2 *Assume that the space* X *is compact and* T *is a metric space and let* $\{P_t : t \in T\}$ *be a family of Markov operators which are contractions in the Fortet–Mourier metric, i.e. for every* $t \in T$ *there exists* $0 \le \lambda_{P_t} < 1$ *such that*

$$d_{FM}(P_t\mu, P_t\nu) \le \lambda_{P_t} d_{FM}(\mu, \nu) \quad \text{for } \mu, \nu \in \mathcal{M}_1(X).$$

Assume that $\mu^*_{P_t}$ *denotes the invariant measure for* P_t. *If for every* $\mu \in \mathcal{M}_1(X)$ *the operator* $T \ni t \longmapsto P_t\mu \in (\mathcal{M}_1(X), d_{FM})$ *is continuous, then*

$$T \ni t \longmapsto \mu^*_{P_t} \in (\mathcal{M}_1(X), d_{FM})$$

is also continuous.

In other theorems, we will not assume the compactness of X. Suppose now that $\mu_0 \in \mathcal{M}_1(X)$ is a fixed measure and define a set $\mathcal{M}(\mu_0)$ by

$$\mathcal{M}(\mu_0) = \big\{\mu \in \mathcal{M}_1(X) : d_H(\mu, \mu_0) < \infty\big\}. \tag{3.2}$$

We start with a result being a slight modification of [20, Theorem 4.1], which will be needed in the proof of the main result of this section.

Proposition 3.1 *Let $P : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ be a Markov operator and assume that there exists $\lambda \in [0, 1)$ such that*

$$d_H(P\mu, P\nu) \le \lambda\, d_H(\mu, \nu) \quad \textit{for } \mu, \nu \in \mathcal{M}(\mu_0), \tag{3.3}$$

where $\mathcal{M}(\mu_0)$ is given by (3.2). Assume moreover that

$$d_H(\mu_0, P\mu_0) < \infty. \tag{3.4}$$

Then the operator P has a unique invariant measure $\mu^* \in \mathcal{M}(\mu_0)$. Furthermore, we have a geometric rate of convergence, i.e

$$d_H(P^n\mu, \mu^*) \le \frac{\lambda^n}{1-\lambda} d_H(\mu, P\mu) \quad \textit{for } n \in \mathbb{N}, \mu \in \mathcal{M}(\mu_0). \tag{3.5}$$

If, additionally,

$$d_{FM}(P\mu, P\nu) \le d_{FM}(\mu, \nu) \quad \textit{for } \mu, \nu \in \mathcal{M}_1(X), \tag{3.6}$$

then the operator P is asymptotically stable.

Proof Assume that P satisfies (3.3) with some $\lambda \in [0, 1)$. Then it is easy to check that condition (3.4) implies $P(\mathcal{M}(\mu_0)) \subset \mathcal{M}(\mu_0)$. According to [20, Theorem 3.3] the metric space $(\mathcal{M}(\mu_0), d_H)$ is complete. The use of the classical Banach fixed point theorem shows that there is an invariant measure $\mu^* \in \mathcal{M}(\mu_0)$ and (3.5) holds.

Assume moreover (3.6). To prove asymptotic stability of P fix $\mu \in \mathcal{M}_1(X)$ and $\varepsilon > 0$. Since the set $\mathcal{M}(\mu_0)$ is a dense subset of the space $\mathcal{M}_1(X)$ with the Fortet–Mourier metric (see [20, Theorem 3.3]; cf. also [25], p. 8) it follows that there is $\nu \in \mathcal{M}(\mu_0)$ such that $d_{FM}(\mu, \nu) < \varepsilon$. By (3.5) we have $d_H(P^n\nu, \mu^*) < \varepsilon$ for large enough $n \in \mathbb{N}$, say $n \ge n_0$. Then for $n \ge n_0$ we obtain

$$d_{FM}(P^n\mu, \mu^*) \le d_{FM}(P^n\mu, P^n\nu) + d_{FM}(P^n\nu, \mu^*) \le d_{FM}(\mu, \nu) + \varepsilon,$$

which ends the proof. □

Consider the family $\mathbf{\Phi}$ of all Markov operators $P : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ such that (3.3) holds with $\lambda = \lambda_P \in [0, 1)$, and (3.4) is fulfilled. On account of Proposition 3.1 operator $P \in \mathbf{\Phi}$ has an invariant measure $\mu^*_P \in \mathcal{M}(\mu_0)$, which is uniquely determined. Therefore we have an operator

$$\mathbf{\Phi} \ni P \longmapsto \mu^*_P \in \mathcal{M}(\mu_0) \tag{3.7}$$

and we are interested in its continuity.

Applying part (ii) of Theorem 3.1 and [20, Lemma 4.3] we obtain the continuity of operator (3.7) on suitable subsets of $\mathbf{\Phi}$.

Theorem 3.3 *Let (T, σ) be a metric space and let $\{P_t : t \in T\}$ be a subset of $\mathbf{\Phi}$ for which $\sup_{t\in T} \lambda_{P_t} < 1$. Assume that for some $M > 0$ and some $0 < \kappa \leq 1$ we have*

$$d_H(P_{t_1}\mu, P_{t_2}\mu) \leq M\big(\sigma(t_1, t_2)\big)^{\kappa} \tag{3.8}$$

for all $\mu \in \mathcal{M}(\mu_0)$ and $t_1, t_2 \in T$. Then

$$T \ni t \longmapsto \mu^*_{P_t} \in (\mathcal{M}(\mu_0), d_H)$$

is continuous.

In next theorems we will not assume the Hölder's condition (3.8). Furthermore, our goal is to obtain some additional properties concerning continuity like estimation of the distance between limit distributions.

From now on, $\mathbb{N}_0$ denotes the set of all nonnegative integers.

Theorem 3.4 *If $P, Q \in \mathbf{\Phi}$, then*

$$d_H(\mu^*_P, \mu^*_Q) \leq \min\left\{\frac{1}{1-\lambda_P}\inf_{\mu\in\mathcal{M}(\mu_0)} \alpha^P_Q(\mu), \frac{1}{1-\lambda_Q}\inf_{\mu\in\mathcal{M}(\mu_0)} \alpha^Q_P(\mu)\right\}, \tag{3.9}$$

where μ^*_P, μ^*_Q is the invariant measure for P, Q, respectively, and,

$$\alpha^Q_P(\mu) = \sup_{n\in\mathbb{N}_0} d_H(QP^n\mu, P^{n+1}\mu), \quad \alpha^P_Q(\mu) = \sup_{n\in\mathbb{N}_0} d_H(PQ^n\mu, Q^{n+1}\mu). \tag{3.10}$$

Proof Assume that $P, Q \in \mathbf{\Phi}$ and fix $\mu \in \mathcal{M}(\mu_0)$. Clearly, P satisfies condition (3.5), by Proposition 3.1. In particular the sequence $(P^n\mu)$ converges in metric d_H to μ^*_P. The same concerns the sequence $(Q^n\mu)$, which tends to μ^*_Q.

We will show that

$$d_H(P^n\mu, Q^n\mu) \leq \sum_{k=1}^{n} \lambda_P^{k-1} d_H\left(P(Q^{n-k}\mu), Q(Q^{n-k}\mu)\right) \tag{3.11}$$

for every $n \in \mathbb{N}$. To do this it is enough to use an induction observing that if the statement (3.11) holds for some arbitrarily fixed n, then it holds for $n + 1$, namely,

$$\begin{aligned} d_H(P^{n+1}\mu, Q^{n+1}\mu) &\leq d_H\left(P^{n+1}\mu, P(Q^n\mu)\right) + d_H\left(P(Q^n\mu), Q^{n+1}\mu\right) \\ &\leq \lambda_P\, d_H\left(P^n\mu, Q^n\mu\right) + d_H\left(P(Q^n\mu), Q(Q^n\mu)\right) \\ &\leq \sum_{k=1}^{n+1} \lambda_P^{k-1} d_H\left(P(Q^{n+1-k}\mu), Q(Q^{n+1-k}\mu)\right). \end{aligned}$$

Therefore by (3.11) we have

$$d_H(P^n\mu, Q^n\mu) \le \sum_{k=1}^{\infty} \lambda_P^{k-1} \alpha_Q^P(\mu) = \frac{1}{1-\lambda_P}\alpha_Q^P(\mu)$$

for every $n \in \mathbb{N}$. By symmetry argument we obtain also

$$d_H(P^n\mu, Q^n\mu) \le \frac{1}{1-\lambda_Q}\alpha_P^Q(\mu).$$

Consequently, passing to the limit we have

$$d_H(\mu_P^*, \mu_Q^*) \le \min\left\{\frac{1}{1-\lambda_P}\alpha_Q^P(\mu), \frac{1}{1-\lambda_Q}\alpha_P^Q(\mu)\right\}.$$

Taking infimum of all $\mu \in \mathcal{M}(\mu_0)$ we get (3.9). □

Remark 3.2 Note that in the proof of Proposition 3.1 as well as in the proof of Theorem 3.4 we do not use linearity of Markov operators. Theorem 3.4 can be easily reformulated to the case of a general abstract contraction mappings acting on any (complete) metric space into itself. However our goal is to analyze random iteration, thus we skip an abstract case.

Remark 3.3 Since

$$\begin{aligned} d_H\left(Q(P^n\mu), P^{n+1}\mu\right) &\le d_H\left(Q(P^n\mu), Q(P^n\mu_P^*)\right) + d_H\left(Q\mu_P^*, \mu_P^*\right) \\ &\quad + d_H\left(P^{n+1}\mu_P^*, P^{n+1}\mu\right) \\ &\le (\lambda_Q + \lambda_P)\, d_H(\mu, \mu_P^*) + d_H\left(Q\mu_P^*, \mu_P^*\right) \end{aligned}$$

it follows that $\alpha_P^Q(\mu)$ is finite for $P, Q \in \mathbf{\Phi}$ and $\mu \in \mathcal{M}(\mu_0)$.

Relying on [6, Remark 1], we note the following.

Remark 3.4 (i) Since $d_{FM}(\mu, \nu) \le \min\{d_H(\mu, \nu), 2\}$, we see that from (3.9) we have

$$\begin{aligned} d_{FM}(\mu_P^*, \mu_Q^*) \le \min\Bigg\{&\frac{1}{1-\lambda_P} \inf_{\mu\in\mathcal{M}(\mu_0)} \alpha_Q^P(\mu), \\ &\frac{1}{1-\lambda_Q} \inf_{\mu\in\mathcal{M}(\mu_0)} \alpha_P^Q(\mu), 2\Bigg\}, \end{aligned} \tag{3.12}$$

where $\alpha_P^Q(\mu)$, $\alpha_Q^P(\mu)$, are given by (3.10).

(ii) Note that the right-hand side of (3.9) and of (3.12) are optimal in the sense that if X has at least two elements, then there are $P, Q \in \mathbf{\Phi}$ such that a distance between μ_P^* and μ_Q^* is positive and inequality in (3.9) as well as in (3.12) must actually be an equality. Indeed, let x, y be different and define $P, Q \in \mathbf{\Phi}$ by $P\mu = \delta_x$, $Q\mu = \delta_y$ for $\mu \in \mathcal{M}_1(X)$. Then $\lambda_P = \lambda_Q = 0$, $\mu_P^* = \delta_x$, $\mu_Q^* = \delta_y$ and

$$\alpha_P^Q(\mu) = \alpha_Q^P(\mu) = d_H(\delta_x, \delta_y) = \rho(x, y), \quad d_{FM}(\mu_P^*, \mu_Q^*) = \min\{\rho(x, y), 2\}.$$

□

Now we will show that the conditions of the above theorem can be expressed in terms of the adjoint operators. To do this let us consider the family $\mathbf{\Psi}$ of all regular Markov operators $P : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ with adjoint operator $P^* : B(X) \to B(X)$ such that

$$|P^*\varphi(x) - P^*\varphi(y)| \le \lambda_P\, \varrho(x, y) \quad \text{for } x, y \in X \text{ and } \varphi \in Lip_1^b(X) \tag{3.13}$$

with $\lambda_P \in [0, 1)$, and (3.4) is fulfilled.

Proposition 3.5 *We have $\mathbf{\Psi} \subset \mathbf{\Phi}$. Moreover, if $P \in \mathbf{\Psi}$, then (3.6) holds and P is asymptotically stable.*

Proof Inclusion $\mathbf{\Psi} \subset \mathbf{\Phi}$ follows from [20, Lemma 4.3]. If $P \in \mathbf{\Psi}$ and $\varphi \in Lip_1(X)$, $||\varphi||_\infty \le 1$, then $P^*\varphi \in Lip_1(X)$, $||P^*\varphi||_\infty \le 1$ and consequently we have

$$\left| \int \varphi dP\mu - \int \varphi dP\nu \right| \le d_{FM}(\mu, \nu) \quad \text{for } \mu, \nu \in \mathcal{M}_1(X),$$

which shows (3.6). To finish the proof it is enough to apply Proposition 3.1. □

Remark 3.6 An assertion concerning asymptotic stability of $P \in \mathbf{\Psi}$ is in fact a part of [20, Theorem 4.1] and of [25, Theorem 3.2]. Proofs of Propositions 3.1, 3.5 are based on ideas from [20].

According to Theorem 3.4 and Proposition 3.5 we have the following corollary, which ensures an estimation of the limit distributions without assumption (3.8).

Corollary 3.1 *If $P, Q \in \mathbf{\Psi}$, then*

$$d_H(\mu_P^*, \mu_Q^*) \le \min\left\{ \frac{1}{1-\lambda_P} \inf_{\mu \in \mathcal{M}(\mu_0)} \alpha_Q^P(\mu), \frac{1}{1-\lambda_Q} \inf_{\mu \in \mathcal{M}(\mu_0)} \alpha_P^Q(\mu) \right\}, \tag{3.14}$$

holds, where μ_P^*, μ_Q^* is the invariant measures for P, Q, respectively, and,

$$\alpha_P^Q(\mu) = \sup\left\{ \left| \int \varphi d(QP^n - P^{n+1})\mu \right| : n \in \mathbb{N}_0, \varphi \in Lip_1^b(X) \right\}, \tag{3.15}$$

$$\alpha_Q^P(\mu) = \sup\left\{ \left| \int \varphi d(PQ^n - Q^{n+1})\mu \right| : n \in \mathbb{N}_0, \varphi \in Lip_1^b(X) \right\}. \tag{3.16}$$

4 Applications to Random-Valued Functions

Assume that $f : X \times \Omega \to X$ is a fixed rv-function. Recall that

$$\pi_n(x, B) = \mathbb{P}^\infty(f^n(x, \cdot) \in B) \quad \text{for } x \in X, n \in \mathbb{N}, B \in \mathcal{B}(X).$$

Note that for any fixed $x \in X$ and $B \in \mathcal{B}(X)$ the function $\pi_1(x, \cdot)$ is a probability distribution on X and $\pi_1(\cdot, B)$ is a Borel-measurable function. In consequence the operator P^* given by

$$P^*\varphi(x) = \int_\Omega \varphi(f(x, \omega))\mathbb{P}(d\omega), \quad \varphi \in B(X), x \in X, \tag{4.1}$$

is adjoint to (2.2), and in addition,

$$\pi_n(x, B) = P^{*n}\mathbf{1}_B(x) = P^n\delta_x(B) \tag{4.2}$$

for $x \in X$ and $B \in \mathcal{B}(X)$.

Following [7] we consider the set $\boldsymbol{\Upsilon}_{\mathbf{rv}}$ of all rv-functions $f : X \times \Omega \to X$ such that

$$\int_\Omega \rho(f(x, \omega), f(y, \omega))\mathbb{P}(d\omega) \le \lambda_f \rho(x, y) \quad \text{for } x, y \in X$$

with a $\lambda_f \in [0, 1)$, and

$$\int_\Omega \rho(f(x_0, \omega), x_0)\mathbb{P}(d\omega) < \infty \tag{4.3}$$

for some (thus for all) $x_0 \in X$.

We say that $f \in \boldsymbol{\Upsilon}_{\mathbf{rv}}$ is a *kernel* of P, if P^* has form (4.1). By $\boldsymbol{\Upsilon}$ we will denote all regular Markov operators with adjoint operator $P^* : B(X) \to B(X)$ given by (4.1) with kernel $f \in \boldsymbol{\Upsilon}_{\mathbf{rv}}$.

In the main result of this section a role of the family $\mathcal{M}(\mu_0)$ given by (3.2) will be played by a family $\mathcal{M}_1^1(X)$ defined as

$$\mathcal{M}_1^1(X) = \left\{\mu \in \mathcal{M}_1(X) : \int \varrho(x, x_0)\mu(dx) < \infty\right\}.$$

Note that the set $\mathcal{M}_1^1(X)$ is independent of x_0. This set consists of all Borel measures on X with the first moment finite. One can show that $\mathcal{M}_1^1(X) = \mathcal{M}(\delta_{x_0})$; see [20, Lemma 3.1].

Theorem 4.1 *We have $\boldsymbol{\Upsilon} \subset \boldsymbol{\Psi}$. If $P, Q \in \boldsymbol{\Upsilon}$, then (3.14) holds, where $\mu_P^*, \mu_Q^* \in \mathcal{M}_1^1(X)$ are the invariant measures for P, Q, respectively, and $\alpha_P^Q(\mu)$, $\alpha_Q^P(\mu)$ are* *defined by* (3.15)*,* (3.16)*. Moreover, if*

$$\alpha_f^g(x,n) = \int_{\Omega^\infty}\int_\Omega \rho\left(g(f^n(x,\omega),\varpi),\, f(f^n(x,\omega),\varpi)\right)\mathbb{P}(d\varpi)\mathbb{P}^\infty(d\omega),$$
$$\alpha_g^f(x,n) = \int_{\Omega^\infty}\int_\Omega \rho\left(f(g^n(x,\omega),\varpi),\, g(g^n(x,\omega),\varpi)\right)\mathbb{P}(d\varpi)\mathbb{P}^\infty(d\omega)$$

for $n \in \mathbb{N}_0$ and $x \in X$, then for any $\mu \in \mathcal{M}_1^1(X)$ we have

$$\alpha_Q^P(\mu) \le \sup_{n\in\mathbb{N}_0}\int_X \alpha_g^f(x,n)\mu(dx) \quad and \quad \alpha_P^Q(\mu) \le \sup_{n\in\mathbb{N}_0}\int_X \alpha_f^g(x,n)\mu(dx), \tag{4.4}$$

where $f, g \in \mathbf{\Upsilon_{rv}}$ is a kernel of P and Q, respectively. In particular,

$$d_H(\mu_P^*, \mu_Q^*) \le \min\left\{\frac{1}{1-\lambda_f}\inf_{x\in X}\alpha_g^f(x),\, \frac{1}{1-\lambda_g}\inf_{x\in X}\alpha_f^g(x)\right\}, \tag{4.5}$$

where

$$\alpha_g^f(x) = sup_{n\in\mathbb{N}_0}\alpha_g^f(x,n) \quad and \quad \alpha_f^g(x) = sup_{n\in\mathbb{N}_0}\alpha_f^g(x,n). \tag{4.6}$$

Proof Assume that $f, g \in \mathbf{\Upsilon_{rv}}$ are as in the statement and fix $\varphi \in Lip_1^b(X)$. Observe that

$$|P^*\varphi(x) - P^*\varphi(y)| \le \int_\Omega \rho(f(x,\omega), f(y,\omega))\mathbb{P}(d\omega) \le \lambda_f\rho(x,y),$$

i.e. (3.13) holds. Moreover,

$$\begin{aligned}\left|\int \varphi dP\delta_{x_0} - \int \varphi d\delta_{x_0}\right| &= \left|P^*\varphi(x_0) - \varphi(x_0)\right| \\ &\le \int_\Omega \left|\varphi(f(x_0,\omega)) - \varphi(x_0)\right|\mathbb{P}(d\omega) \\ &\le \int_\Omega \rho(f(x_0,\omega), x_0)\mathbb{P}(d\omega),\end{aligned}$$

and by (4.3) we obtain (3.4) with $\mu = \delta_{x_0}$. Clearly, the same conclusion can be drawn for any rv-function from $\mathbf{\Upsilon_{rv}}$, which shows that $\mathbf{\Upsilon} \subset \mathbf{\Psi}$.

Due to Corollary 3.1 we have (3.14), where $\lambda_P = \lambda_f, \lambda_Q = \lambda_g$. In addition, an easy calculation shows that

$$\begin{aligned}&\left|\int \varphi dPQ^n\mu - \int \varphi dQ^{n+1}\mu\right| \\ &\qquad = \left|\int_X Q^{n*}(P^*\varphi)(x)\mu(dx) - \int_X Q^{n*}(Q^*\varphi)(x)\mu(dx)\right|\end{aligned}$$

$$
\begin{aligned}
&= \Big| \int_X \int_{\Omega^\infty} P^*\varphi(g^n(x,\omega))\mathbb{P}^\infty(d\omega)\mu(dx) \\
&\quad - \int_X \int_{\Omega^\infty} Q^*\varphi(g^n(x,\omega))\mathbb{P}^\infty(d\omega)\mu(dx) \Big| \\
&= \Big| \int_X \int_{\Omega^\infty} \int_\Omega \varphi\left(f(g^n(x,\omega),\varpi)\right)\mathbb{P}(d\varpi)\mathbb{P}^\infty(d\omega)\mu(dx) \\
&\quad - \int_X \int_{\Omega^\infty} \int_\Omega \varphi\left(g(g^n(x,\omega),\varpi)\right)\mathbb{P}(d\varpi)\mathbb{P}^\infty(d\omega)\mu(dx) \Big|,
\end{aligned}
$$

hence

$$
\begin{aligned}
\alpha_Q^P(\mu) &= \sup\left\{ \Big| \int \varphi d(PQ^n - Q^{n+1})\mu \Big| : n \in \mathbb{N}_0, \varphi \in Lip_1^b(X) \right\} \\
&\le \sup_{n\in\mathbb{N}_0} \int_X \alpha_g^f(x,n)\mu(dx),
\end{aligned}
$$

i.e. (4.4) is fulfilled.

It is obvious that for any $x \in X$ we have

$$
\alpha_Q^P(\delta_x) \le \sup_{n\in\mathbb{N}_0} \alpha_g^f(x,n)
$$

hence,

$$
\inf_{\mu\in\mathcal{M}_1^1(X)} \alpha_Q^P(\mu) \le \inf_{x\in X} \alpha_Q^P(\delta_x) \le \inf_{x\in X} \alpha_g^f(x).
$$

Since roles f and g are symmetrical, the proof is finished. □

To illustrate our theorem, let us state a corollary, which is the main result of [6]. Recall only that by the *convergence in distribution* or *in law* of the sequence of iterates $(f^n(x,\cdot))$ we mean that the sequence $(\pi_n(x,\cdot))$ converges weakly to a probability distribution.

Corollary 4.1 *If $f, g \in \boldsymbol{\Upsilon}_{\mathbf{rv}}$, then the sequences of iterates $(f^n(x,\cdot))$, $(g^n(x,\cdot))$ are convergent in law to the probability distributions $\pi^f, \pi^g \in \mathcal{M}_1^1(X)$, respectively, the limits do not depend on $x \in X$, and*

$$
d_H(\pi^f,\pi^g) \le \frac{1}{1-\min\left\{\lambda_f,\lambda_g\right\}} \sup_{x\in X} \int_\Omega \rho(f(x,\omega),g(x,\omega))\mathbb{P}(d\omega). \tag{4.7}
$$

Proof Note that under above notation operators P, Q with kernels f, g are asymptotically stable and according to (4.2) the sequences of iterates $(f^n(x,\cdot))$, $(g^n(x,\cdot))$ converge in law to probability distributions π^f, π^g, which are in fact invariant measures of P, Q, respectively. Moreover, according to (4.6) for any $y \in X$ we have

$$\alpha_Q^P(\delta_y) \le \alpha_g^f(y) \le \sup_{x\in X} \int_\Omega \rho(f(x,\omega), g(x,\omega))\mathbb{P}(d\omega).$$

In consequence, by Theorem 4.1 we have (4.7). □

Remark 4.2 (i) Originally in [6], instead of metric d_H, there was d_{FM}. This has been strengthened in the paper [7] to the Hutchinson metric in a vector case and the fact that distributions π^f, π^g belong to $\mathcal{M}_1^1(X)$.

(ii) As it was shown in [7, 8] there are many applications of Corollary 4.1. In next two sections we enlarge possible applications of Theorem 4.1 (cf. Remark 5.1).

5 Applications to Random Affine Maps and Perpetuities

Assume now that $(X, ||\cdot||)$ is a separable Banach space. An important class of rv-functions is a family of so called *random affine maps*. Following [15] an rv-function $f : X \times \Omega \to X$ is said to be a random affine map if it has a form

$$f(x,\omega) = \xi_f(\omega)x + \eta_f(\omega) \tag{5.1}$$

for some random variables $\xi_f : \Omega \to \mathbb{R}, \eta_f : \Omega \to X$. We normally omit the argument ω and just write $f(x,\cdot) = \xi_f x + \eta_f$. From now on we will use the symbols $\mathbb{E}$ and $\mathbb{E}_\infty$ to denote expectations with respect to $\mathbb{P}$ and $\mathbb{P}^\infty$, respectively. For any random variable $\zeta : \Delta \to X$ on a probability space (Δ, ν) the symbol $\mathcal{L}(\zeta)$ will be reserved for the distribution of ζ, i.e. $\mathcal{L}(\zeta)(B) = \nu(\zeta \in B)$ for $B \in \mathcal{B}(X)$.

Denote by $\boldsymbol{\Upsilon}_{\mathbf{aff}}$ the set of all random affine maps of the form (5.1) such that

$$\mathbb{E}|\xi_f| < 1, \quad \mathbb{E}||\eta_f|| < \infty. \tag{5.2}$$

Clearly, $\boldsymbol{\Upsilon}_{\mathbf{aff}}$ is included in $\boldsymbol{\Upsilon}_{\mathbf{rv}}$.

Remark 5.1 Suppose that X is unbounded. Let $f(x,\cdot) = \xi_f x + \eta_f$ and $g(x,\cdot) = \eta_g x + \eta_g$ with integrable random variables $\xi_f, \eta_f, \xi_g, \eta_g$. Then, assuming $\mathbb{P}(\xi_f \ne \xi_g) > 0$, we see that

$$\begin{aligned}\sup_{x\in X} &\int_\Omega \left|\left|\xi_f(\omega)x + \eta_f(\omega) - \xi_g(\omega)x - \eta_g(\omega)\right|\right| \mathbb{P}(d\omega)\\ &\ge \mathbb{E}|\xi_f - \xi_g| \sup_{x\in X} ||x|| - \mathbb{E}||\eta_f - \eta_g|| = \infty.\end{aligned}$$

Therefore

$$\sup_{x\in X} \int_\Omega \left|\left|f(x,\omega) - g(x,\omega)\right|\right| \mathbb{P}(d\omega) < \infty$$

if and only if $\xi_f = \xi_g$ a.s. □

Above remark shows that Corollary 4.1 is useful for random affine maps f, g only in the case, where $\xi_f = \xi_g$ with probability 1. Our goal is to omit this restriction by the use of Theorem 4.1. The main result of this section reads as follows.

Theorem 5.1 *If $f, g \in \boldsymbol{\Upsilon}_{\mathbf{aff}}$, then the sequences of iterates $(f^n(x,\cdot))$, $(g^n(x,\cdot))$ are convergent in law to some probability distributions $\pi^f, \pi^g \in \mathcal{M}_1^1(X)$, respectively, the limits do not depend on $x \in X$, and*

$$d_H(\pi^f, \pi^g) \le \min\left\{\frac{1}{1-\mathbb{E}|\xi_f|}\Big(\frac{\mathbb{E}||\eta_g||}{1-\mathbb{E}|\xi_g|}\alpha+\beta\Big), \frac{1}{1-\mathbb{E}|\xi_g|}\Big(\frac{\mathbb{E}||\eta_f||}{1-\mathbb{E}|\xi_f|}\alpha+\beta\Big)\right\}, \tag{5.3}$$

with

$$\alpha = \mathbb{E}|\xi_f - \xi_g|, \quad \beta = \mathbb{E}||\eta_f - \eta_g||.$$

Proof Fix $f, g \in \boldsymbol{\Upsilon}_{\mathbf{aff}}$. On account of Corollary 4.1 the sequences $(f^n(x,\cdot))$, $(g^n(x,\cdot))$ are convergent in law to $\pi^f, \pi^g \in \mathcal{M}_1^1(X)$, respectively. (Moreover these probability distributions are attractive in geometric rate on $\mathcal{M}_1^1(X)$; see Propositions 3.1 and 3.5.)

Let ξ_n, η_n be random variables on Ω^∞ defined as

$$\xi_n(\omega) = \xi_g(\omega_n), \quad \eta_n(\omega) = \eta_g(\omega_n), \quad \omega = (\omega_1, \omega_2, \dots),$$

for $n \in \mathbb{N}$. Obviously, $\mathcal{L}(\xi_n) = \mathcal{L}(\xi_g)$ and $\mathcal{L}(\eta_n) = \mathcal{L}(\eta_g)$, hence

$$\mathbb{E}_\infty|\xi_n| = \mathbb{E}|\xi_g| \quad \text{and} \quad \mathbb{E}_\infty||\eta_n|| = \mathbb{E}||\eta_g|| \quad \text{for } n \in \mathbb{N}. \tag{5.4}$$

By easy induction we have

$$g^n(x,\cdot) = x\prod_{k=1}^n \xi_k + \eta_1 \prod_{k=2}^n \xi_k + \eta_2 \prod_{k=3}^n \xi_k + \dots + \eta_{n-1}\xi_n + \eta_n,$$

and, in particular

$$g^n(0,\cdot) = \sum_{j=1}^n \eta_j \prod_{k=j+1}^n \xi_k$$

for $n \in \mathbb{N}$. Then

$$\begin{aligned} &||f(g^n(0,\omega), \varpi) - g(g^n(0,\omega), \varpi)|| \\ &\qquad \le |\xi_f(\varpi) - \xi_g(\varpi)| \sum_{j=1}^n ||\eta_j(\omega)|| \prod_{k=j+1}^n |\xi_k(\omega)| + ||\eta_f(\varpi) - \eta_g(\varpi)|| \end{aligned}$$

for $n \in \mathbb{N}, \omega \in \Omega^\infty$ and $\varpi \in \Omega$. Since $\eta_j, \xi_{j+1}, \ldots, \xi_n$ are independent, from the above and (5.4) it follows that

$$\begin{aligned}
&\int_{\Omega^\infty} \int_\Omega ||f(g^n(0,\omega),\varpi) - g(g^n(0,\omega),\varpi)||\,\mathbb{P}(d\varpi)\mathbb{P}^\infty(d\omega) \\
&\quad \le \int_{\Omega^\infty} \Big(\int_\Omega |\xi_f(\varpi) - \xi_g(\varpi)|\,\mathbb{P}(d\varpi) \Big) \sum_{j=1}^n ||\eta_j(\omega)|| \prod_{k=j+1}^n |\xi_k(\omega)|\,\mathbb{P}^\infty(d\omega) \\
&\qquad + \int_\Omega ||\eta_f(\varpi) - \eta_g(\varpi)||\,\mathbb{P}(d\varpi) \\
&\quad = \alpha \sum_{j=1}^n \mathbb{E}_\infty ||\eta_j|| \prod_{k=j+1}^n \mathbb{E}_\infty|\xi_k| + \beta = \alpha\,\mathbb{E}||\eta_g|| \sum_{j=1}^n \big(\mathbb{E}|\xi_g|\big)^{n-j} + \beta \\
&\quad = \alpha\,\mathbb{E}||\eta_g||\,\frac{1 - \big(\mathbb{E}|\xi_g|\big)^n}{1 - \mathbb{E}|\xi_g|} + \beta
\end{aligned}$$

for $n \in \mathbb{N}$; it is also true for $n = 0$, since

$$\int_{\Omega^\infty} \int_\Omega ||f(g^0(0,\omega),\varpi) - g(g^0(0,\omega),\varpi)||\,\mathbb{P}(d\varpi)\mathbb{P}^\infty(d\omega) = \beta.$$

By definition (4.6) this implies that

$$\inf_{x \in X} \alpha_g^f(x) \le \alpha_g^f(0) \le \alpha\,\frac{\mathbb{E}||\eta_g||}{1 - \mathbb{E}|\xi_g|} + \beta.$$

Of course the same inequality holds if we replace the subscripts g by f and the superscripts f by g. Combining these two inequalities and applying estimation (4.5) from Theorem 4.1 with $\lambda_f = \mathbb{E}|\xi_f|$ and $\lambda_g = \mathbb{E}|\xi_g|$, we obtain (5.3). $\square$

As we have seen in the preliminary section the sequence of iterates $(f^n(x,\cdot))$ is of forward (or inner) iteration type. Let us consider an operator $\sigma_n : \Omega^\infty \to \Omega^\infty$ given by

$$\sigma_n(\omega_1, \omega_2, \ldots) = (\omega_n, \ldots, \omega_1, \omega_{n+1}, \ldots).$$

The sequence $\big(f^n(x, \sigma_n(\cdot))\big)$ forms backward (known also as outer) iterations and probability distribution of $f^n(x, \sigma_n(\cdot))$ coincides with $\pi_n(x,\cdot)$, because operator σ_n preserves probability $\mathbb{P}^\infty$. The advantage of using backward iterations lies in the fact that they may converge almost sure to a limit. In the case of random affine map (5.1) we have

$$f^n(x, \sigma_n(\cdot)) = x \prod_{k=1}^n \xi_k + \sum_{k=1}^n \eta_k \prod_{j=1}^{k-1} \xi_j,$$

where $\xi_n(\omega) = \xi_f(\omega_n)$, $\eta_n(\omega) = \eta_f(\omega_n)$ for $\omega = (\omega_1, \omega_2, \dots) \in \Omega^\infty$. It can be shown that the sequence $(\prod_{k=1}^n \xi_k)_{n\in\mathbb{N}}$ converges a.s. to zero by the Kolmogorov strong law of large numbers, provided $-\infty < \mathbb{E}\log|\xi_f| < 0$ in case $\mathbb{P}(\xi_f = 0) = 0$. If, additionally $\mathbb{E}\log\max\{||\eta_f||, 1\} < +\infty$, then the sequence $\big(\sum_{k=1}^n \eta_k \prod_{j=1}^{k-1} \xi_j\big)_{n\in\mathbb{N}}$ converges absolutely a.s., by the application of [17, Theorem 2] (cf. [15, Corollary 2.14]) to the i.i.d. sequence $(\xi_n, ||\eta_n||)$. The limit is the sum of the series

$$\sum_{n=1}^{\infty} \eta_n \prod_{k=1}^{n-1} \xi_k, \tag{5.5}$$

which is (in the case $X = \mathbb{R}$) the probabilistic formulation of the actuarial notion of a perpetuity. For more details and interesting examples of perpetuities we refer the reader to [19]; papers [15, 17, 23, 29] provide complete characterizations of its a.s. convergence.

Since distribution of (5.5) coincides with π_f we will be able to use Theorem 5.1 to yield that perpetuities change continuously (of course, assumptions like (5.2) are needed). So let us denote by $\boldsymbol{\Sigma}$ a family of all pairs of random variables $\xi : \Omega \to \mathbb{R}$, $\eta : \Omega \to X$ such that

$$\mathbb{E}|\xi| < 1, \quad \mathbb{E}||\eta|| < \infty. \tag{5.6}$$

We henceforth identify $(\xi, \eta) \in \boldsymbol{\Sigma}$ with the series (5.5), where (ξ_n, η_n), $n \in \mathbb{N}$, is a sequence of independent random variables (on an arbitrary probability space, not necessary on the product Ω^∞), identically distributed as (ξ, η).

Suppose that $(\xi, \eta) \in \boldsymbol{\Sigma}$. Let $\mathbb{P}(\xi = 0) = 0$ holds. Assuming integrability of $\log|\xi|$ by the Jensen inequality we obtain $\mathbb{E}\log|\xi| \le \log\mathbb{E}|\xi| < 0$. Moreover,

$$\mathbb{E}\log\max\{||\eta||, 1\} = \int_{\{||\eta||\ge 1\}} \log||\eta||\, d\mathbb{P} \le \int_{\{||\eta||\ge 1\}} ||\eta||\, d\mathbb{P} \le \mathbb{E}\,||\eta||.$$

Summarizing, we have $-\infty < \mathbb{E}\log|\xi| < 0, \mathbb{E}\log\max\{||\eta||, 1\} < \infty$ and due to perpetuity convergence theorem series (5.5) converges a.s. However, it turns out that to get the convergence of (5.5) it is enough to assume (5.6) without any additional assumptions like $\mathbb{P}(\xi = 0) = 0$ or integrability of $\log|\xi|$. Namely, we have the following lemma, which gives more information.

Lemma 5.1 *Assume that $(\xi, \eta) \in \boldsymbol{\Sigma}$. Let (ξ_n, η_n), $n \in \mathbb{N}$, be a sequence of independent random variables on an arbitrary probability space, say on $(\Omega, \mathcal{A}, \mathbb{P})$, identically distributed as (ξ, η). Then*

$$\mathbb{E}\left(\sum_{n=1}^{\infty} ||\eta_n|| \prod_{k=1}^{n-1} |\xi_k|\right) < \infty. \tag{5.7}$$

In particular series (5.5) converges absolutely a.s.

Proof Obviously, $\eta_{k+1}, \xi_1, \ldots, \xi_k$ are independent, $\mathbb{E}|\xi_n| = \mathbb{E}|\xi|$ and $\mathbb{E}||\eta_n|| = \mathbb{E}||\eta||$ for $k, n \in \mathbb{N}$. Using the Lebesgue monotone convergence theorem we conclude that

$$\begin{aligned}\mathbb{E}\left(\sum_{n=1}^{\infty} ||\eta_n|| \prod_{k=1}^{n-1} |\xi_k|\right) &= \lim_{N\to\infty} \mathbb{E}\left(\sum_{n=1}^{N} ||\eta_n|| \prod_{k=1}^{n-1} |\xi_k|\right) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{E}||\eta_n|| \prod_{k=1}^{n-1} \mathbb{E}|\xi_k| \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{E}||\eta||\big(\mathbb{E}|\xi|\big)^{n-1} = \frac{\mathbb{E}||\eta||}{1-\mathbb{E}|\xi|} < \infty,\end{aligned}$$

i.e. (5.7) holds. Therefore $\sum_{n=1}^{\infty} ||\eta_n|| \prod_{k=1}^{n-1} |\xi_k| < \infty$ a.s. □

It is worth pointing out that under some additional assumptions condition (5.7) is equivalent to (5.6) by [2, Theorem 1.4].

Applying Theorem 5.1 we will show how the Hutchinson distance between perpetuities changes depending on the expected value of some random variables.

Proposition 5.1 *If* $(\xi, \eta), (\zeta, \vartheta) \in \mathbf{\Sigma}$*, then*

$$\begin{aligned} &d_H\left(\mathcal{L}\left(\sum_{n=1}^{\infty} \eta_n \prod_{k=1}^{n-1} \xi_k\right), \mathcal{L}\left(\sum_{n=1}^{\infty} \vartheta_n \prod_{k=1}^{n-1} \zeta_k\right)\right) \\ &\quad \le \min\left\{\frac{1}{1-\mathbb{E}|\xi|}\left(\frac{\mathbb{E}||\vartheta||}{1-\mathbb{E}|\zeta|}\alpha + \beta\right), \frac{1}{1-\mathbb{E}|\zeta|}\left(\frac{\mathbb{E}||\eta||}{1-\mathbb{E}|\xi|}\alpha + \beta\right)\right\},\end{aligned}$$

with

$$\alpha = \mathbb{E}|\xi - \zeta|, \quad \beta = \mathbb{E}||\eta - \vartheta||.$$

Proof Observe that if $(\xi, \eta) \in \mathbf{\Sigma}$, then the probability distribution of series (5.5) depends only on the distributions of ξ and η. (Obviously, this series is a.s. convergent by Lemma 5.1.) Indeed, assume that all ξ_n, η_n are defined on a probability space $(\Theta, \mathbb{B})$ and have the same distribution as ξ, η, respectively, i.e. $\mathbb{B} \circ \xi_n^{-1} = \mathbb{P} \circ \xi^{-1} = \mu, \quad \mathbb{B} \circ \eta_n^{-1} = \mathbb{P} \circ \eta^{-1} = \nu$ for any $n \in \mathbb{N}$. Then

$$\mathbb{B}\left(\eta_n \prod_{k=1}^{n-1} \xi_k \in B\right) = \mathbb{B}\big((\xi_1, \ldots, \xi_{n-1}, \eta_n) \in \Phi^{-1}(B)\big) = \left(\bigotimes_{k=1}^{n-1} \mu\right) \otimes \nu\big(\Phi^{-1}(B)\big),$$

where $\Phi(x_1, \ldots, x_n) = x_1 \cdots x_n$, $B \in \mathcal{B}(X)$, provided $(\xi_n, \eta_n), n \in \mathbb{N}$, are independent. From this it follows that the distribution of series (5.5) is just the same as the distribution of perpetuity with ξ_n, η_n given on the product Ω^∞ by $\xi_n(\omega) = \xi(\omega_n), \eta_n(\omega) = \eta(\omega_n)$. Thus we can apply Theorem 5.1 to get desired result. □

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Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

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