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Qualitative Behaviour of Stochastic Integro-differential Equations with Random Impulses

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Abstract

In this paper, we study the existence and some stability results of mild solutions for random impulsive stochastic integro-differential equations (RISIDEs) with noncompact semigroups in Hilbert spaces via resolvent operators. Initially, we prove the existence of mild solution for the system is established by using Mönch fixed point theorem and contemplating Hausdorff measures of noncompactness. Then, the stability results includes continuous dependence of solutions on initial conditions, exponential stability and Hyers–Ulam stability for the aforementioned system are investigated. Finally, an example is proposed to validate the obtained results.

Keywords Stochastic integro-differential equations · Random impulse · Noncompact semigroup · Hyers–Ulam stability · Mean-square exponential stability

1 Introduction

Impulsive effects are widespread natural phenomena brought on by instantaneous perturbations at certain moments, such as various biological models including thresholds, exploding explosive models in biological theory in medicine, the optimal control model in economics, and other models so on [1]. In the past several decades, differential equations with impulses are frequently used to simulate processes that are subject to abrupt changes at discrete moments, and the dynamics of impulsive differential equations have attracted a large number of experts; for example, see [1–3]. Furthermore, due to a combination of uncertainty and complexity, mathematical models cannot ignore the stochastic aspects since real-world systems and natural phenomena are almost always

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affected by stochastic disturbances. For elementary study of stochastic differential equations (SDEs) reader may refer to [4–14]. (Partial) differential equations driven by stochastic processes or equations with random impulses give a natural and useful way to describe a variety of impulsive occurrences in order to take them into consideration. Impulse in general occurs as deterministic or random models. Nevertheless by natural phenomena, the impulses often occurs at random time points. Many researches have been undergone solving various differential equations with fixed time impulses [1, 15, 16]. In reality, they have been widely used in the fields of medicine, biology, economy, finance and so on. For example, the classical stock price model [17] of the form

$$
\begin{aligned}
d(\mathrm{S}(t)) &= \mathrm{uS}(t)dt + \Theta \mathrm{S}(t)d\mathrm{B}_t,\ t \geq 0,\ t \neq \zeta_k,\\
\mathrm{S}(\zeta_k) &= \mathrm{b}_k \mathrm{S}(\zeta_k^-),\ k = 1, 2, \cdots,\\
\mathrm{S}(0) &= \mathrm{S}_0,
\end{aligned}
$$

is described using an impulsive SDEs. Here, B_t is a Brownian motion or Wiener process, $\mathrm{S}(t)$ represents the price of the stock at time t, and $\{\zeta_k\}$ represents the release time of the important information relating to the stock, $\mathrm{S}(\zeta_k^-) = \lim_{t\to\zeta_k-0} \mathrm{S}(t)$ and $\mathrm{S}_0 \in \mathrm{R}$. In reality, $\{\zeta_k\}$ is a sequence of random variables, which satisfies $0 < \zeta_1 < \zeta_2 < \cdots$,. In particular, Wu and his team [18] first looked at random impulsive differential equations and solved boundedness of solutions through Lyapunov technique. Furthermore, Wu et.al [19] investigated existence and uniqueness of stochastic differential equations with random impulses. Zhou and Wu [20] established the existence and uniqueness of solutions with random impulses under Lipschitz conditions. Recently, in [21] the authors have contributed the existence and Hyers–Ulam stability of mild solutions for random impulsive stochastic functional differential equations by considering the system,

$$
\begin{aligned}
d(x(t)) &= \mathfrak{f}(t, x_t)dt + \mathfrak{g}(t, x_t)d\omega(t), \quad t \geq t_0, \quad t \neq \xi_{\mathrm{k}},\\
x(\xi_{\mathrm{k}}) &= b_{\mathrm{k}}(\delta_{\mathrm{k}})x(\xi_{\mathrm{k}}^-), \qquad \mathrm{k} = 1, 2, \cdots,\\
x_{t_0} &= \xi = \{\xi_\theta : -\delta \leq \theta \leq 0\},
\end{aligned}
$$

and studied using Krasnoselskii's fixed point theorem. It is known that impulsive stochastic differential equations play a vital role in modelling practical processes. Not only from Guassian white noise there are certain other factors that results in the rise of random effects. For articles related to random impulsive stochastic differential system we may refer to the articles [22–24] and the references therein. Gao and Li [25] used a new criteria to prove the mean-square exponential stability of mild solution for the proposed existence of mild solutions for the impulsive stochastic differential equation with noncompact semigroup. Wu and Zhou [26] investigated the existence and uniqueness of stochastic differential equations with random impulses and Markovian switching under non-Lipschitz conditions. The properties of mild solution to integer-order differential equations with random impulses are investigated by many authors [27–29]. Still, the qualitative behaviour of random impulsive stochastic dynamical systems in Hilbert space remains, which serves as a motivation of this present work.

Remarks: Motivated by the previously mentioned literatures [5, 30–32], we add the random impulses into the system and prove the existence of mild solutions to System 1.1 in this paper, which have not been considered before. Moreover, we obtain various types of stability results, whereas only exponential stability was discussed for impulsive stochastic partial differential equations with noncompact semigroups before.

The novelties of this paper are the following aspects:

- Random impulses have been added into the stochastic integro-differential equations with noncompact semigroups. We discover how stochastic integro-differential equations driven by Brownian motions interact with random impulses in the proof of existence of mild solutions by using the Hausdorff measure of noncompactness and Mönch fixed point theorem via resolvent operator.
- Under the influence of both white noise and random impulses, we investigate stability with continuous dependence of initial conditions, Hyers–Ulam stability and mean-square exponential stability of mild solution for the RISIDEs.
- Inspired by the works mentioned earlier [33], to the best of the author's knowledge, there is no research regarding the theoretical approach to qualitative behaviour of RISIDEs with resolvent operator combining the Mönch fixed point theorem and the Hausdorff measure of noncompactness.

Now, we study the following RISIDEs with noncompact semigroups and varying-time delays:

$$\begin{aligned} d\vartheta(\mathfrak{t}) &= \left[\mathfrak{A}\vartheta(\mathfrak{t}) + \int_0^{\mathfrak{t}} \mathfrak{B}(\mathfrak{t}-s)\vartheta(s)ds + \mathfrak{f}(\mathfrak{t}, \vartheta(\mathfrak{t}-\mu(\mathfrak{t})))\right]d\mathfrak{t} \\ &\quad + \mathfrak{g}(\mathfrak{t}, \vartheta(\mathfrak{t}-\rho(\mathfrak{t})))d\omega(\mathfrak{t}), \quad \mathfrak{t} \geq \mathfrak{t}_0, \mathfrak{t} \neq \xi_k, \\ \vartheta(\xi_k) &= \mathfrak{b}_k(\delta_k)\vartheta(\xi_k^-), \quad k = 1, 2, \cdots, \\ \vartheta_{\mathfrak{t}_0} &= \varphi, \end{aligned} \tag{1.1}$$

where $\mathfrak{A}$ is the infinitesimal generator of a strongly continuous semigroup $(\mathfrak{S}(\mathfrak{t}))_{\mathfrak{t}\geq 0}$ of bounded linear operators in a real separable Hilbert space $\mathbb{X}$, $\mathbb{Y}$ be another separable Hilbert space and $\omega(\mathfrak{t})$ be the standard Weiner process on $\mathbb{X}$. Here $(\mathfrak{B}(t))_{t\geq 0}$ is a closed linear operator on $\mathbb{X}$ with domain $\mathcal{D}(\mathfrak{B}) \supset \mathcal{D}(\mathfrak{A})$. $\mu, \rho : [\mathfrak{t}_0, +\infty] \to [0, \delta]$ are delay functions being continuous; $\mathfrak{f} : [\mathfrak{t}_0, +\infty] \times \mathbb{X} \to \mathbb{X}$ and $\mathfrak{g} : [\mathfrak{t}_0, +\infty] \times \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ are all suitable Borel measurable functions, which will be specified in Sect. 2. Let δ_k be random variable from Ω to $\mathscr{D}_k :=^{def} (0, \mathfrak{d}_k)$ with $0 < \mathfrak{d}_k < +\infty$ for $k = 1, 2, \cdots$. δ_i and δ_j be independent as $i \neq j$ for $i, j = 1, 2, \cdots$. The function $\mathfrak{b}_k$ is defined by $\mathscr{D}_k \to \mathbb{X}$ and $\xi_0 = \mathfrak{t}_0$ and $\xi_k = \xi_{k-1} + \delta_k$ for $k = 1, 2, \cdots$, where $\mathfrak{t}_0 \in [\delta, +\infty]$. It is obvious that

$$\mathfrak{t}_0 = \xi_0 < \xi_1 < \cdots < \lim_{k\to\infty} \xi_k = +\infty,$$

ξ_k is a process with independent increments. We may denote $\vartheta(\xi_k^-) = \lim_{\mathfrak{t}\to\xi_k^-} \vartheta(\mathfrak{t})$, the norm $\|\vartheta\|_{\mathfrak{t}} := \sup_{t-\delta\leq s\leq \mathfrak{t}} \|\vartheta(s)\|_{\mathbb{X}}$, and the jump $\Delta\vartheta(\xi_k) := [\mathfrak{b}_k(\delta_k) - 1]\vartheta(\xi_k^-)$

represents the random impulsive effect in the state ϑ at time ξ_k. The initial data $\varphi : [-\delta, 0] \to \mathbb{X}$ is a function with respect to ϑ when $\mathrm{t} = \mathrm{t}_0$. We may assume that $\{\mathscr{N}(\mathrm{t}), \mathrm{t} \geq 0\}$ is a simple counting process generated by $\{\xi_k\}$, $\Im_{\mathrm{t}}^{(1)}$ be the $\sigma-$algebra generated by $\{\mathscr{N}(\mathrm{t}), \mathrm{t} \geq 0\}$ and $\Im_{\mathrm{t}}^{(2)}$ indicates the $\sigma-$algebra generated by $\{\omega(\mathrm{t}) : \mathrm{t} \geq 0\}$ where $\Im_{\infty}^{(1)}, \Im_{\infty}^{(2)}$ and ξ being mutually independent.

The rest of the article is organized as follows: Sect. 2 is devoted to a collection of definitions and known results to be used in the later part of the article. In Sect. 3, for the proposed system (1.1) the existence of mild solutions are proved by Mönch fixed point theorem. Section 4 is devoted to prove the stability results that includes continuous dependence of solutions on initial conditions, Hyers–Ulam stability in Sects. 4.1 and 4.2 respectively. In Sect. 4.3, in order to prove mean square exponential stability results an integral inequality criteria has been established. Finally an example is provided to validate the obtained results.

2 Preliminaries and Notations

Let $\mathbb{X}$ and $\mathbb{Y}$ be real separable Hilbert space with norm $\|.\|$ and $\|.\|_{\mathbb{Y}}$ and $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ denotes the space of bounded linear operators from $\mathbb{Y}$ to $\mathbb{X}$, $(\Omega, \mathfrak{F}, \mathcal{P})$ be a complete filtered probability space provided the filtration $\Im_{\mathrm{t}}^{(1)} \vee \Im_{\mathrm{t}}^{(2)} (\mathrm{t} \geq 0)$ satisfies the usual notation. Let $\{\beta_n(\mathrm{t}), \mathrm{t} \geq 0\}$ be real-valued one dimensional standard Brownian motion mutually independent over probability space $(\Omega, \mathfrak{F}, \mathcal{P})$. $\mathcal{L}^2(\Omega)$ denote the space of square-integrable random variables for the probability measure $\mathcal{P}$. Let $\mathcal{Q} \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ be a positive trace class operator on $\mathcal{L}^2(\mathbb{X})$ and $(\lambda_n, e_n)_n$ symbolizes its spectral elements. The Weiner process $\omega(\mathrm{t})$ is expressed as follows:

$$\omega(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda_n} \beta_n(\mathrm{t}) e_n \quad \text{with} \quad tr\mathcal{Q} = \sum_{n=1}^{+\infty} \lambda_n < +\infty.$$

Then, the $\mathbb{Y}-$valued stochastic process $\omega(\mathrm{t})$ is called a $\mathcal{Q}-$Weiner process.

Definition 2.1 Let $\Xi \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$, we define

$$\|\Xi\|_{\mathcal{L}_2^0}^2 := tr(\Xi \mathcal{Q} \Xi^*) = \left\{ \sum_{n=1}^{+\infty} \|\sqrt{\lambda_n} \Xi e_n\|^2 \right\}.$$

If $\|\Xi\|_{\mathcal{L}_2^0}^2 < +\infty$, then Ξ is called a $\mathcal{Q}-$Hilbert-Schmidt operator and we define $\mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ is the space of all $\mathcal{Q}-$Schmidt operators $\Xi : \mathbb{Y} \to \mathbb{X}$.

Partial Integro-differential Equations

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces such that

$$|y|_{\mathbb{Y}} := |\mathfrak{A} y| + |y| \quad \text{for} \quad y \in \mathbb{Y}.$$

$\mathfrak{A}$ and $\mathfrak{B}(t)$ are closed linear operator on $\mathbb{X}$.

The notations $\mathscr{C}([0, +\infty); \mathbb{Y})$, and $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ denotes the space of continuous functions from $[0, +\infty)$ into $\mathbb{Y}$, the set of all bounded linear operator from $\mathbb{Y}$ into $\mathbb{X}$, respectively.

Let us consider the problem,

$$d\nu(t) = \left(\mathfrak{A}\nu(t) + \int_0^t \mathfrak{B}(t-s)\nu(s)ds\right)dt, \qquad t \geq 0, \tag{2.1}$$
$$\nu(0) = \nu_0 \in \mathbb{X}.$$

Definition 2.2 [34] A resolvent for Eq. 2.1 is a bounded linear operator valued function $\mathfrak{R}(t) \in \mathcal{L}(\mathbb{X})$ for $\mathrm{t} \geq 0$, having the following properties:
(i) $\mathfrak{R}(0) = \mathcal{I}$ and $\|\mathfrak{R}(t)\| \leq \mathscr{H} e^{\lambda \mathrm{t}}$ for some constants $\mathscr{H}$ and λ.
(ii) For each x in $\mathbb{X}$, the function $t \mapsto \mathfrak{R}(t)x$ is strongly continuous for each $t \geq 0$ and for x in $\mathbb{Y}$, $\mathfrak{R}(.)x \in \mathscr{C}^1([0, +\infty); \mathbb{X}) \cap \mathscr{C}([0, +\infty); \mathbb{Y})$ and satisfies

$$\begin{aligned} d\mathfrak{R}(t)x &= \left(\mathfrak{A}\mathfrak{R}(t)x + \int_0^t \mathfrak{B}(t-s)\mathfrak{R}(s)xds\right)dt, \\ &= \left(\mathfrak{R}(t)\mathfrak{A}x + \int_0^t \mathfrak{R}(t-s)\mathfrak{B}(s)xds\right)dt. \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [34]. To deal with the existence of a resolvent operator, we introduce the following assumptions:

(H1) The operator $\mathfrak{A}$ is an infinitesimal generator of a strongly continuous semigroup $(\mathfrak{S}(\mathrm{t}))_{\mathrm{t}\geq 0}$ on $\mathbb{X}$.
(H2) For all $t \geq 0$, $\mathfrak{B}(t)$ denotes a closed continuous linear operator from $\mathscr{D}(\mathfrak{A})$ to $\mathbb{X}$ and $\mathfrak{B}(t) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$. For any $\mathfrak{y} \in \mathbb{Y}$, the map $t \mapsto \mathfrak{B}(t)\mathfrak{y}$ is bounded, differentiable and its derivative $d\mathfrak{B}(t)\mathfrak{y}/dt$ is bounded and uniformly continuous on $[0, \infty)$.

Now consider the conditions that ensure the existence of solutions to the deterministic Integro-differential equation.

$$d\nu(t) = \left(\mathfrak{A}\nu(t) + \int_0^t \mathfrak{B}(t-s)\nu(s)ds + m(t)\right)dt, \quad t \geq 0, \tag{2.2}$$

with $\nu(0) = \nu_0 \in \mathbb{H}$ and $m : [0, +\infty) \to \mathbb{H}$ is a continuous function.

Lemma 2.1 [34] *Suppose the assumptions (H1) and (H2) hold, and if ν is a strict solution of* (2.2)*, then*

$$\nu(t) = \mathfrak{R}(t)\nu_0 + \int_0^t \mathfrak{R}(t-s)m(s)ds, \quad t \geq 0, \tag{2.3}$$

Lemma 2.2 [34] *Assuming (H1), (H2) hold, the resolvent operator* $\mathfrak{R}(t)$ *is continuous for* $t \geq 0$ *on the operator norm, namely for* $t_0 \geq 0$,

$$\lim_{\tau \to 0} \|\mathfrak{R}(t_0 + \tau) - \mathfrak{R}(t_0)\| = 0.$$

Lemma 2.3 [34] *Let (H1)-(H2) be satisfied. Then there is a* $\mathscr{G}$ *such that*

$$\|\mathfrak{R}(t + \epsilon) - \mathfrak{R}(\epsilon)\mathfrak{R}(t)\| \leq \mathscr{G}\epsilon.$$

Lemma 2.4 [33] *If* $\Psi(s)$ *is* $\mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})-$*valued stochastically integrable process in* $[0, \mathtt{T}]$, *then, for every* $\mathfrak{p} \geq 2$, *there exist* $c'_{\mathfrak{p}} = (\mathfrak{p}(\mathfrak{p}-1)/2)^{\mathfrak{p}/2}$ *such that, for every* $\mathrm{t} \geq 0$,

$$\sup_{s \in [0, \mathrm{t}]} \mathbb{E} \left\| \int_0^s \Psi(m) d\omega(m) \right\|^{\mathfrak{p}} \leq \left(\frac{\mathfrak{p}(\mathfrak{p}-1)}{2} \right)^{\frac{p}{2}} \left(\int_0^{\mathrm{t}} \left(\mathbb{E} \|\Psi(s)\|_{\mathcal{L}_2^0}^{\mathfrak{p}} \right)^{\frac{2}{\mathfrak{p}}} ds \right)^{\frac{\mathfrak{p}}{2}}.$$

Let us recall some facts of the Hausdorff measure of noncompactness $\alpha(.)$ defined on a bounded subset $\mathscr{E}$ of a Banach space $\mathbb{X}$ by

$$\alpha(\mathscr{E}) = \inf\{\varepsilon > 0 : \mathscr{E} \text{ has a finite } \varepsilon - \text{net in } \mathbb{X}\}.$$

Lemma 2.5 [33] *Let* $\mathbb{X}$ *be a real Banach space and* $\mathscr{M}, \mathscr{N} \subset \mathbb{X}$ *be bounded. Then we have the following properties:*

(1) $\mathscr{M}$ *is precompact if and only if* $\alpha(\mathscr{M}) = 0$;
(2) $\alpha(\mathscr{M}) = \alpha(\overline{\mathscr{M}}) = \alpha(conv\mathscr{M})$, *where* $\overline{\mathscr{M}}$ *and* $conv\mathscr{M}$ *are the closure and the convex hull of* $\mathscr{M}$, *respectively;*
(3) $\alpha(\mathscr{M}) \leq \alpha(\mathscr{N})$ *when* $\mathscr{M} \subset \mathscr{N}$;
(4) $\alpha(\mathscr{M} + \mathscr{N}) \leq \alpha(\mathscr{M}) + \alpha(\mathscr{N})$, *where* $\mathscr{M} + \mathscr{N} = \{\vartheta + \varpi : \vartheta \in \mathscr{M}, \varpi \in \mathscr{N}\}$;
(5) $\alpha(\mathscr{M} \cup \mathscr{N}) \leq \max\{\alpha(\mathscr{M}), \alpha(\mathscr{N})\}$;
(6) $\alpha(\lambda\mathscr{M}) \leq |\lambda|\alpha(\mathscr{N})$ *for any* $\lambda \in \mathscr{R}$;
(7) *If* $\mathscr{K} \subset \mathscr{C}([0, \mathtt{T}])$ *is bounded, then*

$$\alpha(\mathscr{K}(\mathrm{t})) \leq \alpha(\mathscr{K}) \ \forall \ \mathrm{t} \in [0, \mathtt{T}],$$

where $\mathscr{K}(\mathrm{t}) = \{m(\mathrm{t}) : m \in \mathscr{K} \subset \mathbb{X}\}$. *Further, if* $\mathscr{K}$ *is equicontinuous on* $[0, \mathtt{T}]$, *then* $\mathrm{t} \to \mathscr{K}(\mathrm{t})$ *is continuous on* $[0, \mathtt{T}]$, *and* $\alpha(\mathscr{K}) = \sup\{\mathscr{K}(\mathrm{t}) : \mathrm{t} \in [0, \mathtt{T}]\}$;
(8) *If* $\mathscr{K} \subset \mathscr{C}([0, \mathtt{T}], \mathbb{X})$ *is bounded and equicontinuous, then* $\mathrm{t} \to \alpha(\mathscr{K}(\mathrm{t}))$ *is continuous on* $[0, \mathtt{T}]$ *and* $\alpha\left(\int_0^{\mathrm{t}} \mathscr{K}(s)ds\right) \leq \int_0^{\mathrm{t}} \alpha(\mathscr{K}(s))ds \ \forall \ \mathrm{t} \in [0, \mathtt{T}]$ *where* $\int_0^{\mathrm{t}} \mathscr{K}(s)ds = \{\int_0^{\mathrm{t}} m(s)ds : m \in \mathscr{K}\}$;
(9) *Let* $\{m_n\}_{n=1}^{\infty}$ *be a sequence of Bochner integrable functions from* $[0, \mathtt{T}]$ *to* $\mathbb{X}$ *with* $\|m_n(\mathrm{t})\| \leq \hat{u}(\mathrm{t})$ *for almost all* $\mathrm{t} \in [0, \mathtt{T}]$ *and* $n \geq 1$, *where* $\hat{u} \in \mathcal{L}^1([0, \mathtt{T}], \mathscr{R}^+)$, *then* $\Psi(\mathrm{t}) = \alpha(\{m_n(\mathrm{t})\}_{n=1}^{\infty}) \in \mathcal{L}^1([0, \mathtt{T}], \mathscr{R}^+)$ *and satisfies*

$$\alpha\left(\left\{\int_0^{\mathrm{t}} m_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^{\mathrm{t}} \Psi(s)ds.$$

Lemma 2.6 [33] *If* $\mathscr{K} \subset \mathscr{C}([0, \mathtt{T}], \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X}))$ *and* ω *being a Weiner process, then*

$$\alpha\left(\int_0^{\mathfrak{t}} \mathscr{K}(s)d\omega(s)\right) \leq \sqrt{\mathtt{T}}\alpha(\mathscr{K}(\mathfrak{t})),$$

where,

$$\int_0^{\mathfrak{t}} \mathscr{K}(s)d\omega(s) = \left\{\int_0^{\mathfrak{t}} m(s)d\omega(s) : \ \forall\, m \in \mathscr{K}, \ \mathfrak{t} \in [0, \mathtt{T}]\right\}.$$

Lemma 2.7 [33] *Let* $\mathbb{D}$ *be a closed convex subset of* $\mathbb{X}$ *with* $0 \in \mathbb{D}$. *Suppose* $\Psi : \mathbb{D} \to \mathbb{D}$ *is a continuous map of Mönch type which satisfies:*

$$\begin{gathered}\mathcal{M} \subset \mathbb{D} \ \ countable \ \ and \ \ \mathcal{M} \subset \overline{co}(\{0\} \cup \Psi(\mathcal{M})) \\ implies \ \ that \ \ \mathcal{M} \ \ is \ \ relatively \ \ compact,\end{gathered}$$

then, Ψ *has a fixed point in* $\mathbb{D}$.

3 Existence of Mild Solutions

Definition 3.1 For $\mathtt{T} \in (\mathfrak{t}_0, +\infty)$, an $\mathbb{X}-$valued stochastic process $\{\vartheta(\mathfrak{t}), \mathfrak{t} \in [\mathfrak{t}_0 - \delta, \mathtt{T}]\}$ is said to satisfy the variation of constants formula of (1.1) if

(1) $\vartheta(\mathfrak{t})$ is an $\Im_{\mathfrak{t}}-$adapted process for $\mathfrak{t} \geq \mathfrak{t}_0$;
(2) $\vartheta(\mathfrak{t}) \in \mathbb{X}$ has a cadlag path on $\mathfrak{t} \in [\mathfrak{t}_0, \mathtt{T}]$ almost surely;
(3) $\vartheta(\mathfrak{t}) = \varphi$ if $\mathfrak{t} \in [-\delta, 0]$ and for each $\mathfrak{t} \in [\mathfrak{t}_0, \mathtt{T}]$, we have

$$\begin{aligned}\vartheta(\mathfrak{t}) = \sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathfrak{t}-\mathfrak{t}_0)\varphi(0)\\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds\\ &+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds\\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\\ &+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}),\end{aligned}$$

where $\prod_{j=i}^{k}(.) = 1$ as $i > k$, $\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) = \mathfrak{b}_k(\delta_k)\mathfrak{b}_{k-1}(\delta_{k-1}) \cdots \mathfrak{b}_i(\delta_i)$, and $\mathcal{I}_{\mathfrak{A}}$ represents the indicator function,

$$\mathcal{I}_{\mathfrak{A}}(\mathfrak{t}) = \begin{cases} 1, & if\ \mathfrak{t} \in \mathfrak{A}, \\ 0, & if\ \mathfrak{t} \notin \mathfrak{A}. \end{cases}$$

The following assumptions are taken into consideration to prove our main results:

(A1) There exists a positive constant $\mathscr{H}$, such that for all $\mathfrak{t} \geq 0$, $\|\mathfrak{R}(\mathfrak{t})\| \leq \mathscr{H}$,

(A2) The function $\mathfrak{f} : [\mathfrak{t}_0, \mathtt{T}] \times \mathbb{X} \to \mathbb{X}$ satisfies

- (a) (i) $\mathfrak{f}(., \vartheta) : [\mathfrak{t}_0, \mathtt{T}] \to \mathbb{X}$ is measurable for each $\vartheta \in \mathbb{X}$ and $\mathfrak{f}(\mathfrak{t}, .) : \mathbb{X} \to \mathbb{X}$ is continuous for each $\mathfrak{t} \in [\mathfrak{t}_0, \mathtt{T}]$.
- (b) (ii) There occurs a continuous function $\nu_{\mathfrak{f}} : [\mathfrak{t}_0, \mathtt{T}] \to \mathscr{R}^+$ and a continuous non-decreasing function $\Gamma_{\mathfrak{f}} : \mathscr{R}^+ \to \mathscr{R}^+$ and $\|\vartheta\|^2 \leq \mathfrak{r}$ such that

$$\|\mathfrak{f}(\mathfrak{t}, \vartheta)\|^2 \leq \nu_{\mathfrak{f}}(\mathfrak{t})\Gamma_{\mathfrak{f}}(\|\vartheta\|^2) \leq \nu_{\mathfrak{f}}(\mathfrak{t})\Gamma_{\mathfrak{f}}(\mathfrak{r}).$$

- (c) (iii) There exists a positive function $\mathscr{C}_{\mathfrak{f}}(\mathfrak{t}) \in \mathcal{L}^1([\mathfrak{t}_0, \mathtt{T}]), \mathscr{R}^+$ such that for any bounded subsets $\beta_1 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{f}(\mathfrak{t}, \beta_1(\mathfrak{t}))) \leq \mathscr{C}_{\mathfrak{f}}(\mathfrak{t}) \sup_{\theta \in (-\delta, 0]} \alpha(\beta_1(\theta)).$$

(A3) The function $\mathfrak{g} : [\mathfrak{t}_0, \mathtt{T}] \times \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ satisfies

- (a) (i) $\mathfrak{g}(., \vartheta) : [\mathfrak{t}_0, , \mathtt{T}] \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ is measurable for each $\vartheta \in \mathbb{X}$ and $\mathfrak{g}(\mathfrak{t}, .) : \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ is continuous for each $\mathfrak{t} \in [\mathfrak{t}_0, \mathtt{T}]$.
- (b) (ii) There exist a continuous function $\nu_{\mathfrak{g}} : [\mathfrak{t}_0, \mathtt{T}] \to \mathscr{R}^+$ and a continuous non-decreasing function $\Gamma_{\mathfrak{g}} : \mathscr{R}^+ \to \mathscr{R}^+$ and $\|\vartheta\|^2 \leq \mathfrak{r}$ such that

$$\|\mathfrak{g}(\mathfrak{t}, \vartheta)\|^2 \leq \nu_{\mathfrak{g}}(\mathfrak{t})\Gamma_{\mathfrak{g}}(\|\vartheta\|^2) \leq \nu_{\mathfrak{g}}(\mathfrak{t})\Gamma_{\mathfrak{g}}(\mathfrak{r}).$$

- (c) (iii) There exists a positive function $\mathscr{C}_{\mathfrak{g}} \in \mathcal{L}^1([\mathfrak{t}_0, \mathtt{T}]), \mathscr{R}^+$ such that for any bounded subsets $\beta_2 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{g}(\mathfrak{t}, \beta_2)) \leq \mathscr{C}_{\mathfrak{g}} \sup_{\theta \in (-\delta, 0]} \alpha(\beta_2(\theta)).$$

(A4) $\max_{i,k} \left\{ \prod_{j=i}^{k} \|\mathfrak{b}_j(\delta_j)\| \right\} < +\infty$ and there exist a constant $\mathscr{B} > 0$ such that

$$\mathbb{E}\left(\max_{i,k} \left\{ \prod_{j=i}^{k} \|\mathfrak{b}_j(\delta_j)\| \right\} \right) \leq \mathscr{B} \ \text{ for all } \ \delta_j \in \mathscr{D}_j, \ j \in \mathbb{N}.$$

(A5) $3\max\{1,\mathscr{B}^2\}(\mathrm{T}-\mathfrak{t}_0)\mathscr{H}^2\Big[\lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{f}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathfrak{t}_0}^{\mathfrak{t}}\nu_{\mathfrak{f}}(s)ds+\lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{g}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathfrak{t}_0}^{\mathfrak{t}}\nu_{\mathfrak{g}}(s)$ $ds\Big]\le 1.$

Theorem 3.1 *Assume the conditions (H1),(H2) and (A1)-(A5) hold, then there exists at least one mild solution for* (1.1) *provided:*

$$\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathfrak{t}_0)\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1([\mathfrak{t}_0,\mathrm{T}],\mathscr{R}^+)} + \max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathfrak{t}_0)^{\frac{1}{2}}\|\mathscr{C}_{\mathfrak{g}}\|_{\mathcal{L}^2([\mathfrak{t}_0,\mathrm{T}],\mathscr{R}^+)} < 1. \tag{3.1}$$

Proof Let us introduce the set $\left\{\Upsilon_{\mathrm{T}} : PC\left([\mathfrak{t}_0-\delta,\mathrm{T}],\mathcal{L}^2(\Omega,\mathbb{X})\right)\right\}$ equipped with the norm

$$\|\vartheta\|^2_{\Upsilon_{\mathrm{T}}}=\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\mathbb{E}\|\vartheta\|^2_{\mathfrak{t}}=\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\mathbb{E}\left(\sup_{\mathfrak{t}-\delta\le s\le\mathfrak{t}}\|\vartheta(s)\|^2\right).$$

It is obvious that Υ_{T} is a Banach space and we may define

$$\overline{\Upsilon}_{\mathrm{T}}=\{\vartheta\in\Upsilon_{\mathrm{T}}:\vartheta(s)=\varphi(s),\quad for\ \ s\in[-\delta,0]\}$$

with the norm $\|.\|^2_{\Upsilon_{\mathrm{T}}}$. Thus, (1.1) can be transformed into a fixed point problem. We can define an operator $\Theta:\overline{\Upsilon}_{\mathrm{T}}\to\overline{\Upsilon}_{\mathrm{T}}$ by

$$\begin{aligned}(\Theta\vartheta)(\mathfrak{t})=\sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathfrak{t}-\mathfrak{t}_0)\varphi(0)\\&+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds\\&+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds\\&+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\\&+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}),\quad\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}],\end{aligned}$$

and

$$(\Theta\vartheta)=\varphi(\theta),\quad\mathfrak{t}\in[-\delta,0].$$

Let us divide our proof into several steps:

Step 1: Initially, we have to verify that Θ satisfies the property $\Theta(\mathbb{B}_{\mathfrak{r}})\subset\mathbb{B}_{\mathfrak{r}}$, $\mathbb{B}_{\mathfrak{r}}=\{\vartheta\in\Upsilon_{\mathrm{T}}:\|\vartheta\|^2_{\Upsilon_{\mathrm{T}}}\le\mathfrak{r}\}$. If the result contradicts, for $\vartheta\in\mathbb{B}_{\mathfrak{r}}$, $\Theta(\mathbb{B}_{\mathfrak{r}})\not\subseteq\mathbb{B}_{\mathfrak{r}}$. Thus, we may find $\mathrm{t} \in [\mathrm{t}_0, \mathrm{T}]$ satisfying $\mathbb{E}\|(\Theta\vartheta)(\mathrm{t})\|^2 > \mathfrak{r}$. By the aforementioned assumptions,

$$\begin{aligned}
\mathbb{E}\|(\Theta\vartheta)(\mathrm{t})\|^2 &= \mathbb{E}\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\varphi(0) \\
&\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds \\
&\quad+\int_{\xi_k}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds \\
&\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s) \\
&\quad+\int_{\xi_k}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathrm{t})\Bigg\|^2. \\
&\le 3\mathbb{E}\left(\left(\max_k\{\prod_{i=1}^{k}\|\mathfrak{b}_i(\delta_i)\|\}\right)^2\right)\|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\|^2\mathbb{E}\|\varphi(0)\|^2 \\
&\quad+3\mathbb{E}\left(\max_{i,k}\{\prod_{j=i}^{k}\|\mathfrak{b}_j(\delta_j)\|,1\}\right)^2 \\
&\quad\times\mathbb{E}\left(\|\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds\|^2\right) \\
&\quad+3\mathbb{E}\left(\left(\max_{i,k}\{\prod_{j=i}^{k}\|\mathfrak{b}_j(\delta_j)\|,1\}\right)^2\right) \\
&\quad\times\mathbb{E}\left(\|\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\|^2\right) \\
&\le 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\|\varphi(0)\|^2 \\
&\quad+3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{f}}(s)\Gamma_{\mathfrak{f}}(\mathfrak{r})ds+3\max\{1,\mathscr{B}^2\}\mathscr{H}^2 \\
&\quad\times(\mathrm{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{g}}(s)\Gamma_{\mathfrak{g}}(\mathfrak{r})ds.
\end{aligned}$$

Dividing the above inequality by $\mathfrak{r}$, and letting $\mathfrak{r} \to +\infty$, we have

$$\begin{aligned}
&3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathrm{t}_0) \\
&\left(\lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{f}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{f}}(s)ds+\lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{g}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{g}}(s)ds\right) > 1,
\end{aligned}$$

which contradicts our assumption (A4). Thus $\exists$ some $\vartheta \in \mathbb{B}_{\mathfrak{r}}$ such that $\Theta(\mathbb{B}_{\mathfrak{r}}) \subset \mathbb{B}_{\mathfrak{r}}$.

Step 2: In order to verify the continuity of the operator Θ in $\mathbb{B}_{\mathfrak{r}}$, let $\vartheta, \vartheta_n \in \mathbb{B}_{\mathfrak{r}}$ and $\vartheta_n \to \vartheta$ as $n \to +\infty$. By condition (ii) of (A1),(A2), we have

$$\mathfrak{f}(\mathfrak{t}, \vartheta_n) \to \mathfrak{f}(\mathfrak{t}, \vartheta), \quad n \to +\infty, \|\mathfrak{f}(\mathfrak{t}, \vartheta_n) - \mathfrak{f}(\mathfrak{t}, \vartheta)\|^2 \leq 2\nu_{\mathfrak{f}}(\mathfrak{t})\Gamma_{\mathfrak{f}}(\tau),$$
$$\mathfrak{g}(\mathfrak{t}, \vartheta_n) \to \mathfrak{g}(\mathfrak{t}, \vartheta), \quad n \to +\infty, \|\mathfrak{g}(\mathfrak{t}, \vartheta_n) - \mathfrak{g}(\mathfrak{t}, \vartheta)\|^2 \leq 2\nu_{\mathfrak{g}}(\mathfrak{t})\Gamma_{\mathfrak{g}}(\tau).$$

Using Dominated Convergence theorem and (A3), we may deduce that

$$\begin{aligned}
&\mathbb{E}\,\|(\Theta\vartheta_n)(\mathfrak{t}) - (\Theta\vartheta)(\mathfrak{t})\|^2 \\
&\leq 3\mathbb{E}\left\|\sum_{k=0}^{+\infty}\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathfrak{t}-\mathfrak{t}_0)(\vartheta_n(0)-\vartheta(0))\right\|^2 \\
&\quad +3\mathbb{E}\left\|\sum_{k=0}^{+\infty}\left(\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)[\mathfrak{f}(s,\vartheta_n(s-\mu(s))) - \mathfrak{f}(s,\vartheta(s-\mu(s)))]ds\right.\right. \\
&\quad \left.\left. + \int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)[\mathfrak{f}(s,\vartheta_n(s-\mu(s))) - \mathfrak{f}(s,\vartheta(s-\mu(s)))]ds\right)\mathcal{I}_{[\xi_k,\xi_{k+1})}\right\|^2 \\
&\quad +3\mathbb{E}\left\|\sum_{k=0}^{+\infty}\left(\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)[\mathfrak{g}(s,\vartheta_n(s-\rho(s))) - \mathfrak{g}(s,\vartheta(s-\rho(s)))]d\omega(s)\right.\right. \\
&\quad \left.\left. + \int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)[\mathfrak{g}(s,\vartheta_n(s-\rho(s))) - \mathfrak{g}(s,\vartheta(s-\rho(s)))]d\omega(s)\right)\mathcal{I}_{[\xi_k,\xi_{k+1})}\right\|^2 \\
&\leq 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\,\|\vartheta_n(0)-\vartheta(0)\|^2 \\
&\quad +3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathfrak{t}-\mathfrak{t}_0)\int_{\mathfrak{t}_0}^{\mathfrak{t}}\mathbb{E}\,\|\mathfrak{f}(s,\vartheta_n(s-\mu(s))) - \mathfrak{f}(s,\vartheta(s-\mu(s)))\|^2\,ds \\
&\quad +3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathfrak{t}-\mathfrak{t}_0)\int_{\mathfrak{t}_0}^{\mathfrak{t}}\mathbb{E}\,\|\mathfrak{g}(s,\vartheta_n(s-\rho(s))) - \mathfrak{g}(s,\vartheta(s-\rho(s)))\|^2_{\mathcal{L}_2^0}\,ds \\
&\to 0 \;\; as \;\; n \to +\infty.
\end{aligned}$$

Therefore, Θ is continuous on $\mathbb{B}_{\mathfrak{r}}$.

Step 3: To prove Θ is equicontinuous on $[\mathfrak{t}_0, \mathtt{T}]$, for $\mathfrak{t}_0 < \mathfrak{t}_1 < \mathfrak{t}_2 < \mathtt{T}$ and $\vartheta \in \mathbb{B}_{\mathfrak{r}}$, we have

$$\begin{aligned}
&(\Theta\vartheta)(\mathfrak{t}_2) - (\Theta\vartheta)(\mathfrak{t}_1) \\
&= \sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathfrak{t}_2-\mathfrak{t}_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\right. \\
&\qquad \int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds \\
&\qquad + \int_{\xi_k}^{\mathfrak{t}_2}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds
\end{aligned}$$

$$+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)$$

$$+\int_{\xi_k}^{\mathfrak{t}_2}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}_2)$$

$$-\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathfrak{t}_1-\mathfrak{t}_0)\varphi(0)+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)$$

$$\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}_1-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds$$

$$+\int_{\xi_k}^{\mathfrak{t}_1}\mathfrak{R}(\mathfrak{t}_1-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)$$

$$\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}_1-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)$$

$$+\int_{\xi_k}^{\mathfrak{t}_1}\mathfrak{R}(\mathfrak{t}_1-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}_1)$$

$$=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathfrak{t}_2-\mathfrak{t}_0)\varphi(0)$$

$$+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds$$

$$+\int_{\xi_k}^{\mathfrak{t}_2}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)$$

$$\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)$$

$$+\int_{\xi_k}^{\mathfrak{t}_2}\mathfrak{R}(\mathfrak{t}_2-s)\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)\Bigg]\big(\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}_2)-\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}_1)\big)$$

$$+\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)(\mathfrak{R}(\mathfrak{t}_2-\mathfrak{t}_0)-\mathfrak{R}(\mathfrak{t}_2-\mathfrak{t}_1))\varphi(0)$$

$$+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}(\mathfrak{R}(\mathfrak{t}_2-s)-\mathfrak{R}(\mathfrak{t}_1-s))$$

$$\times\mathfrak{f}(s,\vartheta(s-\mu(s)))ds+\int_{\xi_k}^{\mathfrak{t}_2}(\mathfrak{R}(\mathfrak{t}_2-s)-\mathfrak{R}(\mathfrak{t}_1-s))\mathfrak{f}(s,\vartheta(s-\mu(s)))ds$$

$$+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\times\int_{\xi_{k-1}}^{\xi_k}(\mathfrak{R}(\mathfrak{t}_2-s)-\mathfrak{R}(\mathfrak{t}_1-s))\mathfrak{g}(s,\vartheta(s-\rho(s)))d\omega(s)$$

$$
\begin{aligned}
&\quad + \int_{\xi_k}^{\mathfrak{t}_2} (\mathfrak{R}(\mathfrak{t}_2 - s) - \mathfrak{R}(\mathfrak{t}_1 - s)) \\
&\quad \times \mathfrak{g}(s, \vartheta(s - \rho(s))) d\omega(s) \Big] \mathcal{I}_{[\xi_k, \xi_{k+1})}(\mathfrak{t}_1) \\
&= \mathscr{J}_1 + \mathscr{J}_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E}\,\|\mathscr{J}_1\|^2 = \mathbb{E}\Big\| &\sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \mathfrak{b}_i(\delta_i) \mathfrak{R}(\mathfrak{t}_2 - \mathfrak{t}_0) \varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \\
&\int_{\xi_{k-1}}^{\xi_k} \mathfrak{R}(\mathfrak{t}_2 - s) \mathfrak{f}(s, \vartheta(s - \mu(s))) ds \\
&+ \int_{\xi_k}^{\mathfrak{t}_2} \mathfrak{R}(\mathfrak{t}_2 - s) \mathfrak{f}(s, \vartheta(s - \mu(s))) ds \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\xi_{k-1}}^{\xi_k} \mathfrak{R}(\mathfrak{t}_2 - s) \mathfrak{g}(s, \vartheta(s - \rho(s))) d\omega(s) \\
&+ \int_{\xi_k}^{\mathfrak{t}_2} \mathfrak{R}(\mathfrak{t}_2 - s) \mathfrak{g}(s, \vartheta(s - \rho(s))) d\omega(s) \Big] \left(\mathcal{I}_{[\xi_k, \xi_{k+1})}(\mathfrak{t}_2) - \mathcal{I}_{[\xi_k, \xi_{k+1})}(\mathfrak{t}_1) \right) \Big\|^2, \\
\mathbb{E}\,\|\mathscr{J}_2\|^2 = \mathbb{E}\Big\| &\sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \mathfrak{b}_i(\delta_i) (\mathfrak{R}(\mathfrak{t}_2 - \mathfrak{t}_0) - \mathfrak{R}(\mathfrak{t}_2 - \mathfrak{t}_1)) \varphi(0) \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\xi_{k-1}}^{\xi_k} (\mathfrak{R}(\mathfrak{t}_2 - s) - \mathfrak{R}(\mathfrak{t}_1 - s)) \\
&\times \mathfrak{f}(s, \vartheta(s - \mu(s))) ds + \int_{\xi_k}^{\mathfrak{t}_2} (\mathfrak{R}(\mathfrak{t}_2 - s) - \mathfrak{R}(\mathfrak{t}_1 - s)) \mathfrak{f}(s, \vartheta(s - \mu(s))) ds \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\xi_{k-1}}^{\xi_k} (\mathfrak{R}(\mathfrak{t}_2 - s) - \mathfrak{R}(\mathfrak{t}_1 - s)) \mathfrak{g}(s, \vartheta(s - \rho(s))) d\omega(s) \\
&+ \int_{\xi_k}^{\mathfrak{t}_2} (\mathfrak{R}(\mathfrak{t}_2 - s) - \mathfrak{R}(\mathfrak{t}_1 - s)) \\
&\times \mathfrak{g}(s, \vartheta(s - \rho(s))) d\omega(s) \Big] \mathcal{I}_{[\xi_k, \xi_{k+1})}(\mathfrak{t}_1) \Big\|^2.
\end{aligned}
$$

By treating each term separately,

$$
\begin{aligned}
\mathbb{E}\|\mathscr{J}_1\|^2 \leq 3\mathbb{E} &\left(\max_k \left\{ \prod_{i=1}^{k} \|\mathfrak{b}_i(\delta_i)\|^2 \right\} \right) \|\mathfrak{R}(\mathfrak{t}_2 - \mathfrak{t}_0)\|^2 \\
&\mathbb{E}\|\varphi(0)\|^2 \left(\mathcal{I}_{[\xi_k, \xi_{k+1})}(\mathfrak{t}_2) - \mathcal{I}_{[\xi_k, \xi_{k+1})}(\mathfrak{t}_1) \right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+3\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k}\|\mathfrak{b}_i(\delta_i)\|,1\right\}\right)^2\\
&\mathbb{E}\left(\sum_{k=0}^{+\infty}\int_{\mathrm{t}_0}^{\mathrm{t}_2}\|\mathfrak{R}(\mathrm{t}_2-s)\|^2\,\|\mathfrak{f}(s,\vartheta(s-\mu(s)))\|^2\,ds\right)\\
&\times\left(\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathrm{t}_2)-\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathrm{t}_1)\right)^2+3\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k}\|\mathfrak{b}_i(\delta_i)\|,1\right\}\right)^2\\
&\times\mathbb{E}\left(\sum_{k=0}^{+\infty}\int_{\mathrm{t}_0}^{\mathrm{t}_2}\|\mathfrak{R}(\mathrm{t}_2-s)\|^2\,\|\mathfrak{g}(s,\vartheta(s-\rho(s)))\|^2\,d\omega(s)\right)\\
&\left(\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathrm{t}_2)-\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathrm{t}_1)\right)^2\\
&\to 0\ \ as\ \ \mathrm{t}_2\to\mathrm{t}_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}\|\mathscr{J}_2\|^2 \le{}& 5\mathscr{B}^2\,\|\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_0)-\mathfrak{R}(\mathrm{t}_1-\mathrm{t}_0)\|^2\,\mathbb{E}\|\varphi(0)\|^2+5\max\{1,\mathscr{B}^2\}(\mathrm{t}_1-\mathrm{t}_0)\\
&\times\int_{\mathrm{t}_0}^{\mathrm{t}_1}\|\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\|^2\,\mathbb{E}\,\|\mathfrak{f}(s,\vartheta(s-\mu(s)))\|^2\,ds\\
&+5(\mathrm{t}_2-\mathrm{t}_1)\int_{\mathrm{t}_1}^{\mathrm{t}_2}\|\mathfrak{R}(\mathrm{t}_2-s)\|^2\\
&\times\mathbb{E}\,\|\mathfrak{f}(s,\vartheta(s-\mu(s)))\|^2\,ds+5\max\{1,\mathscr{B}^2\}(\mathrm{t}_1-\mathrm{t}_0)\\
&\int_{\mathrm{t}_0}^{\mathrm{t}_1}\|\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\|^2\\
&\times\mathbb{E}\,\|\mathfrak{g}(s,\vartheta(s-\rho(s)))\|^2\,ds+5(\mathrm{t}_2-\mathrm{t}_1)\\
&\int_{\mathrm{t}_1}^{\mathrm{t}_2}\|\mathfrak{R}(\mathrm{t}_2-s)\|^2\mathbb{E}\,\|\mathfrak{g}(s,\vartheta(s-\rho(s)))\|^2\,ds\\
&\to 0\ \ as\ \ \mathrm{t}_2\to\mathrm{t}_1.
\end{aligned}
$$

Thus we have

$$
\mathbb{E}\,\|(\Theta\vartheta)(\mathrm{t}_2)-(\Theta\vartheta)(\mathrm{t}_1)\|^2\to 0\ \ as\ \ \mathrm{t}_2\to\mathrm{t}_1,
$$

which implies Θ is equicontinuous on $[\mathrm{t}_0,\mathtt{T}]$.

Step 4: Now to verify Mönch condition, let $\gamma\subset\Upsilon_{\mathtt{T}}$ be a nonempty set and $\vartheta_1,\vartheta_2\in\gamma$, by probability 1, we have

$$
d(\Theta\vartheta_1(\mathrm{t}),\Theta\vartheta_2(\mathrm{t}))=d(\overline{\Theta}\vartheta_1(\mathrm{t}),\overline{\Theta}\vartheta_2(\mathrm{t})),
$$

where the term d represents the distance and

$$
\begin{aligned}
(\overline{\Theta}\vartheta)(\mathfrak{t}) &= \max\{1, \mathscr{B}\}\sum_{k=0}^{+\infty}\bigg[\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds \\
&\quad + \int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\vartheta(s-\mu(s)))ds\bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}) \\
&\quad + \max\{1, \mathscr{B}\}\sum_{k=0}^{+\infty}\bigg[\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\vartheta(s-\mu(s)))ds \\
&\quad + \int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\vartheta(s-\mu(s)))ds\bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t}) \\
&= \overline{\Theta}_1 + \overline{\Theta}_2.
\end{aligned}
$$

By the similar procedure used in Lemma 2.3,

$$\alpha((\Theta\vartheta)(\mathfrak{t})) = \alpha((\overline{\Theta})(\mathfrak{t})).$$

Let $\Delta \subset \mathbb{B}_{\mathfrak{r}}$ be countable and $\Delta \subset \overline{co}(\{0\} \cup \Theta(\Delta))$. By proving $\alpha(\Delta) = 0$ the verification of Mönch condition follows. Set $\Delta = \{\vartheta^n\}_{n=1}^{\infty}$ where $\Delta \subset \overline{co}(\{0\} \cup \Theta(\Delta))$ is well defined and equicontinuous on $[\mathfrak{t}_0, \mathrm{T}]$ by the process in step 3.
By Lemmas 2.2 and 2.3,

$$
\begin{aligned}
\alpha(\{\overline{\Theta}_1\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}) &\le \max\{1, \mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathfrak{t}_0)\int_{\mathfrak{t}_0}^{\mathfrak{t}}\mathscr{C}_{\mathfrak{f}}(\mathfrak{t})\sup_{\theta\in(-\delta,0]}\alpha(\{\vartheta^n(\theta-\mu(\theta))\}_{n=1}^{\infty})ds \\
&\le \max\{1, \mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathfrak{t}_0)\left\|\mathscr{C}_{\mathfrak{f}}\right\|_{\mathcal{L}^1([\mathfrak{t}_0,\mathrm{T}],\mathfrak{R}^+)}\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\alpha(\{\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}), \\
\alpha(\{\overline{\Theta}_2\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}) &\le \max\{1, \mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathfrak{t}_0)^{\frac{1}{2}}\left\|\mathscr{C}_{\mathfrak{g}}\right\|_{\mathcal{L}^2([\mathfrak{t}_0,\mathrm{T}],\mathfrak{R}^+)}\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\alpha(\{\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}).
\end{aligned}
$$

By using Lemma 2.3,

$$
\begin{aligned}
\alpha(\{\Theta_1\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}) &= \alpha(\{\overline{\Theta}_1\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}) \\
&\le \alpha(\{\overline{\Theta}_1\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}) + \alpha(\{\overline{\Theta}_2\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}) \\
&\le \bigg[\max\{1, \mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathfrak{t}_0)\left\|\mathscr{C}_{\mathfrak{f}}\right\|_{\mathcal{L}^1([\mathfrak{t}_0,\mathrm{T}],\mathfrak{R}^+)} + \max\{1, \mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathfrak{t}_0)^{\frac{1}{2}} \\
&\quad \times \left\|\mathscr{C}_{\mathfrak{g}}\right\|_{\mathcal{L}^2([\mathfrak{t}_0,\mathrm{T}],\mathfrak{R}^+)}\bigg]\alpha(\{\vartheta^n(\mathfrak{t})\}_{n=1}^{\infty}).
\end{aligned}
$$

It follows that

$$\alpha(\Delta) \le \alpha(\overline{co}(\{0\} \cup \Theta(\Delta))) = \alpha(\Theta(\Delta)) \le \alpha(\Delta),$$

implying $\alpha(\Delta) = 0$ and then Δ is a relatively compact set. Thus Θ has a fixed point in Δ which is the mild solution of (1.1). This completes the proof. □

4 Stability Results

4.1 Continuous Dependence of Solutions on Initial Conditions

To prove the stability results, let us assume the following assumptions:

(A5) There exists a constants $\mathscr{C}_1, \mathscr{C}_2$ such that

$$\|\mathfrak{f}(\mathfrak{t}, \vartheta) - \mathfrak{f}(\mathfrak{t}, \varpi)\| \le \mathscr{C}_1 \|\vartheta - \varpi\|, \quad \|\mathfrak{g}(\mathfrak{t}, \vartheta) - \mathfrak{g}(\mathfrak{t}, \varpi)\|_{\mathcal{L}_2^0} \le \mathscr{C}_2 \|\vartheta - \varpi\|_{\mathcal{L}_2^0}.$$

Theorem 4.1 *Let $\vartheta(\mathfrak{t})$ and $\overline{\vartheta}(\mathfrak{t})$ be mild solutions for (1.1) with initial values $\varphi(0)$ and $\overline{\varphi}(0)$ respectively. Assuming (A3), (A5) hold, then the mild solution of (1.1) is stable in the mean square.*

Proof $\mathbb{E}\left\|\vartheta - \overline{\vartheta}\right\|_{\mathfrak{t}}^2$

$$\begin{aligned}
&\le 3\mathbb{E}\left\|\sum_{k=0}^{+\infty}\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\right\|^2 \|\mathfrak{R}(\mathfrak{t}-\mathfrak{t}_0)\|^2\,\mathbb{E}\,\|\varphi(0)-\overline{\varphi}(0)\|^2\\
&\quad +3\mathbb{E}\left\|\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\Big(\mathfrak{f}(s,\vartheta(s-\mu(s)))\right.\right.\\
&\quad -\mathfrak{f}(s,\overline{\vartheta}(s-\mu(s)))\Big)ds+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\,(\mathfrak{f}(s,\vartheta(s-\mu(s)))\\
&\quad \left.\left.-\mathfrak{f}(s,\overline{\vartheta}(s-\mu(s))))\,ds\right]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t})\right\|^2\\
&\quad +3\mathbb{E}\left\|\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\,(\mathfrak{g}(s,\vartheta(s-\rho(s)))\right.\right.\\
&\quad -\mathfrak{g}(s,\overline{\vartheta}(s-\rho(s))))\,ds\\
&\quad \left.\left.+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\,\big(\mathfrak{g}(s,\vartheta(s-\rho(s)))-\mathfrak{g}(s,\overline{\vartheta}(s-\rho(s)))\big)\,ds\right]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t})\right\|^2\\
&\le 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\,\|\varphi(0)-\overline{\varphi}(0)\|^2+3\max\{1,\mathscr{B}^2\}(\mathrm{T}-\mathfrak{t}_0)\\
&\quad \left[\int_{\mathfrak{t}_0}^{\mathfrak{t}}\mathbb{E}\left\|\mathfrak{f}(s,\vartheta(s-\mu(s)))-\mathfrak{f}(s,\overline{\vartheta}(s-\mu(s)))\right\|^2 ds\right.\\
&\quad \left.+\int_{\mathfrak{t}_0}^{\mathfrak{t}}\mathbb{E}\left\|\mathfrak{g}(s,\vartheta(s-\rho(s)))-\mathfrak{g}(s,\overline{\vartheta}(s-\rho(s)))\right\|^2 ds\right]
\end{aligned}$$

which implies

$$\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\mathbb{E}\left\|\vartheta-\overline{\vartheta}\right\|_{\mathfrak{t}}^2 \le 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\,\|\varphi(0)-\overline{\varphi}(0)\|^2$$

$$+3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathtt{T}-\mathtt{t}_0)(\mathscr{C}_1+\mathscr{C}_2)\int_{\mathtt{t}_0}^{\mathtt{t}}\sup_{s\in[\mathtt{t}_0,\mathtt{t}]}\mathbb{E}\left\|\vartheta-\overline{\vartheta}\right\|_s^2 ds.$$

By Gronwall's inequality,

$$\sup_{\mathtt{t}\in[\mathtt{t}_0,\mathtt{T}]}\mathbb{E}\left\|\vartheta-\overline{\vartheta}\right\|_{\mathtt{t}}^2 \le 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\left\|\varphi(0)-\overline{\varphi}(0)\right\|^2 \exp\left\{3\mathscr{H}^2\max\{1,\mathscr{B}^2\}(\mathtt{T}-\mathtt{t}_0)(\mathscr{C}_1+\mathscr{C}_2)\right\}.$$

For $\epsilon>0$, there exists a positive number

$$\tau=\frac{\epsilon}{3\mathscr{B}^2\mathscr{H}^2\exp\{3\mathscr{H}^2\max\{1,\mathscr{B}^2\}(\mathtt{T}-\mathtt{t}_0)(\mathscr{C}_1+\mathscr{C}_2)\}}>0$$

such that $\mathbb{E}\left\|\varphi(0)-\overline{\varphi}(0)\right\|^2<\tau$, then

$$\sup_{\mathtt{t}\in[\mathtt{t}_0,\mathtt{T}]}\mathbb{E}\left\|\vartheta-\overline{\vartheta}\right\|_{\mathtt{t}}^2\le\epsilon.$$

This completes the proof. □

4.2 Hyers–Ulam Stability

Definition 4.1 Suppose that $\varpi(\mathtt{t})$ is a $\mathbb{Y}-$valued stochastic process and there exists a real number $\mathscr{C}>0$ such that for arbitrary $\epsilon>0$ holds

$$\begin{aligned}\mathbb{E}\Bigg\|\varpi(\mathtt{t})-\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathtt{t}-\mathtt{t}_0)\varphi(0)+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathtt{t}-s)\mathfrak{f}(s,\varpi(s-\mu(s)))ds\\+\int_{\xi_k}^{\mathtt{t}}\mathfrak{R}(\mathtt{t}-s)\mathfrak{f}(s,\varpi(s-\mu(s)))ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathtt{t}-s)\mathfrak{g}(s,\varpi(s-\rho(s)))d\omega(s)\\+\int_{\xi_k}^{\mathtt{t}}\mathfrak{R}(\mathtt{t}-s)\mathfrak{g}(s,\varpi(s-\rho(s)))d\omega(s)\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathtt{t})\Bigg\|^2\le\epsilon,\qquad\forall\ \mathtt{t}\in[\mathtt{t}_0,\mathtt{T}].\end{aligned}\tag{4.1}$$

For each solution $\varpi(\mathtt{t})$ with the initial value $\varpi_{\mathtt{t}_0}=\vartheta_{\mathtt{t}_0}=\varphi$, if $\exists$ a solution $\vartheta(\mathtt{t})$ of (1.1) with $\mathbb{E}\left\|\varpi(\mathtt{t})-\vartheta(\mathtt{t})\right\|^2\le\mathscr{C}\epsilon$, for $\mathtt{t}\in[\mathtt{t}_0,\mathtt{T}]$, then (1.1) has Hyers–Ulam Stability.

Theorem 4.2 *Assume conditions (H1), (H2), (A3) and (A5) are satisfied, then (1.1) has the Hyers–Ulam Stability.*

Proof Let $\vartheta(\mathtt{t})$ be a mild solution of (1.1) and $\varpi(\mathtt{t})$ a $\mathbb{Y}-$valued stochastic process satisfy (4.1). Obviously, $\mathbb{E}\left\|\varpi(\mathtt{t})-\vartheta(\mathtt{t})\right\|^2=0$ for $\mathtt{t}\in[-\delta,0]$. Moreover, for $\mathtt{t}\in[\mathtt{t}_0,\mathtt{T}]$, we have

$$
\begin{aligned}
&\mathbb{E}\,\|\varpi-\vartheta\|_{\mathfrak{t}}^{2}\\
&\quad\le 2\mathbb{E}\Bigg\|\varpi(\mathfrak{t})-\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathfrak{t}-\mathfrak{t}_0)\varphi(0)+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\\
&\qquad\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\varpi(s-\mu(s)))ds\\
&\qquad+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{f}(s,\varpi(s-\mu(s)))ds\\
&\qquad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\varpi(s-\rho(s)))d\omega(s)\\
&\qquad+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\mathfrak{g}(s,\varpi(s-\rho(s)))d\omega(s)\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t})\Bigg\|^2\\
&\qquad+2\mathbb{E}\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\\
&\qquad\times\left(\mathfrak{f}(s,\varpi(s-\mu(s)))-\mathfrak{f}(s,\vartheta(s-\mu(s)))\right)ds\\
&\qquad+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\left(\mathfrak{f}(s,\varpi(s-\mu(s)))-\mathfrak{f}(s,\vartheta(s-\mu(s)))\right)ds\\
&\qquad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\left(\mathfrak{g}(s,\varpi(s-\rho(s)))-\mathfrak{g}(s,\vartheta(s-\rho(s)))\right)ds\\
&\qquad+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\left(\mathfrak{g}(s,\varpi(s-\rho(s)))-\mathfrak{g}(s,\vartheta(s-\rho(s)))\right)ds\Bigg]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t})\Bigg\|^2\\
&\quad\le 2\epsilon+2\mathbb{E}\|\mathscr{J}\|^2.
\end{aligned}
$$

Now, we consider

$$
\begin{aligned}
&\mathbb{E}\|\mathscr{J}\|^2\\
&=2\mathbb{E}\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\left(\mathfrak{f}(s,\varpi(s-\mu(s)))\right.\\
&\quad\left.-\mathfrak{f}(s,\vartheta(s-\mu(s)))\right)ds\\
&\quad+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\left(\mathfrak{f}(s,\varpi(s-\mu(s)))-\mathfrak{f}(s,\vartheta(s-\mu(s)))\right)ds\\
&\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k}\mathfrak{R}(\mathfrak{t}-s)\times\left(\mathfrak{g}(s,\varpi(s-\rho(s)))\right.\\
&\quad\left.-\mathfrak{g}(s,\vartheta(s-\rho(s)))\right)ds+\int_{\xi_k}^{\mathfrak{t}}\mathfrak{R}(\mathfrak{t}-s)\Big(\mathfrak{g}(s,\varpi(s-\rho(s)))
\end{aligned}
$$

$$-\mathfrak{g}(s,\vartheta(s-\rho(s)))\Big)ds\Big]\mathcal{I}_{[\xi_k,\xi_{k+1})}(\mathfrak{t})\Big\|^2$$
$$\leq 2\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathfrak{t}_0)\int_{\mathfrak{t}_0}^{\mathfrak{t}}\mathbb{E}\,\|\mathfrak{f}(s,\varpi(s-\mu(s)))-\mathfrak{f}(s,\vartheta(s-\mu(s)))\|^2\,ds$$
$$+2\max\{1,\mathscr{B}^2\}\mathscr{H}^2\int_{\mathfrak{t}_0}^{\mathfrak{t}}\mathbb{E}\,\|\mathfrak{g}(s,\varpi(s-\rho(s)))-\mathfrak{g}(s,\vartheta(s-\rho(s)))\|^2\,ds.$$

Taking supremum on both sides and using (A5),

$$\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\mathbb{E}\,\|\varpi-\vartheta\|_{\mathfrak{t}}^2\leq 2\epsilon+4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathfrak{t}_0)\mathscr{C}_1\int_{\mathfrak{t}_0}^{\mathfrak{t}}\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\mathbb{E}\,\|\varpi-\vartheta\|_s^2\,ds$$
$$+4\max\{1,\mathscr{B}^2\}\mathscr{H}^2\mathscr{C}_2\int_{\mathfrak{t}_0}^{\mathfrak{t}}\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\mathbb{E}\,\|\varpi-\vartheta\|_s^2\,ds.$$

By following Gronwall's inequality, there occurs a constant

$$\mathscr{C}:=2\exp\{\max\{1,\mathscr{B}^2\}\mathscr{H}^2[(\mathrm{T}-\mathfrak{t}_0)\mathscr{C}_1+\mathscr{C}_2]\}>0.$$

This implies that

$$\sup_{\mathfrak{t}\in[\mathfrak{t}_0,\mathrm{T}]}\mathbb{E}\,\|\varpi-\vartheta\|_{\mathfrak{t}}^2\leq\mathscr{C}\epsilon,$$

From which follows, the Hyers–Ulam stability of (1.1). Thus the proof is complete. □

4.3 Mean-Square Exponential Stability

Now we will analyze the exponential stability in the mean square moment for the mild solution to system 1.1. We need to impose some additional assumption and lemma:

(A6) The resolvent operator $\mathfrak{R}(\mathfrak{t})_{\mathfrak{t}\geq 0}$ satisfies the further condition: There exists a constant $\mathscr{H}>0$ and a real number $\varsigma>0$ such that $\|\mathfrak{R}(\mathfrak{t})\|\leq\mathscr{H}e^{\varsigma\mathfrak{t}}$, $\mathfrak{t}\geq 0$.

In order to prove the theorem we can take into consideration the following lemma:

Lemma 4.1 [33] *For $\varsigma>0$, $\exists$ some positive constants $\upsilon,\upsilon'>0$ such that if $\upsilon'<\varsigma$, and*

$$\varpi(\mathfrak{t})=\begin{cases}\upsilon e^{-\varsigma(\mathfrak{t}-\mathfrak{t}_0)}, & \mathfrak{t}\in[-\delta,0]\\ \upsilon e^{-\varsigma(\mathfrak{t}-\mathfrak{t}_0)}+\upsilon'\displaystyle\int_{\mathfrak{t}_0}^{\mathfrak{t}}e^{-\varsigma(\mathfrak{t}-s)}\sup_{\theta\in(-\delta,0]}\varpi(s+\theta)ds, & \mathfrak{t}\geq\mathfrak{t}_0\end{cases}$$

hold. Then we have $\varpi(\mathfrak{t})\leq\mathscr{F}e^{-\tau(\mathfrak{t}-\mathfrak{t}_0)}$, where $\tau>0$ satisfying

$$\frac{\upsilon'}{\varsigma-\tau}e^{\tau(\delta+\mathfrak{t}_0)}=1$$

and

$$\mathscr{F} = \max\{\frac{\upsilon}{\upsilon'}(\varsigma - \tau)e^{-\tau\delta}, \varsigma\}.$$

Theorem 4.3 *Assume (H1), (H2),(A3), (A5), (A6) are satisfied, then the mild solution of* (1.1) *is mean-square exponentially stable.*

Proof Using with the assumed hypotheses and Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E}\|\vartheta(\mathrm{t})\|^2 \leq\ & 3\mathbb{E}\left(\max_k \left\{\prod_{i=1}^{k} \|\mathfrak{b}_i(\delta_i)\|^2\right\}\right)^2 \|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\|^2\, \mathbb{E}\|\varphi(0)\|^2 \\
&+3\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\right\}, 1\right)^2 \\
&\times\mathbb{E}\left(\int_{\mathrm{t}_0}^{\mathrm{t}} \|\mathfrak{R}(\mathrm{t}-s)\|\, \|\mathfrak{f}(s, \vartheta(s-\mu(s)))\|\, ds\right)^2 \\
&+3\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\right\}, 1\right)^2 \\
&\times\mathbb{E}\left(\int_{\mathrm{t}_0}^{\mathrm{t}} \|\mathfrak{R}(\mathrm{t}-s)\|\, \|\mathfrak{g}(s, \vartheta(s-\rho(s)))\|\, d\omega(s)\right)^2 \\
\leq\ & 3\mathscr{B}^2\mathscr{H}^2 e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\varphi(0)\|^2 + 3\max\{1, \mathscr{B}^2\}\mathscr{H}^2 \\
&\int_{\mathrm{t}}^{\mathrm{t}_0} e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\mathfrak{f}(s, \vartheta(s-\mu(s)))\|^2\, ds \\
&\times\int_{\mathrm{t}_0}^{\mathrm{t}} e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)} ds + 3\max\{1, \mathscr{B}^2\}\mathscr{H}^2 \int_{\mathrm{t}_0}^{\mathrm{t}} e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)} ds \\
&\int_{\mathrm{t}}^{\mathrm{t}_0} e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\mathfrak{g}(s, \vartheta(s-\rho(s)))\|^2\, ds \\
\leq\ & 3\mathscr{B}^2\mathscr{H}^2 e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\varphi(0)\|^2 + 3\max\{1, \mathscr{B}^2\}\frac{\mathscr{H}^2(\mathscr{C}_1+\mathscr{C}_2)}{\varsigma} \\
&\int_{\mathrm{t}_0}^{\mathrm{t}} \sup_{\theta\in[-\delta,0]} \mathbb{E}\,\|\vartheta(s+\theta)\|^2\, ds \\
\leq\ & \mathscr{F} e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}, \ \ \forall\ \mathrm{t}\in[-\delta, 0],
\end{aligned}$$

where $\mathscr{F} = \max\{3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\|\varphi(0)\|^2, \sup_{\theta\in[-\delta,0]} \mathbb{E}\|\varphi\|^2\}$.

Thus by Lemma 4.1, $\forall\ \mathrm{t}\in[\mathrm{t}_0-\delta, +\infty]$,

$$\mathbb{E}\|\vartheta(\mathrm{t})\|^2 \leq \mathscr{F} e^{-\tau\mathrm{t}}.$$

This completes the proof. □

Remark 1 The technique used here can be extended to investigate a class of nonlocal RISIDEs driven by Poisson jump of the form:

$$\begin{aligned} d\vartheta(\mathfrak{t}) &= \left[\mathfrak{A}\vartheta(\mathfrak{t}) + \int_0^{\mathfrak{t}} \mathfrak{B}(\mathfrak{t}-s)\vartheta(s)ds + \mathfrak{f}(\mathfrak{t}, \vartheta(\mathfrak{t}-\mu(\mathfrak{t})))\right]d\mathfrak{t} \\ &\quad + \mathfrak{g}(\mathfrak{t}, \vartheta(\mathfrak{t}-\rho(\mathfrak{t})))d\omega(\mathfrak{t}) \\ &\quad + \int_{\mathrm{Z}} \mathfrak{h}(\mathfrak{t}, \vartheta(\mathfrak{t}-\sigma(\mathfrak{t})), \mathrm{z})\widetilde{N}(d\mathfrak{t}, dz), \quad \mathfrak{t} \geq \mathfrak{t}_0, \mathfrak{t} \neq \xi_k, \\ \vartheta(\xi_k) &= \mathfrak{b}_k(\delta_k)\vartheta(\xi_k^-), \quad k = 1, 2, \cdots, \\ \vartheta_{\mathfrak{t}_0} + \mathfrak{q}(\mathfrak{t}) &= \varphi, \end{aligned} \tag{4.2}$$

Here, the functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{b}_k$ are defined as in Eq. (1.1) and also define $\mathfrak{h} : [\mathfrak{t}_0, +\infty] \times \mathbb{X} \times \mathrm{Z} \to \mathbb{X}$, $\mathfrak{q} : [\mathfrak{t}_0, +\infty] \to \mathbb{X}$ are suitable functions. Hence, all the hypotheses of Theorems 3.1, 4.1, 4.2 and 4.3 are satisfied.

Remark 2 Every semigroup is a resolvent operator but resolvent operator is not a semigroup. However, the resolvent operator does not satisfy semigroup properties (see, for instance [35]) and our objective in the present paper is to apply the theory developed by Grimmer [34] because it is valid for generators of strongly continuous semigroup, not necessarily analytic. The main contribution of this manuscript is that it proposes a framework for studying the mild solution of RISIDEs.

Remark 3 Blood cell production model: Leukemia is a type of cancer that develops in the bone marrow's blood-forming cells. It is nothing more than the comparatively mature bone marrow's production of an increase in aberrant white blood cells. As a result, more aberrant white blood cells are produced than would be expected at the normal rate. Treatments for leukaemia patients may include chemotherapy, monoclonal antibodies, supportive care, leukapheresis, surgery, and radiotherapy, and they may be curative (with fixed or random time periods). The efficacy of the treatments is demonstrated by the subsequent random impulsive model. For example, the blood cell production model can be represented by nonlinear first order random impulsive control system of the form

$$\begin{aligned} \dot{x}(\mathfrak{t}) &= -ax(\mathfrak{t}) + \frac{bx(\mathfrak{t}-r)}{1+x^c(\mathfrak{t}-r)}, \ \zeta_m^{'} < t < \zeta_{m+1}^{'}, \ \mathfrak{t} \geq \mathfrak{t}_0, \\ x(\zeta_m^{'+}) &= w_m x(\zeta_m^{'-}), \ \mathfrak{t} = \zeta_m^{'}, \ m \in \mathbb{Z}_+, \\ x(\theta) &= \phi(\theta), \ \theta \in [-r, 0], \end{aligned}$$

where $x(\mathfrak{t})$ represnts the density of mature cells at time t, $x(\mathfrak{t} - r)$ represents the density of abnormal white blood cells and r is the time-delay between the production of abnormal white blood cells in the bone marrow and their release of the mature cells in to the blood streams. For additional details on the above model, we refer the reader to [36, 37].

5 Illustration

To illustrate the abstract theory, let us consider the system on a bounded domain $\Omega \subset \mathscr{R}^n$ with the boundary $\partial\Omega$:

$$\begin{aligned} \frac{dz(\mathfrak{t},\vartheta)}{\partial \mathfrak{t}} &= \frac{\partial^2}{\partial\vartheta^2}z(\mathfrak{t},\vartheta) + \int_{-\mathfrak{r}}^{\mathfrak{t}} \alpha(\mathfrak{t}-s)\frac{\partial^2}{\partial\vartheta^2}z(s,\vartheta)ds + \int_{-\mathfrak{r}}^{\mathfrak{t}} \kappa_1(\theta)z(\mathfrak{t}+\theta,\vartheta(\mathfrak{t}-\mu(\mathfrak{t})))d\theta \\ &\quad + [\kappa_2(\theta)z(\mathfrak{t}+\theta,\vartheta(\mathfrak{t}-\rho(\mathfrak{t})))d\theta]\,d\omega(\mathfrak{t}), \qquad \mathfrak{t}\geq\delta,\ \ \mathfrak{t}\neq\xi_k, \\ z(\xi_k,\vartheta) &= \mathfrak{h}(k)\delta_k z(\xi_k^-,\vartheta), \qquad \vartheta\in\Omega \\ z(\mathfrak{t}_0,\vartheta) &= \varphi(\theta,\vartheta), \qquad \vartheta\in\Omega, \theta\in[-\delta,0] \\ z(\mathfrak{t},\vartheta) &= 0, \qquad \vartheta\in\partial\Omega. \end{aligned} \tag{5.1}$$

Let $\mathscr{X} = \mathcal{L}^2(\Omega)$, $\alpha : \mathscr{R}_+ \to \mathscr{R}_+$. κ_1, κ_2 be positive functions from $[-\delta, 0]$ to $\mathscr{R}$. Assuming δ_k to be a random variable defined on $\mathscr{D}_k = (0, \mathfrak{d}_k)$ with $0 < \mathfrak{d}_k < +\infty$ for $k = 1, 2, \cdots$. Without loss of generality, we may assume that $\{\delta_k\}$ follows Erlang distribution. δ_i, δ_j are mutually independent with $i \neq j$ for $i, j = 1, 2, \cdots$. $\mathfrak{h}$ be a function of k, $\xi_k = \xi_{k-1} + \delta_k$ where $\{\xi_k\}$ forms a strictly increasing process with independent increments and $\mathfrak{t}_0 \in [0, \mathrm{T}]$ be an arbitrary real number.
Let $\mathfrak{A}$ be an operator on $\mathscr{X}$ by $\mathfrak{A}z = \frac{\partial^2 z}{\partial\vartheta}$ with the following domain

$$\mathscr{D}(\mathfrak{A}) = \{z \in \mathscr{X} : z \textit{ and } z_\vartheta \text{ are absolutely continuous},\ \ z_{\vartheta\vartheta} \in \mathscr{X},\ z = 0 \textit{ on } \partial\Omega\}.$$

Also, let the map $\mathfrak{B} : \mathscr{D}(\mathfrak{A}) \subset \mathscr{X} \to \mathscr{X}$ be the operator defined by

$$\mathfrak{B}(\mathfrak{t})(z) = \alpha(\mathfrak{t})\mathfrak{A}z \ \text{ for }\ \mathfrak{t} \geq 0 \ \text{ and }\ z \in \mathscr{D}(\mathfrak{A}).$$

The operator $\mathfrak{A}$ can be expressed as

$$\mathfrak{A}z = \sum_{n=1}^{\infty} \mathfrak{n}^2 \langle z, z_\mathfrak{n}\rangle z_\mathfrak{n}, \quad z \in \mathscr{D}(\mathfrak{A}),$$

where $z_\mathfrak{n}(\varpi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}$ is the corresponding eigenvectors of $\mathfrak{A}$. Obviously, $z_\mathfrak{n}(\varpi)$ form an orthonormal system in $\mathbb{X}$. Moreover, $\mathfrak{A}$ is the infinitesimal generator of an analytic semigroup $(\mathfrak{T}(\mathfrak{t}))_{\mathfrak{t}\geq 0}$ in $\mathbb{X}$, satisfying

$$\|\mathfrak{T}(\mathfrak{t})\| \leq \exp\{-\pi^2(\mathfrak{t} - \mathfrak{t}_0)\}, \quad \mathfrak{t} \geq \mathfrak{t}_0.$$

Also, we have the following additional conditions:

(i) $\displaystyle\int_{-\delta}^{0} \kappa_1(\theta)^2 d\theta < \infty, \int_{-\delta}^{0} \kappa_2(\theta)^2 d\theta < \infty,$

(ii) $\displaystyle\mathbb{E}\left(\max_{i,k}\{\prod_{j=i}^{k} \left\|\mathfrak{h}(j)(\delta_j)\right\|\}^2\right) < \infty.$

Using the aforementioned conditions, (5.1) can be modelled as the abstract RISIDEs of the form (1.1),

$$\begin{aligned}\mathfrak{f}(t, z_{\mathfrak{t}}) &= \int_{-\mathfrak{r}}^{\mathfrak{t}} \kappa_1(\theta) z(\mathfrak{t}+\theta, \vartheta(\mathfrak{t}-\mu(\mathfrak{t})))d\theta,\\ \mathfrak{g}(t, z_{\mathfrak{t}}) &= \int_{-\mathfrak{r}}^{\mathfrak{t}} \kappa_2(\theta) z(\mathfrak{t}+\theta, \vartheta(\mathfrak{t}-\rho(\mathfrak{t})))d\theta,\\ \mathfrak{b}_k(\delta_k) &= \mathfrak{h}(k)\delta_k.\end{aligned}$$

Condition (i) implies (A5) holds with

$$\mathscr{C}_i = \int_{\mathfrak{r}}^{0} \kappa_i^2(\theta)d\theta, \quad for \;\; i = 1, 2,$$

along with Condition (ii) implying (A3). This depicts that (5.1) has a mild solution. Moreover achieving stability results [continuous dependence of solution on initial conditions and Hyers Ulam Stability] as in Sect. 4. Finally, if $\lambda' \leq \tau$, (i.e)

$$3\max\{1, \mathscr{B}^2\}(\mathscr{C}_1 + \mathscr{C}_2)/(\pi^2) \leq \pi^2,$$

then (5.1) is mean square exponentially stable under the assumptions (A3) and (A5).

6 Conclusion

In this paper, we have obtained the existence and various types of stability results for RISIDEs by means of functional analysis and stochastic analysis method. Many evolution processes from fields as diverse as physics, population dynamics, aeronautics, economics, telecommunications and engineering are characterized by the fact that they undergo abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of process act instantaneously in the form of impulses. The impulses may be deterministic or random.In addition, it is of great interest for future research to study RISIDEs including more complicated stochastic factors,such as stochastic processes driven by fractional Brownian motions, or G-Brownian motions, and Rosenblatt process, which describe some stochastic phenomena more precisely, see [9, 38] for more details.

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