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Viral Infection Model with Diffusion and Distributed Delay: Finite-Dimensional Global Attractor

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Abstract
We study a virus dynamics model with reaction-diffusion, logistic growth terms and a general non-linear infection rate functional response. The model has a distributed delay, including the case of state-selective delay. We construct a dynamical system in a Hilbert space and prove the existence of a finite-dimensional global attractor.

Keywords Delay equations · Reaction-diffusion · Evolution equations · Attractor · Virus infection model

Mathematics Subject Classification Primary 93C23; Secondary 34D45 · 35K57

1 Introduction

We are interested in qualitative properties of mathematical models of viral infections. Such models attract much attention during last years, especially after wide spread of viral diseases, including COVID-19, HIV, hepatitis B and C. Many viruses continue to be a major global public health issues.

World Health Organization (WHO) reports [41, p.3] "As of 20 April 2022, more than 504.4 million confirmed COVID-19 cases and over 6.2 million related deaths had been reported to WHO." Moreover [41, p.2] "the long-term impact of infection on people's health is not yet fully understood." "A recent systematic review reported a prevalence of persistent symptoms in patients after mild COVID-19 infection ranged from 10% to 35% . Cognitive impairment, various neuropsychiatric symptoms, fatigue, headaches and other complaints are among the conditions reported four or more weeks after the

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initial infection. A detailed understanding of these long-lasting symptoms has not yet been achieved."

According to WHO '354 million people globally live with a hepatitis B or C infection' (296 million with chronic hepatitis B and 58 million with chronic hepatitis C).[1] More data (2017) are collected in [40].

In such a situation, any step toward understanding the qualitative (particularly, long-time asymptotic) behaviour of viral infection models is important.

The classical models [21, 22] contain ordinary differential equations (without delay) for three variables: susceptible host cells T, infected host cells T^* and free virus particles V. The intracellular delay is an important property of the biological problem, so we start discussion with the delay problem (for a particular case of DeAngelis-Beddington functional response f see e.g. [9, 15])

$$\begin{cases} \dot{T}(t) = \lambda - dT(t) - f(T(t), V(t)), \\ \dot{T}^*(t) = e^{-\omega h} f(T(t-h), V(t-h)) - \delta T^*(t), \\ \dot{V}(t) = N\delta T^*(t) - cV(t). \end{cases} \tag{1.1}$$

In (1.1), susceptible cells T are produced at a rate λ, die at rate dT, and become infected at rate $f(T, V)$. Properties and examples of incidence function f are discussed below. Infected cells T^* die at rate δT^*, free virions V are produced by infected cells at rate $N\delta T^*$ and are removed at rate $cV(t)$. In (1.1) $h > 0$ denotes the delay between the time a virus particle contacts a target susceptible cell and the time the cell becomes actively infected (start to produce new virions). It is clear that the constancy of the *discrete* delay is an extra assumption which essentially simplifies the analysis, but has no biological background.

Since the precise value of the discrete delay h may be difficult to find, viral models with a *distributed delay* may provide another way to study the dynamics of a disease. For ODE cases and motivations see e.g. [12, 20, 37] and references therein. For motivations for a state-selective delay see Introduction in [23].

For discussion of different viral infection models we refer to [12–15, 19, 28, 29, 31, 36, 37]. PDE models which take into account spatial mobility of cells and virus, possibility of cell-to-cell transmission of the infection (see e.g. [3]) as well as natural time delay effects are discussed in many papers (see e.g. [18, 30, 38, 39] and references therein).

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain with a smooth boundary. Let $T(t,x)$, $T^*(t,x)$, $V(t,x)$ represent the densities of uninfected cells, infected cells and free virions at position $x \in \Omega$ at time t.

In this note we are interested in the following PDEs system with delay

$$\begin{cases} \dot{T}(t,x) = rT(t,x)\left(1 - \frac{T(t,x)}{T_K}\right) - dT(t,x) - f(T(t,x), V(t,x)) + d^1 \Delta T(t,x), \\ \dot{T}^*(t,x) = e^{-\omega h} \int_{-h}^{0} f(T(t+\theta),x), V(t+\theta),x))\xi(\theta, x, u_t)\, d\theta - \delta T^*(t,x) \\ +d^2 \Delta T^*(t,x), \\ \dot{V}(t,x) = N\delta T^*(t,x) - cV(t,x) + d^3 \Delta V(t,x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

[1] https://www.who.int/health-topics/hepatitis.

Here the dot over a function denotes the partial derivative with respect to time i.e, $\dot{T}(t,x) = \frac{\partial T(t,x)}{\partial t}$, all the constants $d, \delta, N, c, r, N, \omega$ including $d^i, i = 1, 2, 3$ (diffusion coefficients) are positive. We consider a general functional response $f(T, V)$ satisfying natural assumptions presented below. In earlier models (with constant or without delay) the study was started in case of bilinear $f(T, V) = \text{const} \cdot TV$ and then extended to more general classes of non-linearities. For more details and discussion see [18, 30].

As usual, for a delay system we denote $u_t = u_t(\theta) \equiv u(t+\theta)$ for $\theta \in [-h, 0]$, $h > 0$. For general theory on delay equations see [8, 10, 16, 33, 42].

We need initial conditions for the delay problem (1.2):

$$u(\theta, x) = \varphi(\theta, x) \equiv (T(\theta, x), T^*(\theta, x), V(\theta, x)), \quad \theta \in [-h, 0] \tag{1.3}$$

or shortly $u_0 = \varphi$.

In the papers cited above, the main goal was to study asymptotic stability of stationary solutions. In case of a globally asymptotically stable stationary solution, the long-time behaviour of the system looks very simple. We are interested in a more general case when the existence of a global attractor (for the corresponding dynamical system) may, in general, allow the co-existence of multiple stationary solutions, periodic orbits, invariant manifolds. In case when the existence of a global attractor is proved, the important question arises if the attractor is finite-dimensional (for general facts on attractors see e.g. [4, 34]). The existence of a finite-dimensional global attractor forms a theoretical basis for finite-dimensional approximations which reflect all the asymptotic behaviour of the original system.

Our main mathematical tool in studying of the asymptotic behaviour of solutions is the *quasi-stability method* developed by I.D.Chueshov (for more details and definitions see [6]). For applications of this metod to delay PDEs see [5] where a model with discrete state-dependent delay is studied. For connections between PDEs with a discrete state-dependent delay [25–27] and considered in the current paper PDEs with a distributed state-selective delay see [23, 24].

To the best of our knowledge, the existence of a finite-dimensional global attractor for a viral infection model has not been investigated before (except the simple case of a globally asymptotically stable stationary solution). It is important to mention that the problem under consideration is infinite-dimensional in both space coordinate (as PDE [4, 34]) and time coordinate (as delay problem [8, 10, 16]).

2 Main Results

We combine two lines of investigations, one is in a Hilbert space, while the other is in a Banach space.

2.1 Study in $L^2(\Omega)$. Part 1

We start with the *Hilbert space* approach.

Define the following linear operator

$$-\mathcal{A} = diag\left(d^1\Delta - \frac{d}{2}, d^2\Delta - \frac{\delta}{2}, d^3\Delta - \frac{c}{2}\right) \text{ in } H \equiv [L^2(\Omega)]^3 \qquad (2.1)$$

with domain $D(\mathcal{A}) \equiv D(d^1\Delta) \times D(d^2\Delta) \times D(d^3\Delta)$. Here we set $D(d^i\Delta) \equiv \{v \in L^2(\Omega) : \Delta v \in L^2(\Omega), \frac{\partial v(x)}{\partial n}|_{\partial\Omega} = 0\}$. We consider operator $-\Delta$ in $L^2(\Omega)$ with the Neumann boundary conditions. This type of conditions is more adequate to biological nature of the problem.

Operator $\mathcal{A}$ is a positive self-adjoint operator in $[L^2(\Omega)]^3$. It is a positive self-adjoint operator with *discrete spectrum* i.e. there exists an othonormal basis $\{e_k\}_{k=1}^{\infty}$ of H, where e_k are eigenvectors of $\mathcal{A} : \mathcal{A}e_k = \lambda_k e_k, 0 < \lambda_1 \le \lambda_2 \le ..., \lim_{k\to\infty} \lambda_k = +\infty$ (see e.g., [6, Definition 4.1.1]). Hence we can define spaces $H_\alpha \equiv D(\mathcal{A}^\alpha)$, $H_0 = H$.

Let $C_\alpha \equiv C([-h, 0]; H_\alpha) \subset C_0 = C \equiv C([-h, 0]; H)$ for $\alpha \in [0, 1)$.

We write, the system (1.2) in the following abstract form

$$\frac{d}{dt}u(t) + \mathcal{A}u(t) = F(u_t), \qquad t > 0. \qquad (2.2)$$

The non-linear mapping $F : C \equiv C_0 \to H$ is defined by

$$F(\varphi)(x) = \begin{pmatrix} r\,\varphi^1(0,x)\left(1 - \frac{\varphi^1(0,x)}{T_K}\right) - \frac{d}{2}\varphi^1(0,x) - f(\varphi^1(0,x), \varphi^3(0,x)) \\ B(\varphi, x) - \frac{\delta}{2}\varphi^2(0,x) \\ N\delta\varphi^2(0,x) - \frac{c}{2}\varphi^3(0,x) \end{pmatrix}. \qquad (2.3)$$

Here $\varphi = (\varphi^1, \varphi^2, \varphi^3) \in C$ and the nonlinear distributed delay term has the following form

$$B(\varphi, x) \equiv e^{-\omega h} \int_{-h}^{0} f(\varphi^1(\theta, x), \varphi^3(\theta, x))\xi(\theta, x, \varphi)\, d\theta, \quad x \in \Omega. \qquad (2.4)$$

We assume the following

$(H1)$ Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous, $\xi : C \to L^1(-h, 0; L^\infty(\Omega))$ be continuous and bounded, i.e.

$$\int_{-h}^{0} ||\xi(\theta, \cdot, \varphi) - \xi(\theta, \cdot, \psi)||_{L^\infty(\Omega)} d\theta \to 0, \quad \text{as } ||\varphi - \psi||_C \to 0,$$

$$\int_{-h}^{0} ||\xi(\theta, \cdot, \varphi)||_{L^\infty(\Omega)} d\theta \le M_{\xi,1}, \quad \forall \varphi \in C.$$

Considering $||B(\varphi, \cdot) - B(\psi, \cdot)||_{L^2(\Omega)}$ one can check that assumptions (H1) imply continuity $B : C \to L^2(\Omega)$. It implies continuity $B : C_\alpha \to L^2(\Omega)$ for $\alpha \in [0, 1)$.

Hence, the form (2.3) and the above continuity of B give that F is a nonlinear continuous mapping from C_α into H for all $\alpha \in [0, 1)$ and is bounded on bounded sets in C_α.

We use the standard

Definition 2.1 We call a function $u \in C([-h, T]; H_\alpha)$ a *mild solution* (H_α-mild) of the problem (2.2), (1.3) if $u_0 = \varphi$ and

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-\tau)A} F(u_\tau)\, d\tau, \quad t \in [0, T]. \tag{2.5}$$

The local existence of a mild solution to (2.2), (1.3) is standard due to the continuity of F, its boundedness on bounded sets and Schauder's fixed point theorem (see [35]).

Now we assume the nonlinear term B has the form (2.4) and

$(H2)$ f is Lipschitz, ξ is Lipschitz and bounded in the following norms

$$\int_{-h}^0 ||\xi(\theta, \cdot, \varphi) - \xi(\theta, \cdot, \psi)||_{L^\infty(\Omega)} d\theta \le L_{\xi,1}||\varphi - \psi||_C, \tag{2.6}$$

$$\int_{-h}^0 ||\xi(\theta, \cdot, \varphi)||_{L^\infty(\Omega)} d\theta \le M_{\xi,1}, \quad \forall \varphi \in C. \tag{2.7}$$

Theorem 2.2 *Let f, ξ satisfy $(H2)$. Then, for any initial $\varphi \in C_\alpha$ with $\alpha \in [0, 1)$ there exist $T = T_\varphi > 0$ and an unique mild solution to (2.2), (1.3) on $[-h, T]$. The solution continuously depends on initial function φ i.e. for two mild solutions $||u^i(t)||_\alpha \le R$, $i = 1, 2$ one has*

$$||u^1(t) - u^2(t)||_\alpha \le C_{\widehat{T},R}||\varphi^1 - \varphi^2||_{C_\alpha}, \quad t \in [0, \widehat{T}], \quad \widehat{T} \equiv \min\{T_{\varphi^1}; T_{\varphi^2}\}. \tag{2.8}$$

Proof To prove Theorem 2.2 we need only to show that $F : C_\alpha \to H$ is locally Lipschitz. We start with the following

Lemma 2.3 *Assume $(H2)$ is satisfied. Then the nonlinear distributed delay term B (2.4) is locally Lipschitz continuous*

$$||B(\varphi, \cdot) - B(\psi, \cdot)||_{L^2(\Omega)} \le L_B(R)||\varphi - \psi||_C, \quad \forall ||\varphi||_C, ||\psi||_C \le R \tag{2.9}$$

with $L_B(R) = e^{-\omega h} L_f \left(2M_{\xi,1} + L_{\xi,1} \cdot R\right)$.

Proof of Lemma Consider the difference

$$\begin{aligned} &B(\varphi, x) - B(\psi, x) \\ &\quad = e^{-\omega h} \int_{-h}^0 \Big(f(\varphi^1(\theta, x), \varphi^3(\theta, x)) - f(\psi^1(\theta, x), \psi^3(\theta, x)) \Big)\, \xi(\theta, x, \varphi)\, d\theta \\ &\qquad + e^{-\omega h} \int_{-h}^0 f(\psi^1(\theta, x), \psi^3(\theta, x))\, (\xi(\theta, x, \varphi) - \xi(\theta, x, \psi))\, d\theta. \end{aligned}$$

We have $||B(\varphi,\cdot)-B(\psi,\cdot)||_{L^2(\Omega)}$

$$
\begin{aligned}
&\le e^{-\omega h}\int_{-h}^{0}||(f(\varphi^1(\theta,\cdot),\varphi^3(\theta,\cdot))-f(\psi^1(\theta,\cdot),\psi^3(\theta,\cdot)))\xi(\theta,\cdot,\varphi)\,||_{L^2(\Omega)}\,d\theta\\
&+e^{-\omega h}\int_{-h}^{0}||f(\psi^1(\theta,\cdot),\psi^3(\theta,\cdot))\,(\xi(\theta,\cdot,\varphi)-\xi(\theta,\cdot,\psi))\,||_{L^2(\Omega)}\,d\theta\equiv I.
\end{aligned}
$$

To estimate the $L^2(\Omega)$-norm in the first term we consider (notice the square of the norm)

$$
\begin{aligned}
&\int_\Omega|f(\varphi^1(\theta,x),\varphi^3(\theta,x))-f(\psi^1(\theta,x),\psi^3(\theta,x))|^2|\xi(\theta,x,\varphi)|^2\,dx\\
&\le|||f(\varphi^1(\theta,\cdot),\varphi^3(\theta,\cdot))-f(\psi^1(\theta,\cdot),\psi^3(\theta,\cdot))|^2||_{L^1(\Omega)}|||\xi(\theta,\cdot,\varphi)|^2||_{L^\infty(\Omega)}\\
&\le 2L_f^2||\varphi-\psi||_C^2\cdot||\xi(\theta,\cdot,\varphi)||^2_{L^\infty(\Omega)}.
\end{aligned}
$$

Here we used properties $\int_\Omega|a(x)b(x)|\,dx\le||a||_{L^1(\Omega)}||b||_{L^\infty(\Omega)}$, $|||b|^2||_{L^\infty(\Omega)}=||b||^2_{L^\infty(\Omega)}$ and the Lipschitz property of f. Similar properties for the second term give

$$
\begin{aligned}
&\int_\Omega|f(\psi^1(\theta,\cdot),\psi^3(\theta,\cdot)|^2|\xi(\theta,\cdot,\varphi)-\xi(\theta,\cdot,\psi)|^2\,dx\\
&\quad\le L_f^2||\psi||_C^2\cdot||\xi(\theta,\cdot,\varphi)-\xi(\theta,\cdot,\psi)||^2_{L^\infty(\Omega)}.
\end{aligned}
$$

These estimates and (H2) allow to continue

$$
I\le e^{-\omega h}L_f\left(2M_{\xi,1}+L_{\xi,1}\cdot||\psi||_C\right)||\varphi-\psi||_C\le L_B(R)||\varphi-\psi||_C,
$$

$\forall||\varphi||_C,||\psi||_C\le R$ with $L_B(R)=e^{-\omega h}L_f\left(2M_{\xi,1}+L_{\xi,1}\cdot R\right)$. It completes the proof of lemma.

Next, we notice that for $\alpha>0$ and $v\in H_\alpha$ one has $||v||\le\lambda_1^{-\alpha}||v||_\alpha$. Hence (2.9) implies similar Lipschitz property in smaller space

$$
||B(\varphi,\cdot)-B(\psi,\cdot)||_{L^2(\Omega)}\le L_{B,\alpha}(R)||\varphi-\psi||_{C_\alpha},\quad\forall||\varphi||_{C_\alpha},||\psi||_{C_\alpha}\le R \tag{2.10}
$$

with $L_{B,\alpha}(R)=\lambda_1^{-\alpha}e^{-\omega h}L_f\left(2M_{\xi,1}+L_{\xi,1}\cdot R\right)$. Finally, (2.10) and (2.3) give the local Lipschitz property of F (since all the other terms are polynomials): for every $R>0$ there exists $L_{F,\alpha}(R)$ such that

$$
||F(\varphi)-F(\psi)||_{L^2(\Omega)}\le L_{F,\alpha}(R)||\varphi-\psi||_{C_\alpha},\quad\forall||\varphi||_{C_\alpha},||\psi||_{C_\alpha}\le R. \tag{2.11}
$$

The rest of the proof is standard (see e.g. [35], [6, theorem 6.1.6]). We do not repeat it here. □

2.2 Study in $C(\overline{\Omega})$

We use the basic functional framework described in [17] and applied in [30].

Define the following linear operator $-\mathcal{A}^0 = diag\,(d^1\Delta - \frac{d}{2}, d^2\Delta - \frac{\delta}{2}, d^3\Delta - \frac{c}{2})$ in $C(\overline{\Omega}; \mathbb{R}^3)$ with $D(\mathcal{A}^0) \equiv D(d^1\Delta) \times D(d^2\Delta) \times D(d^3\Delta)$. Here, for $d^i \neq 0$ we set $D(d^i\Delta) \equiv \{v \in C^2(\overline{\Omega}) : \frac{\partial v(x)}{\partial n}|_{\partial\Omega} = 0\}$ and $D(d^j\Delta) \equiv C(\overline{\Omega})$ for $d^j = 0$. We omit the space coordinate x, for short, for unknown $u(t) = (T(t), T^*(t), V(t)) \in X \equiv [C(\overline{\Omega})]^3 \equiv C(\overline{\Omega}; \mathbb{R}^3)$. It is well-known that the closure $-\mathcal{A} = -\mathcal{A}_C$ (in X) of the operator $-\mathcal{A}^0$ generates a C_0-semigroup $e^{-\mathcal{A}t}$ on X which is analytic and nonexpansive [17, p.5]. We denote the space of continuous functions by $C_X \equiv C([-h, 0]; X)$ equipped with the sup-norm $||\psi||_{C_X} \equiv \max_{\theta\in[-h,0]} ||\psi(\theta)||_X$.

We can use the abstract form (2.2) and nonlinear map (2.3), changing linear operator ($\mathcal{A} = \mathcal{A}_C$ instead of $\mathcal{A}$, see (2.1)) and corresponding spaces.

Definition 2.4 We call a function $u \in C([-h, T]; X)$ a *mild solution* ($C(\overline{\Omega})$-mild) of the problem (2.2), (1.3) if $u_0 = \varphi$ and (2.5) holds with $\mathcal{A} = \mathcal{A}_C$ instead of $\mathcal{A}$.

We notice that Definitions 2.1 and 2.4 give different notions of mild solutions (belong to different spaces and use different semigroups $e^{-t\mathcal{A}}$ on H_α and $e^{-t\mathcal{A}_C}$ on X).

Now we assume the nonlinear term B has the form (2.4) and (c.f. $(H2)$)

$(H3)$ f is Lipschitz, $\xi \geq 0$ is Lipschitz and bounded in the following norms

$$\int_{-h}^{0} ||\xi(\theta, \cdot, \varphi) - \xi(\theta, \cdot, \psi)||_{C(\overline{\Omega})} d\theta \leq L_{\xi,C} ||\varphi - \psi||_{C_X}, \tag{2.12}$$

$$\int_{-h}^{0} ||\xi(\theta, \cdot, \varphi)||_{C(\overline{\Omega})} d\theta \leq M_{\xi,C}, \quad \forall \varphi \in C_X. \tag{2.13}$$

We need further assumptions on Lipschitz function f :

$$(\mathbf{Hf_1}+) \quad \begin{cases} f(T, 0) = f(0, V) = 0, \quad \text{and } f(T, V) > 0 \text{ for all } T > 0,\ V > 0; \\ f \text{ is strictly increasing in both coordinates for all } T > 0,\ V > 0; \\ \text{there exists } \mu > 0 \text{ such that } |f(T, V)| \leq \mu |T| \text{ for all } T, V \in \mathbb{R}. \end{cases} \tag{2.14}$$

Define the set

$$\Omega^{log} \equiv \Big\{ \varphi = (\varphi^1, \varphi^2, \varphi^3) \in C = C_X : \quad 0 \leq \varphi^1(\theta) \leq M^1 \equiv \frac{r}{2d} T_K,$$
$$0 \leq \varphi^2(\theta) \leq M^2 \equiv e^{-\omega h} \frac{\mu r}{d\delta} T_K M_{\xi,C}, 0 \leq \varphi^3(\theta) \leq M^3 \equiv 2e^{-\omega h} \frac{N\mu r}{dc} T_K M_{\xi,C} \Big\} \tag{2.15}$$

where $\theta \in [-h, 0]$, μ is defined in (Hf_1+) and all the inequalities hold pointwise w.r.t. $x \in \overline{\Omega}$.

We have the following result

Theorem 2.5 *Let non-linear Lipschitz function f satisfy (Hf_1+) (see (2.14)), ξ satisfy $(H3)$. Then Ω^{log} is invariant i.e. for any $\varphi \in \Omega^{log}$ the unique $C(\overline{\Omega})$-mild solution to problem (2.2), (1.3) exists and satisfies $u_t \in \Omega^{log}$ for all $t \geq 0$.*

Proof We start with the local Lipschitz property of $B : C_X \to X$. Assumptions $(H3)$ give

$$||B(\varphi,\cdot) - B(\psi,\cdot)||_{C(\overline{\Omega})} \leq L_{B,C}(R)||\varphi - \psi||_{C_X}, \quad \forall ||\varphi||_{C_X}, ||\psi||_{C_X} \leq R \tag{2.16}$$

with $L_{B,C}(R) = e^{-\omega h} L_f \left(M_{\xi,C} + L_{\xi,C} \cdot R \right)$.

One can check that $F : C_X \to X$ is locally Lipschitz. The existence and uniqueness of a mild solution $u \in C([-h, T]; X)$ to the problem (2.2), (1.3) is standard. The proof of the invariance part follows the invariance result of [17] with the use of the Lipschitz property of nonlinearity F. The estimates (for the subtangential condition) are the same as for the constant delay case, see e.g. [18, Theorem 2.2].

Consider $\rho \geq 0$ and $\varphi \in \Omega^{log}$.

$$\varphi(0,x) + \rho F(\varphi,x) = \begin{pmatrix} \varphi^1(0,x) + \rho r\, \varphi^1(0,x)\left(1 - \frac{\varphi^1(0,x)}{T_K}\right) - \rho \frac{d}{2}\varphi^1(0,x) \\ \quad -\rho f(\varphi^1(0,x), \varphi^3(0,x)) \\ \varphi^2(0,x) + \rho B(\varphi,x) - \rho\frac{\delta}{2}\varphi^2(0,x) \\ \varphi^3(0,x) + \rho N\delta\varphi^2(0,x) - \rho\frac{c}{2}\varphi^3(0,x) \end{pmatrix}$$

We use notation $F = (F^1, F^2, F^3)^T$ and estimate separately each of three coordinates above.

(a) We notice that the logistic term (see the first equation in (1.2)) $rT\left(1 - \frac{T}{T_K}\right)$ has its maximum at point $T = T_K/2$, so $rT\left(1 - \frac{T}{T_K}\right) \leq \frac{1}{4} r T_K$ for all $T \in \mathbb{R}$.

Hence, for small enough $\rho \geq 0$ and $\varphi^1(0,x) \in [0, M^1]$ (see (2.15)) we have

$$\varphi^1(0,x) + \rho F^1(\varphi,x) \leq \varphi^1(0,x) + \rho\left(\frac{1}{4} r T_K - \frac{d}{2}\varphi^1(0,x)\right) \leq M^1.$$

(b) For the second coordinate we use (see (2.13) and (2.14))

$$B(\varphi,x) \leq e^{-\omega h}\int_{-h}^{0} \mu|\varphi^1(\theta,x)|\xi(\theta,x,\varphi)d\theta \leq e^{-\omega h}\mu\frac{r}{2d} T_K M_{\xi,C}.$$

This estimate gives for small enough $\rho \geq 0$ and $\varphi^2(0,x) \in [0, M^2]$ (see (2.15)):

$$\varphi^2(0,x) + \rho F^2(\varphi,x) \leq \varphi^2(0,x) + \rho e^{-\omega h}\mu\frac{r}{2d} T_K M_{\xi,C} - \rho\frac{\delta}{2}\varphi^2(0,x) \leq M^2.$$

(c) For small enough $\rho \geq 0$ and $\varphi^3(0,x) \in [0, M^3]$ (see (2.15)) one has

$$\varphi^3(0,x) + \rho F^3(\varphi,x) \leq \varphi^3(0,x) + \rho N\delta M^2 - \rho\frac{c}{2}\varphi^3(0,x) \leq N\delta M^2\frac{2}{c} = M^3.$$

Combining the estimates above, one can check that for small enough $\rho \geq 0$ and $\varphi \in \Omega^{log}$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leq \varphi(0,x) + \rho F(\varphi, x) \leq \begin{pmatrix} M^1 \\ M^2 \\ M^3 \end{pmatrix} \equiv M$$

or shortly $\varphi(0,x) + \rho F(\varphi,x) \in [0, M] \subset \mathbb{R}^3$ for all $x \in \Omega$.

The above implies $\lim_{\rho\to 0+} dist\{\varphi(0,\cdot) + \rho F(\varphi,\cdot); [0,M]_X\} = 0, \quad \forall\varphi \in \Omega^{log} \subset C_X$.

It gives the subtangential condition and allows to apply the invariance result of [17, 32].

The proof of Theorem 2.5 is complete. □

2.3 Study in $L^2(\Omega)$. Part 2

In this section we continue our study of H_α-mild solutions and use results of Theorem 2.5 obtained for $C(\overline{\Omega})$-mild solutions. The key point here is the Sobolev imbedding theorem [1, p.85] which suggests values of α for which the imbedding $H_\alpha \to C(\overline{\Omega})$ holds.

Let us remind the part we need of the Sobolev imbedding theorem [1, p.85].

Let Ω be a domain in $\mathbb{R}^n$. Let $j \geq 0, m \geq 1$ be integers and let $1 \leq p < \infty$. Suppose Ω satisfies the strong loc.Lipschitz condition. If $mp > n > (m-1)p$, then $W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega})$ for $0 < \lambda \leq m - \frac{n}{p}$.

In our case, $n = 3$, $p = 2$. We have (see condition $mp > n > (m-1)p$) that $m \in (\frac{3}{2}, \frac{5}{2})$. We are interested in $m < 2$, so consider $m \in (\frac{3}{2}, 2)$.

In case $A = -\Delta$, the condition $m \in (\frac{3}{2}, 2)$ corresponds to $\alpha \in (\frac{3}{4}, 1)$.

We notice the importance of the restriction $\alpha \in (\frac{3}{4}, 1)$ which guaranties $u(t) \in X = [C(\overline{\Omega})]^3$.

Combining this property with the uniqueness results for both H_α-mild solutions and $C(\overline{\Omega})$-mild solutions (both for the same initial function φ) one has the following key property: for any initial $\varphi \in C_\alpha, \alpha \in (\frac{3}{4}, 1)$ the H_α-mild solution is $C(\overline{\Omega})$-mild solution.

Our goal is to construct a dynamical system in phase space $\Omega_\alpha^{log} \equiv C_\alpha \cap \Omega^{log}, \alpha \in (\frac{3}{4}, 1)$. On this space we define evolution operator $S_t\varphi = u_t, t \geq 0$, where u is the unique mild solution of problem (2.2), (1.3).

Now we need the following space

$$Y_\beta \equiv \{v \in C_\alpha : |v|_{Y_\beta} < \infty\},$$

where $\beta \in (\alpha, 1)$ and

$$|v|_{Y_\beta} \equiv \max_{\theta\in[-h,0]} ||\mathcal{A}^\beta v(\theta)|| + \max_{\theta_1,\theta_2\in[-h,0],\theta_1\neq\theta_2} \frac{||\mathcal{A}^\alpha(v(\theta_1) - v(\theta_2))||}{|\theta_1 - \theta_2|^{\beta-\alpha}}. \quad (2.17)$$

We remind the following result (formulated for an abstract equation of the form (2.2)).

Proposition 2.6 [6, p. 293] *Let $\mathcal{A}$ be a linear positive self-adjoint operator with discrete spectrum on H. Let $F : C_\alpha \to H$ be a locally Lipschitz mapping i.e. for every $R > 0$ (2.11) holds. Assume that the problem (2.2), (1.3) generates a dynamical system (C_α, S_t). Let D be a forward invariant bounded set in C_α.*

Then

(1) For every $t > h$ the set $S_t D$ is bounded in Y_β for arbitrary $\beta \in (\alpha, 1)$. Moreover, for every $\delta > 0$ there exists R_δ such that

$$S_t D \subset B_\beta = \{u \in Y_\beta : |u|_{Y_\beta} \le R_\delta\} \quad \text{for all} \quad t \ge \delta + h. \tag{2.18}$$

In particular, this means that the dynamical system (C_α, S_t) is conditionally compact and thus asymptotically smooth.

(2) The mapping S_t is Lipschitz from D into Y_β. Moreover, for every $h < a < b < +\infty$ there exists a constant $M_D(a, b)$ such that

$$|S_t\varphi - S_t\psi|_{Y_\beta} \le M_D(a, b)||\varphi - \psi||_{C_\alpha}, \qquad t \in [a, b], \varphi, \psi \in D. \tag{2.19}$$

In particular, this means that the dynamical system (C_α, S_t) is quasi-stable at any time $t \in [a, b]$.

We remind (see, e.g., [4, 34])

Definition 2.7 A *global attractor* of the dynamical system (C_α, S_t) is defined as a bounded closed set $U \subset C_\alpha$ which is invariant ($S_t U = U$ for all $t > 0$) and uniformly attracts all bounded sets

$$\lim_{t\to+\infty} \sup\{\mathrm{dist}_{C_\alpha}(S_t y, U) : \ y \in B\} = 0 \quad \text{for any bounded set } B \text{ in } C_\alpha.$$

Our main result is the following

Theorem 2.8 *Let $\alpha \in (\frac{3}{4}, 1)$, non-linear Lipschitz function f satisfy (Hf_1+) (see (2.14)), ξ satisfy $(H2)$, $(H3)$. Then the pair $(S_t; \Omega_\alpha^{log})$ constitutes a dynamical system constructed by problem (2.2), (1.3). This dynamical system possesses a finite-dimensional global attractor.*

Proof The well-posedness of the problem (2.2), (1.3) (the existence, uniqueness and continuous dependence on initial function $\varphi \in C_\alpha$) is given by Theorem 2.2.

First we remind an important estimate (and its derivation) which is a part of the property (2.18). For more details, see [6, p.294]. Let D be a forward invariant bounded set in C_α (for this part $\alpha \ge 0$). Consider $\beta \in (\alpha, 1)$ and a mild solution $u(t) = S_t\varphi$, see (2.5). We use property $||\mathcal{A}^\alpha e^{-t\mathcal{A}}|| \le \left(\frac{\alpha}{et}\right)^\alpha$, $t > 0, \alpha \ge 0$ (with the rule $0^0 = 1$) to get

$$||u(t)||_\beta \le \left(\frac{\beta - \alpha}{e(t - s)}\right)^{\beta-\alpha} ||u(s)||_\alpha + \int_s^t \left(\frac{\beta}{e(t - \tau)}\right)^\beta ||F(u_\tau)|| \, d\tau$$

for all $t > s \geq 0$. Since $S_t\varphi \in D$ for all $t \geq 0$ one has $||u(t)||_\alpha \leq C_D, \forall t \geq 0$. So

$$||u(t)||_\beta \leq \left(\frac{\beta-\alpha}{e(t-s)}\right)^{\beta-\alpha} C_D + K_D(F)\left(\frac{\beta}{e}\right)^{\beta}\frac{|t-s|^{1-\beta}}{1-\beta}$$

for all $t > s \geq 0$, where $K_D(F) = \sup\{||F(v)|| : v \in D\}$. If we choose $s = t - \delta$, then

$$S_t D \subset \{u \in C_\beta : ||u||_\beta \leq R^*_\delta\}, \quad \text{for all} \quad t \geq \delta + h, \tag{2.20}$$

where

$$R^*_\delta \equiv \left(\frac{\beta-\alpha}{e\delta}\right)^{\beta-\alpha} C_D + K_D(F)\left(\frac{\beta}{e}\right)^{\beta}\frac{\delta^{1-\beta}}{1-\beta}.$$

This estimate is a part of the property (2.18), see (2.17).

Since parameter α is a smoothness parameter of the space C_α and phase space Ω^{log}_α, we change notations in (2.20) to adopt it for the proof of the dissipativeness (the existence of a bounded absorbing set). More precisely, we consider a solution $||u(t)||_\gamma \leq C_D, \forall t \geq 0$. Here $\gamma \geq 0$ instead of α. Now we estimate $||u(t)||_\alpha, \alpha \in (\gamma, 1)$ instead of $||u(t)||_\beta$. The estimate, similar to (2.20) gives

$$S_t D \subset \{u \in C_\alpha : ||u||_\alpha \leq \widehat{R}^*_\delta\}, \quad \text{for all} \quad t \geq \delta + h, \tag{2.21}$$

where

$$\widehat{R}^*_\delta \equiv \left(\frac{\alpha-\gamma}{e\delta}\right)^{\alpha-\gamma} C_D + K_D(F)\left(\frac{\alpha}{e}\right)^{\alpha}\frac{\delta^{1-\alpha}}{1-\alpha}.$$

Notice that for any $v \in [C(\overline{\Omega})]^3 \subset [L^2(\Omega)]^3$ one has $||v||_0 = ||v||_{[L^2(\Omega)]^3} \leq ||v||_{[C(\overline{\Omega})]^3} \cdot |\Omega|$ with $|\Omega| \equiv \int_\Omega 1dx$.

We apply the above property (2.21) for $\gamma = 0$ and $D = \Omega^{log}_\alpha \subset \Omega^{log}$ (bounded in C_X). Hence for $\gamma = 0$ the property $||u(t)||_0 \leq C_D, \forall t \geq 0$ holds. As a result, (2.21) implies (a) mild solutions are global (defined for all $t \geq -h$) and (b) the dissipativeness of the dynamical system $(S_t; \Omega^{log}_\alpha)$ for each $\alpha \in (\frac{3}{4}, 1)$.

Now by Proposition 2.6 [6, p.293] our dynamical system $(S_t; \Omega^{log}_\alpha)$ is quasi-stable.

We can apply [6, Theorem 6.1.12] to the dynamical system $(S_t; \Omega^{log}_\alpha)$ to get the main result - the existence of a finite-dimensional global attractor.

It completes the proof of Theorem 2.8. □

2.4 Examples of the Distributed Delay Term

Consider the nonlinear delay term B of the form (2.4). We present a simple example of function $\xi : [-h, 0] \times \Omega \times C \to \mathbb{R}$ (c.f. [23, 24])

$$\xi(\theta, x, \varphi) = e^{-\sigma(-\eta(\varphi)-\theta)^2} g(x), \quad \sigma > 0,$$

where

(i) $\eta : C \to [0, h]$ is Lipschitz continuous and $g \in L^\infty(\Omega)$ to satisfy $(H2)$ and
(ii) $\eta : C_X \to [0, h]$ is Lipschitz continuous and $g \in C(\overline{\Omega})$ to satisfy $(H3)$.

For motivations for such a state-selective delay see e.g. [23]. The profile function $e^{-\sigma(c-\theta)^2}$ was chosen for simplicity to show that the delay term of the form $\int_{-h}^{0} e^{-\sigma(c-\theta)^2} \phi(\theta) d\theta$ has the maximal historical impact in a neighbourhood of the time moment c (the maximum of function $e^{-\sigma(c-\theta)^2}$ at point $\theta = c$). In our example this maximum point can be *state-selective* [23] (state-dependent) $c = -\eta(\varphi) \in [-h, 0]$.

We mention some well-known examples of non-linear functions f used for viral infection models. The first one is the DeAngelis-Bendington [2, 7] functional response $f(T, V) = \frac{kTV}{1+k_1T+k_2V}$, with $k, k_1 \geq 0, k_2 > 0$. We also mention that the functional response includes as a special case ($k_1 = 0$) the *saturated incidence* rate $f(T, V) = \frac{kTV}{1+k_2V}$. Another example of the nonlinearity is the Crowley-Martin incidence rate $f(T, V) = \frac{kTV}{(1+k_1T)(1+k_2V)}$, with $k \geq 0, k_1, k_2 > 0$ and more general the Hattaf-Yousfi functional response of the form $\frac{kTV}{k_0+k_1T+k_2V+k_3TV}$ [11]. For more general class of functions f see, e.g. [11, 18, 29]. We notice that, in contrast to [11, 18], we do not assume here the differentiability of f.

We also mention that our assumptions on f are naturally less restrictive comparing to the ones in the mentioned above works where asymptotic stability of stationary solutions are discussed.

Conclusion

In this paper we study a virus dynamics model with reaction-diffusion, logistic growth terms and a general non-linear infection rate functional response. The model has a distributed delay, including the case of state-selective delay which is a distributed 'analog' to a discrete state-dependent delay.

Our main mathematical tool in studying of the asymptotic behaviour of solutions is the *quasi-stability method* developed by I.D.Chueshov [6]. We construct a dynamical system in a Hilbert space and prove the existence of a *finite-dimensional* global attractor. To prove the natural for a virus dynamics model dissipativness of the dynamical system we conduct a parallel study in a Banach space.

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