Escaping Orbits Are also Rare in the Almost Periodic Fermi–Ulam Ping-Pong

Henrik Schließauf¹

Received: 3 September 2019 / Accepted: 31 October 2019 / Published online: 11 December 2021
© The Author(s) 2021

Abstract

We study the one-dimensional Fermi–Ulam ping-pong problem with a Bohr almost periodic forcing function and show that the set of initial condition leading to escaping orbits typically has Lebesgue measure zero.

Keywords Almost periodic forcing · Fermi acceleration · Escaping set

Mathematics Subject Classification 37J25 · 37E40 · 37A05 · 43A60 · 70K40 · 70H11 · 70H14

1 Introduction

The Fermi–Ulam ping-pong is a model describing how charged particles bounce off magnetic mirrors and thus gain energy. They undergo the so called Fermi acceleration and one central question is whether the particles velocities can get close to the speed of light that way. The model was introduced by Fermi [13] in order to explain the origin of high energy cosmic radiation. A common one-dimensional mathematical formulation of this problem is as follows: The point particle bounces completely elastically between two vertical plates of infinite mass, one fixed at $x = 0$ and one moving in time as $x = p(t)$ for some forcing function $p = p(t) > 0$. The particle alternately hits the walls and experiences no external force in between the collisions. The motion can be described by the successor map $f : (t_0, v_0) \mapsto (t_1, v_1)$, mapping the time $t_0 \in \mathbb{R}$ of an impact at the left plate $x = 0$ and the corresponding velocity $v_0 > 0$ right after the collision to (t_1, v_1) , representing the subsequent impact at $x = 0$. Since one is interested in the long term behavior, we study the forward iterates $(t_n, v_n) = f^n(t_0, v_0)$ for $n \in \mathbb{N}$ and in particular the ‘escaping set’

✉ Henrik Schließauf
hschlies@math.uni-koeln.de

¹ Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany

$$E = \left\{ (t_0, v_0) : \lim_{n \rightarrow \infty} v_n = \infty \right\},$$

consisting of initial data, which lead to infinitely fast particles. The most studied case is that of a periodic forcing $p(t)$. Ulam [27] conjectured an increase in energy with time on the average. Based on some numerical simulations, he however realized that rather large fluctuations and no clear gain in energy seemed to be the typical behavior. Two decades later, the development of KAM theory allowed to prove that the conjecture is indeed false. If the forcing p is sufficiently smooth, all orbits stay bounded in the phase space, since the existence of invariant curves prevents the orbits from escaping [17, 23]. The proofs are based on Moser's twist theorem [18], which relies on the higher regularity. And indeed, Zharnitsky [29] showed the existence of escaping orbits if only continuity is imposed on p . In the non-periodic case, one can even find C^∞ -forcings with this behavior [15]. More recently, Dolgopyat and De Simoi developed a new approach. They consider the periodic case and study some maps which are basically approximations of the successor map f . This way they could prove several results regarding the Lebesgue measure of the escaping set E [9–11, 25].

Finally, Zharnitsky [30] investigated the case of a quasi-periodic forcing function whose frequencies satisfy a Diophantine inequality. Again, using an invariant curve theorem, he was able to show that the velocity of every particle is uniformly bounded in time. Since no such theorem is available if the Diophantine condition is dropped, a different approach is necessary in this case. This was done by Kunze and Ortega in [16]. They apply a refined version of the Poincaré recurrence theorem due to Dolgopyat [8] to the set of initial condition leading to unbounded orbits, and thereby show that most orbits are recurrent. Thus, typically the escaping set E will have Lebesgue measure zero. Now, in this work we will give an affirmative answer to the question raised in [16] whether this result can be generalized to the almost periodic case. Indeed, most of their arguments translate naturally into the language of Bohr almost periodic functions. Our main theorem (Theorem 5.1) states that the escaping set E is most likely to have measure zero, provided the almost periodic forcing p is sufficiently smooth.

In order to explain more precisely what we mean by ‘most likely’, we first need to introduce some properties and notation regarding almost periodic functions. This is done in Sect. 2. Subsequently we will study measure-preserving successor maps of a certain type and their iterations. We end this part by stating Theorem 3.1, a slightly generalized version of a theorem by Kunze and Ortega [16], which describes conditions under which the escaping set typically will have measure zero. This will be the most important tool and its proof will be given in the following section. Then, in the last section we discuss the ping-pong model in more detail and finally state and prove the main theorem.

2 Almost Periodic Functions and Their Representation

2.1 Compact Topological Groups and Minimal Flows

Let Ω be a commutative topological group, which is metrizable and compact. We will consider the group operation to be additive. Moreover, suppose there is a continuous

homomorphism $\psi : \mathbb{R} \rightarrow \Omega$, such that the image $\psi(\mathbb{R})$ is dense in Ω . This function ψ induces a canonical flow on Ω , namely

$$\Omega \times \mathbb{R} \rightarrow \Omega, \quad \omega \cdot t = \omega + \psi(t).$$

This flow is minimal, since

$$\overline{\omega \cdot \mathbb{R}} = \overline{\omega + \psi(\mathbb{R})} = \omega + \overline{\psi(\mathbb{R})} = \Omega$$

holds for every $\omega \in \Omega$. Let us also note that in general ψ can be nontrivial and periodic, but this happens if and only if $\Omega \cong \mathbb{S}^1$ [21].

Now consider the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and a continuous homomorphism $\varphi : \Omega \rightarrow \mathbb{S}^1$. Such functions φ are called characters and together with the point wise product they form a group, the so called dual group Ω^* . Its trivial element is the constant map with value 1. It is a well known fact that nontrivial characters exist, whenever Ω is nontrivial [22]. Also non-compact groups admit a dual group. Crucial to us will be the fact that

$$\mathbb{R}^* = \{t \mapsto e^{i\alpha t} : \alpha \in \mathbb{R}\}.$$

Now, for a nontrivial character $\varphi \in \Omega^*$ we define

$$\Sigma = \ker \varphi = \{\omega \in \Omega : \varphi(\omega) = 1\}.$$

Then Σ is a compact subgroup of Ω . If in addition $\Omega \not\cong \mathbb{S}^1$, it can be shown that Σ is perfect [21]. This subgroup will act as a global cross section to the flow on Ω . Concerning this, note that since $\varphi \circ \psi$ describes a nontrivial character of $\mathbb{R}$, there is a unique $\alpha \neq 0$ such that

$$\varphi(\psi(t)) = e^{i\alpha t}$$

for all $t \in \mathbb{R}$. Therefore, the minimal period of this function,

$$S = \frac{2\pi}{|\alpha|},$$

can be seen as a returning time on Σ in the following sense. If we denote by $\tau(\omega)$ the unique number in $[0, S)$ such that $\varphi(\omega) = e^{i\alpha\tau(\omega)}$, then one has

$$\varphi(\omega \cdot t) = \varphi(\omega + \psi(t)) = \varphi(\omega)\varphi(\psi(t)) = e^{i\alpha\tau(\omega)}e^{i\alpha t}$$

and thus

$$\omega \cdot t \in \Sigma \Leftrightarrow t \in -\tau(\omega) + S\mathbb{Z}.$$

Fig. 1 On the 2-torus $\mathbb{T}^2$, intersections of $\Sigma = \{0\} \times \mathbb{T}$ and the orbit of $\psi(t)$ are separated by time intervals of length $S = 1/\nu_1$

Also τ as defined above is a function $\tau : \Omega \rightarrow [0, S)$ that is continuous where $\tau(\omega) \neq 0$, i.e. on $\Omega \setminus \Sigma$. From this we can derive that the restricted flow

$$\Phi : \Sigma \times [0, S) \rightarrow \Omega, \quad \Phi(\sigma, t) = \sigma \cdot t,$$

is a continuous bijection. Like $\tau(\omega)$, its inverse

$$\Phi^{-1}(\omega) = (\omega \cdot (-\tau(\omega)), \tau(\omega))$$

is continuous only on $\Omega \setminus \Sigma$. Therefore, Φ describes a homeomorphism from $\Sigma \times (0, S)$ to $\Omega \setminus \Sigma$.

Example 2.1 One important example for such a group Ω is the N -Torus $\mathbb{T}^N$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We will denote classes in $\mathbb{T}^N$ by $\bar{\theta} = \theta + \mathbb{Z}$. Then, the image of the homomorphism

$$\psi(t) = (\overline{\nu_1 t}, \dots, \overline{\nu_N t})$$

winds densely around the torus $\mathbb{T}^N$, whenever the frequency vector $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{R}^N$ is nonresonant, i.e. rationally independent. It is easy to verify that the dual group of $\mathbb{T}^N$ is given by

$$(\mathbb{T}^N)^* = \{(\bar{\theta}_1, \dots, \bar{\theta}_N) \mapsto e^{2\pi i(k_1\theta_1 + \dots + k_N\theta_N)} : k \in \mathbb{Z}^N\}.$$

Therefore, one possible choice for the cross section would be

$$\Sigma = \{(\bar{\theta}_1, \dots, \bar{\theta}_N) \in \mathbb{T}^N : e^{2\pi i\theta_1} = 1\} = \{0\} \times \mathbb{T}^{N-1},$$

so $\varphi(\bar{\theta}_1, \dots, \bar{\theta}_N) = e^{2\pi i\theta_1}$. In this case, consecutive intersections of the flow and Σ would be separated by an interval of the length $1/\nu_1$.

Example 2.2 Lets consider another important topological group. Let $\Omega = \mathcal{S}_p$ be the p -adic solenoid, where $p = (p_i)_{i \in \mathbb{N}}$ is a sequence of prime numbers. $\mathcal{S}_p$ is defined as the projective limit of the inverse limit system

$$\mathcal{S}_p : \mathbb{S}^1 \xleftarrow{z^{p_1}} \mathbb{S}^1 \xleftarrow{z^{p_2}} \mathbb{S}^1 \xleftarrow{z^{p_3}} \dots,$$

where $\mathbb{S}^1 \xleftarrow{z^{p_i}} \mathbb{S}^1$ denotes the mapping $z \mapsto z^{p_i}$ of the circle $\mathbb{S}^1$ into itself. A point $z \in \mathcal{S}_p$ has the form $z = (z_0, z_1, z_2, \dots)$, where $z_{k-1} = z_k^{p_k}$ for $k \in \mathbb{N}$. Moreover, if we take the coordinatwise multiplication as the action, $\mathcal{S}_p$ becomes a compact abelian group with neutral element $(1, 1, \dots)$ [14, Theorem 10.13]. It can be endowed with the metric

$$d(z, w) = \sum_{k=0}^{\infty} \frac{d_{\mathbb{S}^1}(z_k, w_k)}{q_k},$$

where $d_{\mathbb{S}^1}$ denotes the canonical metric on $\mathbb{S}^1$ and $q_0 = 1$, $q_k = p_1 \cdots p_k$ for $k \in \mathbb{N}$. For $\lambda > 0$, the map

$$\psi(t) = (e^{2\pi i \lambda t / q_0}, e^{2\pi i \lambda t / q_1}, e^{2\pi i \lambda t / q_2}, \dots)$$

provides a minimal flow on $\mathcal{S}_p$. A cross section with return time $S = 1/\lambda$ is then given by

$$\Sigma = \{z \in \mathcal{S}_p : z_0 = 1\}.$$

Geometrically, $\mathcal{S}_p$ can be described as the intersections of a sequence of solid tori $T_1 \supset T_2 \supset \dots$ in $\mathbb{R}^3$, where T_{k+1} is wrapped p_k times longitudinally inside T_k without self intersecting. See [3] for a nice description of the construction in the case of the dyadic solenoid $\mathcal{S}_2$.

2.2 Almost Periodic Functions

The notion of almost periodic functions was introduced by H. Bohr as a generalization of strictly periodic functions [5]. A function $u \in C(\mathbb{R})$ is called (Bohr) *almost periodic*, if for any $\epsilon > 0$ there is a relatively dense set of ϵ -almost-periods of this function. By this we mean, that for any $\epsilon > 0$ there exists $L = L(\epsilon)$ such that any interval of length L contains at least number T such that

$$|u(t + T) - u(t)| < \epsilon \quad \forall t \in \mathbb{R}.$$

Later, Bochner [4] gave an alternative but equivalent definition of this property: For a continuous function u , denote by $u_\tau(t)$ the translated function $u(t + \tau)$. Then u is (Bohr) almost periodic if and only if every sequence $(u_{\tau_n})_{n \in \mathbb{N}}$ of translations of u has a subsequence that converges uniformly.

There are several other characterizations of almost periodicity, as well as generalizations due to Stepanov [26], Weyl [28] and Besicovitch [1]. In this work we will only consider the notion depicted above and therefore call the corresponding functions just *almost periodic (a.p.)*. We will however introduce one more way to describe a.p. functions using the framework of the previous section:

Consider (Ω, ψ) as above and a function $U \in C(\Omega)$. Then, the function defined by

$$u(t) = U(\psi(t)) \quad (2.1)$$

is almost periodic. This can be verified easily with the alternative definition due to Bochner. Since $U \in C(\Omega)$, any sequence $(u_{\tau_n})_{n \in \mathbb{N}}$ will be uniformly bounded and equicontinuous. Hence the Arzelà-Ascoli theorem guarantees the existence of a uniformly convergent subsequence. We will call any function obtainable in this manner *representable over* (Ω, ψ) . Since the image of ψ is assumed to be dense, it is clear that the function $U \in C(\Omega)$ is uniquely determined by this relation. As an example take $\Omega \cong \mathbb{S}^1$, then ψ is periodic. Thus (2.1) gives rise to periodic functions. Conversely it is true, that any almost periodic function can be constructed this way. For this purpose we introduce the notion of hull. The hull $\mathcal{H}_u$ of a function u is defined by

$$\mathcal{H}_u = \overline{\{u_\tau : \tau \in \mathbb{R}\}},$$

where the closure is taken with respect to uniform convergence on the whole real line. Therefore if u is a.p. then $\mathcal{H}_u$ is a compact metric space. If one uses the continuous extension of the rule

$$u_\tau * u_s = u_{\tau+s} \quad \forall \tau, s \in \mathbb{R}$$

onto all of $\mathcal{H}_u$ as the group operation, then the hull becomes a commutative topological group with neutral element u . (For $v, w \in \mathcal{H}_u$ with $v = \lim_{n \rightarrow \infty} u_{\tau_n}^v$ and $w = \lim_{n \rightarrow \infty} u_{\tau_n}^w$ we have

$$v * w = \lim_{n \rightarrow \infty} u_{\tau_n^v + \tau_n^w}^{v+w}, \quad -v = \lim_{n \rightarrow \infty} u_{-\tau_n^v}^{-v}.$$

These limits exist by Lemma 6.1 from the appendix. The continuity of both operations can be shown by a similar argument.) If we further define the flow

$$\psi_u(\tau) = u_\tau, \quad (2.2)$$

then the pair $(\mathcal{H}_u, \psi_u)$ matches perfectly the setup of the previous section. Now, the representation formula (2.1) holds for $U \in C(\mathcal{H}_u)$ defined by

$$U(w) = w(0) \quad \forall w \in \mathcal{H}_u. \quad (2.3)$$

This function is sometimes called the ‘extension by continuity’ of the almost periodic function $u(t)$ to its hull $\mathcal{H}_u$. This construction is standard in the theory of a.p. functions and we refer the reader to [20] for a more detailed discussion.

For a function $U : \Omega \rightarrow \mathbb{R}$ let us introduce the derivative along the flow by

$$\partial_\psi U(\omega) = \lim_{t \rightarrow 0} \frac{U(\omega + \psi(t)) - U(\omega)}{t}.$$

Let $C_\psi^1(\Omega)$ be the space of continuous functions $U : \Omega \rightarrow \mathbb{R}$ such that $\partial_\psi U$ exists for all $\omega \in \Omega$ and $\partial_\psi U \in C(\Omega)$. The spaces $C_\psi^k(\Omega)$ for $k \geq 2$ are defined accordingly. Let us also introduce the norm $\|U\|_{C_\psi^k(\Omega)} = \|U\|_\infty + \sum_{n=1}^k \|\partial_\psi^{(n)} U\|_\infty$. Now consider $U \in C(\Omega)$ and assume the almost periodic function $u(t) = U(\psi(t))$ is continuously differentiable. Then $\partial_\psi U$ exists on $\psi(\mathbb{R})$ and we have

$$u'(t) = \partial_\psi U(\psi(t)) \quad \text{for all } t \in \mathbb{R}.$$

Lemma 2.3 *Let $U \in C(\Omega)$ and $u \in C(\mathbb{R})$ be such that $u(t) = U(\psi(t))$. Then we have $u \in C^1(\mathbb{R})$ and $u'(t)$ is a.p. if and only if $U \in C_\psi^1(\Omega)$.*

One part of the equivalence is trivial. The proof of the other part can be found in [21, Lemma 13]. We also note that the derivative $u'(t)$ of an almost periodic function is itself a.p. if and only if it is uniformly continuous. This, and many other interesting properties of a.p. functions are demonstrated in [1].

Example 2.4 Let us continue Example 2.1, where $\Omega = \mathbb{T}^N$. For $U \in C(\mathbb{T}^N)$ consider the function

$$u(t) = U(\psi(t)) = U(\overline{v_1 t}, \dots, \overline{v_N t}).$$

Such functions are called *quasi-periodic*. In this case, ∂_ψ is just the derivative in the direction of $v \in \mathbb{R}^N$. So if U is in the space $C^1(\mathbb{T}^N)$ of functions in $C^1(\mathbb{R}^N)$, which are 1-periodic in each argument, then

$$\partial_\psi U = \sum_{i=1}^N v_i \partial_{\theta_i} U.$$

Note however, that in general $C^1(\mathbb{T}^N)$ is a proper subspace of $C_\psi^1(\mathbb{T}^N)$.

Example 2.5 The so called limit periodic functions are another important subclass of the a.p. functions. Here, a map $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *limit periodic* if it is the uniform limit of continuous periodic functions. Now, in continuation of Example 2.2, let $\Omega = S_p$ and consider $U \in C(S_p)$. Then, the function $u(t)$ defined by

$$u(t) = U(\psi(t)) = U(e^{2\pi i \lambda t / q_0}, e^{2\pi i \lambda t / q_1}, e^{2\pi i \lambda t / q_2}, \dots)$$

is limit periodic, since it is the uniform limit of a sequence of q_k -periodic functions u_k given by

$$u_k(t) = U(e^{2\pi i \lambda t / q_0}, \dots, e^{2\pi i \lambda t / q_k}, 1, 1, \dots).$$

Vice versa it is true, that for suitable $\mathbf{p}$ and $\lambda > 0$ any limit periodic function $v(t)$ can be obtained in this manner. To see this, first note that v can be expanded in a uniformly convergent series of continuous 1-periodic functions,

$$v(t) = \sum_{k=0}^{\infty} v_k(t/T_k),$$

with $T_0 > 0$ and T_k such that $T_k/T_{k-1} \in \mathbb{N}$ for all $k \in \mathbb{N}$ (cf. [6]). W.l.o.g. we can assume that $p_k := \frac{T_k}{T_{k-1}}$ is a prime number for all k . Moreover, one can show that the hull $\mathcal{H}_v$ of $v(t)$ consists of those functions $w_\phi(t)$ which can be written in the form

$$w_\phi(t) = \sum_{k=0}^{\infty} v_k((t + \phi_k)/T_k),$$

where ϕ_k is an angle defined modulo T_k such that $\phi_{k-1} \equiv \phi_k \pmod{T_{k-1}}$ for all $k \in \mathbb{N}$ (see [19] for a more detailed discussion in the case of a specific example). We can then define $z_k = e^{2\pi i \phi_k / T_k}$ to obtain a series $(z_k)_{k \in \mathbb{N}_0}$ in $\mathbb{S}^1$ so that

$$z_k^{p_k} = e^{2\pi i \phi_k / T_{k-1}} = e^{2\pi i \phi_{k-1} / T_{k-1}} = z_{k-1}.$$

In other words, we have $z \in \mathcal{S}_{\mathbf{p}}$, where $\mathbf{p} = (p_k)_{k \in \mathbb{N}}$. We write $\eta : \mathcal{S}_{\mathbf{p}} \rightarrow \mathcal{H}_v$ for the resulting continuous map $(z_k) = (e^{2\pi i \phi_k / T_k}) \mapsto w_\phi(t)$. The translation flow restricted to $\mathcal{H}_v$ as described in (2.2) then corresponds to ψ as above with $\lambda = 1/T_0$, that is $\psi(\tau) = (e^{2\pi i \tau / T_0}, e^{2\pi i \tau / T_1}, \dots)$. Indeed, defining $V = U \circ \eta$, where U is the extension by continuity as in (2.3), yields

$$V(\psi(\tau)) = U(w_\tau) = w_\tau(0) = \sum_{k=0}^{\infty} v_k(\tau/T_k) = v(\tau).$$

2.3 Haar Measure and Decomposition Along the Flow

It is a well known fact, that for every compact commutative topological group Ω there is a unique Borel probability measure μ_Ω , which is invariant under the group operation, i.e. $\mu_\Omega(\mathcal{D} + \omega) = \mu_\Omega(\mathcal{D})$ holds for every Borel set $\mathcal{D} \subset \Omega$ and every $\omega \in \Omega$. This measure is called the *Haar measure* of Ω . (This follows from the existence of the invariant *Haar integral* of Ω and the Riesz representation theorem. Proofs can be found in [22] and [14], respectively.) For Example if $\Omega = \mathbb{S}^1$ we have

$$\mu_{\mathbb{S}^1}(\mathcal{B}) = \frac{1}{2\pi} \lambda\{t \in [0, 2\pi) : e^{it} \in \mathcal{B}\},$$

where λ is the Lebesgue measure on $\mathbb{R}$. Let ψ , Σ and Φ be as in Sect. 2.1. Then Φ defines a decomposition $\Omega \cong \Sigma \times [0, S)$ along the flow. Since Σ is a subgroup, it has a Haar measure μ_Σ itself. Also the interval $[0, S)$ naturally inherits the probability measure

$$\mu_{[0, S)}(I) = \frac{1}{S}\lambda(I).$$

As shown in [7], the restricted flow $\Phi : \Sigma \times [0, S) \rightarrow \Omega$, $\Phi(\sigma, t) = \sigma \cdot t$ also allows for a decomposition of the Haar measure μ_Ω along the flow.

Lemma 2.6 *The map Φ is an isomorphism of measure spaces, i.e.*

$$\mu_\Omega(\mathcal{B}) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi^{-1}(\mathcal{B})) \quad (2.4)$$

holds for every Borel set $\mathcal{B} \subset \Omega$.

Before we prove this lemma, let us begin with some preliminaries. Consider the function $\chi : \Sigma \times [0, \infty) \rightarrow \Sigma \times [0, S)$ defined by

$$\chi(\sigma, t) = \Phi^{-1}(\sigma \cdot t) = \Phi^{-1}(\sigma + \psi(t)). \quad (2.5)$$

Since Φ is just the restricted flow, we have $\chi = \text{id}$ on $\Sigma \times [0, S)$. This yields

$$\begin{aligned} \chi(\sigma, t) &= \Phi^{-1}(\sigma + \psi(t)) = \Phi^{-1}\left(\sigma + \psi\left(\left\lfloor \frac{t}{S} \right\rfloor S\right) + \psi\left(t - \left\lfloor \frac{t}{S} \right\rfloor S\right)\right) \\ &= \left(\sigma + \psi\left(\left\lfloor \frac{t}{S} \right\rfloor S\right), t - \left\lfloor \frac{t}{S} \right\rfloor S\right) \end{aligned}$$

for every $(\sigma, t) \in \Sigma \times \mathbb{R}$, where $\lfloor \cdot \rfloor$ indicates the floor function. This representation shows that χ is measure-preserving on every strip $\Sigma \times [t, t + S)$ of width S , since μ_Σ and λ are invariant under translations in Σ and $\mathbb{R}$, respectively. Moreover, the equality

$$\chi(\Phi^{-1}(\omega) + \Phi^{-1}(\tilde{\omega})) = \Phi^{-1}(\omega + \tilde{\omega}) \quad \forall \omega, \tilde{\omega} \in \Omega \quad (2.6)$$

follows directly from the definition of χ .

Proof of Lemma 2.6 First we show that Φ^{-1} is Borel measurable. To prove this, it suffices to show that the image $\Phi(A \times I)$ of every open rectangle $A \times I \subset \Sigma \times [0, S)$ is a Borel set. If $0 \notin I$ this image is open in $\Omega \setminus \Sigma$, since Φ^{-1} is continuous. But if $0 \in I$, again $\Phi(A \times (I \setminus \{0\}))$ is open and $\Phi(A \times \{0\}) = A$ is it as well.

Now, consider the measure μ_Φ on Ω defined by

$$\mu_\Phi(\mathcal{B}) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi^{-1}(\mathcal{B})). \quad (2.7)$$

Fig. 2 Let $\chi(\sigma, t) = (\tilde{\sigma}, s)$. The map χ ‘divides out’ every complete period of $\varphi \circ \psi$, i.e. $s = t \bmod S$, while preserving the relation $\tilde{\sigma} \cdot s = \omega = \sigma \cdot t$

Since $\mu_\Phi(\Omega) = 1$, this is a Borel probability measure. We will show that μ_Φ is also invariant under addition in the group. For this purpose, let $\mathcal{B} \subset \Omega$ be a Borel set and let $\omega_0 \in \Omega$. Then, by (2.6) we have

$$\mu_\Phi(\mathcal{B} + \omega_0) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi^{-1}(\mathcal{B} + \omega_0)) \quad (2.8)$$

$$= \frac{1}{S}(\mu_\Sigma \otimes \lambda) \left(\chi(\Phi^{-1}(\mathcal{B}) + \Phi^{-1}(\omega_0)) \right). \quad (2.9)$$

Denoting $\Phi^{-1}(\omega_0) = (\sigma_0, s_0)$, we get $\Phi^{-1}(\mathcal{B}) + \Phi^{-1}(\omega_0) \subset \Sigma \times [s_0, s_0 + S)$. So it is contained in a strip of width S and therefore

$$\frac{1}{S}(\mu_\Sigma \otimes \lambda) \left(\chi(\Phi^{-1}(\mathcal{B}) + (\sigma_0, s_0)) \right) = \frac{1}{S}(\mu_\Sigma \otimes \lambda) \left(\Phi^{-1}(\mathcal{B}) + (\sigma_0, s_0) \right)$$

But the product measure $\mu_\Sigma \otimes \lambda$ is invariant under translations in $\Sigma \times \mathbb{R}$. Thus, in total we have

$$\mu_\Phi(\mathcal{B} + \omega_0) = \frac{1}{S}(\mu_\Sigma \otimes \lambda) \left(\Phi^{-1}(\mathcal{B}) \right) = \mu_\Phi(\mathcal{B}). \quad (2.10)$$

Therefore, μ_Φ is a Borel probability measure on Ω which is invariant under group action. Since the Haar measure is unique, it follows $\mu_\Omega = \mu_\Phi$. $\square$

3 A Theorem About Escaping Sets

3.1 Measure-Preserving Embeddings

From now on we will consider functions

$$f : \mathcal{D} \subset \Omega \times (0, \infty) \rightarrow \Omega \times (0, \infty),$$

where $\mathcal{D}$ is an open set. We will call such a function *measure-preserving embedding*, if f is continuous, injective and furthermore

$$(\mu_\Omega \otimes \lambda)(f(\mathcal{B})) = (\mu_\Omega \otimes \lambda)(\mathcal{B})$$

holds for all Borel sets $\mathcal{B} \subset \mathcal{D}$, where λ denotes the Lebesgue measure of $\mathbb{R}$. It is easy to show that under these conditions, $f : \mathcal{D} \rightarrow \hat{\mathcal{D}}$ is a homeomorphism, where $\hat{\mathcal{D}} = f(\mathcal{D})$.

Since we want to use the iterations of f , we have to carefully construct a suitable domain on which these forward iterations are well-defined. We initialize $\mathcal{D}_1 = \mathcal{D}$, $f^1 = f$ and set

$$\mathcal{D}_{n+1} = f^{-1}(\mathcal{D}_n), \quad f^{n+1} = f^n \circ f \quad \text{for } n \in \mathbb{N}.$$

This way f^n is well-defined on $\mathcal{D}_n$. Clearly, f^n is a measure-preserving embedding as well. Also inductively it can be shown that $\mathcal{D}_{n+1} = \{(\omega, r) \in \mathcal{D} : f(\omega, r), \dots, f^n(\omega, r) \in \mathcal{D}\}$ and therefore $\mathcal{D}_{n+1} \subset \mathcal{D}_n \subset \mathcal{D}$ for all $n \in \mathbb{N}$. Initial conditions in the set

$$\mathcal{D}_\infty = \bigcap_{n=1}^{\infty} \mathcal{D}_n \subset \Omega \times (0, \infty)$$

correspond to complete forward orbits, i.e. if $(\omega_0, r_0) \in \mathcal{D}_\infty$, then

$$(\omega_n, r_n) = f^n(\omega_0, r_0)$$

is defined for all $n \in \mathbb{N}$. It could however happen that $\mathcal{D}_\infty = \emptyset$ or even $\mathcal{D}_n = \emptyset$ for some $n \geq 2$. The set of initial data leading to unbounded orbits is denoted by

$$\mathcal{U} = \left\{ (\omega_0, r_0) \in \mathcal{D}_\infty : \limsup_{n \rightarrow \infty} r_n = \infty \right\}. \quad (3.1)$$

Complete orbits such that $\lim_{n \rightarrow \infty} r_n = \infty$ will be called *escaping orbits*. The corresponding set of initial data is

$$\mathcal{E} = \left\{ (\omega_0, r_0) \in \mathcal{D}_\infty : \lim_{n \rightarrow \infty} r_n = \infty \right\}.$$

3.2 Almost Periodic Successor Maps

Now, consider a measure-preserving embedding $f : \mathcal{D} \subset \Omega \times (0, \infty) \rightarrow \Omega \times (0, \infty)$, which has the special structure

$$f(\omega, r) = (\omega + \psi(F(\omega, r)), r + G(\omega, r)), \quad (3.2)$$

where $F, G : \mathcal{D} \rightarrow \mathbb{R}$ are continuous. For $\omega \in \Omega$ we introduce the notation $\psi_\omega(t) = \omega + \psi(t) = \omega \cdot t$ and define

$$D_\omega = (\psi_\omega \times \text{id})^{-1}(\mathcal{D}) \subset \mathbb{R} \times (0, \infty).$$

On this open set, consider the map $f_\omega : D_\omega \subset \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R} \times (0, \infty)$ given by

$$f_\omega(t, r) = (t + F(\psi_\omega(t), r), r + G(\psi_\omega(t), r)). \quad (3.3)$$

Then f_ω is continuous and meets the identity

$$f \circ (\psi_\omega \times \text{id}) = (\psi_\omega \times \text{id}) \circ f_\omega \quad \text{on } D_\omega,$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & f(\mathcal{D}) \subset \mathbb{T}^N \times (0, \infty) \\ \psi_\omega \times \text{id} \uparrow & & \psi_\omega \times \text{id} \uparrow \\ D_\omega & \xrightarrow{f_\omega} & f_\omega(D_\omega) \subset \mathbb{R} \times (0, \infty) \end{array}$$

Therefore f_ω is injective as well. Again we define $D_{\omega,1} = D_\omega$ and $D_{\omega,n+1} = f_\omega^{-1}(D_{\omega,n})$ to construct the set

$$D_{\omega,\infty} = \bigcap_{n=1}^{\infty} D_{\omega,n} \subset \mathbb{R} \times (0, \infty),$$

where the forward iterates $(t_n, r_n) = f_\omega^n(t_0, t_0)$ are defined for all $n \in \mathbb{N}$. Analogously, unbounded orbits are generated by initial conditions in the set

$$U_\omega = \left\{ (t_0, r_0) \in D_{\omega,\infty} : \limsup_{n \rightarrow \infty} r_n = \infty \right\}$$

and escaping orbits originate in

$$E_\omega = \left\{ (t_0, r_0) \in D_{\omega,\infty} : \lim_{n \rightarrow \infty} r_n = \infty \right\}.$$

These sets can also be obtained through the relations

$$D_{\omega,\infty} = (\psi_\omega \times \text{id})^{-1}(\mathcal{D}_\infty), \quad U_\omega = (\psi_\omega \times \text{id})^{-1}(\mathcal{U}), \quad E_\omega = (\psi_\omega \times \text{id})^{-1}(\mathcal{E}).$$

Finally we are in position to state the theorem [16, Theorem 3.1]:

Theorem 3.1 *Let $f : \mathcal{D} \subset \Omega \times (0, \infty) \rightarrow \Omega \times (0, \infty)$ be a measure-preserving embedding of the form (3.2) and suppose that there is a function $W = W(\omega, r)$ satisfying $W \in C^1_\psi(\Omega \times (0, \infty))$,*

$$0 < \beta \leq \partial_r W(\omega, r) \leq \delta \text{ for } \omega \in \Omega, \quad r \in (0, \infty), \quad (3.4)$$

with some constants $\beta, \delta > 0$, and furthermore

$$W(f(\omega, r)) \leq W(\omega, r) + k(r) \text{ for } (\omega, r) \in \mathcal{D}, \quad (3.5)$$

where $k : (0, \infty) \rightarrow \mathbb{R}$ is a decreasing and bounded function such that $\lim_{r \rightarrow \infty} k(r) = 0$. Then, for almost all $\omega \in \Omega$, the set $E_\omega \subset \mathbb{R} \times (0, \infty)$ has Lebesgue measure zero.

Here, $C^1_\psi(\Omega \times (0, \infty))$ denotes the space of continuous functions $U(\omega, r)$ such that both derivatives $\partial_\psi U$ and $\partial_r U$ exist on $\Omega \times (0, \infty)$ and $\partial_\psi U, \partial_r U \in C(\Omega \times (0, \infty))$. The function W can be seen as a generalized adiabatic invariant, since any growth will be slow for large energies.

4 Proof of Theorem 3.1

The proof of Theorem 3.1 is based on the fact, that almost all unbounded orbits of f are recurrent. In order to show this, we will apply the Poincaré recurrence theorem to the set $\mathcal{U}$ of unbounded orbits and the corresponding restricted map $f|_{\mathcal{U}}$. We will use it in the following form [16, Lemma 4.2].

Lemma 4.1 *Let $(X, \mathcal{F}, \mu)$ be a measure space such that $\mu(X) < \infty$. Suppose that there exists a measurable set $\Gamma \subset X$ of measure zero and a map $T : X \setminus \Gamma \rightarrow X$ which is injective and so that the following holds:*

- (a) *T is measurable, in the sense $T(B), T^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{F}$, and*
- (b) *T is measure-preserving, in the sense that $\mu(T(B)) = \mu(B)$ for $B \in \mathcal{F}$.*

Then for every measurable set $B \subset X$ almost all points of B visit B infinitely many times in the future (i.e. T is infinitely recurrent).

Since we can not guarantee that $\mathcal{U}$ has finite measure, we will also need the following refined version of the recurrence theorem due to Dolgopyat [8, Lemma 4.3].

Lemma 4.2 *Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose that the map $T : X \rightarrow X$ is injective and such that the following holds:*

- (a) *T is measurable, in the sense $T(B), T^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{F}$,*
- (b) *T is measure-preserving, in the sense that $\mu(T(B)) = \mu(B)$ for $B \in \mathcal{F}$, and*
- (c) *there is a set $A \in \mathcal{F}$ such that $\mu(A) < \infty$ with the property that almost all points from X visit A in the future.*

Then for every measurable set $B \subset X$ almost all points of B visit B infinitely many times in the future (i.e. T is infinitely recurrent).

For the sake of completeness let us state the proof.

Proof of Lemma 4.2 Let $\Gamma \subset X$ be measurable such that $\mu(\Gamma) = 0$ and all points of $X \setminus \Gamma$ visit A in the future. Thus, the first return time $r(x) = \min\{k \in \mathbb{N} : T^k(x) \in A\}$ is well-defined for $x \in X \setminus \Gamma$. It induces a map $S : X \setminus \Gamma \rightarrow A$ defined by $S(x) = T^{r(x)}(x)$. The restriction $S|_{A \setminus \Gamma}$ is injective: Assume $S(x) = S(y)$ for distinct points $x, y \in A \setminus \Gamma$ and suppose $r(x) > r(y)$, then $T^{r(x)-r(y)}(x) = y \in A$ is a contradiction to the minimality of $r(x)$. It is also measure-preserving [12, cf. Lemma 2.43]. Now, consider a measurable set $B \subset X$ and define $B_j = \{y \in B \setminus \Gamma : r(y) \leq j\}$ as well as

$$A_j = S(B_j) = \bigcup_{k=1}^j (T^k(B) \cap A) \subset A \quad \forall j \in \mathbb{N}.$$

But since $\mu(A) < \infty$ by assumption, the Poincaré recurrence theorem (Lemma 4.1) applies to A_j . Thus we can find measurable sets $\Gamma_j \subset A_j$ with measure zero, such that every point $x \in A_j \setminus \Gamma_j$ returns to A_j infinitely often (via S). Now consider the set

$$F = B \cap \left(\Gamma \cup \bigcup_{j \in \mathbb{Z}} S^{-1}(\Gamma_j) \right).$$

Then $\mu(F) = 0$ and every point $y \in B \setminus F$ returns to B infinitely often in the future. To see this, select $j \in \mathbb{N}$ such that $r(y) \leq j$, i.e. $y \in B_j$. Then $x = S(y) \in A_j \setminus \Gamma_j$. Hence there exist infinitely many $k \in \mathbb{N}$ so that $k \geq j$ and $S^k(x) \in A_j$. Let us fix one of these k . Then $S^k(x) = S(z)$ for some $z \in B_j$. So in total we have

$$T^{r(z)}(z) = S(z) = S^k(x) = S^{k+1}(y) = T^{\sum_{j=1}^k r(S^j(y))}(y).$$

Now, since $\sum_{j=1}^k r(S^j(y)) \geq k+1 > j \geq r(z)$, this yields $T^m(y) = z \in B_j \subset B$, where $m = \sum_{j=1}^k r(S^j(y)) - r(z) \in \mathbb{N}$. $\square$

One way to construct such a set A of finite measure is given by the next lemma [16]. It is based on the function $W(\omega, r)$ introduced in Theorem 3.1 and in fact is the only reason to assume the existence of W in the first place.

Lemma 4.3 Let $f : \mathcal{D} \subset \Omega \times (0, \infty) \rightarrow \Omega \times (0, \infty)$ be a measure-preserving embedding and suppose that there is a function $W = W(\omega, r)$ satisfying $W \in C_\psi^1(\Omega \times (0, \infty))$, (3.4) and (3.5). Let $(\epsilon_j)_{j \in \mathbb{N}}$ and $(W_j)_{j \in \mathbb{N}}$ be sequences of positive numbers with the properties $\sum_{j=1}^\infty \epsilon_j < \infty$, $\lim_{j \rightarrow \infty} W_j = \infty$ and $\lim_{j \rightarrow \infty} \epsilon_j^{-1} k(\frac{1}{4\gamma} W_j) = 0$. Denote

$$\mathcal{A} = \bigcup_{j \in \mathbb{N}} \mathcal{A}_j, \quad \mathcal{A}_j = \{(\omega, r) \in \Omega \times (0, \infty) : |W(\omega, r) - W_j| \leq \epsilon_j\}. \quad (4.1)$$

Then $\mathcal{A}$ has finite measure and every unbounded orbit of f enters $\mathcal{A}$. More precisely, if $(\omega_0, r_0) \in \mathcal{U}$, where $\mathcal{U}$ is from (3.1), and if $(\omega_n, r_n)_{n \in \mathbb{N}}$ denotes the forward orbit under f , then there is $K \in \mathbb{N}$ so that $(\omega_K, r_K) \in \mathcal{A}$.

Proof First let us show that $\mathcal{A}$ has finite measure. By Fubini's theorem,

$$(\mu_\Omega \otimes \lambda)(\mathcal{A}_j) = \int_\Omega \lambda(\mathcal{A}_{j,\omega}) d\mu_\Omega(\omega)$$

holds for the sections $\mathcal{A}_{j,\omega} = \{r \in (0, \infty) : (\omega, r) \in \mathcal{A}_j\}$. Now, consider the diffeomorphism $w_\omega : r \mapsto W(\omega, r)$. Its inverse w_ω^{-1} is Lipschitz continuous with constant β^{-1} , due to (3.4). But then, $\mathcal{A}_{j,\omega} = w_\omega^{-1}((W_j - \epsilon_j, W_j + \epsilon_j))$ implies $\lambda(\mathcal{A}_{j,\omega}) \geq 2\beta^{-1}\epsilon_j$. Thus in total we have

$$(\mu_\Omega \otimes \lambda)(\mathcal{A}) \leq \sum_{j=1}^{\infty} (\mu_\Omega \otimes \lambda)(\mathcal{A}_j) \leq \sum_{j=1}^{\infty} \frac{2\epsilon_j}{\beta} < \infty.$$

Next we will prove the recurrence property. To this end, let $(\omega_0, r_0) \in \mathcal{U}$ be fixed and denote by (ω_n, r_n) the forward orbit under f . We will start with some preliminaries. Using (3.4) and the mean value theorem, we can find $\hat{r}$ such that

$$\frac{\beta}{2} \leq \frac{W(\omega, r)}{r} \leq 2\delta \quad \forall (\omega, r) \in \Omega \times (\hat{r}, \infty). \quad (4.2)$$

Furthermore, by assumption we can find an index $j_0 \geq 2$ such that

$$W_{j_0} > \max\{W(\omega_1, r_1), \|k\|_\infty + \max_{\omega \in \Omega} W(\omega, \hat{r}), 2\|k\|_\infty\} \quad \text{and} \quad k\left(\frac{1}{4\gamma} W_{j_0}\right) \leq \epsilon_{j_0}.$$

Moreover we have $\limsup_{n \rightarrow \infty} W(\omega_n, r_n) = \infty$: Due to $\limsup_{n \rightarrow \infty} r_n = \infty$, (3.4) implies

$$W(\omega_n, r_n) \geq \beta(r_n - r_1) + W(\omega_n, r_1)$$

for n sufficiently large. But then $\limsup_{n \rightarrow \infty} W(\omega_n, r_n) = \infty$ follows from the compactness of Ω . Now, since $W(\omega_1, r_1) < W_{j_0}$ we can select the first index $K \geq 2$ such that $W(\omega_K, r_K) > W_{j_0}$. So in particular this means $W(\omega_{K-1}, r_{K-1}) \leq W_{j_0}$. Since (3.5) yields $W(\omega_K, r_K) \leq W(\omega_{K-1}, r_{K-1}) + k(r_{K-1})$, we can derive the following inequality:

$$\begin{aligned} W(\omega_{K-1}, r_{K-1}) &\geq W(\omega_K, r_K) - \|k\|_\infty > W_{j_0} - \|k\|_\infty \geq \max_{\omega \in \Omega} W(\omega, \hat{r}) \\ &\geq W(\omega_{K-1}, \hat{r}) \end{aligned}$$

Then, the monotonicity of $w_{\omega_{K-1}}$ implies $r_{K-1} > \hat{r}$. Hence we can combine (4.2) with the previous estimate to obtain

$$r_{K-1} \geq \frac{1}{2\delta} W(\omega_{K-1}, r_{K-1}) \geq \frac{1}{2\delta} (W_{j_0} - \|k\|_\infty) \geq \frac{1}{4\delta} W_{j_0}.$$

Finally, since $k(r)$ is decreasing, $W(\omega_K, r_K) > W_{j_0} \geq W(\omega_{K-1}, r_{K-1})$ yields

$$\begin{aligned} |W(\omega_K, r_K) - W_{j_0}| &\leq W(\omega_K, r_K) - W(\omega_{K-1}, r_{K-1}) \leq k(r_{K-1}) \\ &\leq k\left(\frac{1}{4\delta} W_{j_0}\right) \leq \epsilon_{j_0}, \end{aligned}$$

which implies $(\omega_K, r_K) \in \mathcal{A}_{j_0}$. $\square$

Now, we are ready to prove the theorem.

Proof of Theorem 3.1 Consider the set

$$\mathcal{U} = \left\{ (\omega_0, r_0) \in \mathcal{D}_\infty : \limsup_{n \rightarrow \infty} r_n = \infty \right\}.$$

We will assume that $\mathcal{U} \neq \emptyset$, since otherwise the assertion would be a direct consequence. Step 1: Almost all unbounded orbits are recurrent. We will prove the existence of a set $\mathcal{Z} \subset \mathcal{U}$ of measure zero such that if $(\omega_0, r_0) \in \mathcal{U} \setminus \mathcal{Z}$, then

$$\liminf_{n \rightarrow \infty} r_n < \infty.$$

In particular, we would have $\mathcal{E} \subset \mathcal{Z}$. To show this, we consider the restriction $T = f|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}$. This map is well-defined, injective and, like f , measure-preserving. We will distinguish three cases:

- (i) $(\mu_\Omega \otimes \lambda)(\mathcal{U}) = 0$,
- (ii) $0 < (\mu_\Omega \otimes \lambda)(\mathcal{U}) < \infty$, and
- (iii) $(\mu_\Omega \otimes \lambda)(\mathcal{U}) = \infty$.

In the first case $\mathcal{Z} = \mathcal{U}$ is a valid choice. In case (ii) we can apply the Poincaré recurrence theorem (Lemma 4.1), whereas in case (iii) the modified version of Dolgopyat (Lemma 4.2) is applicable due to Lemma 4.3. Now, let us cover $\Omega \times \mathbb{R}$ by the sets $\mathfrak{B}_j = \Omega \times (j-1, j+1)$ for $j \in \mathbb{N}$. Then, for $\mathcal{B}_j = \mathfrak{B}_j \cap \mathcal{U}$ one can use the recurrence property to find sets $\mathcal{Z}_j \subset \mathcal{B}_j$ of measure zero such that every orbit $(\omega_n, r_n)_{n \in \mathbb{N}}$ starting in $\mathcal{B}_j \setminus \mathcal{Z}_j$ returns to $\mathcal{B}_j$ infinitely often. But this implies $\liminf_{n \rightarrow \infty} r_n \leq r_0 + 2 < \infty$. Therefore, the set $\mathcal{Z} = \bigcup_{j \in \mathbb{N}} \mathcal{Z}_j \subset \mathcal{U}$ has all the desired properties.

Step 2: We will show the existence of a subgroup $\Sigma \subset \Omega$ such that E_σ has Lebesgue measure zero for almost all $\sigma \in \Sigma$. Since $\mathcal{E} \subset \mathcal{Z}$ by construction, the inclusion

$$E_\omega = (\psi_\omega \otimes \text{id})^{-1}(\mathcal{E}) \subset (\psi_\omega \otimes \text{id})^{-1}(\mathcal{Z})$$

holds for all $\omega \in \Omega$. To $j \in \mathbb{Z}$ we can consider the restricted flow

$$\Phi_j : \Sigma \times [jS, (j+1)S) \rightarrow \Omega, \quad \Phi_j(\sigma, t) = \sigma \cdot t = \psi_\sigma(t).$$

It is easy to verify that just like $\Phi = \Phi_0$ of Lemma 2.6 those functions are isomorphisms of measure spaces. In other words, Φ_j is bijective up to a set of measure zero, both Φ_j and Φ_j^{-1} are measurable, and for every Borel set $\mathcal{B} \subset \Omega$ we have

$$\mu_\Omega(\mathcal{B}) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi_j^{-1}(\mathcal{B})). \quad (4.3)$$

This clearly implies

$$(\mu_\Omega \otimes \lambda)(B) = \frac{1}{S}(\mu_\Sigma \otimes \lambda^2)(\Phi_j^{-1} \times \text{id})(B) \quad (4.4)$$

for every Borel set $B \subset \Omega \times (0, \infty)$. Let

$$\begin{aligned} C_j &= \{(\sigma, t, r) \in \Sigma \times [jS, (j+1)S) \times (0, \infty) : (\Phi_j(\sigma, t), r) \in \mathcal{Z}\} \\ &= (\Phi_j^{-1} \times \text{id})(\mathcal{Z}). \end{aligned}$$

Since $\mathcal{Z}$ has measure zero, (4.4) yields $(\mu_\Sigma \otimes \lambda^2)(C_j) = 0$. Next we consider the cross sections

$$C_{j,\sigma} = \{(t, r) \in [jS, (j+1)S) \times (0, \infty) : (\sigma, t, r) \in C_j\}.$$

Then, $\lambda^2(C_{j,\sigma}) = 0$ for μ_Σ -almost all $\sigma \in \Sigma$ follows from Fubini's theorem. So for every $j \in \mathbb{Z}$ there is a set $M_j \subset \Sigma$ with $\mu_\Sigma(M_j) = 0$ such that $\lambda^2(C_{j,\sigma}) = 0$ for all $\sigma \in \Sigma \setminus M_j$. Thus $M = \bigcup_{j \in \mathbb{Z}} M_j$ has measure zero as well and

$$\lambda^2\left(\bigcup_{j \in \mathbb{Z}} C_{j,\sigma}\right) = 0$$

for all $\sigma \in \Sigma \setminus M$. But we have

$$\bigcup_{j \in \mathbb{Z}} C_{j,\sigma} = \{(t, r) \in \mathbb{R} \times (0, \infty) : (\psi_\sigma(t), r) \in \mathcal{Z}\} = (\psi_\sigma \times \text{id})^{-1}(\mathcal{Z}),$$

and recalling that $E_\sigma \subset (\psi_\sigma \times \text{id})^{-1}(\mathcal{Z})$, we therefore conclude $\lambda^2(E_\sigma) = 0$ for all $\sigma \in \Sigma \setminus M$.

Step 3: Concluding from Σ to Ω . If we denote by $T_s(t, r) = (t+s, r)$ the translation in time, then clearly

$$f_{\omega \cdot s} = T_{-s} \circ f_\omega \circ T_s \quad \text{on } D_{\omega \cdot s}$$

holds for all $\omega \in \Omega$ and $s \in \mathbb{R}$. But this implies $T_s(E_{\omega \cdot s}) = E_\omega$, since the identity above stays valid under iterations. In particular we have

$$\lambda^2(E_{\omega \cdot s}) = \lambda^2(E_\omega), \quad \forall \omega \in \Omega, s \in \mathbb{R}.$$

Again, we consider the restricted flow $\Phi : \Sigma \times [0, S) \rightarrow \Omega$, $\Phi(\omega, t) = \omega \cdot t$. Using $M \subset \Sigma$ of Step 2 we define $Z_* = \Phi(M \times [0, S)) \subset \Omega$. Then, (4.3) and $\mu_\Sigma(M) = 0$ imply that also Z_* has measure zero. Now let $\omega \in \Omega \setminus Z_*$ be fixed and let $(\sigma, \tau) = \Phi^{-1}(\omega)$. Then $\sigma \in \Sigma \setminus M$ and $\sigma \cdot \tau = \omega$. Therefore, Step 2 implies

$$\lambda^2(E_\omega) = \lambda^2(E_{\sigma \cdot \tau}) = \lambda^2(E_\sigma) = 0,$$

which proves the assertion. $\square$

5 Statement and Proof of the Main Result

We start with a rigorous description of the ping-pong map. To this end, let p be a forcing such that

$$p \in C^2(\mathbb{R}), \quad 0 < a \leq p(t) \leq b \quad \forall t \in \mathbb{R}, \quad \|p\|_{C^2} = \|p\|_\infty + \|\dot{p}\|_\infty + \|\ddot{p}\|_\infty < \infty. \quad (5.1)$$

Now, we consider the map

$$(t_0, v_0) \mapsto (t_1, v_1),$$

which sends a time t_0 of impact to the left plate $x = 0$ and the corresponding velocity $v_0 > 0$ immediately after the impact to their successors t_1 and v_1 describing the subsequent impact to $x = 0$. If we further denote by $\tilde{t} \in (t_0, t_1)$ the time of the particle's impact to the moving plate, then we can determine $\tilde{t} = \tilde{t}(t_0, v_0)$ implicitly through the equation

$$(\tilde{t} - t_0)v_0 = p(\tilde{t}), \quad (5.2)$$

since this relation describes the distance that the particle has to travel before hitting the moving plate. With that we derive a formula for the successor map:

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{v_1}, \quad v_1 = v_0 - 2\dot{p}(\tilde{t}) \quad (5.3)$$

To ensure that this map is well defined, we will assume that

$$v_0 > v_* := 2 \max \left\{ \sup_{t \in \mathbb{R}} \dot{p}(t), 0 \right\}. \quad (5.4)$$

This condition guarantees that v_1 is positive and also implies that there is a unique solution $\tilde{t} = \tilde{t}(t_0, v_0) \in C^1(\mathbb{R} \times (v_*, \infty))$ to (5.2). Thus we can take $\mathbb{R} \times (v_*, \infty)$ as the domain of the ping-pong map (5.3). Now, we are finally ready to state the main theorem.

Theorem 5.1 Assume $0 < a < b$ and $P \in C_{\psi}^2(\Omega)$ are such that

$$a \leq P(\omega) \leq b \quad \forall \omega \in \Omega. \quad (5.5)$$

Consider the family $\{p_{\omega}\}_{\omega \in \Omega}$ of almost periodic forcing functions defined by

$$p_{\omega}(t) = P(\omega + \psi(t)), \quad t \in \mathbb{R}. \quad (5.6)$$

Let $v_* = 2 \max\{\max_{\varpi \in \Omega} \partial_{\psi} P(\varpi), 0\}$ and denote by

$$E_{\omega} = \left\{ (t_0, v_0) \in \mathbb{R} \times (v_*, \infty) : (t_n, v_n)_{n \in \mathbb{N}} \text{ is well defined and } \lim_{n \rightarrow \infty} v_n = \infty \right\}$$

the escaping set for the ping-pong map with forcing function $p(t) = p_{\omega}(t)$. Then, for almost all $\omega \in \Omega$, the set $E_{\omega} \subset \mathbb{R}^2$ has Lebesgue measure zero.

Remark 5.2 The notation $v_* = 2 \max\{\max_{\varpi \in \Omega} \partial_{\psi} P(\varpi), 0\}$ is consistent with (5.4), since for every $\omega \in \Omega$ the set $\omega \cdot \mathbb{R}$ lies dense in Ω and thus

$$\sup_{t \in \mathbb{R}} \dot{p}_{\omega}(t) = \sup_{t \in \mathbb{R}} \partial_{\psi} P(\omega + \psi(t)) = \max_{\varpi \in \Omega} \partial_{\psi} P(\varpi).$$

We will give some further preliminaries before starting the actual proof. First we note, that the ping-pong map $(t_0, v_0) \mapsto (t_1, v_1)$ is not symplectic. To remedy this defect, we reformulate the model in terms of time t and energy $E = \frac{1}{2}v^2$. In these new coordinates the ping-pong map becomes

$$\mathcal{P} : (t_0, E_0) \mapsto (t_1, E_1), \quad (5.7)$$

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{\sqrt{2E_1}}, \quad E_1 = E_0 - 2\sqrt{2E_0}\dot{p}(\tilde{t}) + 2\dot{p}(\tilde{t})^2 = \left(\sqrt{E_0} - \sqrt{2}\dot{p}(\tilde{t})\right)^2, \quad (5.8)$$

where $\tilde{t} = \tilde{t}(t_0, E_0)$ is determined implicitly through the relation $\tilde{t} = t_0 + \frac{p(\tilde{t})}{\sqrt{2E_0}}$. This map is defined for $(t_0, E_0) \in \mathbb{R} \times (\frac{1}{2}v_*^2, \infty)$. Since it has a generating function [15, Lemma 3.7], it is measure-preserving. Furthermore, from the inverse function theorem we can derive that $\mathcal{P}$ is locally injective. Note however, that in general $\mathcal{P}$ fails to be injective globally (see Appendix 6.2).

Now, we will demonstrate that $W(t_0, E_0) = p(t_0)^2 E_0$ acts as an adiabatic invariant for the ping-pong map. For this purpose we will cite the following lemma [16, Lemma 5.1]:

Lemma 5.3 There is a constant $C > 0$, depending only upon $\|p\|_{C^2}$ and $a, b > 0$ from (5.1), such that

$$|p(t_1)^2 E_1 - p(t_0)^2 E_0| \leq C \Delta(t_0, E_0) \quad \forall (t_0, E_0) \in \mathbb{R} \times (v_*^2/2, \infty),$$

where $(t_1, E_1) = \mathcal{P}(t_0, E_0)$ denotes the ping-pong map for the forcing p , and $\Delta(t_0, E_0) = E_0^{-1/2} + \sup\{|\ddot{p}(t) - \ddot{p}(s)| : t, s \in [t_0 - C, t_0 + C], |t - s| \leq C E_0^{-1/2}\}$.

So far we have depicted the case of a general forcing function p . Now we will replace $p(t)$ by $p_\omega(t)$ from (5.6) and study the resulting ping-pong map. First we note that due to $P \in C_\psi^2(\Omega)$ we have $p_\omega \in C^2(\mathbb{R})$. Also $0 < a \leq p_\omega(t) \leq b$ holds for all $\omega \in \Omega$ by assumption. Furthermore, since $\omega \cdot \mathbb{R}$ lies dense in Ω it is

$$\|p_\omega\|_\infty = \|P\|_\infty, \quad \|\dot{p}_\omega\|_\infty = \|\partial_\psi P\|_\infty, \quad \|\ddot{p}_\omega\|_\infty = \|\partial_\psi^2 P\|_\infty.$$

In particular this means $\|p_\omega\|_{C^2(\mathbb{R})} = \|P\|_{C_\psi^2(\Omega)}$ for all $\omega \in \Omega$. Therefore all considerations above apply with uniform constants. As depicted in Remark 5.2, also the threshold $v_* = 2 \max\{\max_{\varpi \in \Omega} \partial_\psi P(\varpi), 0\}$ is uniform in ω . Finally, since $\ddot{p}_\omega(t) = \partial_\psi^2 P(\omega + \psi(t))$, the function $\Delta(t_0, E_0)$ can be uniformly bounded by

$$\Delta(E_0) = E_0^{-1/2} + \sup\{|\partial_\psi^2 P(\varpi) - \partial_\psi^2 P(\varpi')| : \varpi, \varpi' \in \Omega, \|\varpi - \varpi'\| \leq C E_0^{-1/2}\}.$$

Hence, from Lemma 5.3 we obtain

Lemma 5.4 *There is a constant $C > 0$, uniform in $\omega \in \Omega$, such that*

$$|p(t_1)^2 E_1 - p(t_0)^2 E_0| \leq C \Delta(E_0) \quad \forall (t_0, E_0) \in \mathbb{R} \times (v_*^2/2, \infty),$$

where $(t_0, E_0) \mapsto (t_1, E_1)$ denotes the ping-pong map $\mathcal{P}$ for the forcing function $p_\omega(t)$.

Consider the equation

$$\tau = \frac{1}{\sqrt{2E_0}} P(\omega_0 + \psi(\tau)). \quad (5.9)$$

Since $P \in C_\psi^1(\Omega)$ and $1 - (2E_0)^{-1/2} \partial_\psi P(\omega_0 + \psi(\tau)) \geq \frac{1}{2} > 0$ for $E_0 > \frac{1}{2} v_*^2$, equation (5.9) can be solved implicitly for $\tau = \tau(\omega_0, E_0) \in C(\Omega \times (v_*^2/2, \infty))$ (cf. [2] for a suitable implicit function theorem). For $\omega \in \Omega$ and $t_0 \in \mathbb{R}$ one can consider (5.9) with $\omega_0 = \omega + \psi(t_0)$. Then $P \in C_\psi^1(\Omega)$ and the classical implicit function theorem yield $\tau \in C_\psi^1(\Omega \times (v_*^2/2, \infty))$. Moreover, comparing this to the definition of $\tilde{t}$, we observe the following relation:

$$\tilde{t}(t_0, E_0) = t_0 + \tau(\omega + \psi(t_0), E_0). \quad (5.10)$$

Now we will give the proof of the main theorem, in which we will link the ping-pong map corresponding to $p_\omega(t)$ to the setup of Sect. 3.

Proof of Theorem 5.1 Let $\mathcal{D} = \Omega \times (E^*, \infty)$, where $E^* = \max\{\frac{1}{2} v_*^2, E_{**}\}$ and E_{**} will be determined below. Consider $f : \mathcal{D} \subset \Omega \times (0, \infty) \rightarrow \Omega \times (0, \infty)$, $f(\omega_0, E_0) = (\omega_1, E_1)$, given by

$$\omega_1 = \omega_0 + \psi(F(\omega_0, E_0)), \quad E_1 = E_0 + G(\omega_0, E_0),$$

where

$$F(\omega_0, E_0) = \left(\frac{1}{\sqrt{2E_0}} + \frac{1}{\sqrt{2E_1}} \right) P(\omega_0 + \psi(\tau)),$$

$$G(\omega_0, E_0) = -2\sqrt{2E_0} \partial_\psi P(\omega_0 + \psi(\tau)) + 2\partial_\psi P(\omega_0 + \psi(\tau))^2,$$

for $\tau = \tau(\omega_0, E_0)$. Then f has special form (3.2) and therefore we can study the family $\{f_\omega\}_{\omega \in \Omega}$ of planar maps defined by (3.3). But plugging (5.10) into the definition of $\mathcal{P}$ shows, that f_ω is just the ping-pong map $\mathcal{P}$ in the case of the forcing $p_\omega(t)$. Independently of ω , these maps are defined on $D_\omega = (\psi_\omega \times \text{id})^{-1}(\mathcal{D}) = \mathbb{R} \times (E^*, \infty)$.

Let us show that f is injective on $\Omega \times (E_{**}, \infty)$, if E_{**} is sufficiently large. Therefore suppose $f(\omega_0, E_0) = (\omega_1, E_1) = f(\tilde{\omega}_0, \tilde{E}_0)$. Since $\omega_0 + \iota(F(\omega_0, E_0)) = \tilde{\omega}_0 + \iota(F(\tilde{\omega}_0, \tilde{E}_0))$ there is $\omega \in \Omega$ and $t_0, \tilde{t}_0 \in \mathbb{R}$ such that $\omega_0 = \omega + \psi(t_0)$ and $\tilde{\omega}_0 = \omega + \psi(\tilde{t}_0)$. Implicit differentiation yields $\partial_{t_0} \tau(\omega + \psi(t_0), E_0) = O(E_0^{-1/2})$ and $\partial_{E_0} \tau(\omega + \psi(t_0), E_0) = O(E_0^{-3/2})$. Moreover, $E_1 = O(E_0)$ implies

$$D_{f_\omega}(t_0, E_0) = \begin{pmatrix} 1 + O(E_0^{-1/2}) & O(E_0^{-3/2}) \\ O(E_0^{1/2}) & 1 + O(E_0^{-1/2}) \end{pmatrix}$$

for the Jacobian matrix of f_ω . Throughout this paragraph C will denote positive constants depending on E_{**} and $\|P\|_{C_\psi^2(\Omega)}$, which will not be further specified. Without loss of generality we may assume $E_0 \leq \tilde{E}_0$. Then, applying the mean value theorem yields $|t_0 - \tilde{t}_0| \leq C E_0^{-1/2} |t_0 - \tilde{t}_0| + C E_0^{-3/2} |E_0 - \tilde{E}_0|$ and $|E_0 - \tilde{E}_0| \leq C \tilde{E}_0^{1/2} |t_0 - \tilde{t}_0| + C E_0^{-1/2} |E_0 - \tilde{E}_0|$, provided E_{**} is sufficiently big. Thus, for large E_{**} we get $|t_0 - \tilde{t}_0| \leq C E_0^{-3/2} |E_0 - \tilde{E}_0|$ and $|E_0 - \tilde{E}_0| \leq C \tilde{E}_0^{1/2} |t_0 - \tilde{t}_0|$. Now, combining these inequalities gives us $|t_0 - \tilde{t}_0| \leq C E_0^{-3/2} \tilde{E}_0^{1/2} |t_0 - \tilde{t}_0|$. But since $E_1 = O(E_0)$ and also $\tilde{E}_0 = O(E_1)$, we can conclude $|t_0 - \tilde{t}_0| \leq C E_0^{-1} |t_0 - \tilde{t}_0|$. In turn, this implies $t_0 = \tilde{t}_0$ and $E_0 = \tilde{E}_0$ for E_{**} sufficiently large, which proves the injectivity of f_ω and f .

Next we want to show that f is also measure-preserving. To this end, consider the maps $g : \Sigma \times [0, S) \times (E^*, \infty) \rightarrow \Sigma \times [0, \infty) \times (0, \infty)$ defined by

$$g(\sigma, s, E) = (\sigma, f_\sigma(s, E))$$

and $\chi : \Sigma \times [0, \infty) \rightarrow \Sigma \times [0, S)$, $\chi(\sigma, t) = \Phi^{-1}(\sigma \cdot t)$ from (2.5). Then, the identity

$$f = (\Phi \times \text{id}) \circ (\chi \times \text{id}) \circ g \circ (\Phi^{-1} \times \text{id})$$

holds on $\mathcal{D}$. This can be illustrated as follows:

$$\begin{array}{ccccc} (\omega_0, E_0) & \xrightarrow{f} & (\omega_1, E_1) & & \\ \Phi^{-1} \times \text{id} \downarrow & & & \nwarrow \Phi \times \text{id} & \\ (\sigma_0, s_0, E_0) & \xrightarrow{g} & (\sigma_0, s_1, E_1) & \xrightarrow{\chi \times \text{id}} & (\sigma_1, s'_1, E_1) \end{array}$$

Recalling Lemma 2.6 and the fact that f_ω has a generating function, it suffices to show that $\chi \times \text{id}$ preserves the measure of any Borel set $\mathcal{B} \subset g((\Phi^{-1} \times \text{id})(\mathcal{D}))$. Therefore, consider the sets

$$\mathcal{B}_k = \mathcal{B} \cap (\Sigma \times [(k-1)S, kS) \times (0, \infty)), \quad k \in \mathbb{N}.$$

Then we have

$$(\mu_\Sigma \otimes \lambda^2)((\chi \times \text{id})(\mathcal{B}_k)) = (\mu_\Sigma \otimes \lambda^2)(\mathcal{B}_k),$$

as depicted in Sect. 2.3. Moreover, the injectivity of f implies the injectivity of $\chi \times \text{id}$ on $\mathcal{B}$ and thus the sets $(\chi \times \text{id})(\mathcal{B}_k)$ are mutually disjoint. Since $\mathcal{B} = \cup_{k \in \mathbb{N}} \mathcal{B}_k$, this yields $(\mu_\Sigma \otimes \lambda^2)((\chi \times \text{id})(\mathcal{B})) = (\mu_\Sigma \otimes \lambda^2)(\mathcal{B})$.

Finally, we need to find a function $W \in C^1_\psi(\Omega \times (0, \infty))$ such that (3.4) and (3.5) are verified. For this define

$$W(\omega_0, E_0) = P(\omega_0)^2 E_0.$$

Conditions (3.4) clearly holds if we take $\beta = a^2$ and $\delta = b^2$ with a, b from (5.5). Moreover, the definition of f yields

$$\begin{aligned} W(f(\omega_0, E_0)) - W(\omega_0, E_0) &= P(\omega_1)^2 E_1 - P(\omega_0)^2 E_0 \\ &= P(\omega_0 + \psi(F(\omega_0, E_0)))^2 E_1 - P(\omega_0)^2 E_0 \\ &= p_{\omega_0}(F(\omega_0, E_0))^2 E_1 - p_{\omega_0}(0)^2 E_0. \end{aligned}$$

Now let $t_0 = 0$ and $(t_1, E_1) = f_{\omega_0}(t_0, E_0)$. Then $t_1 = F(\omega_0, E_0)$ and thus Lemma 5.4 yields

$$W(f(\omega_0, E_0)) - W(\omega_0, E_0) = p_{\omega_0}(t_1)^2 E_1 - p_{\omega_0}(t_0)^2 E_0 \leq C \Delta(E_0),$$

where $C > 0$ is uniform in ω_0 . But then taking $k(E_0) = C \Delta(E_0)$ proves (3.5), since $\lim_{r \rightarrow \infty} \Delta(r) = 0$ follows from $\partial_\psi^2 P \in C(\Omega)$.

Now we have validated all conditions of Theorem 3.1 for the map $f : \mathcal{D} \rightarrow \Omega \times (0, \infty)$. Applying it yields $\lambda^2(\hat{E}_\omega) = 0$ for almost all $\omega \in \Omega$, where $\hat{E}_\omega = \{(t_0, E_0) \in \hat{D}_{\omega, \infty} : \lim_{n \rightarrow \infty} E_n = \infty\}$ and $\hat{D}_{\omega, \infty}$ is defined as in Sect. 3.2. This can be translated back to the original coordinates $(t, v) = (t, \sqrt{2E})$: Let us denote by g_ω the ping-pong map $(t_0, v_0) \mapsto (t_1, v_1)$ from (5.3) for the forcing $p(t) = p_\omega(t)$ and let

$$\tilde{D}_\omega = \mathbb{R} \times (\sqrt{2E^*}, \infty), \quad \tilde{D}_{\omega,1} = \tilde{D}_\omega, \quad \tilde{D}_{\omega,n+1} = g_\omega^{-1}(\tilde{D}_{\omega,n}), \quad \tilde{D}_{\omega,\infty} = \bigcap_{n=1}^{\infty} \tilde{D}_{\omega,n}.$$

Then $\lambda^2(\tilde{E}_\omega) = 0$ for almost all $\omega \in \Omega$, where $\tilde{E}_\omega = \{(t_0, v_0) \in \tilde{D}_{\omega,\infty} : \lim_{n \rightarrow \infty} v_n = \infty\}$. Now, consider the escaping set E_ω from the theorem and take

$(t_0, v_0) \in E_\omega$. Since $\lim_{n \rightarrow \infty} v_n = \infty$, there is $n_0 \in \mathbb{N}$ such that $v_n > \sqrt{2E^*}$ for all $n \geq n_0$. But this just means $(t_n, v_n) \in \tilde{E}_\omega$ for $n \geq n_0$. In particular, this implies $E_\omega \subset \bigcup_{n \in \mathbb{N}} g_\omega^{-n}(\tilde{E}_\omega)$. Considering that g_ω is area-preserving, this proves the assertion: $\lambda^2(E_\omega) = 0$ for almost all $\omega \in \Omega$. $\square$

Remark 5.5 Let us also point out that the framework developed in the present paper can be applied to a lot of other dynamical systems. A famous example of such a system is given by the so called Littlewood boundedness problem. There, the question is whether solutions of an equation $\ddot{x} + G'(x) = p(t)$ stay bounded in the $(x, \dot{x})$ -phase space if the potential G satisfies some superlinearity condition. In [24] it is shown that the associated escaping set E typically has Lebesgue measure zero for $G'(x) = |x|^{\alpha-1}x$ with $\alpha \geq 3$ and a quasi-periodic forcing function $p(t)$. Indeed, this result can be improved to the almost periodic case in a way analogous to the one presented here (for the ping-pong problem).

Acknowledgements The funding was provided by Deutsche Forschungsgemeinschaft (CRC/TRR 191).

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

6 Appendix

6.1 The Hull of an Almost Periodic Function

Lemma 6.1 *Let $u \in C(\mathbb{R})$ be almost periodic. If the sequences $\{u_{\tau_n}\}, \{u_{s_n}\}$ are uniformly convergent, then $\{u_{\tau_n - s_n}\}$ is uniformly convergent as well.*

Proof Let $\epsilon > 0$ be given. Since $\{u_{\tau_n}\}, \{u_{s_n}\}$ are Cauchy sequences, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$|u_{\tau_n}(-s_n + t) - u_{\tau_m}(-s_n + t)| < \frac{\epsilon}{2}$$

and

$$|u_{s_n}(\tau_m - s_n - s_m + t) - u_{s_m}(\tau_m - s_n - s_m + t)| < \frac{\epsilon}{2},$$

where $t \in \mathbb{R}$ is arbitrary. Together this yields

$$|u(\tau_n - s_n + t) - u(\tau_m - s_m + t)| < \epsilon.$$

for all $n, m \geq N$ and $t \in \mathbb{R}$, and thus proves the assertion. $\square$

6.2 Ping-Pong Map

The map $\mathcal{P}$ from (5.7) can fail to be injective globally. For this, suppose there are $\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}$ with $\tilde{t}_1 < \tilde{t}_2$ such that the derivative $\dot{p}(t)$ reaches its maximum at both $\tilde{t}_1$ and $\tilde{t}_2$, and moreover $p(\tilde{t}_1) > p(\tilde{t}_2)$. For the sake of simplicity, let us consider the original coordinates (t, v) . Let $v_1 > 0$ be the unique number so that $\tilde{t}_1 + \frac{p(\tilde{t}_1)}{v_1} = \tilde{t}_2 + \frac{p(\tilde{t}_2)}{v_1}$. Now, we define $v_0 = v_1 + 2\dot{p}(\tilde{t}_1) = v_1 + 2\dot{p}(\tilde{t}_2)$ and $t_{0,i} = \tilde{t}_i - \frac{p(\tilde{t}_i)}{v_1}$ for $i = 1, 2$. From $p(\tilde{t}_1) > p(\tilde{t}_2)$ we can derive $t_{0,1} < t_{0,2}$. But $v_0 = v_1 + 2 \sup_{t \in \mathbb{R}} \dot{p}(t) > v_*$ implies that $(t_{0,i}, v_0)$ are in the domain of $\mathcal{P}$ and furthermore $\mathcal{P}(t_{0,i}, v_0) = (t_1, v_1)$, where $t_1 = \tilde{t}_1 + \frac{p(\tilde{t}_1)}{v_1}$.

References

1. Besicovitch, A.S.: On generalized almost periodic functions. *Proc. Lond. Math. Soc.* **s2-25**(1), 495–512 (1926)
2. Biasi, C., Gutierrez, C., dos Santos, E.L.: The implicit function theorem for continuous functions. *Topol. Methods Nonlinear Anal.* **32**(1), 177–185 (2008)
3. Bell, H., Meyer, K.: Limit periodic functions, adding machines, and solenoids. *J. Dyn. Differ. Equat.* **7**, 409–422 (1995)
4. Bochner, S.: Beiträge zur Theorie der fastperiodischen Funktionen. *Math. Ann.* **96**(1), 119–147 (1927)
5. Bohr, H.: Zur Theorie der fast periodischen Funktionen. *Acta Math.* **45**, 29–127 (1925)
6. Chulaevsky, V.A.: *Almost Periodic Operators and Related Nonlinear Integrable Systems*. Manchester University Press, Manchester (1989)
7. Campos, J., Tarallo, M.: Nonmonotone equations with large almost periodic forcing terms. *J. Differ. Equ.* **254**(2), 686–724 (2013)
8. Dolgopyat, D.: Lectures on Bouncing Balls. <https://www.math.umd.edu/~dolgov/BBNotes2.pdf>. Online; Accessed 6 Aug 2019
9. Dolgopyat, D.: Bouncing balls in non-linear potentials. *Discrete Contin. Dynam. Syst.* **22**(1), 165–182 (2008)
10. Dolgopyat, D.: *Geometric and Probabilistic Structures in Dynamics*, chapter Fermi acceleration, pp. 149–166. AMS, Providence (2008)
11. Dolgopyat, D., de Simoi, J.: Dynamics of some piecewise smooth Fermi-Ulam models. *Chaos: An Interdisciplinary Journal of Nonlinear Science* **22**(2), 026124 (2012)
12. Einsiedler, M., Ward, T.: *Ergodic Theory*. Springer, London (2011)
13. Fermi, E.: On the origin of the cosmic radiation. *Phys. Rev.* **75**, 1169–1174 (1949)
14. Hewitt, E., Ross, K.: *Abstract Harmonic Analysis I*. Springer, New York (1979)
15. Kunze, M., Ortega, R.: Complete orbits for twist maps on the plane: extensions and applications. *J. Dyn. Differ. Equ.* **23**(3), 405–423 (2010)
16. Kunze, M., Ortega, R.: Escaping orbits are rare in the quasi-periodic Fermi-Ulam ping-pong. *Ergodic Theory Dyn. Syst.* 1–17 (2018)
17. Laederich, S., Levi, M.: Invariant curves and time-dependent potentials. *Ergodic Theory Dyn. Syst.* **11**(02), (1991)
18. Moser, J.: On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen* **II**, 1–20 (1962)
19. Meyer, K., Sell, G.: Melnikov transforms, bernoulli bundles, and almost periodic perturbations. *Trans. Am. Math. Soc.* **314**, 07 (1989)
20. Nemytskii, V.V., Stepanov, V.V.: *Qualitative Theory of Differential Equations*. Princeton Univ Press, Princeton (1960)

21. Ortega, R., Tarallo, M.: Almost periodic linear differential equations with non-separated solutions. *J. Funct. Anal.* **237**(2), 402–426 (2006)
22. Pontryagin, L.S.: *Topological Groups*. Gordon & Breach, New York (1966)
23. L.D. Pustyl'nikov. On Ulam's problem. *Theor. Math. Phys.* **57**(1), 1035–1038 (1983)
24. Schließauf, H.: Escaping orbits are rare in the quasi-periodic Littlewood boundedness problem. *Non-linear Differ. Equ. Appl. NoDEA*, **26**, (2019)
25. de Simoi, J.: Fermi acceleration in anti-integrable limits of the standard map. *Commun. Math. Phys.* **321**(3), 703–745 (2013)
26. Stepanoff, W.: Über einige Verallgemeinerungen der fast periodischen Funktionen. *Mathematische Annalen*, **95**(1), 473–498 (1926)
27. Ulam, S.M.: On Some Statistical Properties of Dynamical Systems. In: *Proceedings of the Fourth Berkeley Symposium on Math. Statistics and Probability, Volume 3: Contributions to Astronomy, Meteorology, and Physics*, pp. 315–320. University of California Press, Berkeley (1961)
28. Weyl, H.: Integralgleichungen und fastperiodische Funktionen. *Math. Ann.* **97**(1), 338–356 (1927)
29. Zharnitsky, V.: Instability in Fermi–Ulam ping-pong problem. *Nonlinearity* **11**(6), 1481–1487 (1998)
30. Zharnitsky, V.: Invariant curve theorem for quasiperiodic twist mappings and stability of motion in Fermi–Ulam problem. *Nonlinearity* **13**(4), 1123–1136 (2000)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.